Solving Permutation Expressions for *n, r*

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The general permutation expression is of form

$$nPr = \frac{n!}{(n-r)!} = k$$

One of the problems you are asked to solve in Principles of Math 12 involves making either n or r the unknown, and then solving for it, given the other values. No formulae are given for doing so, but I insisted formulae had to exist and I decided to try and derive them myself.

SOLVING FOR *n* GIVEN *r*

The first case, solving for n, always yields a polynomial equation of degree r. To see that this is so: we know that when n! is divided by (n—r)!, we will have

$$(n)(n-1)(n-1)\ldots(n-r+1)$$

Expressed using pi notation, this becomes:

$$\prod_{j=0}^{r-1}(n-j)$$

The number of terms is then the *(upper bound—lower bound + 1)*, which is simply r.

Since this new expression is still equal to k, subtracting k from both sides gives us

$$(n)(n-1)(n-1)\ldots(n-r+1)-k=0$$

which is clearly a polynomial equation.

Since there is no general formula for the general polynomial equation, there is also no general formula for n given any value of r. However, we can have formulae for certain given values of r.

(In the special case when r is 0, n can be any whole number, since ${}_nP_0 = 1$for any whole number value of n. This case is of little practical importance in Math 12, since it never shows up.)

If r is 1, the polynomial equation is

$$n - k = 0$$

and the solution is clearly

$$n = k.$$

If r is 2, the polynomial equation is

$$(n)(n-1) - k = n^2 - n - k = 0$$

Applying the quadratic formula gives us

$$n = \frac{1 \pm \sqrt{1+4k}}{2}$$

However, the "minus" solution must be rejected. To see why, consider that since k is always a natural number, $\sqrt{1+4k}$ will always be greater than 1. Thus, using the "minus" solution yields a negative answer, which is clearly not a possible value for n. Thus the (only) solution is:

$$n = \frac{1 + \sqrt{1+4k}}{2}$$

If r is 3, the polynomial equation is:

$$(n)(n-1) - k = n^3 - 3n^2 + 2n - k = 0$$

and the solution, by application of the cubic formula, is

$$n = 1 - \frac{1}{3}\left(\sqrt[3]{\frac{1}{2}(-27k + \sqrt{729k^2 - 108})} + \sqrt[3]{\frac{1}{2}(-27k - \sqrt{729k^2 - 108})}\right)$$

If r is 4, the polynomial equation is:

$$(n)(n-1) - k = n^4 - 6n^3 + 11n^2 - 6n - k = 0$$

and the solution, by application of the quartic formula, is

$$n = \frac{\sqrt{3\sqrt{k+1} + 5} + 3}{2}$$

(Surprised at its simplicity compared to the cubic?)

These formulae are not so complicated that their use is impractical, but it may be easier to simply use the rational root theorem (which will always work, since n has to be a natural number) for polynomial equations of degree 3 and above.

Note: polynomial equations of degree 5+ have no general solution in the realm of elementary algebra—this is the Abel-Ruffini theorem.

SOLVING FOR *r* GIVEN *n*

Solving for r is, in some ways, easier than solving for n. Unlike the case with n, there is *one* formula for r that will work for all values, but it requires the invention of a new function—the inverse factorial function.

The *inverse factorial function* (which I denote using invfact(x), but you're quite free to make up your own notation) is simply the function that "undoes" a factorial operation. For example, 3! is 6, so the inverse factorial of 6 is 3.

To actually perform the inverse factorial operation, this is the algorithm I came up with

Divide the argument by each consecutive natural number (beginning from 1) until only 1 is left. At this point, the last number you divided by is the inverse factorial.

For example, to find the inverse factorial of 120, we divide it by 1 (120 is left), then 2 (60 is left), then 3 (20 is left), then 4 (5 is left), then 5 (1 is left)—and since 1 is left, the last number we divided by, 5, is the inverse factorial of 120.

The actual formula for r itself is simply:

$$r = n - invfact\left(\frac{n!}{k}\right)$$

One minor snag: since both 0! and 1! are 1, the inverse factorial of 1 is ambiguous. The solution here is to simply return both, and consider both cases separately (i.e. n-0 and n-1), since there will always be two answers (except when n is 0). I know this means that the inverse factorial function won't technically be a *function* in the sense that there will be only one output value for every valid input value, but that's a minor nitpick.