Solving the Quasi-Symmetric Quartic

The *quasi-symmetric quartic* is a quartic equation of form

$$ax^4+bx^3+cx^2+bmx+am^2=0$$

where *a* is nonzero and *a, b, c*, and *m* are freely chosen in such a way as to match the coefficients of any given quasi-symmetric quartic.

Such a quartic can be solved following the manner of the general quartic, but here, we explore the use of a very neat trick which greatly simplifies the algebra required.

To begin with, we divide by x^2, such that

$$ax^2+bx+c+\frac{bm}{x}+\frac{am^2}{x^2}=0$$

We then make the definition

$$z=x+\frac{m}{x}$$

which means

$$z^2=x^2+2m+\frac{m^2}{x^2}$$

Examining our original equation, we find that we can factor it so that we have

$$a\left(x^2+2m+\frac{m^2}{x^2}\right)+b\left(x+\frac{m}{x}\right)+c-2am=0$$

which becomes

$$az^2+bz+(c-2am)=0$$

a simple quadratic equation that is trivially solvable with the quadratic formula:

$$z=\frac{-b\pm\sqrt{b^2-4a(c-2am)}}{2a}=\frac{-b\pm\sqrt{b^2-4ac+8a^2m}}{2a}$$

As we have values for *z* purely in terms of the original coefficients, we now only need express *x* in terms of *z*. Since

$$z=x+\frac{m}{x}$$

we can rearrange this to find

$$x^2 - zx + m = 0$$

which is trivially solved as

$$x = \frac{z \pm \sqrt{z^2 - 4m}}{2}$$

Substituting in *z* and simplifying gives us

$$x = \frac{-b \pm \sqrt{b^2 - 4ac + 8a^2 m} \pm \sqrt{2b^2 - 4ac - 8a^2 m - 2b\sqrt{b^2 - 4ac + 8a^2 m}}}{4a}$$

which, after using every possible sign combination, gives us the four solutions to the original quasi-symmetric quartic.