Friedrich-Alexander-Universität
Technische Fakultät

Machine Learning
Data Analytics

FAU

Kalman Filter

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MLTS Exercise, 21.11.2024

MLTS Exercise – Organization

~~Introduction (31.10.2024)~~

~~Bayesian Linear Regression (07.11.2024)~~

~~Bayesian Linear Regression (14.11.2024)~~

Kalman Filter (21.11.2024)

Kalman Filter (28.11.2024)

Dynamic Time Warping (05.12.2024)

Dynamic Time Warping (12.12.2024)

No exercise planned (19.12.2024)

Holiday

RNN + LSTM (09.01.2025)

RNN + LSTM (16.01.2025)

Transformers (23.01.2025)

Transformers (30.01.2025)

Kalman Filter Overview

Kalman filters (KF) are used to estimate the state of a dynamical system

- Trajectory estimation
- Indirect measurements (Rocket thruster temperature)

Assumptions:

- Linear Gaussian system with additive noise
  - Optimal estimator

Kalman Filter Overview

Two important steps:

1. Prediction
    - Given an initial state, we leverage our knowledge of the process to produce an estimate of the current state – along with its uncertainty

2. Correction (or Filtering)
    - We update our belief based on new sensory information

Handy example - Prediction

x is the position of a rider

$\dot{x}$ is the rider‘s velocity

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$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ is the system‘s state

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Process model:

We can write down the Newtonian laws of motion for the system

$$x_n = x_{n-1} + \dot{x}_{n-1}\Delta t + \frac{1}{2}\frac{f}{m}(\Delta t)^2 + r_{1_n}$$

$$\dot{x}_n = \dot{x}_{n-1} + \frac{f}{m}\Delta t + r_{2_n}$$

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$$\begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ \dot{x}_{n-1} \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} f + \begin{bmatrix} r_{1_n} \\ r_{2_n} \end{bmatrix}$$

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$$z_n = Fz_{n-1} + Bu_n + r_n$$

r_n Gaussian Noise
F State Transition Matrix
B Additional Information

Handy example - Filtering

x is the position of a rider
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$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ is the system‘s state

Measurement model:

Let's assume that we measure the position directly and that this measurement is subject to random noise

$$y_n = x_n + q_n$$

Handy example - Filtering

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Let's assume that we measure the position directly and that this measurement is subject to random noise

$$y_n = x_n + q_n$$

$$y_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ \dot{x_n} \end{bmatrix} + \begin{bmatrix} q_{1_n} \\ q_{2_n} \end{bmatrix}$$

Handy example - Filtering

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$$y_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ \dot{x_n} \end{bmatrix} + \begin{bmatrix} q_{1_n} \\ q_{2_n} \end{bmatrix}$$

$$y_n = H z_n + q_n$$

Handy example - Initialization

x is the position of a rider

$\dot{x}$ is the rider‘s velocity

$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ is the system‘s state

Initial conditions:

We need to define initial conditions for the state and the error covariance

$$z_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \qquad \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice: *We also need to define covariance matrices associated with r and q.*

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Algorithmic view

- **Prediction (Time update)**

$$\bar{z}_n = F_n z_{n-1} + B_n u_n$$

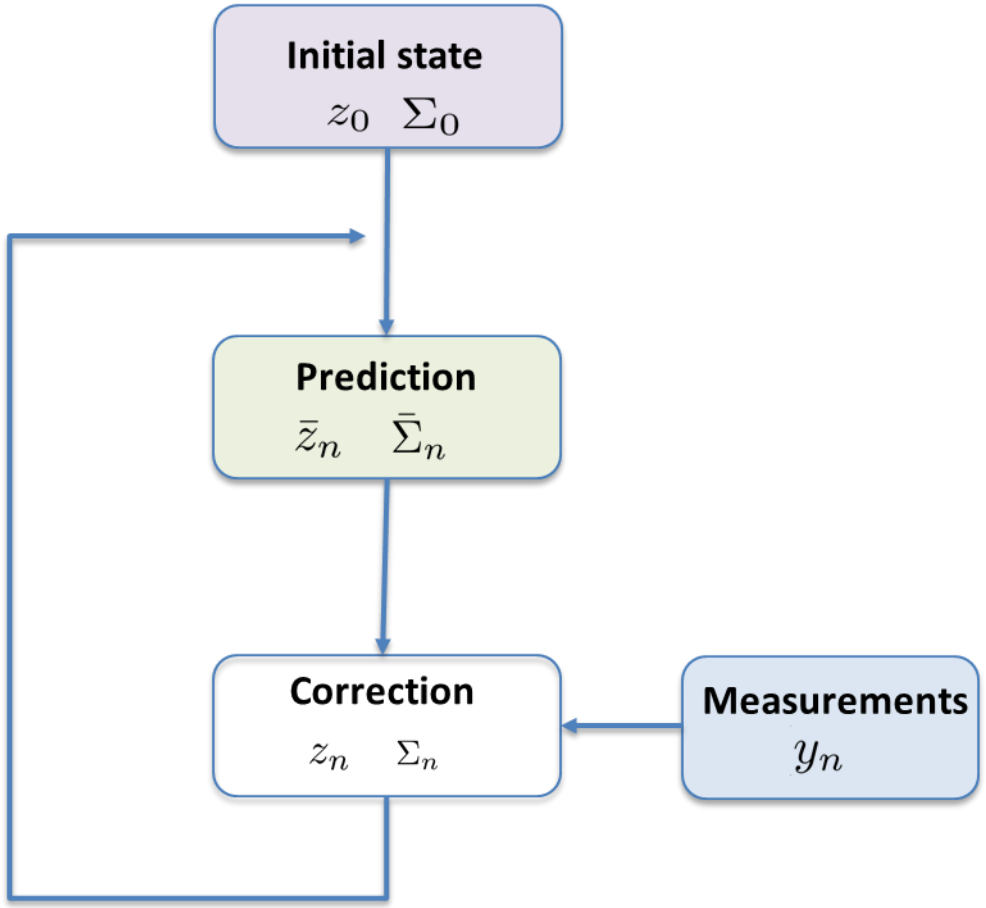
$$\bar{\Sigma}_n = F_n \Sigma_{n-1} F_n^T + R_n$$

- **Correction (Measurements update)**

$$K_n = \bar{\Sigma}_n H_n^T \left(H_n \bar{\Sigma}_n H_n^T + Q_n \right)^{-1}$$

$$z_n = \bar{z}_n + K_n \left(y_n - H_n \bar{z}_n \right)$$

$$\Sigma_n = \left(1 - K_n H_n \right) \bar{\Sigma}_n$$



Practice Questions

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Does the prediction step of the Kalman Filter increases or decreases the uncertainty of the state estimate?

Practice Questions

Does the correction step of the Kalman Filter increases or decreases the uncertainty of the state estimate?

Algorithmic view

- **Prediction (Time update)**

$$\bar{z}_n = F_n z_{n-1} + B_n u_n$$

$$\bar{\Sigma}_n = F_n \Sigma_{n-1} F_n^T + R_n$$

- **Correction (Measurements update)**

$$K_n = \bar{\Sigma}_n H_n^T \left(H_n \bar{\Sigma}_n H_n^T + Q_n \right)^{-1}$$

$$z_n = \bar{z}_n + K_n \left(y_n - H_n \bar{z}_n \right)$$

$$\Sigma_n = \left(1 - K_n H_n \right) \bar{\Sigma}_n$$

$\bar{z}$:	a priori state
$\bar{\Sigma}$:	a priori covariance
F:	state transition matrix
B:	user input transformation matrix
u:	user input
R:	process noise
z:	a posteriori state
Σ:	a posteriori covariance
K:	Kalman gain (intermediate variable)
H:	measurement matrix
Q:	measurement noise
y:	measurements

Kalman Filter Limitations

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Remember KF assumptions!

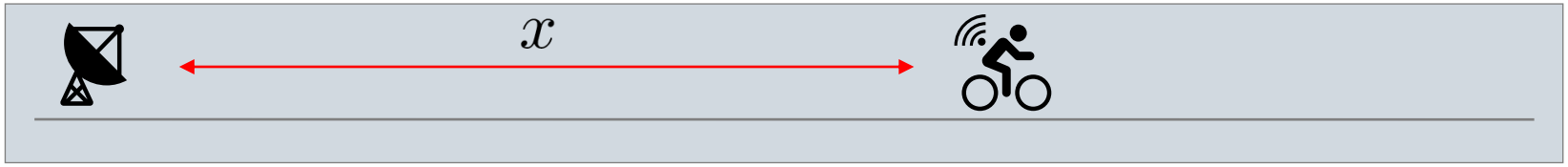
- Linear state transition model
- Linear measurement model
- Gaussian noise

If this assumptions do not hold, we apply other methods:

- Extended Kalman Filter (non-linear models, Gaussian noise)
- Particle Filtering (non-linear models, any noise distribution)

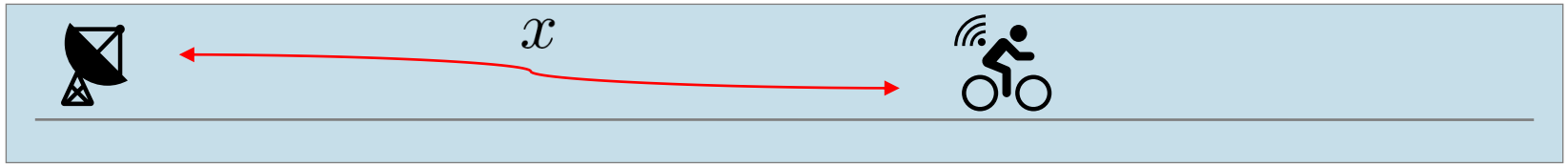
Extended Kalman Filter (EKF)

Extended Kalman Filter Example



$$z_n = F z_{n-1} + B u_n + r_n$$

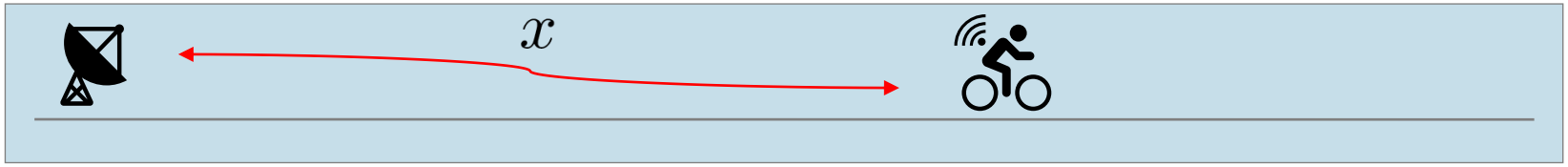
$$y_n = H z_n + q_n$$



$$z_n = f(z_{n-1}, u_n) + r_n$$

$$y_n = h(z_n) + q_n$$

Extended Kalman Filter Example

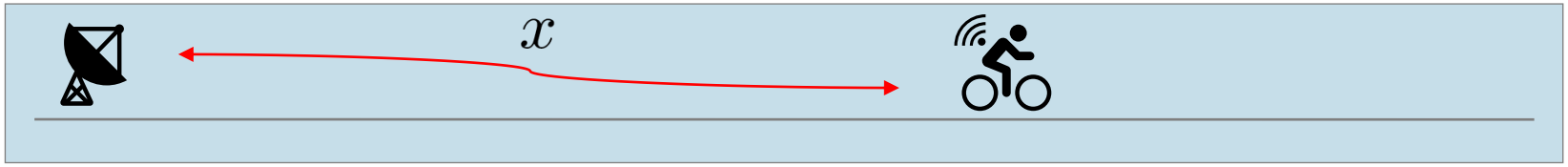


$$z_n = f(z_{n-1}, u_n) + r_n$$

$$y_n = h(z_n) + q_n$$

Assumption: Non-linear but differentiable State-transition model and/or Measurement model

Extended Kalman Filter Example



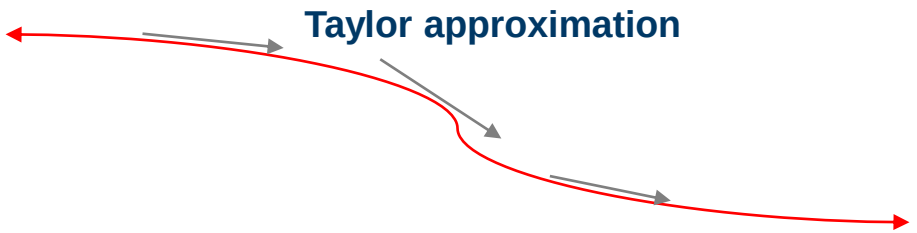
$$z_n = f(z_{n-1}, u_n) + r_n$$

$$y_n = h(z_n) + q_n$$

Linearization with 1st order Taylor Expansion around the current estimates:

$$J_{n-1}^f = \nabla f|_{z_{n-1}, u_n}$$

$$J_n^h = \nabla h|_{z_n}$$



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Algorithmic view

Prediction (Time update)

$$\bar{z}_n = f(z_{n-1}, u_n)$$

$$\bar{\Sigma}_n = J_n^f \Sigma_{n-1} J_n^{f\,T} + R_n$$

Correction (Measurements update)

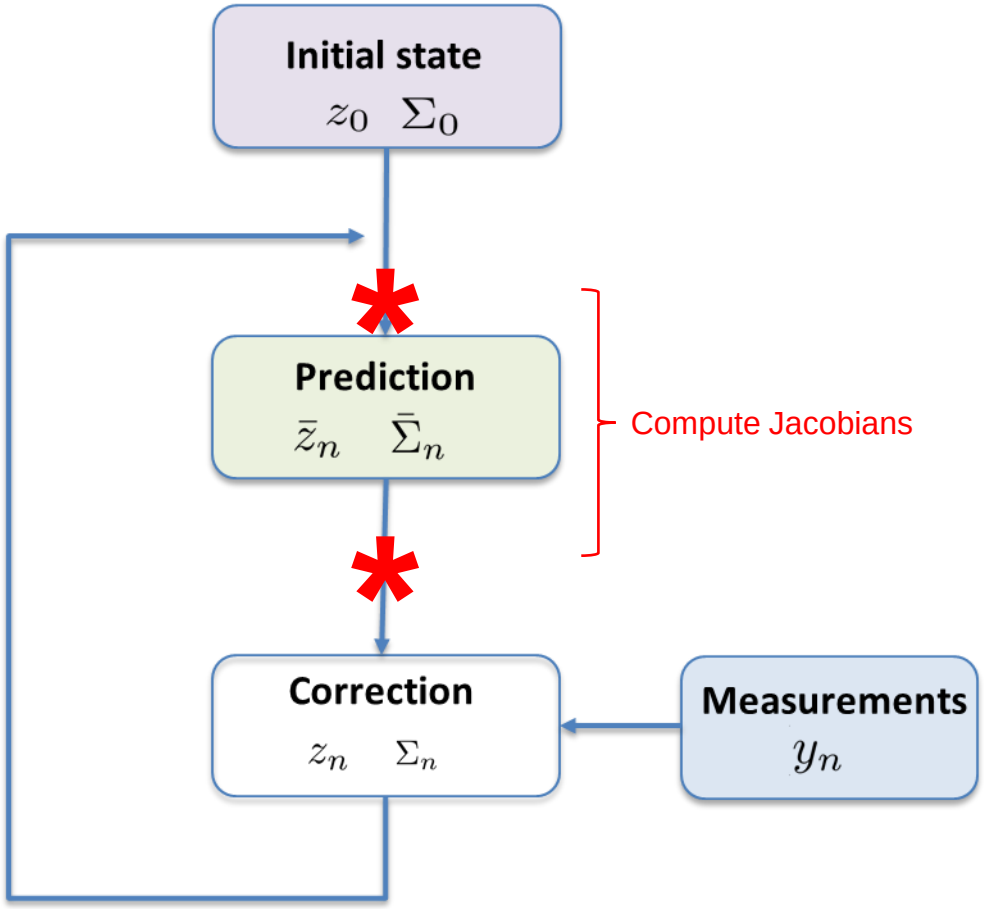
$$K_n = \bar{\Sigma}_n J_n^{h\,T} \left(J_n^h \bar{\Sigma}_n J_n^{h\,T} + Q_n \right)^{-1}$$

$$z_n = \bar{z}_n + K_n \left(y_n - h(\bar{z}_n) \right)$$

$$\Sigma_n = \left(1 - K_n J_n^h \right) \bar{\Sigma}_n$$

$\bar{z}$:	a priori state
$\bar{\Sigma}$:	a priori covariance
f:	non-linear state transition function
J^f:	Jacobian of f
R:	process noise
z:	a posteriori state
Σ:	a posteriori covariance
K:	Kalman gain (intermediate variable)
h:	non-linear measurement function
J^h:	Jacobian of h
Q:	measurement noise
y:	measurements

Algorithmic view



Advantages:

- Deals with non-linear state-transition model and/or measurement model

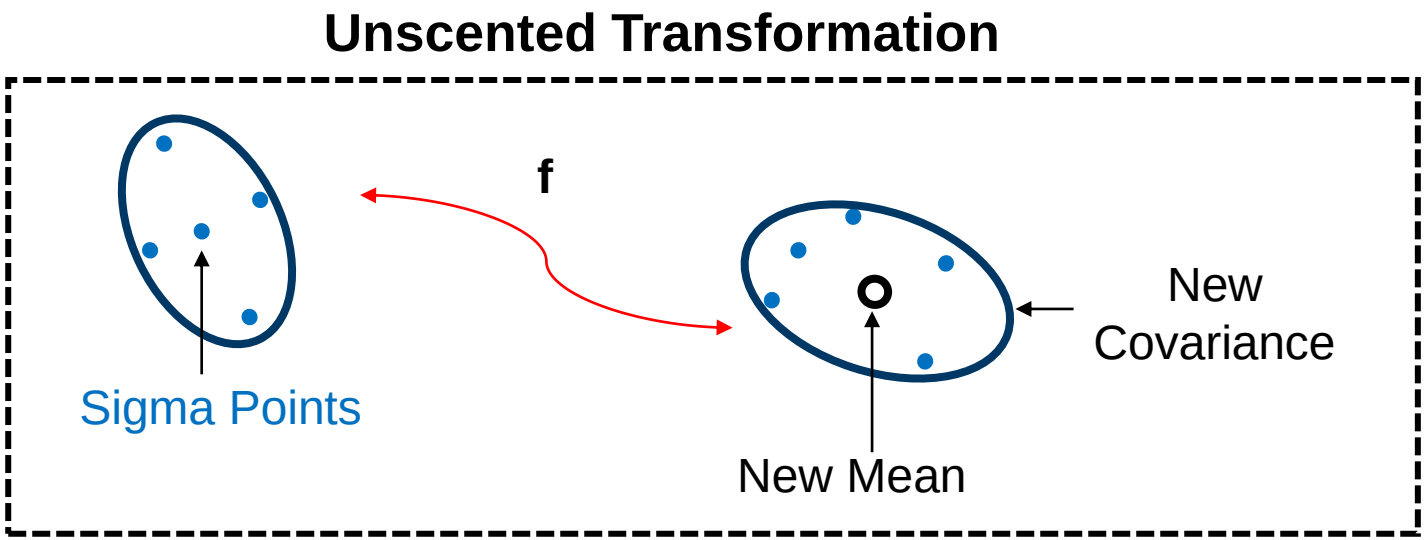
Disadvantages:

- Higher computational cost
- Non-optimal estimator (optimal only in the linear case, as for the KF)
- If the initial estimate is wrong, the system may diverge soon
- It does not work if the models are highly non-linear

Unscented Kalman Filter (UKF)

Unscented Kalman Filter

1. Makes use of deterministic sampling technique, namely **unscented transformation**
   - Pick up minimal sample of **sigma points**
2. Sigma points are **propagated** through a non-linear function **f**
   - New mean and covariance estimates



Sigma points (simplest choice)

$$\{s^0, ..., s^{2D}\}_{n-1}$$

$$s^0_{n-1} = z_{n-1}$$

$$s^i_{n-1} = z_{n-1} + \sqrt{\frac{D}{1-w_0}} A_i, i = 1, ..., D$$

$$s^{D+i}_{n-1} = z_{n-1} - \sqrt{\frac{D}{1-w_0}} A_i, i = 1, ..., D$$

$$w_i = \frac{1-w_0}{2D}, i = 1, ..., 2D$$

$$AA^T = \Sigma_{n-1}$$

Algorithmic view

Prediction (Time update)

$$\{s^0, ..., s^{2D}\}_{n-1} \qquad \bar{z}_n = \sum_{i=0}^{2D} w_i f(s_{n-1}^i) \qquad \bar{\Sigma}_n = \sum_{i=0}^{2D} w_i \left(f(s_{n-1}^i) - \bar{z}_n\right) \left(f(s_{n-1}^i) - \bar{z}_n\right)^T + R_n$$

Correction (Measurements update)

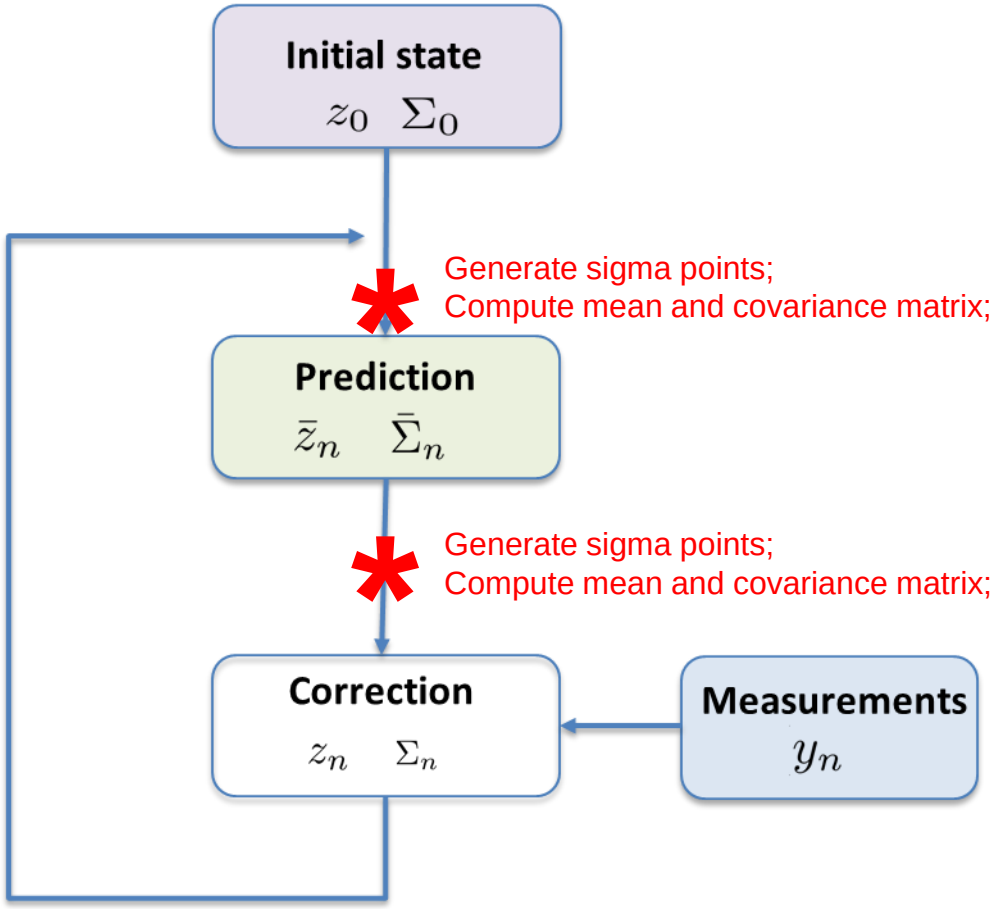
$$\{\bar{s}^0, ..., \bar{s}^{2D}\}_{n-1} \qquad \bar{y}_n = \sum_{i=0}^{2D} w_i h(\bar{s}_{n-1}^i) \qquad \bar{S}_n = \sum_{i=0}^{2D} w_i \left(h(\bar{s}_{n-1}^i) - \bar{y}_n\right) \left(h(\bar{s}_{n-1}^i) - \bar{y}_n\right)^T + Q_n$$

$$\bar{\Sigma}_n^{z,y} = \sum_{i=0}^{2D} w_i \left(\bar{s}_{n-1}^i - \bar{z}_n\right) \left(h(\bar{s}^i) - \bar{y}_n\right)^T$$

$$K_n = \bar{\Sigma}_n^{z,y} \bar{S}_n^{-1}$$

$$z_n = \bar{z}_n + K_n \left(y_n - \bar{y}_n\right) \qquad \Sigma_n = \bar{\Sigma}_n - K_n \bar{S}_n K_n^T$$

Algorithmic view



Unscented Kalman Filter

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Advantages:

- For certain systems, UKF provides better estimates of mean and covariance
- No need to compute Jacobians
  (which is expansive or even not possible for difficult non-linear functions)

Disadvantages:

- Increased computational cost

Critical comparison

Critical comparison

Estimator	State-transition / Measurement models assumptions	Assumed noise distribution	Computational cost
Kalman Filter	Linear	Gaussian	Low
Extended Kalman Filter	Non-linear (but locally linear)	Gaussian	Low / Medium (depending on the difficulty of computing the Jacobian)
Unscented Kalman Filter	Non-linear	Gaussian	Medium

Practice Questions

MLTS Exercise 06

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With a linear function, the extended Kalman filter is just a Kalman filter?

Practice Questions

MLTS Exercise 06

Unscented and Extended Kalman filter are the same and only differ in their formulations?

Critical comparison

Estimator	State-transition / Measurement models assumptions	Assumed noise distribution	Computational cost
Kalman Filter	Linear	Gaussian	Low
Extended Kalman Filter	Non-linear (but locally linear)	Gaussian	Low / Medium (depending on the difficulty of computing the Jacobian)
Unscented Kalman Filter	Non-linear	Gaussian	Medium
Particle Filter	**Non-linear**	**Non-Gaussian**	

Kalman Filter Task

FilterPy

FilterPy is a Python library that implements a number of Bayesian filters, most notably Kalman filters. I am writing it in conjunction with my book *Kalman and Bayesian Filters in Python* [1], a free book written using Ipython Notebook, hosted on github, and readable via nbviewer. However, it implements a wide variety of functionality that is not described in the book.

https://filterpy.readthedocs.io/en/latest/

Any questions?