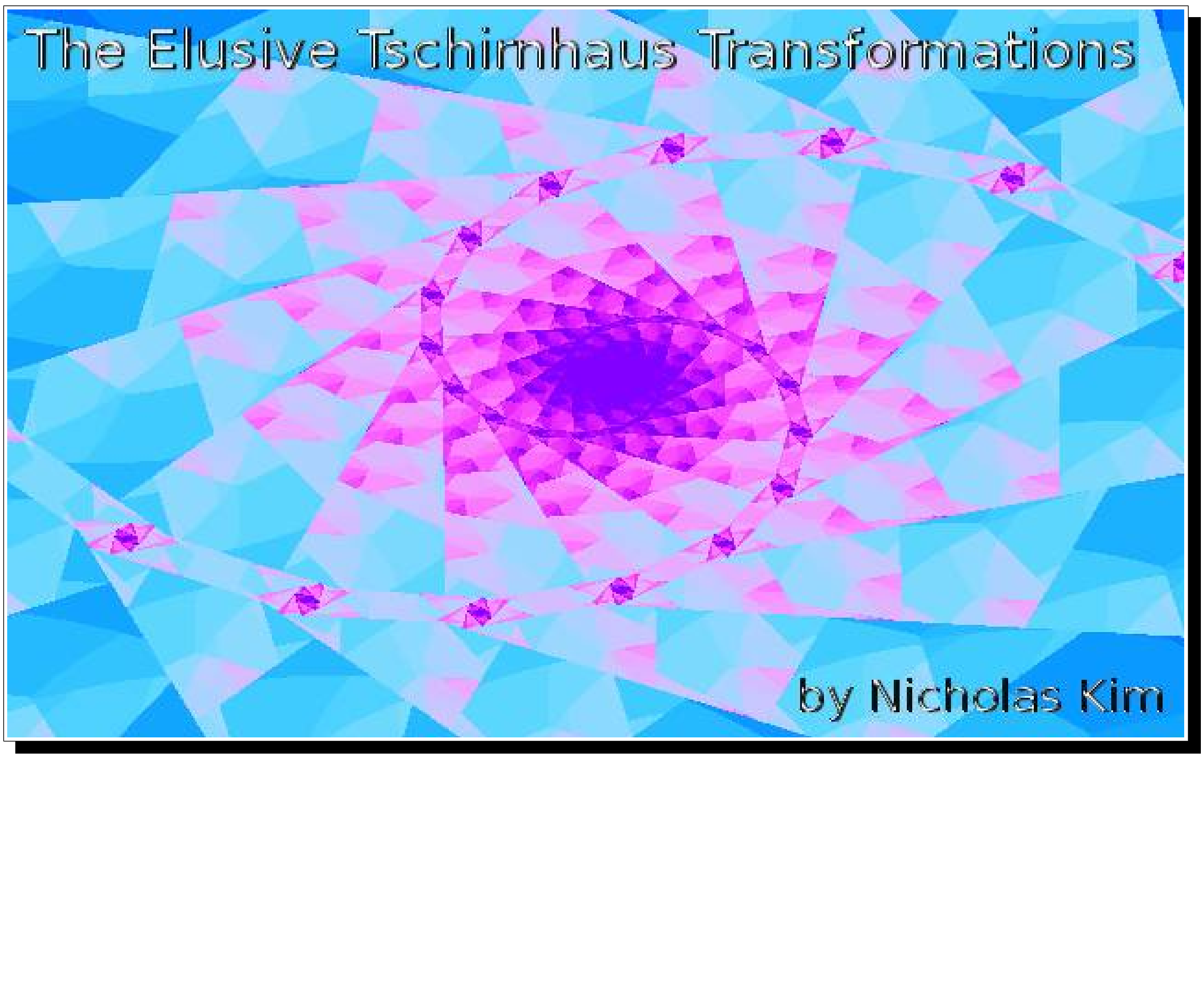
The Elusive Tschirnhaus Transformations
by Nicholas Kim

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Preamble

Mathematical papers on analytical solutions to polynomial equations, including my previous paper "On the Algebraic Solution of Polynomial Equations" (hereafter simply referred to as "my previous paper" for conciseness), generally make some reference to Tschirnhaus transformations, claiming that using them is a useful technique for reducing the number of terms in a polynomial equation. Yet, in all my time of searching, I have yet to find an article or paper published online that clearly explains what these transformations are and how exactly they may be utilized, which prompted the writing of this short paper, "The Elusive Tschirnhaus Transformations", in order to explain what the transformations are in a manner accessible to those with a background in elementary algebra. (As such, this paper will omit proofs and attempt to favor clarity over conciseness and mathematical elegance.)

Obviously, there is more to the transformations than can be explained at this level of mathematics. The hope here is for one to gain an elementary understanding of how these transformations are used, and, perhaps, to present another, possibly more intuitive method for solving the quadratic, cubic, and quartic algebraically.

– Nicholas Kim

What is a Tschirnhaus Transformation?

In the 16th century, Italian mathematicians finally succeeded in giving algebraic solutions for third- and fourth-degree polynomial equations, prompting a surge of interest in the topic in Europe. Many mathematicians, famous and otherwise, began a search for a similar solution to the quintic, and while it continued to elude them all, many other interesting results were obtained in the meantime, as mathematicians such as Lagrange began to propose unifying methods of solving polynomial equations, in the hopes that a general solution to the *n*th-degree polynomial equation could be obtained.

It was during this time that Ehrenfried Walther von Tschirnhaus (1651-1708) lived, and he eventually came to propose a technique for solving polynomial equations through a transformation that now bears his name. His reasoning was something like the following:

Solving a polynomial equation is, in general, difficult. However, if some of the coefficients can be turned into zeroes, that makes life easier – and the more coefficients disappear, the better. In fact, ideally, we can make *all* coefficients (except the first and last) disappear, such that we have a polynomial equation of form

$$x^n + k = 0$$

Tschirnhaus proposed that we use other polynomials to transform the original equation into simpler forms, ideally in the above form (since that is trivially solvable).

Linear Tschirnhaus Transformations

Before jumping straight into discussion of general Tschirnhaus transformations (which actually gets a bit complicated), let's first look at a special case: the linear transformations.

The linear transformations are readily understandable and had been known to the Renaissance Italians, who used it in their solutions of the cubic and quartic, and Descartes, who formalized the transformation for general polynomial equations. If you have studied the solutions to the cubic and quartic, you should already have a solid understanding of linear Tschirnhaus transformations; nevertheless, a short discussion follows.

Suppose we have a general polynomial equation

$$x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$

(Remember that the leading coefficient can always assumed to be 1 without loss of generality, since we can always divide all terms by it.)

While nominally difficult to solve, it would certainly be easier (or at least, not harder) *if the second-highest degree coefficient were zero*. Mathematically, this means we want a new polynomial equation

$$y^n + b_{n-2} y^{n-2} + \ldots + b_2 y^2 + b_1 y + b_0 = 0$$

where the *b* coefficients can be expressed in terms of the *a* coefficients, and the coefficient of the second-highest degree term is zero.

To accomplish this, we use a linear Tschirnhaus transformation: a transformation of form

$$y = x + \alpha$$

where α is to be chosen such that

$$b_{n-1} = 0$$

and it turns out that the appropriate value for α is

$$\alpha = \frac{a_{n-1}}{n}$$

(where *n* is the degree of the polynomial). And so all we have to do is solve the polynomial equation in *y*, which gives us values for *y*. We plug those values into

$$y = x + \alpha$$

and solve for *x*, giving us the solutions to our original polynomial equation.

So now the only question is, how do we determine what the *b* coefficients are? Well, there are several ways to go about this, but the simplest way is to simply make a substitution; that is, since *y=x+α*, we can rearrange this to get *x=y-α*, and for each *x* that we have in our original polynomial equation, we simply substitute *y-α* in, and expand.

Example

Let's solve the quadratic equation

$$x^2 - 7x + 12 = 0$$

using a linear Tschirnhaus transformation. (We already know how to do it with factoring, or the quadratic formula.)

First, we need to know what substitution to make. This is easy; since we're dealing with a quadratic equation, *n=2*, and we have

$$y = x + \frac{-7}{2}$$

using the given formula (-7 is the second-highest degree coefficient, which we're trying to get rid of). This means that our substitution is

$$x = y + \frac{7}{2}$$

and we perform our substitution:

$$\left(y + \frac{7}{2}\right)^2 - 7\left(y + \frac{7}{2}\right) + 12 = 0$$
$$y^2 + 7y + \frac{49}{4} - 7y - \frac{49}{2} + 12 = 0$$
$$y^2 - \frac{1}{4} = 0$$

The resulting equation is trivially solvable:

$$y = \pm\frac{1}{2}$$

Plugging these values of *y* back to obtain our values for *x*, we get

$$x = \frac{1}{2} + \frac{7}{2} = 4$$
$$x = -\frac{1}{2} + \frac{7}{2} = 3$$

which are, of course, the solutions we were expecting.

Applying a linear Tschirnhaus transformation to a polynomial is also known as *depressing* the polynomial, and a polynomial that has been so transformed is said to be in *depressed form*.

Newton's Identities

Now that we've looked at linear Tschirnhaus transformations, we can begin to look at higher-degree Tschirnhaus transformations. But before we do that, we need some new math material, known as *Newton's identities*.

Newton's identities will probably look mathematically complicated, but the basic idea is actually quite simple:

Newton's identities give us the sum of the roots (raised to any natural number power) of a polynomial equation, without having to actually find the roots. (For an analogy, think of the discriminant of a polynomial, which tells us what the roots will be like, without having to actually find them.)

The Identities

Let's say we have a polynomial of degree *n*, whose roots are

$$x_{1,} x_{2,} \ldots x_n$$

(they will always exist, by the Fundamental Theorem of Algebra, though they may not all be real and/or distinct).

What we *want to know* – that is, what Newton's identities tells us – is the sum of these roots (raised to any natural number power). So, for instance, we might want to know

$$S_1(x_k) = \sum_{k=1}^{n} x_k$$

which is the sum of all the roots. Or, we might want to know

$$S_2(x_k) = \sum_{k=1}^{n} x_k^2$$

which is the sum of all the *squared* roots (i.e. each root squared). In general, we wish to know

$$S_i(x_k) = \sum_{k=1}^{n} x_k^i$$

for any natural number *i*.

Newton's identities tell us that, for a polynomial equation

$$x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

(note that the leading coefficient *must* be 1 here),

$$S_i(x_k) = -i\, a_{n-i} - \sum_{k=1}^{i-1} S_{i-k}\, a_{n-k}$$

where

$$a_k = 0$$

for all *k* < *0*.

Despite the definition being very complicated, we should note that the definition is *recursive*; that is, the definition can be built one step at a time. Thus, at least in theory, the definition is easy to apply.

In any case, we will not worry too much about the general definition, and limit our attention to the specific cases described in later sections. The important thing is to know that the formula exists; how to use it will be demonstrated shortly.

General Tschirnhaus Transformations

We now turn our attention to more general Tschirnhaus transformations. We have already seen that a linear Tschirnhaus transformation gives us one *degree of freedom* – one parameter which we are free to choose – which enabled us to get rid of one term in a general polynomial equation.

Following this, Tschirnhaus reasoned that a quadratic Tschirnhaus transformation, which gives us two degrees of freedom, would enable us to get rid of two terms in a general polynomial equation. (He proved his reasoning correct.) Then, at the end, we just solve the quadratic transformation polynomial (which we already know how to do, of course).

He continued to reason that a cubic transformation would get rid of three terms, and in general, an *n*th-degree transformation would get rid of *n* terms. Therefore, he reasoned, all we need to do is use an *(n-1)*th-degree transformation for a polynomial of degree *n*, which will get rid of all terms except the first and the last (and the resulting equation is, of course, trivially solvable). By doing this, we solve all polynomial equations, by reducing the problem to one of lower degree each time!

...or so he thought. Although not known in his time, his method could not possibly work, since all polynomial equations of degree five and higher cannot be solved only using radicals (which are the functions Tschirnhaus was expecting to use). Indeed, long before Ruffini and Abel proved this, Leibniz noted that, in fact, Tschirnhaus' idea yields a polynomial of degree *(n-1)!*, making the method useless for high enough values of *n*.

Despite the failure of his attempt to solve all polynomial equations, Tschirnhaus nevertheless came up with a very important technique for solving polynomial equations, and his work was expanded upon by many others, including Erland Bring (who managed to remove three terms from the quintic) and George Jerrard (who generalized Bring's result for general polynomial equations).

The final part of this paper will demonstrate the utility of Tschirnhaus transformations by solving the quadratic, cubic, and quartic in a different manner from that described in my previous paper.

Solving the Quadratic with Tschirnhaus Transformations

The general quadratic equation

$$ax^2+bx+c=0$$

is trivial to solve using Tschirnhaus' techniques, as we only need to get rid of a single term (the linear term), so a linear transformation will suffice.

First, we divide all terms by the leading coefficient:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Following the section “Linear Tschirnhaus Transformations”, the transformation we want is

$$y=x+\frac{b}{2a}$$

We rearrange this, such that we have

$$x=y-\frac{b}{2a}$$

and substitute this to get

$$\left(y-\frac{b}{2a}\right)^2+\frac{b}{a}\left(y-\frac{b}{2a}\right)+\frac{c}{a}=0$$

Expanding this gives us

$$y^2-\frac{b}{a}y+\frac{b^2}{4a^2}+\frac{b}{a}y-\frac{b^2}{2a^2}+\frac{c}{a}=0$$

$$y^2+\frac{c}{a}=\frac{b^2}{4a^2}$$

$$y^2=\frac{b^2-4ac}{4a^2}$$

$$y=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Substituting these two values of *y* back into our original substitution gives us

$$x=\frac{\pm\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

which is exactly the quadratic formula.

Solving the Cubic with Tschirnhaus Transformations

The cubic had first been solved in its entirety by Tartaglia (although del Ferro was the first to solve the depressed case), but Tschirnhaus demonstrated the power of his transformation techniques by solving the cubic in a rather different, and perhaps more intuitive, manner.

He first depressed the cubic (the details of which are in my previous paper, and as such, will be omitted here), using another linear transformation; as such, we assume, without loss of generality, that our cubic is in the form

$$x^3+px+q=0$$

Following the manner of Tschirnhaus, we want to be rid of the linear term as well. Since this requires forcing two terms (quadratic and linear) to be zero, we require a quadratic transformation, offering two parameters, of form

$$y=x^2+\alpha x+\beta$$

Because the quadratic and linear terms are gone, we will be left with a cubic of form

$$y^3+k=0$$

which is, of course, trivially solvable; we use the resulting values of *y* and substitute them into the above transformation, which gives us a quadratic equation in *x*.

Unlike as is the case with linear transforms, we **do not try and substitute this into our equation**, as the resulting algebraic work would be far too complicated. Instead, we exploit Newton's identities to *indirectly* determine what ɑ, β, and *k* should be.

To do this, let us first note down what Newton's identities say the powersums of the roots are. Using the recursive definition above, where

$$x_{1,}x_{2,}x_3$$

are the roots of the depressed cubic (I will spare you the algebraic work, so you can take my word for it that these are correct):

$$S_1(x_k)=x_1+x_2+x_3=0$$

$$S_2(x_k)=x_1^2+x_2^2+x_3^2=-2p$$

$$S_3(x_k)=x_1^3+x_2^3+x_3^3=-3q$$

$$S_4(x_k)=x_1^4+x_2^4+x_3^4=2p^2$$

$$S_5(x_k)=x_1^5+x_2^5+x_3^5=5pq$$

$$S_6(x_k)=x_1^6+x_2^6+x_3^6=3q^2-2p^3$$

Similarly, the powersums for the roots of the principal (as it would be formally known) cubic

$$y^3+k=0$$

are

$$S_1(y_k)=y_1+y_2+y_3=0$$

$$S_2(y_k)=y_1^2+y_2^2+y_3^2=0$$

$$S_3(y_k)=y_1^3+y_2^3+y_3^2=-3k$$

Now for the magic trick (which won't stick a pencil through anyone's eye, I promise): notice that the *y*'s are substitutable with *x*'s, using the transformation equation

$$y=x^2+\alpha x+\beta$$

Therefore, we can say that

$$S_1(y_k)=y_1+y_2+y_3=(x_1^2+\alpha x_1+\beta)+(x_2^2+\alpha x_2+\beta)+(x_3^2+\alpha x_3+\beta)=0$$

and we can rearrange this to get

$$x_1^2+x_2^2+x_3^2+\alpha(x_1+x_2+x_3)+3\beta=0$$

but, as we can see, these *x*'s are exactly the expressions seen in the powersum formulas for *x*. We can therefore replace them what they're equal to, namely:

$$(-2p)+\alpha(0)+3\beta=0$$
$$\beta=\frac{2}{3}p$$

and we have a value for β!

Now let's turn our attention to the second powersum formula for *y*. We get

$$(x_1^2+\alpha x_1+\beta)^2+(x_2^2+\alpha x_2+\beta)^2+(x_3^2+\alpha x_3+\beta)^2=0$$

Expanding these (again, more tedious algebra):

$$\begin{gathered} x_1^4+2\alpha x_1^3+(\alpha^2+2\beta)x_1^2+2\alpha\beta x_1+\beta^2+ \\ x_2^4+2\alpha x_2^3+(\alpha^2+2\beta)x_2^2+2\alpha\beta x_2+\beta^2+ \\ x_3^4+2\alpha x_3^3+(\alpha^2+2\beta)x_3^2+2\alpha\beta x_3+\beta^2=0 \end{gathered}$$

and rearranging (even more tedious algebra):

$$x_1^4+x_2^4+x_3^4+2\alpha(x_1^3+x_2^3+x_3^3)+(\alpha^2+2\beta)(x_1^2+x_2^2+x_3^2)+2\alpha\beta(x_1+x_2+x_3)+3\beta^2=0$$

Substituting the powersum expressions for *x* gives us

$$(2p^2)+2\alpha(-3q)+(\alpha^2+2\beta)(-2p)+2\alpha\beta(0)+3\beta^2=0$$

and simplifying this, substituting in our value for β, gives us

$$p\alpha^2+3q\alpha-\frac{p^2}{3}=0$$

which we may solve for α to get

$$\alpha=\frac{-3q\pm\sqrt{9q^2-\frac{4}{3}p^3}}{2p}$$

of which either may be used in the transformation polynomial

$$y=x^2+\alpha x+\beta$$

In any case, all that is left is to determine a value for k. To do this, we use the final powersum formula for y:

$$S_3(y_k) = y_1^3 + y_2^3 + y_3^3 = -3k$$

Substituting in and expanding as before (all the tedious algebra will be omitted), the final result is

$$k = -q^3 + \frac{2}{3}p^3 - 5pq\alpha - 2p^2\alpha^2 - 2p^2\beta + q\alpha^3 + 6q\alpha\beta + 2p\alpha^2\beta + 2p\beta^2 - \beta^3$$

and we can, of course, substitute our values of α and β in. We then solve the principal cubic (easy to do), use the solutions to solve the quadratic Tschirnhaus polynomial (again, easy to do), and we have all the solutions to the depressed (and by extension, general) cubic!

Of course, as I'm sure you've noticed by now, this method is horrendously complicated, far more so than Tartaglia's, and what's worse, because quadratics have two solutions, we will get a total of six solutions to the depressed cubic, of which only three are valid (there is no way to know which ones are valid without numerical testing).

It is also important to note that we don't need to depress the cubic before applying the quadratic transformation, since Tschirnhaus' transformation controls both the quadratic and linear coefficients at once; it merely saves us some tedious algebra if we depress the cubic first (since the quadratic coefficient can already be assumed to be zero, the powersum formulas are simpler).

Solving the Quartic with Tschirnhaus Transformations

Nominally, the quartic may be solved in a manner following Tschirnhaus' solution of the cubic; that is, by using a cubic transformation to eliminate the cubic, quadratic, and linear terms, leaving one with a trivial quartic of form

$$y^4 + k = 0$$

As the resulting algebra would be horrendously complicated, far more so than that seen in Ferrari's solution of the quartic, I will not expand on the details here. However, in principle, the solution method is identical to that used for the cubic – which precisely fulfills Tschirnhaus' goal of giving a *general* solution of polynomial equations (up to the quartic, at least).

Future Extensions

Tschirnhaus transformations, while resulting in horribly complex algebraic expressions, do succeed at one important thing – giving a unified method of solving polynomial equations of degree four and lower. As a more elegant alternative, we may examine Lagrange's own attempts to do so – which also serve as a precursor to modern abstract algebra.