On the Algebraic Solution of the Quartic, version 2

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Preamble

Since I have the impression that my paper "On the Algebraic Solution of Polynomial Equations" wasn't particularly well-received (due to the terse and highly mathematical nature of the paper), I decided to rewrite the algebraic solution of the quartic in a manner which I hope is a bit more intuitively accessible.

The Solution of the Quartic, Part 1

The general quartic equation is of form

$$ax^4+bx^3+cx^2+dx+e=0$$

As with the cubic, we first seek to be rid of the second-highest power (the fewer terms, the better, right?), which in this case is the cubic term. If we divide everything by *a* and then make the substitution

$$y=x+\frac{b}{4a}$$

we eventually end up with

$$y^4+py^2+qy+r=0$$

which is a *depressed quartic equation*.

The Solution of the Quartic, Part 2

The really nice thing about solving the quartic is that the process is a lot more intuitive than that used for the cubic, because we can capitalize on one very intuitive idea: what if we factor the depressed quartic into two quadratics? If we can do that, our job is basically done, since we already know how to solve quadratic equations.

So let's do that. In mathematical terms, we want two quadratics of form

$$(y^2+\alpha y+\beta)(y^2+\gamma y+\delta)=0$$

and our task is to determine what these coefficients will be. To do that, let's expand this expression out:

$$y^4+(\alpha+\gamma)y^3+(\beta+\delta+\alpha\gamma)y^2+(\alpha\delta+\beta\gamma)y+\beta\delta=0$$

Matching up coefficients with the depressed quartic:

$$\begin{aligned}\alpha+\gamma&=0\\ \beta+\delta+\alpha\gamma&=p\\ \alpha\delta+\beta\gamma&=q\\ \beta\delta&=r\end{aligned}$$

The first equation immediately gives us

$$\gamma=-\alpha$$

which simplifies the other three equations to

$$\begin{aligned}\beta+\delta-\alpha^2&=p\\ \alpha\delta-\beta\alpha&=q\\ \beta\delta&=r\end{aligned}$$

Now let's try and eliminate some more variables. From the first two equations, we can rearrange them to get

$$\begin{aligned}\beta+\delta&=p+\alpha^2\\ \alpha(\delta-\beta)&=q\end{aligned}$$

and then rearrange once more to get

$$\begin{aligned}\delta+\beta&=p+\alpha^2\\ \delta-\beta&=\frac{q}{\alpha}\end{aligned}$$

Solving this system of linear equations gives us

$$\begin{aligned}\delta&=\frac{1}{2}\left(p+\alpha^2+\frac{q}{\alpha}\right)\\ \beta&=\frac{1}{2}\left(p+\alpha^2-\frac{q}{\alpha}\right)\end{aligned}$$

We can now use these for the last equation to solve for α:

$$\begin{aligned}\beta\delta&=r\\ \left(\frac{1}{2}\right)\left(p+\alpha^2+\frac{q}{\alpha}\right)\left(\frac{1}{2}\right)\left(p+\alpha^2-\frac{q}{\alpha}\right)&=r\end{aligned}$$

Expanding this out and simplifying gives us

$$p^2+2p\alpha^2+\alpha^4-\frac{q^2}{\alpha^2}=4r$$

and then multiplying through by α^2 and rearranging gives us

$$\alpha^6+2p\alpha^4+(p^2-4r)\alpha^2-q^2=0$$

which is a sixth-degree polynomial equation in α. But because all of the powers of α are even, this is really only a third-degree polynomial equation once we make the substitution

$$z=\alpha^2$$

to get

$$z^3+2pz^2+(p^2-4r)z-q^2=0$$

Now use the cubic formula to solve for *z* (any solution will do). Then take the square root (either the positive or negative solution will do, as the quadratics have nearly identical coefficients except for signs, so picking one or the other merely swaps the order of the quadratics) to get α.

Use this value of α to get the values of β, γ, and δ. With these, the coefficients of the quadratics are now all known, so proceed to use the quadratic formula on each of the quadratic factors, which solves the depressed quartic. Then finish with the final backwards substitutions to solve the original, general quartic.

(In the general case – that is, if one were to derive the quartic formula in its entirety – the result is exponentially more complex than the cubic formula, and I'm not even going to bother trying to write it down here. My original paper has it, or you can run a Google search for "quartic formula".)