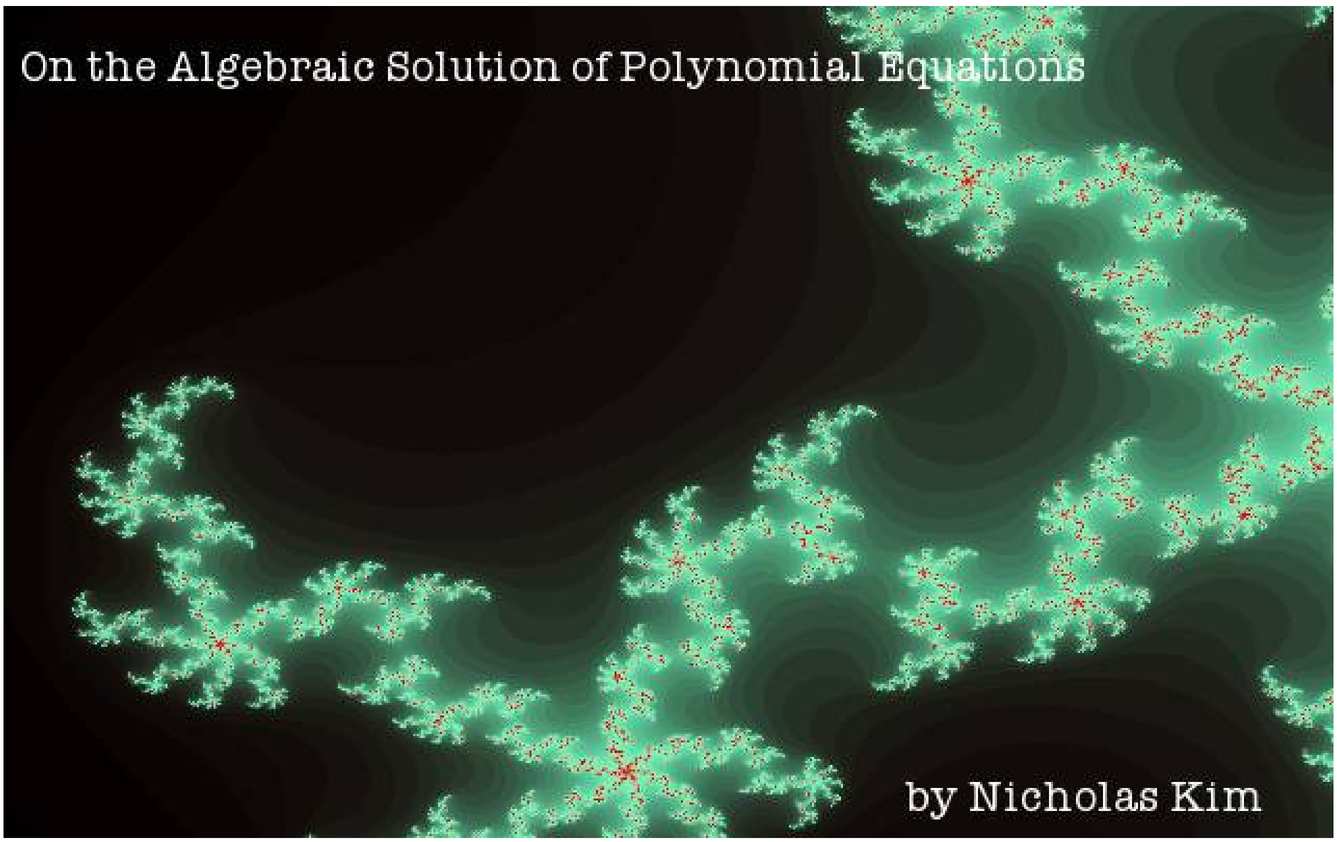
On the Algebraic Solution of Polynomial Equations
by Nicholas Kim

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# Preamble

*"It is a well-known experience that the only truly enjoyable and profitable way of studying mathematics is the method of 'filling in details' by one's own efforts."*
*– Cornelius Lanczos,* Applied Analysis

Ever since I began serious study of mathematics, my favorite topic in math has been, and continues to be, polynomials – the central topic of elementary algebra.

My first real progress on the study of polynomials was in senior high school math classes, when I first learned the quadratic formula, as well as heuristic techniques for solving higher-degree polynomial equations. As it was well within the grasp of a sufficiently curious student, I quickly learned for myself the derivation of the quadratic formula, and, inspired by the ease and elegance of the derivation, began to search for a cubic formula, an attempt that, unfortunately, failed miserably (I was attempting to 'complete the cube'; see the section on cubic equations). One of my math teachers gave me a stack of notes on how to derive the cubic formula upon request, but it would not be until much later that I would be able to make sense of them all.

While my interest in polynomials has risen and declined sporadically over the years since then, I nevertheless steadily accumulated the mathematical foundations required to fully understand the algebraic, analytic solutions of polynomial equations. So, in a sense, this document is the result of several *years'* worth of study and work; a collection of all the knowledge I possess on how to solve polynomial equations algebraically, and explained in a manner that I hope is clearer than other sources found on the Internet (which I nevertheless owe my thanks to, for guiding me on the right paths).

As always, I hope this work proves to be an interesting and worthwhile read. Please offer any feedback you can, either on my website or in person.

– Nicholas Kim

## Structure

This document is organized by degree of polynomial equation (from linear to quartic), with Appendix A describing, and/or proving, various theorems and concepts related to the solution of polynomial equations, and Appendix B briefly describing quintic equations in further detail. It is assumed that the reader has a solid grounding in elementary algebra; linear and quadratic equations are covered at a much faster pace than that of cubic and quartic equations.

Although there are further possibilities, this document is restricted to solutions of polynomial equations that can be derived strictly using elementary algebraic operations (arithmetic, exponentiation, radicals), meaning that polynomial equations of degree five and higher are automatically excluded, by the Abel-Ruffini theorem (see "To Infinity and Beyond"); trigonometric/hyperbolic solutions, as well as those using more advanced, non-elementary functions, are strictly outside of the scope of this work (with the exception of the Bring radical, which is briefly touched upon in Appendix B).

**Most importantly, this document is not a tutorial, but rather more of a reference.** While I have made reasonable efforts to explain my work as clearly as possible, particularly in the solving of cubics and quartics, this work primarily emphasizes mathematical precision and elegance; examples are relatively few and terse.

## Acknowledgements

Many sources on the Internet have contributed to my work; while it would be tedious to list them all individually, Wikipedia and Wolfram MathWorld were my primary sources of information.

The partial formulae listed in the appendix are my own research (with the exception of the partial cubic formula given two solutions; that is also on Wikipedia). Most derivations and proofs were done by me, although they are generally not my original ideas.

This document was created using LibreOffice 3.4, running on Mac OS X version 10.6 ("Snow Leopard"). The title image was created using the GIMP.

# Linear Equations

*"Numbers exist only in our minds. There is no physical entity that is number 1. If there were, 1 would be in a place of honor in some great museum of science, and past it would file a steady stream of mathematicians gazing at 1 in wonder and awe."*
*– Fraleigh/Beauregard,* Linear Algebra

The most basic type of polynomial equation, a *linear equation* is a polynomial equation of degree 1, so named due to the fact that graphs of linear functions are always straight lines on the Cartesian coordinate plane.

The general linear equation takes the form

$$ax+b=0$$

(where *a* is nonzero) which can be trivially solved to give what may be called the *linear formula*:

$$x=-\frac{b}{a}$$

In practice, of course, this formula is never used, since linear equations are faster to solve directly.

## Closure Under the Rational Numbers

Uniquely among polynomial equations, linear equations are algebraically closed under the rational numbers. In other words, given

$$ax+b=0$$

if *a* and *b* are both rational numbers, *x* is likewise guaranteed to be rational. The proof is easy: since a rational number is defined as the ratio of two integers, we may say that

$$a=\frac{a_1}{a_2}$$

and

$$b=\frac{b_1}{b_2}$$

where all the new variables are integers. Then, by the linear formula, the solution for *x* is

$$x=-\frac{\frac{b_1}{b_2}}{\frac{a_1}{a_2}}$$

and since a division of fractions may be treated as a multiplication of the divisor's reciprocal,

$$x=-\frac{b_1}{b_2}\times\frac{a_2}{a_1}=-\frac{b_1 a_2}{b_2 a_1}$$

By the closure property of integer multiplication, both $b_1a_2$ and $b_2a_1$ must be integers, and so the result is indeed a rational number. □

# Quadratic Equations

*"But to tell you the truth, I see nothing but a scientific extravagance in all these calculations. That which is neither useful nor agreeable is worthless. And as for useful things, they have all been discovered; and to those which are agreeable, I hope that good taste will not admit algebra among them."*
*– Frederick the Great,* Letters of Voltaire and Frederick the Great

A *quadratic equation* is a polynomial equation of degree 2. The word *quadratic* is derived from the Latin word for square, *quadratum*, and applied to second-degree polynomial equations since the highest power of $x$ is *squared* – a bit of a bad math joke.

The general quadratic equation takes the form

$$ax^2+bx+c=0$$

(where $a$ is nonzero). Graphs of quadratics always form *parabolas*, a symmetrical curve that grows increasingly rapidly as it gets further away from its *vertex*, or center point.

Unlike linear equations, quadratic equations, and indeed all higher-order equations, cannot be solved directly, as the presence of both an $x$ and an $x^2$ term render any direct attempts to isolate $x$ futile. It is therefore necessary to develop techniques that will condense the terms in such a way that only a single $x$ remains.

## Factoring

The simplest technique for attacking a quadratic equation exploits the fact that a quadratic equation can factor into two linear factors (see "Factor Theorem" in Appendix A). Suppose we have

$$(x+m)(x+n)=0$$

Expanding this gives us

$$x^2+(m+n)x+mn=0$$

which forms a general quadratic equation. (It is general, despite the leading coefficient being 1, because all terms may be divided by the leading coefficient, as it is guaranteed not to be 0.)

Now, why is the first equation easier to solve? The answer lies in the zero-multiplication property of real numbers: if $ab=0$, then at least one of $a$ and $b$ has to be zero. Applying this to the first equation:

If $(x+m)(x+n)=0$, then at least one of $(x+m)$ and $(x+n)$ has to be zero. **Thus, $x$ must be either $-m$, or $-n$.**

(It is important to note that *both* are solutions to the quadratic. In general, a quadratic has *two* solutions, and in general, an $n$th-degree polynomial equation has $n$ solutions (though they need not all be distinct, or even real – see "the Fundamental Theorem of Algebra" in Appendix A).)

We can now see that solving a quadratic equation has been reduced to the problem of factoring a quadratic into such linear factors. From the expanded equation, we can see that, given a quadratic

$$x^2+bx+c=0$$

we must come up with two numbers $m$ and $n$ such that

$$\begin{aligned} m+n&=b \\ mn&=c \end{aligned}$$

Unfortunately, we cannot solve this system of equations directly, as doing so will merely yield another quadratic equation. Therefore, the numbers must be guessed, which, in general, does not work. However, there are many simple cases where it does, and if successful, that particular quadratic will have been solved.

## Example 1

Suppose we wish to solve the quadratic equation

$$x^2 - 7x + 12 = 0$$

From before, we seek two numbers *m* and *n* that satisfy

$$\begin{aligned} m + n &= -7 \\ mn &= 12 \end{aligned}$$

By trial and error, we may find that -4 and -3 are the answers. Thus, the quadratic factors into

$$(x-4)(x-3) = 0$$

which means one of

$$\begin{aligned} x - 4 &= 0 \\ x - 3 &= 0 \end{aligned}$$

must be satisfied. Hence, the solutions are 4 and 3. □

## Completing the Square

Since the factoring technique is rather unreliable, we must look for alternative methods. So, let us consider the following identity:

$$x^2 + 2hx + h^2 = (x+h)^2$$

We know how to create the left-hand side; we simply expand the right. But now, we might ask ourselves, can we not do the reverse? Indeed, if we do, the quadratic has been solved, since the right-hand expression contains only a single $x$, which is trivial to isolate. And the left-hand side looks like a quadratic equation – therefore, we have a very promising means for which to solve quadratics.

Unfortunately, not every quadratic equation can be collapsed into a perfect square. However, this is easily remedied; consider that the constant $h$ can be freely chosen, such that $2h$ = the linear coefficient of the quadratic. Then $h^2$, the constant term, is dependent on that, but we can easily add another free variable, say $k$, such that

$$x^2 + 2hx + h^2 + k = (x+h)^2 + k$$

and the right-hand side is still trivially solvable. But since $h$ and $k$ can now be chosen to perfectly match the coefficients of any quadratic equation (again, the leading coefficient can be assumed to be 1 without loss of generality), the quadratic has been solved.

## Example 1

Suppose we wish to solve the quadratic equation

$$x^2 + 4x + 1 = 0$$

As described above, we aim to factor it into form

$$(x+h)^2 + k = 0$$

for some value of $h$ and $k$. From above, we know that since $2h$ = the linear coefficient, $h=4/2=2$. And since $h^2+k$ = the constant, $k$ must equal -3 since $h^2$ is 4 and the constant is 1. Hence the quadratic factors into

$$(x+2)^2-3=0$$

From here, the rest is basic algebra. The final solution is

$$x=\pm\sqrt{3}-2$$

and we are done. □

## The Quadratic Formula

Let us formalize the method of completing the square in order to solve the quadratic. Consider the general quadratic equation once more:

$$ax^2+bx+c=0$$

Without loss of generality, the leading coefficient can be assumed to be 1. Let us make it so:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Now consider that, because *2h* = the linear coefficient,

$$2h=\frac{b}{a}$$
$$h=\frac{b}{2a}$$
$$h^2=\frac{b^2}{4a^2}$$

Thus, to create the perfect square, we have

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}=\frac{b^2}{4a^2}$$

(We add the same expression to both sides to obey the Golden Rule of Equations.) Moving *c/a* to the right-hand side and combining it with the other expression, we have

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

The left-hand side is now a perfect square:

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

And now the rest is basic algebra:

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(or alternatively, expressing each solution separately)

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$
$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Thus the general quadratic has been solved, with the result known as the *quadratic formula*. □

## Discriminant

The expression under the radical in the quadratic formula, $b^2$-$4ac$, is a special expression known as the *discriminant*. It tells us about the nature of the solutions to the quadratic equation without needing to explicitly find the solutions.

In particular, if the coefficients are all real, we may note that:

- If the discriminant is equal to 0, the radical vanishes, and we are left with $x=-b/2a$ as the only solution (a double root).
- If the discriminant is greater than 0, the radical yields a positive number, and we have two distinct real solutions.
- If the discriminant is less than 0, the radical is undefined over the real numbers, and the quadratic equation can be said to have two complex, but non-real, solutions (see Appendix A).

Further information about discriminants may be found in Appendix A.

# Cubic Equations

*"Numbers are the ruler of forms and ideas, and the cause of gods and demons."*
*– Pythagoras*

A *cubic equation* is a polynomial equation of degree 3, so named since the highest power is cubed.

The general cubic equation takes the form

$$ax^3+bx^2+cx+d=0$$

(where $a$ is nonzero).

## A Brief History of Cubics

Techniques for the solution of quadratics have been known to humans before the Pythagorean theorem (although, of course, negative, irrational, and imaginary solutions were usually not accepted or recognized). However, cubics, as we will soon see, are an entirely different beast, and the algebraic solution of them is widely considered to be the pinnacle of 16th-century mathematics. (Techniques for approximating solutions to cubics were known to the Persians many centuries earlier, but they tend to be overlooked by eurocentric views of math – nevertheless, the algebraic solutions are arguably much more impressive.)

Our story begins in the early 16th century, with a little-known Italian mathematician by the name of Scipione del Ferro. He happened to chance upon an algebraic method for solving *depressed cubics* – namely, cubics of form

$$x^3+mx+n=0$$

(i.e. lacking a quadratic term). As was rather customary of the time, he kept his achievement secret, only passing it on to his student Antonio Fiore just before his death.

In contrast to his old teacher, Fiore was not very good at keeping secrets, and rumors started to circulate about the cubic having been solved. Such rumors prompted another Italian mathematician, named Nicollò Tartalgia, to figure out how to solve cubics as well. He eventually managed to solve a slightly different class of cubics, namely those lacking a linear term (as opposed to depressed cubics, which lack a quadratic term). He announced his achievement rather openly, unlike del Ferro, and was soon challenged by Fiore to a problem-solving contest. Tartalgia won rather easily, as he had discovered the solution to all forms of cubics a few days before the contest, while Fiore was stuck with forms he could not solve.

Later, another mathematician, Girolamo Cardano, persuaded Tartalgia to reveal his secrets for solving cubic equations to him. Tartalgia agreed, under the condition that Cardano not publish them until Tartalgia had. However, as Cardano later discovered del Ferro's knowledge of solving cubics, he decided that he could get around Tartalgia's promise, and promptly published *Ars Magna*, one of the most well-known math textbooks in history, describing, in detail, the algebraic solution of the cubic. Tartalgia was unsurprisingly rather furious, and challenged Cardano to a contest, a challenge later accepted and won by Lodovico Ferrari, Cardano's student (and the discoverer of the algebraic solution to the quartic).

## Completing the Cube

Since the quadratic is solved by completing the square, it seems very reasonable to ask if the cubic can be solved by “completing the cube”. Let us try this approach.

The key to solving the quadratic was that *all* quadratic equations could be written in the form

$$x^2+2hx+h^2+k=(x+h)^2+k$$

since $h$ and $k$ can be freely chosen in such a way that the coefficients will always match up (and the leading coefficient may always be assumed to be 1 if convenient). So, then, the objective is to see if we write the general cubic equation in the form

$$(x+h)^3+k$$

since any cubic of this form is clearly solvable. To do this, let us expand this expression:

$$x^3+3hx^2+3h^2x+h^3+k$$

Yet again, $h$ and $k$ may be freely chosen. We can clearly see that the quadratic coefficient can always be matched, since $3h$ = the quadratic coefficient. Thus $h$ is now bound, but $k$, still being free, lets us match up the constant term to whatever we need it to be as well.

But what about the linear coefficient? Since the value of $h$ was fixed from the quadratic, the linear coefficient is forced to be dependent on the quadratic coefficient – and thus it cannot always be matched up to whatever we need it to be. Therefore, we have reached an impasse, and must resign ourselves to the fact that such a neat trick, which worked so well for quadratics, looks horribly inadequate for cubics.

## Depressing the Cubic: The Linear Tschirnhaus Transformation

Since completing the cube does not work, we must ask ourselves: what else can we do?

To answer that, let's return to the quadratic equation. Suppose we didn't know about completing the square. How might we approach the problem, then? Well, if we can't tackle the general quadratic, we can certainly tackle the special case when the linear coefficient is zero, can we not? Indeed, the solution is trivially

$$x=\pm\sqrt{-\frac{c}{a}}$$

in that case. So one might reasonably be inclined to ask: is there any way we can *reduce* the general case to this special one? And in fact, there is. Returning to the solution of the quadratic, we found that the general quadratic could be (after moving all terms to one side) written in the form

$$\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}=0$$

We may note that this is equal to

$$y^2+k=0$$

where

$$y=x+\frac{b}{2a}$$

and

$$k=-\frac{b^2-4ac}{4a^2}$$

which is exactly the form of the trivial case we know how to solve directly. Seeing these results, we should ask ourselves: since *y=x+b/2a*, this means *x=y-b/2a*. What happens if we make this substitution in the general quadratic? Well, we get

$$\begin{aligned} a\left(y-\frac{b}{2a}\right)^2+b\left(y-\frac{b}{2a}\right)+c=0 \\ ay^2-by+\frac{b^2}{4a}+by-\frac{b^2}{2a}+c=0 \\ y^2-\frac{b^2-4ac}{4a^2}=0 \end{aligned}$$

which is exactly the same result as we found before – and using this result, we can solve for *y* directly, then resubstitute *x* to obtain the quadratic formula! Seeing the success of this, then, should motivate us to try a similar substitution on the general cubic, using *h* to denote that we don't quite know what that substitution should be yet:

$$x=y+h$$

The algebra is omitted for brevity, but suffice to say at the end, if we pick *h* to be *-b/3a*, the quadratic coefficient becomes 0, and we get

$$y^3+py+q=0$$

where

$$p=\frac{3ac-b^2}{3a^2}$$

and

$$q=\frac{2b^3-9abc+27a^2d}{27a^3}$$

In other words, **it is enough to solve the depressed cubic**, because we have just shown how the general cubic can be reduced to a depressed one. This makes del Ferro's discovery all the more crucial, for now, by combining his method and this transformation technique, we can solve *all* cubics.

This powerful transformation technique is called a *linear Tschirnhaus transformation*, so named after the mathematician who popularized this technique for general polynomial equations, and the fact that the substitution involves the use of a linear polynomial. **In general, when making a linear Tschirnhaus transformation of form *y=x+h*, choosing *h=b/na* for a polynomial equation of degree *n* will eliminate the second-highest degree term.**

(Higher-degree Tschirnhaus transformations do exist, but they are much less elegant and primarily used in more advanced mathematics, which is beyond the scope of this work.)

## Del Ferro's Solution of the Depressed Cubic

Now that we know how to transform a general cubic into a depressed one, let us see how del Ferro (and Tartaglia/Cardano) went about solving the depressed cubic:

$$y^3+py+q=0$$

The first thing del Ferro did was to make another substitution, this time of form

$$y=u+v$$

Making this substitution, we get

$$\begin{aligned}(u+v)^3+p(u+v)+q&=0\\ u^3+3u^2v+3uv^2+v^3+p(u+v)+q&=0\\ u^3+3uv(u+v)+v^3+p(u+v)+q&=0\\ u^3+v^3+(3uv+p)(u+v)+q&=0\end{aligned}$$

Now if we impose the condition

$$3uv+p=0$$

which is equivalent to

$$u^3v^3=-\frac{p^3}{27}$$

(we will see the motivation for taking the cube shortly), then the term *(3uv+p)(u+v)* becomes 0 and disappears, leaving us with

$$u^3+v^3+q=0$$

which is equivalent to

$$u^3+v^3=-q$$

Now, how does this help us? Well, we have a pair of simultaneous equations

$$\begin{aligned}u^3+v^3&=-q\\ u^3v^3&=-\frac{p^3}{27}\end{aligned}$$

which, at first glance, doesn't seem any easier to solve than the depressed cubic. But suppose we actually try to solve this system:

$$v^3 = -q - u^3$$
$$u^3(-q-u^3) = -\frac{p^3}{27}$$
$$u^6 + qu^3 - \frac{p^3}{27} = 0$$

We have a sextic (sixth-degree polynomial) equation, which seems to make things even worse. But upon closer examination, all the powers of *u* are multiples of 3. So, using one final substitution:

$$z = u^3$$

we get

$$z^2 + qz - \frac{p^3}{27} = 0$$

which is **a quadratic equation**! Therefore, solving the cubic has been reduced to the solution of an *auxiliary (resolvent) quadratic*, which we already know how to solve – hence, the cubic has been successfully solved! All that is now required is to solve this resolvent quadratic, and then make all the necessary backward substitutions.

## The Cubic Formula

The *cubic formula* is now simple to obtain by solving the auxiliary quadratic, and performing all the backward substitutions. Let us do so; from the quadratic formula,

$$z = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$$

$$u^3 = z = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Then since

$$v^3 = -q - u^3$$

we get

$$v^3 = -\frac{q}{2} \mp \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Since *u* and *v* are interchangeable (as $y=u+v$ from before, and addition is commutative), we only need one value of *u* and *v*; arbitrarily, let us pick

$$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$
$$v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Now comes a slightly tricky part. Here, we take the cube roots, but this only gives us one value for *u* and *v*, and thus only one solution; to obtain the other two solutions, we must use *complex* cube roots. (The full details may be found elsewhere.) Since

$$u^3+v^3=-q$$
$$y=u+v$$

we get, for each possible value of *u* and *v*,

$$y_1=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$
$$y_2=\frac{-1+i\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\frac{-1-i\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$
$$y_3=\frac{-1-i\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\frac{-1+i\sqrt{3}}{2}\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

which are the general solutions to the depressed cubic. All that is left to do is to backwards substitute for the original coefficients of the general cubic:

$$x=y-\frac{b}{3a}$$
$$p=\frac{3ac-b^2}{3a^2}$$
$$q=\frac{2b^3-9abc+27a^2d}{27a^3}$$

The algebra is tedious and omitted for brevity; suffice to say that the final result, known as the *cubic formula*, becomes

$$x_1=-\frac{b}{3a}$$
$$+\sqrt[3]{\frac{-2b^3+9abc-27a^2d+\sqrt{(2b^3-9abc+27a^2d)^2-4(b^2-3ac)^3}}{54a^3}}$$
$$+\sqrt[3]{\frac{-2b^3+9abc-27a^2d-\sqrt{(2b^3-9abc+27a^2d)^2-4(b^2-3ac)^3}}{54a^3}}$$

$$x_2=-\frac{b}{3a}$$
$$+\frac{-1+i\sqrt{3}}{2}\sqrt[3]{\frac{-2b^3+9abc-27a^2d+\sqrt{(2b^3-9abc+27a^2d)^2-4(b^2-3ac)^3}}{54a^3}}$$
$$+\frac{-1-i\sqrt{3}}{2}\sqrt[3]{\frac{-2b^3+9abc-27a^2d-\sqrt{(2b^3-9abc+27a^2d)^2-4(b^2-3ac)^3}}{54a^3}}$$

$$x_3 = -\frac{b}{3a}$$
$$+\frac{-1-i\sqrt{3}}{2}\sqrt[3]{\frac{-2b^3+9abc-27a^2d+\sqrt{(2b^3-9abc+27a^2d)^2-4(b^2-3ac)^3}}{54a^3}}$$
$$+\frac{-1+i\sqrt{3}}{2}\sqrt[3]{\frac{-2b^3+9abc-27a^2d-\sqrt{(2b^3-9abc+27a^2d)^2-4(b^2-3ac)^3}}{54a^3}}$$

While 'beautiful', in a very specific sense of the word (an analytic solution purely in terms of arithmetic operations, exponentiation, and radicals), the formula's considerable length and complexity makes it obvious why the formula is never taught, nor used, in practice. We therefore give preference to root-testing heuristics (see Appendix A), or root-approximation algorithms (consult a calculus tutorial).

### Casus Irreducibilis

One notable problem with the cubic formula, besides its complexity, is a phenomenon known as *casus irreducibilis* ("the irreducible case"). To explore the problem, recall that an auxiliary quadratic needs to be solved in the derivation of the cubic formula:

$$z^2+qz-\frac{p^3}{27}=0$$

When we previously solved this resolvent equation, we implicitly assumed the quadratic, in fact, had solutions – which it is guaranteed to, if we consider complex numbers as solutions (see "the Fundamental Theorem of Algebra" in Appendix A). However, if we are only interested in real solutions, then the discriminant of this quadratic must be non-negative – in other words,

$$q^2+\frac{4p^3}{27}\geq 0$$

Now, why exactly *is* this an issue? Well, the answer lies in the final cubic formula: it turns out that, if the original cubic has three (distinct) real solutions, **the solutions to the auxiliary quadratic will be complex, and there is no way to get rid of the complex numbers algebraically.** (Ironically, if the original cubic has only a single real solution, the solutions to the auxiliary quadratic will be real.) This leads to the 'problem' of having real solutions being able to be expressed only through the use of complex numbers – the aforementioned 'casus irreducibilis'.

### Vietè's Solution of the Depressed Cubic

Some time after del Ferro's discovery of the solution of the depressed cubic, mathematicians discovered a slightly more elegant method, known as *Vietè's substitution*: given the depressed cubic

$$y^3+py+q=0$$

make the substitution

$$y=z-\frac{p}{3z}$$

which gives way to the resolvent quadratic directly.

## *Quartic Equations*

*"By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced*

*problems, and in effect increases the mental power of the race."*
*– Alfred North Whitehead,* A History of Mathematical Notations

A *quartic equation* is a polynomial equation of degree 4.

The general quartic equation takes the form

$$ax^4+bx^3+cx^2+dx+e=0$$

(where *a* is nonzero).

## A Brief History of Quartics

Soon after the algebraic solution to the cubic was discovered (see "A Brief History of Cubics"), Cardano encouraged his student, Lodovico Ferrari, to try and see if he could come up with a similar solution to the quartic. Ferrari succeeded at the task, and his work was published by Cardano in *Ars Magna*.

## Depressing the Quartic

As with the cubic, a direct attack of the general quartic equation is fruitless, so we instead opt to perform some preliminary simplifications first; in particular, as with the general cubic, we may perform the linear Tschirnhaus transformation to eliminate the second-highest power (in this case the cubic term):

$$x=y-\frac{b}{4a}$$

We then substitute this into the general quartic, which eventually (after much tedious algebra) yields the *depressed quartic*:

$$y^4+py^2+qy+r=0$$

where

$$p=\frac{8ac-3b^2}{8a^2}$$

$$q=\frac{8a^2d-4abc+b^3}{8a^3}$$

$$r=\frac{256a^3e-64a^2bd+16ab^2c-3b^4}{256a^4}$$

## Ferrari's Solution of the Depressed Quartic

In a strange sense, the algebraic solution of the quartic is arguably 'easier' (or at least more intuitive) than that of the cubic (even if the results are algebraically more tedious). The reason is because the standard solutions to the quartic borrow many ideas from the solution of the quadratic, which we already know how to solve.

Ferrari is likely to have thought of this when thinking of how to approach the quartic. After depressing it, he suggested a very intuitive solution: suppose we try and fold up the depressed quartic into a perfect square (not unlike the technique of completing the square to solve the quadratic). In particular, we may note that

$$(y^2+p)^2=y^4+2py^2+p^2$$

which means we may rewrite the depressed quartic as (after a little moving of terms)

$$(y^2+p)^2=py^2-qy+p^2-r$$

This initially appears to be no improvement, as we are still stuck with a *y* on both sides. However, as may already be obvious from this arrangement of terms, what if we mold the right-hand side into a perfect square? If we do so, we can collapse it and take square roots of both sides, which reduces the solution of the quartic to the solution of a quadratic.

Here, however, we encounter a slight snag: the problem is that directly attempting to complete the square on the right-hand side of the above equation complicates the left-hand side as well (which we do not want, as described by our perfect squares strategy above), due to the Golden Rule of Equations. Fortunately, this is not too difficult to get around; suppose we introduce an auxiliary variable *u* and add it into our left-hand folded perfect square:

$$(y^2+p+u)^2$$

Since

$$(y^2+p+u)^2-(y^2+p)^2=2uy^2+2up+u^2$$

(which can easily be verified using a few quick expansions) all we need to do to balance out this modification is to add the right-hand side of this identity to the former equation:

$$(y^2+p+u)^2=(p+2u)y^2-qy+(u^2+2up+p^2-r)$$

All that is left to do now is to choose a value of *u* such that the right-hand side becomes a perfect square. This is rather easily done; the right-hand side is a quadratic function in *y*, which means that it is a perfect square if and only if its discriminant is zero (see the section “Discriminant” in “Quadratic Equations”):

$$(-q)^2-4(p+2u)(u^2+2up+p^2-r)=0$$

Expanding this out, we eventually get

$$8u^3+20pu^2+(16p^2-8r)u+(4p^3-4pr-q^2)=0$$

which is known as the *auxiliary (resolvent) cubic*. From here on, the rest is trivial: solve for *u* using the cubic formula, substitute that into the previous equation, take square roots of both sides, use the quadratic formula to solve for *y*, then perform the back substitutions to solve for *x*.

## The Quartic Formula

(N.B. because of the monstrous length of these formulas, as well as the tedium and error-proneness in deriving them, I have used WolframAlpha, a free online computer algebra system tool, to derive the final formulae.)

It should go without saying that we should expect the quartic formula to be exponentially more complex than the cubic formula, just as the cubic formula is exponentially more complex than the quadratic formula (since the cubic and the quartic formulae depend on quadratic and cubic resolvent polynomials, respectively). Nevertheless, the formula can, in principle, easily be derived.

To begin, let us solve our auxiliary cubic via substitution of values into the cubic formula. After some tedious algebra, we get

$$u=-\frac{5}{6}p+\sqrt[3]{\frac{848p^3-11520r+6192pr+1728q^2+\sqrt{(-848p^3+11520r-6192pr-1728q^2)^2-4(-364p^2+192r)^3}}{27648}}+\sqrt[3]{\frac{848p^3-11520r+6192pr+1728q^2-\sqrt{(-848p^3+11520r-6192pr-1728q^2)^2-4(-364p^2+192r)^3}}{27648}}$$

(We only require one solution since any value of *u* will generate a perfect square, so we may as well opt for the simplest version of the cubic formula.)

Now that we have this value of *u* (which we will denote *u** to save writing space), we may substitute it into the perfect squares equation to create perfect squares on both sides:

$$(y^2+p+u*)^2=(p+2u*)\left(y-\frac{q}{2(p+2u*)}\right)^2=\left(\sqrt{(p+2u*)^2}\,y-\frac{q}{2\sqrt{p+2u*}}\right)^2$$

Now taking square roots of both sides:

$$y^2+p+u*=\pm\left(\sqrt{(p+2u*)^2}\,y-\frac{q}{2\sqrt{p+2u*}}\right)$$

We have two quadratic equations, one for each sign taken:

$$y^2-\sqrt{(p+2u*)^2}\,y+p+u*+\frac{q}{2\sqrt{p+2u*}}=0 \quad (1)$$

$$y^2+\sqrt{(p+2u*)^2}\,y+p+u*-\frac{q}{2\sqrt{p+2u*}}=0 \quad (2)$$

Solving these with the quadratic formula gives us

$$y_1=\frac{\sqrt{(p+2u*)^2}+\sqrt{-3p-2u*-\frac{2q}{\sqrt{p+2u*}}}}{2}$$
$$y_2=\frac{\sqrt{(p+2u*)^2}-\sqrt{-3p-2u*-\frac{2q}{\sqrt{p+2u*}}}}{2}$$
$$y_3=\frac{-\sqrt{(p+2u*)^2}+\sqrt{-3p-2u*+\frac{2q}{\sqrt{p+2u*}}}}{2}$$
$$y_4=\frac{-\sqrt{(p+2u*)^2}-\sqrt{-3p-2u*+\frac{2q}{\sqrt{p+2u*}}}}{2}$$

which, after substituting in *u**, give us the solutions to the depressed quartic.

We finish off with the back substitutions for the general quartic. The algebra grinding is omitted, as it is monstrously tedious to slog through; suffice to say that the final result, the *quartic formula*, is

$$
\begin{aligned}
x_1 = -\frac{b}{4a}
&-\frac{1}{2}\sqrt{\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}} \\
&-\frac{1}{2}\sqrt{\frac{b^2}{2a^2}-\frac{-\frac{b^3}{a^3}+\frac{4bc}{a^2}-\frac{8d}{a}}{4\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}}} \\
&-\frac{1}{3\sqrt[3]{2}}\left[\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{4c}{3a}\right]
\end{aligned}
$$

$$
\begin{aligned}
x_2 = -\frac{b}{4a}
&-\frac{1}{2}\sqrt{\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}} \\
&+\frac{1}{2}\sqrt{\frac{b^2}{2a^2}-\frac{-\frac{b^3}{a^3}+\frac{4bc}{a^2}-\frac{8d}{a}}{4\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}}} \\
&-\frac{1}{3\sqrt[3]{2}}\left[\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{4c}{3a}\right]
\end{aligned}
$$

$$
\begin{aligned}
x_3 = -\frac{b}{4a}
&+\frac{1}{2}\sqrt{\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}} \\
&-\frac{1}{2}\sqrt{\frac{b^2}{2a^2}+\frac{-\frac{b^3}{a^3}+\frac{4bc}{a^2}-\frac{8d}{a}}{4\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}}} \\
&-\frac{1}{3\sqrt[3]{2}}\left[\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{4c}{3a}\right]
\end{aligned}
$$

$$
\begin{aligned}
x_4 = -\frac{b}{4a}
&+\frac{1}{2}\sqrt{\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}} \\
&+\frac{1}{2}\sqrt{\frac{b^2}{2a^2}+\frac{-\frac{b^3}{a^3}+\frac{4bc}{a^2}-\frac{8d}{a}}{4\left(\frac{\frac{b^2}{4a^2}+1}{3\sqrt[3]{2}a}\right)\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{2c}{3a}}} \\
&-\frac{1}{3\sqrt[3]{2}}\left[\left[\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}+\frac{\sqrt[3]{2}(12ae-3bd+c^2)}{\left[3a\sqrt{(-72ace+27ad^2+27b^2e-9bcd+2c^3)^2-4(12ae-3bd+c^2)^3}-72ace+27ad^2+27b^2e-9bcd+2c^3\right]^{1/3}}-\frac{4c}{3a}\right]
\end{aligned}
$$

(zoom in to read the text underneath the radicals) simultaneously extremely 'beautiful' (an algebraic solution to the general quartic purely in terms of its coefficients and elementary algebraic operations) and an utter monstrosity. Even computers are hard-pressed to make use of this formula, as the operations required are often not stable and computationally very expensive.

## Factoring the Depressed Quartic

An alternative, and perhaps more intuitive, solution to the quartic depends on factoring the quartic into two quadratics (much as a quadratic is factored into two linear equations).

Suppose we opt to try and factor the depressed quartic

$$y^4+py^2+qy+r=0$$

as

$$(y^2+\alpha\, y+\beta)(y^2+\gamma\, y+\delta)=0$$

Expanding this, we get

$$y^4+(\alpha+\gamma)\, y^3+(\beta+\delta+\alpha\,\gamma)\, y^2+(\alpha\,\delta+\beta\,\gamma)\, y+\beta\,\delta=0$$

This means that we have the following system of equations to solve (by matching up coefficients):

$$\begin{aligned} \alpha+\gamma&=0 \\ \beta+\delta+\alpha\,\gamma&=p \\ \alpha\,\delta+\beta\,\gamma&=q \\ \beta\,\delta&=r \end{aligned}$$

From the first equation, we may immediately infer

$$\gamma=-\alpha$$

which simplifies the other three equations to

$$\begin{aligned} \beta+\delta-\alpha^2&=p \\ \alpha(\delta-\beta)&=q \\ \beta\,\delta&=r \end{aligned}$$

Now suppose we try and determine the value of $\alpha$. We may note, with a few rearrangements, that

$$\begin{aligned} \delta+\beta&=p+\alpha^2 \\ \delta-\beta&=\frac{q}{\alpha} \end{aligned}$$

We therefore get, with a simple solving of linear equations by elimination of variables,

$$\delta=\frac{1}{2}\left(p+\alpha^2+\frac{q}{\alpha}\right)$$

and

$$\beta=\frac{1}{2}\left(p+\alpha^2-\frac{q}{\alpha}\right)$$

Now since

$$\beta\,\delta=r$$

we may substitute our values of $\beta$ and $\delta$ to find

$$\frac{1}{4}\left(p+\alpha+\frac{q}{\alpha}\right)\left(p+\alpha-\frac{q}{\alpha}\right)=r$$

Expanding and simplifying, we get

$$p^2+2p\alpha^2+\alpha^4-\frac{q^2}{\alpha^2}=4r$$

and then, after more rearranging,

$$\alpha^6+2p\alpha^4+(p^2-4r)\alpha^2-q^2=0$$

which is a sixth-degree equation in $\alpha$. But because all of the powers of $\alpha$ are even, we may make the substitution

$$z=\alpha^2$$

which gives us a resolvent cubic, as before.

This can be solved to find *z*, and then $\alpha$, $\beta$, and $\delta$, which then become the coefficients of the quadratic factors; each quadratic can then be solved with the quadratic formula to yield the four solutions of the depressed quartic, and then the final back substitutions can be performed to give the solutions to the general quartic – the quartic formula.

## To Infinity And Beyond

*"$10^{50}$ is a long way away from infinity."*
– *Daniel Shanks,* Solved and Unsolved Problems in Number Theory

While the techniques used to solve each degree of polynomial equation thus far are rather ingenious, justification of each step, particularly the choice of auxiliary variables, is an overarching issue throughout, and more significantly, each degree of polynomial equation requires its own battery of techniques – what works well for one class of equations fails miserably for another (completing the square is a good example; it works a charm on quadratics, but the equivalent technique fails miserably on cubics).

In the late 18th century, Lagrange began the groundwork for remedying this issue, by unifying the techniques for solving these equations. He succeeded in producing a single, unified method, based on the theory of permutations and what are known as *Lagrange resolvents* – both precursors to modern abstract algebra. However, his new techniques, while successful in solving quadratics, cubics, and quartics, failed to solve any higher-degree equations – in particular, the general quintic equation could not be reduced to the solution of a quartic, as the resolvent polynomial was of higher degree than the original quintic. (Consider that, in contrast, quartics are reduced to resolvent cubics and cubics are reduced to resolvent quadratics.)

Soon, a little known mathematician by the name of Ruffini came along and published an extremely long and complex proof attempting to show that the algebraic solution of the quintic was, in fact, *impossible*. While mostly correct, it was largely ignored, until another, equally obscure mathematician, Abel, came along a little while later and published a much shorter, more elegant version of the proof (which also corrected the small deficiency present in Ruffini's proof). The result, known as the *Abel-Ruffini theorem*, states that no polynomial equation of degree five or greater has an elementary algebraic solution.

Note that the theorem does *not* state that such equations are unsolvable – in fact, *every* polynomial equation has solutions, by the fundamental theorem of algebra. Nor does the theorem preclude the possibility of analytic solutions in general – in fact, many analytic solutions for the quintic equation, and even higher-degree equations, have been developed. The theorem only states that any analytic solutions cannot be written in purely elementary algebraic terms – that is, using only the arithmetical operators, exponentiation, and radicals. At least one other function (one simple example is the *ultraradical* – the inverse of $x^5+x$) must be included for the analytic solution of the quintic (and higher-degree equations) to be possible. (See Appendix B for a brief explanation.)

---

In the modern age, the importance of analytical solutions to polynomial equations has largely declined, as techniques from calculus, combined with computers, provide a very efficient means to approximate solutions, both real and complex, to any degree of accuracy we desire. However, such solutions continue to fascinate pure mathematicians and, despite initially being a topic of elementary algebra, are now almost exclusively analyzed using more advanced branches of mathematics, which are beyond the scope of this work.

---

We have now been brought to, and seen, the very limits of elementary algebra; as has been proven, no more can be accomplished without leaving this realm of math. Nevertheless, the solutions of linear, quadratic, cubic, and quartic polynomial equations still stand today as a symbol of the tenacity and ingenuity of early mathematicians, and a testament to the sheer power of such a 'limited' and 'elementary' branch of mathematics.

# *Appendix A: General Concepts and Techniques*

Listed here are several concepts, techniques, and theorems that are often useful for the purposes of solving general polynomial equations. The list is by no means complete or comprehensive, but these are the major analytic techniques available from elementary algebra. Proofs of these techniques often require calculus, number theory, and/or more advanced forms of algebra, all of which are beyond the scope of this work.

## Imaginary and Complex Numbers

Although the real numbers are sufficient for the purposes of solving many polynomial equations, we are still left unable to solve many equations, such as $x^2+1=0$.

To remedy this, we can create a number *i*, and give it this definition:

$$i=\sqrt{-1}$$

All numbers of the form *bi*, then, where *b* is a real number, are defined as *imaginary numbers*, so called primarily due to historical resistance against their usage (and to distinguish them from real numbers). We can then combine them with real numbers, we can create the general *complex number*, *a*+*bi*, where *a* and *b* are real numbers.

---

Addition, subtraction, and multiplication on complex numbers is easily performed by treating *i* like any ordinary variable and using standard rules for adding, subtracting, and multiplying binomials. (The only point to remember is to replace any occurrences of $i^2$ with -1.)

Division of complex numbers is a little trickier. To divide complex numbers, we must first explore the concept of a *complex conjugate*, which is simply a complex number with the sign of its imaginary part reversed – that is, given a complex number *a*+*bi*, its conjugate is *a*-*bi*.

The importance of a conjugate shows when you multiply a complex number by its conjugate – after the algebraic dust settles, the end result is $a^2+b^2$, which has the interesting (and useful) property of being a non-negative real number. Thus, to divide two complex numbers, simply multiply the dividend and divisor by the divisor's conjugate – the divisor will then become a real number, which can then be split amongst the real and imaginary parts of the dividend.

---

A more verbose, detailed treatment of complex numbers, and further information, may be obtained from other sources.

Note: in many engineering disciplines, the imaginary unit is denoted with *j*, to avoid confusion with the symbol for electrical current (also *i*).

## Remainder Theorem

*Theorem: if a polynomial function* f(x) *is divided by a linear factor of form* x-r*, the remainder is* f(r)*.*

Proof: the division statement for this division may be written as

$$f(x)=(x-r)g(x)+h$$

where *g(x)* is the quotient polynomial and *h* is the remainder. Now suppose we substitute *r* for *x*:

$$f(r)=(r-r)g(r)+h=h$$

Hence the remainder is equivalent to *f(r)*. □

## Factor Theorem

*Theorem: if there exists a number* k *such that* f(k)=0 *for some polynomial function* f(x) *– that is,* k *is a solution of the polynomial equation* f(x)=0 *– then* x-k *is a factor of the polynomial.*

Proof: by the remainder theorem, when *f(x)* is divided by *x-k*, the remainder must be 0, since *f(k)=0*. Hence the division statement reads

$$f(x)=(x-k)g(x)$$

since the remainder is 0, and we are done. □

*Corollary: every polynomial function of degree* n*, if it has* n *solutions, can be factored into* n *linear factors of form* (x-k)*, where* k *is a solution to the polynomial equation* f(x)=0*.*

Proof: By induction. The statement is obviously true for a linear function. Now suppose it is true for a polynomial function of degree *n*:

$$f(x)=(x-k_1)(x-k_2)\dots(x-k_n)$$

Now for a polynomial function of degree *n+1*:

$$g(x)=(x-k_1)h(x)$$

by the Factor Theorem, where *h(x)* is a polynomial function of degree *n* by polynomial division. (The solution is guaranteed to exist since the assumption is that the polynomial equation has *n+1* solutions.) Then by the induction hypothesis,

$$g(x)=(x-k_1)(x-k_2)\dots(x-k_n)(x-k_{n+1})$$

as required. □

## Rational Root Theorem

*Theorem: if a polynomial equation only contains integer coefficients, all rational solutions, if any exist, are of form* p/q*, where* p *is a factor of the constant term and* q *is a factor of the leading coefficient. (Either is, of course, permitted to be negative.)*

The proof is outside of the scope of elementary algebra.

## Solving Polynomial Equations by Factoring

The remainder, factor, and rational root theorems can be combined to formulate a technique for tackling general polynomial equations (with strictly rational coefficients). While rather tidy and elementary, the technique is severely lacking in generality, and thus, as a general rule of thumb, unreliable; nevertheless, as many polynomial equations that arise in practice do have strictly rational coefficients, the technique is worth exploring.

The technique: suppose we have a polynomial equation of form

$$f(x)=a_n x^n+a_{n-1}x^{n-1}+\dots++a_2x^2+a_1x+a_0=0$$

where all the coefficients are rational numbers.

1. If the polynomial equation is linear or quadratic, solve it directly.
2. Otherwise, we now assume the polynomial equation is of degree 3 or greater.
3. If the polynomial's coefficients are rational but not integral, multiply the polynomial by the lowest common denominator of the rational coefficients, such that the new polynomial equation has strictly integral coefficients.
4. Use the rational root theorem to guess all possible rational solutions to the polynomial equation. If one cannot be found, this technique is useless; henceforth, we now assume one has been found.
5. Divide the polynomial equation by the corresponding linear factor (synthetic division is a good choice) to create a new polynomial of degree *n-1*.
6. Repeat steps 2-5 until the resulting polynomial equation is quadratic. Then finish off with the quadratic formula.

## Example 1

Suppose we aim to solve the cubic equation

$$x^3 - 15x - 4 = 0$$

The coefficients are already integral, so we do not need to modify them.

Now, by the rational root theorem, any rational solution must have a numerator which is a factor of 4, and a denominator which is a factor of 1. As the only factor of 1 is 1, the denominator must be 1 – in other words, any rational solution must also be integral. This limits our possible choices to 1, -1, 2, -2, 4, and -4 (the positive and negative factors of 4). Trying all of these in some random order, we will find that 4 is a solution. Hence, by the factor theorem, we may now factor the cubic equation as

$$(x+4)g(x) = 0$$

where *g(x)* is a quadratic polynomial whose coefficients we do not know yet. Obtaining them requires dividing the original equation by *x+4*; we may do so with synthetic division.

$$\begin{array}{r|rrrr} -4 & 1 & 0 & -15 & 4 \\ & \downarrow & -4 & 16 & -4 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

Thus

$$g(x) = x^2 - 4x + 1$$

and since this is a quadratic, we can solve it with the quadratic formula. We find

$$x = 2 \pm \sqrt{3}$$

and we are done. □

## Descartes' Rule of Signs

In order to streamline the process of solving equations by factoring, one can make use of *Descartes' rule of signs*, which gives hints as to the number of positive and negative solutions (useful in conjunction with the rational root testing).

*Theorem: if given a polynomial equation* f(x)=0*, of form*

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

*where all the coefficients are real, the number of positive solutions is equal to the number of sign changes between consecutive, nonzero coefficients, or less than it by a multiple of 2. Similarly, the number of negative solutions is equal to the number of non-sign changes between consecutive, nonzero coefficients, or less than it by a multiple of 2.*

The proof is omitted.

## Example 1

Suppose we have the polynomial equation

$$x^3 - x^2 + x - 1 = 0$$

The sign changes here are
- $x^3$ to $x^2$ (+ to -)
- $x^2$ to $x$ (- to +)
- $x$ to 1 (+ to -)

and there are no non-sign changes. Therefore:
- There must be 3 or 1 positive solutions (since the number of positive solutions equals the number of sign changes, or possibly less by a multiple of 2).
- There cannot be any negative solutions (since the sign never remains the same).
- Since a cubic equation has three solutions (by the fundamental theorem of algebra), and either 1 or 3 of the solutions are real and positive, either 0 or 2 solutions are complex.

In fact, factoring this polynomial gives us

$$(x^2+1)(x-1)=0$$

indicating one positive real solution (1) and two complex solutions ($i$, $-i$).

## Partial Cubic and Quartic Formulae

In order to streamline the process of solving equations by factoring, one can derive *partial* cubic and quartic formulae to immediately solve a cubic or quartic equation, if one is given some of the roots in advance.

## Partial Cubic Formula (given two solutions)

Suppose our two solutions are called $r_1$ and $r_2$, respectively. We seek the third solution, $r_3$, to the general cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

By the Factor Theorem, we may factor this into

$$(x-r_1)(x-r_2)(mx+n)=0$$

where $m$ and $n$ are the coefficients of the unknown factor. To determine these coefficients, a simple expansion is sufficient:

$$mx^3 + (n - m(r_1+r_2))x^2 + (mr_1r_2 - n(r_1+r_2))x + nr_1r_2 = 0$$

Then matching coefficients gives us

$$m=a$$

$$n-m(r_1+r_2)=b$$
$$n=b+a(r_1+r_2)$$

Thus the solution to the linear factor, and the final solution to the cubic, is

$$r_3=-\frac{b}{a}-(r_1+r_2)$$

## Partial Cubic Formula (given one solution)

As with the partial cubic formula for two given solutions, we will factor the cubic; however, this time, as we only know one solution ($r_1$), the cubic will be factored into a linear and a quadratic polynomial:

$$(x-r_1)(mx^2+nx+p)=0$$

As before, determining the coefficients is done through expansion:

$$mx^3+(n-mr_1)x^2+(p-nr_1)x-pr_1=0$$

Equating coefficients:

$$m=a$$

$$n-mr_1=b$$
$$n=b+ar_1$$

$$p-nr_1=c$$
$$p=c+(b+ar_1)r_1$$
$$p=c+br_1+ar_1^2$$

Now that we have the coefficients, the quadratic formula gives us

$$r_2=\frac{-b-ar_1+\sqrt{-3a^2r_1^2-2abr_1+b^2-4ac}}{2a}$$
$$r_3=\frac{-b-ar_1-\sqrt{-3a^2r_1^2-2abr_1+b^2-4ac}}{2a}$$

as the other two solutions to the cubic

$$ax^3+bx^2+cx+d=0$$

## Partial Quartic Formula (given three solutions)

The process for finding a partial quartic formula for the general quartic equation

$$ax^4+bx^3+cx^2+dx+e=0$$

is very similar to that used for the cubic (given two solutions), so the derivation will be omitted. The final result is

$$r_4 = -\frac{b}{a} - (r_1 + r_2 + r_3)$$

## Partial Quartic Formula (given two solutions)

If we only have two solutions, the quadratic formula can be used to find the last two solutions (again, the derivation is omitted since the process for finding them is similar to that used for the cubic). The partial quartic formula in this case becomes

$$r_3 = \frac{-b - a(r_1 + r_2) + \sqrt{b^2 - 4ac + (-2ab - 4a^2)(r_1 + r_2) - 3a^2(r_1^2 + r_2^2) - 6a^2 r_1 r_2}}{2a}$$

$$r_4 = \frac{-b - a(r_1 + r_2) - \sqrt{b^2 - 4ac + (-2ab - 4a^2)(r_1 + r_2) - 3a^2(r_1^2 + r_2^2) - 6a^2 r_1 r_2}}{2a}$$

## Partial *n*th-degree Polynomial Formula (given *n-1* solutions)

We can easily generalize the partial cubic and quartic formulas given two and three solutions, respectively.

*Theorem: for a polynomial equation of degree* n*, if given* n-1 *solutions of form* $r_k$*, for* k *between 1 and n-1 inclusively, the final solution is*

$$r_n = -\frac{b}{a} - \sum_{k=1}^{n-1} r_k$$

*where* a *is the leading coefficient and* b *is the coefficient of the second-highest power.*

Proof: by the Factor Theorem, we may factor the polynomial as

$$(x - r_1)(x - r_2)\ldots(x - r_{n-1})(mx + n) = 0$$

for some unknown *m* and *n*. Now if we expand this out, the leading coefficient *a* must be equal to *m*, since the leading term is created solely through all the *x*'s multiplied together. As for the second-highest power, it combines

- when *n* is multiplied by all the *x*'s
- when *mx* is multiplied by all the other *x*'s, except for one, where it will instead be multiplied by one of the *r*'s

So we get that the coefficient of the second-highest power is

$$b = n - m(r_1 + r_2 + \ldots + r_{n-1})$$

(the subtraction is simply a result of factoring all the negative signs out of the *r*'s). Since we already know the value of *m*, the value of *n* becomes

$$n = b + a(r_1 + r_2 + \ldots + r_{n-1})$$

So the solution to the final linear factor is

$$r_n = \frac{-n}{m} = -\frac{b}{a} - \sum_{k=1}^{n-1} r_k$$

as required. □

## Discriminant

While normally thought of as a property exclusive to quadratic polynomials, the concept of a discriminant extends to all polynomials.

The *discriminant* of a polynomial, usually denoted Δ, is an expression, composed of terms using the coefficients of the polynomial equation, that determines the nature of the roots of the polynomial. In particular, **it is equal to zero if and only if the polynomial has at least one multiple root.**

More specific information can generally be obtained if the coefficients are all real (which then gives hints as to the number of real and complex solutions).

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The discriminant of a linear equation

$$ax+b=0$$

is

$$1$$

which states, rather redundantly, that no linear equation has a multiple root.

---

The discriminant of a quadratic equation

$$ax^2+bx+c=0$$

is

$$b^2-4ac$$

If the coefficients are all real, the following information may be obtained from the discriminant:

- If $\Delta > 0$, the equation has two distinct real solutions.
- If $\Delta = 0$, the equation has two identical real solutions (a double root).
- If $\Delta < 0$, the equation has two distinct complex solutions, which are conjugates of each other (as per the complex conjugate solution theorem; see the appropriate section).

---

The discriminant of a cubic equation

$$ax^3+bx^2+cx+d=0$$

is

$$b^2c^2-4b^3d-4ac^3+18abcd-27a^2d^2$$

If the coefficients are all real, the following information may be obtained from the discriminant:

- If $\Delta > 0$, the equation has three distinct real solutions.
- If $\Delta = 0$, the equation has either a double root (and another, distinct single root), or a triple root. In either case, the roots are all real.
- If $\Delta < 0$, the equation has a single real solution, and two distinct complex solutions, which are conjugates of each other (as per the complex conjugate solution theorem; see the appropriate section).

---

The discriminant of a quartic equation

$$ax^4+bx^3+cx^2+dx+e=0$$

is

$$b^2c^2d^2 - 4b^3d^3 - 4ac^3d^2 + 18abcd^3 - 27a^2d^4 + 256a^3e^3$$
$$+e\left(-4b^2c^3 + 18b^3cd + 16ac^4 - 80abc^2d - 6ab^2d^2 + 144a^2cd^2\right)$$
$$+e^2\left(-27b^4 + 144ab^2c - 128a^2c^2 - 192a^2bd\right)$$

If the coefficients are all real, the following information may be obtained from the discriminant:

- If $\Delta > 0$, the equation has one of
  - four distinct real solutions
  - four distinct complex solutions (which will form two pairs of complex conjugates, as per the complex conjugate solution theorem; see the appropriate section).
- If $\Delta = 0$, the equation has one of
  - a double real root, and two other, distinct real/complex solutions (if complex, they will be conjugates of each other)
  - a triple real root, and one other, distinct real solution
  - a quadruple real root
- If $\Delta < 0$, the equation has two distinct real solutions, and two distinct complex solutions, which will be conjugates of each other.

---

A general expression for discriminants of polynomial equations may be formed through use of linear and abstract algebra, which is beyond the scope of this work.

## The Fundamental Theorem of Algebra

One of the primary motivations for extending the real number system into the complex number system is that unlike the real numbers, the complex numbers are *algebraically closed*. In elementary terms, this states that every polynomial equation, *even if its coefficients are all complex numbers*, can only have solutions that are likewise complex numbers. The same is not true for the real numbers, as, even if the coefficients of a polynomial equation are all real, the solutions need not be likewise real (the archetypical example being $x^2+1=0$).

This algebraical closure property of the complex numbers is known as the *fundamental theorem of algebra*, which, in its most concise form, states that the field of complex numbers is algebraically closed. More verbosely, but equivalently, it asserts that every polynomial equation with complex coefficients has at least one complex solution.

Usually, however, the theorem is stated as follows:

**Every polynomial equation (with complex coefficients) of degree *n* has at most *n* solutions, and in fact, is guaranteed *n* solutions if we count solutions up to their multiplicity and include complex solutions.**

Although this appears to be a stronger statement than the others, it is a direct consequence of the fact that all degree-*n* polynomials can theoretically be factored into *n* linear factors (see the section “Factor Theorem”).

It might be noted that, despite the name, the theorem is neither 'fundamental', nor a 'theorem of algebra'. The theorem's name was given at a time when 'algebra' only meant 'elementary algebra' (i.e. it is fundamental to elementary algebra, but not algebra has a whole), and furthermore, all proofs of it require at least a small measure of analysis (at a minimum, the intermediate value theorem).

## Radical Conjugate Solutions

*Theorem: if one solution of a polynomial equation* f(x)=0 *with strictly rational coefficients is of form*

$$a+\sqrt{b}$$

*where* a *and* b *are rational and* √b *is irrational, then*

$$a-\sqrt{b}$$

*is also a solution.*

Proof: let us define a polynomial *d(x)* as follows:

$$d(x)=(x-(a+\sqrt{b}))(x-(a-\sqrt{b}))=(x-a)^2-b$$

Then by polynomial division,

$$f(x)=d(x)q(x)+cx+e$$

for some quotient polynomial *q(x)* and a remainder, which is a linear polynomial of form *cx+e*, where *c* and *e* are rational numbers (they must be, since all of *f*'s coefficients are rational). Our objective will be to show that *c* and *e* must equal zero (for then, by the Factor Theorem, *d(x)* is a factor of *f(x)*, so *d(x)*'s factors are also factors, and thus solutions, of *f(x)=0*).

To do this, consider what happens when we substitute

$$a+\sqrt{b}$$

for *x*. As it is a factor of *d(x)*, only the remainder is left, so our polynomial equation becomes

$$f(a+\sqrt{b})=c(a+\sqrt{b})+e=0$$

since $a+\sqrt{b}$ is a factor of *f*. Now suppose *e* is not zero. It must then be equal to

$$-c(a+\sqrt{b})$$

which is irrational, a contradiction of the fact that *e* is known to be rational. So *e* must be zero. But then *c* must be zero as well in order to preserve the equality of the polynomial equation. □

## Complex Conjugate Solutions

*Theorem: if one solution of a polynomial equation with strictly real coefficients is a complex number, that number's conjugate is also a solution.*

Proof: Suppose one solution to the polynomial equation *f(x)=0*, with real coefficients, has a complex solution *a+bi*. We aim to show that *a-bi* is also a solution. Therefore, let us define a polynomial *g(x)* as follows:

$$g(x)=(x-(a+bi))(x-(a-bi))=(x-a)^2+b^2$$

By polynomial division,

$$f(x)=g(x)q(x)+cx+d$$

for some quotient polynomial *q(x)* and a linear remainder of form *cx+d*, where *c* and *d* must be real (since *f*'s coefficients are real). We aim to show that *c* and *d* are zero, which establishes *g(x)* as a factor of *f(x)*, meaning that the factors of *g* are likewise factors (and thus solutions) of *f(x)=0*.

So suppose we substitute *a+bi* for *x*. We obtain (since it is a factor of *g(x)*, the middle term disappears)

$$f(a+bi)=c(a+bi)+d=0$$

Now suppose *d* is nonzero. Then it equals *-c(a+bi)*, a non-real number, contradicting the real nature of *d*. So *d* is zero. But then *c* is forced to be zero as well to maintain equality. □

## Real Solutions to Odd-Degree Polynomial Equations

*Theorem: if an odd-degree polynomial equation only contains real coefficients, at least one of the solutions to the equation must be real.*

### Proof 1

By the complex conjugate solution theorem (see the appropriate section), all complex solutions must be paired with their conjugates. Thus, with an odd-degree polynomial equation, which factors into an odd number of linear factors (by the fundamental theorem of algebra), at least one of the factors must be left unpaired, requiring that particular solution to be real if the polynomial's coefficients are all real. □

### Proof 2

Since there are an even number of coefficients for an odd-degree polynomial equation (some of them may be zero), there are likewise an even number of signs (+ or -) attached to each coefficient; therefore, there are an odd number of potential sign changes.

Now suppose the number of sign changes is odd. Then the number of positive solutions is odd, by Descartes' rule of signs (which we may use since all the coefficients are real), and thus there cannot be no positive solutions, as zero is an even number; hence we are guaranteed a real solution. Otherwise, if the number of sign changes is even, the number of negative solutions is odd (because if there are an even number of sign changes, there must be an odd number of non-sign changes, as the only way to add two numbers to get an odd number is if one is odd and the other is even), and thus there cannot be no negative solutions; hence, yet again, we are guaranteed a real solution. □

# *Appendix B: Quintic Equations*

A *quintic equation* is a polynomial equation of degree 5, taking the general form

$$ax^5+bx^4+cx^3+dx^2+ex+f=0$$

(where *a* is nonzero). As the section “To Infinity and Beyond” mentions, the general quintic equation is *unsolvable* in the realm of elementary algebra. (Of course, many specific quintics *can* be solved using only elementary algebra.) However, a brief glimpse of the work surrounding these fifth-degree polynomials will be shown here.

## Elimination of Terms

While the general quintic cannot be solved completely using only elementary algebraic operations, it can be remarkably reduced with them.

### Depressed Quintic

The general quintic is easily depressed (i.e. second-highest degree term removed) via a linear Tschirnhaus transformation, in an identical fashion to that used by the solutions to the cubic and quartic. The substitution to make, of course, is $x=y-b/5a$.

### Principal Quintic

A *quadratic* Tschirnhaus transformation can be used to eliminate both the second- and third-highest degree terms, leaving one with a quintic of form

$$y^5+py^2+qy+r=0$$

which is known as a quintic in *principal form*.

### Bring-Jerrard Quintic

From the principal quintic, it is possible to simplify even further and remove the quadratic term, bringing one to what is known as the *Bring-Jerrard form* of the quintic:

$$z^5+uz+v=0$$

As a linear Tschirnhaus transformation removes the quartic term and a quadratic Tschirnhaus transformation the cubic term, it would seem, intuitively, that a cubic Tschirnhaus transformation would do the trick for reducing a quintic to this form – but, unfortunately, that is not the case (as Tschirnhaus himself found out when attempting to solve the quintic). Instead, a *quartic* Tschirnhaus transformation must be used for the reduction to this form.

## Bring Normal Quintic

A simple substitution of variables enables one last simplification. Suppose we have the Bring-Jerrard quintic

$$z^5+pz+q=0$$

We can make the substitution

$$w=\frac{z}{\sqrt[4]{p}}$$

(assuming *p* is not zero; if it is, the quintic is trivially solvable) which brings us to a quintic of form

$$w^5+w+k=0$$

where

$$k=\frac{q}{(\sqrt[4]{p})^5}$$

and the linear coefficient is guaranteed to be 1. Hence, elementary algebra can bring us all the way down to this form, the *Bring normal form* – but to go any further and solve the quintic itself requires at least one additional, non-elementary function. The simplest example is given in the next section.

## Ultraradicals and the Quintic Formula

An *ultraradical* (also known as a Bring radical) is a non-elementary function, which has the property of being the inverse of the function

$$f(x)=x^5+x$$

analogous to the normal radical of 5th degree, which is the inverse of the function

$$f(x)=x^5$$

It is clear that the ultraradical permits us to solve the equation

$$w^5+w+k=0$$

Indeed, the solution is trivially

$$w=BR(-k)$$

where BR denotes the ultraradical (Bring radical). By back substitution, the solution to the general Bring-Jerrard quintic

$$z^5+pz+q=0$$

is

$$z = (\sqrt[4]{p})\, BR\left(-\frac{q}{(\sqrt[4]{p})^5}\right)$$

and further back substitutions will eventually yield the solution to the general quintic – a quintic formula, in terms of elementary algebraic operations and the ultraradical. However, needless to say, the formula is indescribably tedious to derive and write out.

# *End of Document*