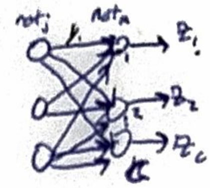
② Consider a three-layer neural network for classification with output units employing softmax, trained with 0-1 signals

(a) Derive the learning rule if the criterion function is SSE, that is,

$$J(w) = \frac{1}{2}\sum_{k=1}^{c}(t_k - z_k)^2$$

$j = 1 \ldots$ # of weights in unit k

Derivative in terms of output unit weights w_{kj}:



$$\frac{\delta J}{\delta w_{kj}} = \frac{\delta J}{\delta z_k}\frac{\delta z_k}{\delta net_k}\frac{\delta net_k}{\delta w_{kj}}$$

$$= (t_k - z_k)(-1)\cdot\frac{e^{net_k}}{\sum e^{net_k}}\left(1 - \frac{e^{net_k}}{\sum e^{net_k}}\right)\cdot y_j$$

$$\boxed{\begin{array}{l}\text{Update rule for } w_{kj}: \\ w_{kj} \leftarrow w_{kj} - \alpha\dfrac{\delta J}{\delta w_{kj}}\end{array}}$$

Derivative in terms of ~~output~~ hidden-to-output (y_j) $f(net)$

$$\frac{\delta J}{\delta y_j} = \sum_k \frac{\delta J}{\delta z_k}\frac{\delta z_k}{\delta net_k}\frac{\delta net_k}{\delta y_j}$$

$$= \sum_k (z_k - t_k)\cdot\frac{e^{net_k}}{\sum e^{net_k}}\left(1 - \frac{e^{net_k}}{\sum e^{net_k}}\right)\cdot w_{kj}$$

Derivative of loss function in terms of hidden unit weights

$$\frac{\delta J}{\delta w_{ji}} = \frac{\delta J}{\delta y_j}\cdot\frac{\delta y_j}{\delta net_j}\cdot\frac{\delta net_j}{\delta w_{ji}}$$

$$= \left[\sum_k (z_k - t_k)\cdot\frac{e^{net_k}}{\sum e^{net_k}}\left(1 - \frac{e^{net_k}}{\sum e^{net_k}}\right)\cdot w_{kj}\right]\cdot f'(net_j)\cdot x_i$$

$$\boxed{\begin{array}{l}\text{Update rule for } w_{ji}: \\ w_{ji} \leftarrow w_{ji} - \alpha\dfrac{\delta J}{\delta w_{ji}}\end{array}}$$

SEE Q1 @BACK!

(b) Repeat for cross-entropy criterion function

$$J_{ce}(W) = \sum_{k=1}^{c} t_k \ln\frac{t_k}{z_k}$$

$$\frac{\delta J_{ce}}{\delta z_k} = \frac{\delta}{\delta z_k} t_k \ln\frac{t_k}{z_k} = t_k\cdot\cancel{\frac{z_k}{t_k}}\cdot(-1)\frac{t_k}{z_k^2} = \boxed{-\frac{t_k}{z_k}}$$

The rest follows directly from (a)

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CSE569 HW4

Q3

Suppose there is a one-dimensional mixture density consisting of two Gaussian Components, each centered at the origin.

$$p(x|\theta) = P(\omega_1)\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-x^2/(2\sigma_1^2)} + (1-P(\omega_1))\frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-x^2/(2\sigma_2^2)}$$

and $\theta = (P(\omega_1), \sigma_1, \sigma_2)^T$

(a) Show that under these conditions, the density is completely unidentifiable.

If the variance of the Gaussian [redacted] components is equal, we get [redacted] $\frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} (\cancel{P(\omega_1)} + (1-\cancel{P(\omega_1)}))$.

Now for any value of $P(\omega_1)$, the density is the same, making it unidentifiable.

(b) [redacted]

Suppose the value of $P(\omega_1)$ is fixed and known. do the model identifiable?

If the prior $=0$, we cannot recover σ_1

" " " $=1$, we cannot recover σ_2

" " " $=0.5$, we can't distinguish between σ_1 and σ_2

otherwise, the model is identifiable.

(c) Suppose σ_1 & σ_2 are [redacted] known, but $P(\omega_1)$ is not. Is the resulting model identifiable?

Following from (a), if $\sigma_1 = \sigma_2$, then $P(\omega_1)$ is not identifiable

If $\sigma_1 \neq \sigma_2$, then the model is identifiable.

Q4. On the k-means algorithm

(1) Will the algo still work if μ_i for $i=1,\dots,c$ to the same initial value?

Yes, though in practice its results will probably depend on how empty sets are handled.

When the centroids are initialized + then a single iteration occurs, the centroids will likely change, at which point there is no longer a concern of them having been initialized at the same point.

(2) What is the complexity of the algorithm in terms of N (the # of samples), C (the # of clusters), and T (the total # of iterations until convergence?

$O(NCT)$

(3) Consider the case of N data points drawn from a mixture density model w/ C normal densities with means at μ_i, $i=1,\dots,C$, respectively. After running the k-means algorithm on this data set, will you get the respective mean vectors μ_i as the outcome?

You will get some vectors back, but with finite data points there is no guarantee of getting the true parameters θ, and even with infinite data points the ground-truth parameters of the mixture density would still play a role in determining if θ is identifiable.

q1.py

```python
#Q1
# (A)
# # Generate training data and labels
# training_w1_data = multivariate_normal(mean_w1, covar_w1, size=(50))
# training_w1_labels = np.zeros(training_w1_data.shape[0])
# training_w2_data = multivariate_normal(mean_w2, covar_w2, size=(50))
# training_w2_labels = np.ones(training_w2_data.shape[0])
#
# # Generate test data and labels
# test_w1_data = multivariate_normal(mean_w1, covar_w1, size=(20))
# test_w1_labels = np.zeros(test_w1_data.shape[0])
# test_w1_labels = test_w1_labels.reshape((-1, 1))
# test_w2_data = multivariate_normal(mean_w2, covar_w2, size=(20))
# test_w2_labels = np.ones(test_w2_data.shape[0])
# test_w2_labels = test_w2_labels.reshape((-1, 1))
# training_data = np.r_[training_w1_data, training_w2_data]
# training_labels = np.r_[training_w1_labels, training_w2_labels]
# training = unison_shuffled_copies(training_data, training_labels)
# training[1] = training[1].reshape((-1, 1))
# test_data = np.r_[test_w1_data, test_w2_data]
# test_labels = np.r_[test_w1_labels, test_w2_labels]
# test = unison_shuffled_copies(test_data, test_labels)
# test[1] = test[1].reshape((-1, 1))

# (B) Best validation error achieved: 22%
#     n*_H = 4
# (C) 10%
# (D) In part B, we fit the data too strongly to the training data. In doing so,
#     the validation accuracy decreased.
#     In Part C, we stopped training at the minimum value, causing us to avoid
#     overfitting.


import numpy as np, math
import argparse
import pickle
import matplotlib

matplotlib.use('agg')
from matplotlib import pyplot as plt

import os, sys
from tensorflow.examples.tutorials.mnist import input_data

alpha = 0.1


class NeuralNetwork:
    def __init__(self, *args, dropout=False):
        # Layers of the neural network (doesn't include an input layer)
        self.loss = []
        self.layers = []

        self.image_count = 0

        current_wire = None
        for i, (layerType, units) in enumerate(*args):
            # For each layer defined, create the layer and attach it to the output wires
            # of the preceding layer

            if i == 0:
                # Create wires going from inputs to first hidden layer
                current_wire = Wire(units)
            else:
                new_layer = {
                    'sigmoid': sigmoidLayer(current_wire, units, dropout=dropout),
                    'softmax': softmaxLayer(current_wire, units, dropout=dropout),
                    'relu': reluLayer(current_wire, units, dropout=dropout)
                }.get(layerType, None)
                current_wire = new_layer.outputWire
                self.layers.append(new_layer)

    def train_and_validate(self, train_data, validation_data, epochs):
        global alpha
        alpha = 0.1
        for epoch in range(epochs):
            print("Epoch:", epoch)
            self.train(train_data)
            self.test(validation_data)
            alpha *= 1

    def train(self, data):
        train_loss = []
        for i, (input, label) in enumerate(zip(data[0], data[1])):
            output = self.forward_pass(input, training=True)
            errorGradient = output-label  # TODO: Modify this when gradient calculation is resolved
            errorGradient = errorGradient.reshape((-1, 1))
            print("training", "output:", output, "label:", label, "errorGradient:", errorGradient)
            train_loss.append(-label.T.dot(output))
            self.backward_pass(errorGradient)
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        self.loss.append(sum(train_loss) / float(len(train_loss)))
        plt.clf()
        plt.plot(self.loss)
        #         plt.savefig('images/training_' + str(self.image_count) + '.png')
        self.image_count += 1

    def test(self, data):
        correct_count = 0
        for i, (input, label) in enumerate(zip(data[0], data[1])):

            output = self.forward_pass(input)
            print(input, output, label)
            if np.round(output) == label:
                correct_count += 1

        print(correct_count / data[0].shape[0])
        print()

    def forward_pass(self, input, training=False):
        current_output = input
        for i, layer in enumerate(self.layers):
            current_output = layer.forward(current_output, training=training)
        return current_output

    def calculate_error(self, prediction, label):
        return 0.5 * (prediction - label) ** 2  # TODO: Modify when gradient issue resolved

    def backward_pass(self, errorGradient):
        for i, layer in enumerate(self.layers[::-1]):
            if i == 0:
                layer.outputWire.gradients = errorGradient
                layer.backward()  # TODO: Modify when gradient issue resolved
            else:
                layer.backward()

    def saveModel(self):
        pickle.dump(self, open("model", "wb"))

    @staticmethod
    def load_model():
        return pickle.load(open("model", "rb"))


class Wire:
    def __init__(self, input_dimensions):
        self.input_dimensions = input_dimensions

    def initialize(self, units, fn):
        self.dropout_proba = 0.5
        self.units = units
        self.weights = fn(self.input_dimensions + 1, units)
        self.gradients = None
        self.velocity = np.zeros(self.weights.shape)

    def generate_dropout_variables(self):
        self.dropout_vector = np.random.choice([0, 1], size=(self.units,),
                                               p=[self.dropout_proba, 1 - self.dropout_proba])
        self.dropout_matrix = np.diag(self.dropout_vector)


class Layer:
    def __init__(self, inputWire, units, dropout=False, weight_fn=lambda x, y: np.random.randn(x, y) / np.sqrt(x)):
        if not inputWire:
            raise TypeError("Must initialize layer with Wire")
        self.dropout = dropout
        self.inputWire = inputWire
        self.inputWire.initialize(units, fn=weight_fn)

        self.outputWire = Wire(units)

    def forward(self, x, training=False):
        raise NotImplementedError("Must implement forward-pass function -- forward( self,  )")

    def backward(self):
        raise NotImplementedError("Must implement backward-pass function -- backward( self,  )")


class sigmoidLayer(Layer):
    def __init__(self, inputWire, units, dropout=False):
        super().__init__(inputWire, units, dropout)

    def forward(self, x, training=False):
        self.inputs = x
        # Add input value of "1" to the input array for the bias weight (w_0*x_0 + ...)
        self.inputs = np.insert(x, 0, 1.0, 0)
        self.linear_outputs = self.inputWire.weights.T.dot(self.inputs)

        self.outputs = self.sigmoid(self.linear_outputs)
        if training and self.dropout:
            self.inputWire.generate_dropout_variables()
            # self.outputs = self.outputs.dot(self.inputWire.dropout_matrix)
            self.outputs = self.outputs * self.inputWire.dropout_vector / self.inputWire.dropout_proba
        return self.outputs
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    def sigmoid(self, x):
        # Calculate sigmoid on vector cells
        self.sig_values = 1 / (1 + np.exp(-x))
        return self.sig_values

    def backward(self):
        d_p_wrt_f = self.sig_values * (1 - self.sig_values)
        if self.dropout:
            d_p_wrt_f = d_p_wrt_f * self.inputWire.dropout_vector
        self.d_E_wrt_f = np.sum(self.outputWire.gradients) * d_p_wrt_f if self.outputWire.gradients.shape[
                                                                             0] > 1 else self.outputWire.gradients * d_p_wrt_f
        d_E_wrt_x = self.inputWire.weights.dot(
            np.diag(self.d_E_wrt_f))  # Multiply outgoing gradients by the dE/df of their respective node in this layer
        self.inputWire.gradients = d_E_wrt_x
        # print("backward gradient:", d_E_wrt_x)

        self.update()

    def update(self):
        global alpha
        # if self.dropout:
        #     self.inputWire.weights -= alpha*(np.outer(self.inputs, self.inputWire.dropout_matrix.dot(self.d_E_wrt_f)))
        # self.inputWire.velocity = self.inputWire.momentum*self.inputWire.velocity - alpha * (np.outer(self.inputs, self.d_E_wrt_f))
        # self.inputWire.weights += self.inputWire.velocity
        self.inputWire.weights -= alpha * (np.outer(self.inputs, self.d_E_wrt_f))


class reluLayer(Layer):
    def __init__(self, inputWire, units, dropout=False):
        super().__init__(inputWire, units, dropout, weight_fn=lambda x, y: np.random.randn(x, y) / np.sqrt(x / 2))

    def forward(self, x, training=False):
        self.inputs = x
        # Add input value of "1" to the input array for the bias weight (w_0*x_0 + ...)
        self.inputs = np.insert(x, 0, 1.0, 0)
        self.linear_outputs = self.inputWire.weights.T.dot(self.inputs)

        self.outputs = self.relu(self.linear_outputs)
        if training and self.dropout:
            self.inputWire.generate_dropout_variables()
            # self.outputs = self.outputs.dot(self.inputWire.dropout_matrix)
            self.outputs = self.outputs * self.inputWire.dropout_vector / self.inputWire.dropout_proba
        return self.outputs

    def relu(self, x):
        # Calculate sigmoid on vector cells
        self.relu_values = np.maximum(0, x)
        return self.relu_values

    def backward(self):
        d_p_wrt_f = np.zeros(self.relu_values.shape)
        d_p_wrt_f[self.relu_values > 0] = 1
        if self.dropout:
            d_p_wrt_f = d_p_wrt_f * self.inputWire.dropout_vector
        self.d_E_wrt_f = np.sum(self.outputWire.gradients[1:], 1) * d_p_wrt_f
        d_E_wrt_x = self.inputWire.weights.dot(
            np.diag(self.d_E_wrt_f))  # Multiply outgoing gradients by the dE/df of their respective node in this layer
        self.inputWire.gradients = d_E_wrt_x
        # print("backward gradient:", d_E_wrt_x)

        self.update()

    def update(self):
        global alpha
        # if self.dropout:
        #     self.inputWire.weights -= alpha*(np.outer(self.inputs, self.inputWire.dropout_matrix.dot(self.d_E_wrt_f)))
        # self.inputWire.velocity = self.inputWire.momentum*self.inputWire.velocity - alpha * (np.outer(self.inputs, self.d_E_wrt_f))
        # self.inputWire.weights += self.inputWire.velocity
        self.inputWire.weights -= alpha * (np.outer(self.inputs, self.d_E_wrt_f))


class softmaxLayer(Layer):
    def __init__(self, inputWire, units, dropout=False):
        super().__init__(inputWire, units, dropout)

    def forward(self, x, training=False):
        self.inputs = x
        # Add input value of "1" to the input array for the bias weight (w_0*x_0 + ...)
        self.inputs = np.insert(x, 0, 1, 0)
        self.outputs = self.softmax(self.inputWire.weights.T.dot(self.inputs))
        return self.outputs

    def softmax(self, x):
        # Calculate sigmoid on vector cells
        exponentials = np.exp(x - np.max(x))
        self.softmax_values = exponentials / np.sum(exponentials)
        return self.softmax_values

    def backward(self):
        self.inputWire.gradients = None

        d_p_wrt_f = self.softmax_values * (1 - self.softmax_values)
        self.d_E_wrt_f = self.outputWire.gradients * d_p_wrt_f  # single incoming gradient
        d_E_wrt_x = self.inputWire.weights.dot(
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            np.diag(self.d_E_wrt_f))  # Multiply outgoing gradients by the dE/df of their respective node in this layer
        self.inputWire.gradients = d_E_wrt_x

        self.update()

    def backward(self, y):
        self.inputWire.gradients = None

        self.d_E_wrt_f = self.outputs - y  # single incoming gradient
        d_E_wrt_x = self.inputWire.weights.dot(
            np.diag(self.d_E_wrt_f))  # Multiply outgoing gradients by the dE/df of their respective node in this layer
        self.inputWire.gradients = d_E_wrt_x

        self.update()

    def update(self):
        global alpha
        # TODO: Refactor away from the outer() function to allow for mini-batch
        self.inputWire.weights -= alpha * (np.outer(self.inputs, self.d_E_wrt_f))

from numpy.random import multivariate_normal

def unison_shuffled_copies(a, b):
    assert len(a) == len(b)
    p = np.random.permutation(len(a))
    return [a[p], b[p]]

mean_w1 = np.array([0, 0])
covar_w1 = np.array([[1,0], [0,1]])
mean_w2 = np.array([1, 0.5])
covar_w2 = np.array([[3, 1],[1, 2]])

# Generate training data and labels
training_w1_data = multivariate_normal(mean_w1, covar_w1, size=(50))
training_w1_labels = np.zeros(training_w1_data.shape[0])
training_w2_data = multivariate_normal(mean_w2, covar_w2, size=(50))
training_w2_labels = np.ones(training_w2_data.shape[0])

# Generate test data and labels
test_w1_data = multivariate_normal(mean_w1, covar_w1, size=(20))
test_w1_labels = np.zeros(test_w1_data.shape[0])
test_w1_labels = test_w1_labels.reshape((-1, 1))
test_w2_data = multivariate_normal(mean_w2, covar_w2, size=(20))
test_w2_labels = np.ones(test_w2_data.shape[0])
test_w2_labels = test_w2_labels.reshape((-1, 1))
training_data = np.r_[training_w1_data, training_w2_data]
training_labels = np.r_[training_w1_labels, training_w2_labels]
training = unison_shuffled_copies(training_data, training_labels)
training[1] = training[1].reshape((-1, 1))
test_data = np.r_[test_w1_data, test_w2_data]
test_labels = np.r_[test_w1_labels, test_w2_labels]
test = unison_shuffled_copies(test_data, test_labels)
test[1] = test[1].reshape((-1, 1))

EPOCHS = 100
nn_architecture = [("input", 2), ("sigmoid", 5), ("sigmoid", 1)]
nn = NeuralNetwork(nn_architecture, dropout=False)
nn.train_and_validate(training, training, EPOCHS)
```