Name: AVINASH SRIVASTAVA ID: 2018AC04564

MFDS Assignment

Ans1) Let $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$, $\{\beta_1, \beta_2 \ldots \beta_m\}$ be the two bases of finite dimensional vector space V.

$\because$ $\{\alpha_1, \alpha_2 \ldots, \alpha_n\}$ is a basis of V & $\{\beta_1, \beta_2 \ldots \beta_m\}$ is a linearly independent set of vectors in V, $m \leq n$

Since $\{\beta_1, \beta_2 \ldots \beta_m\}$ is a basis of V & $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a linearly independent set of vectors in V, $n \leq m$.

So, $m \leq n$ & $n \leq m \Rightarrow m = n$

This proves that any two different basis of V will have the same cardinality.

Ans 2>

Let u be an arbitarary vector in V. Since V is the linear ~~space~~ span of vectors $x_1, x_2, \ldots x_m$ hence there exist scalars $a_1, a_2, \ldots a_m$ such that

$$u = a_1x_1 + a_2x_2 + \ldots a_ix_i + \ldots a_mx_m$$

Further since x_i is a linear combination of $x_1, x_2 \ldots x_{i-1}, x_{i+1}, \ldots x_m$ hence there exist scalars $b_1, b_2 \ldots b_{i-1}, b_{i+1} \ldots b_m$ such that
$x_i = b_1x_1 + b_2x_2 + \ldots b_{i-1}x_{i-1} + b_{i+1}x_{i+1} + \ldots b_mx_m$.

Then $u = a_1x_1 + a_2x_2 + \ldots a_ix_i + \ldots a_mx_m = a_1x_1 + a_2x_2 + \ldots a_{i-1}x_{i-1} + a_i(b_1x_1 + b_2x_2 + \ldots b_{i-1}x_{i-1} + b_{i+1}x_{i+1} + \ldots b_mx_m) + \ldots a_mx_m =$
$(a_1 + a_ib_1)x_1 + (a_2 + a_ib_2)x_2 + \ldots + (a_{i-1} + a_ib_{i-1})x_{i-1} + (a_{i+1} + a_ib_{i+1})x_{i+1} + \ldots (a_m + a_ib_m)x_m$.
Thus, u is a linear combination of vectors $x_1, x_2 \ldots x_{i-1}, x_{i+1} \ldots x_m$.
Since u is arbitrary vector in V, hence the set $\{x_1, x_2, \ldots x_{i-1}, x_{i+1}, \ldots x_m\}$ also spans V.

Ans 3)

Let V be finite dimensional vector space &
let dim $V = n$

Let $S = \{x_1, \dots x_m\}$ be any linear Independent subset of V.

Let $B = \{y_1, \dots y_n\}$ be basis of V.

Consider $S \cup B = \{x_1, \dots x_m, y_1 \dots y_n\}$

Then $L(S \cup B) = L(S) + L(\beta) = L(S) + V = V$

Since $x_i \in S \subseteq V$, $\therefore$ each y_i can be written as L.I linear combination of elements of basis B.

i.e $x_i = \alpha_1 y_1 + \alpha_2 y_2 + \dots \alpha_n y_n$ for some $\alpha_i \in F$
$1 \le i \le n$

$\Rightarrow$ $x_i = 0x_1 + \dots 0x_m + \alpha_1 y_1 + \alpha_2 y_2 + \dots \alpha_n y_n$

i.e an element $S \cup B$ can be written as linear combination of other elements of $S \cup B$.

$\Rightarrow$ $S \cup B$ is L.D. set

$\therefore$ Some elements of $S \cup B$ must be a linear combination of its preceding elements & this element cannot be ~~outer~~ any one of the x_i because S is L.I.

Thus some y_i is expressible as a linear combination of its preceding elements of & let it be y_k.

Then $y_k = \sum_{i=1}^{m} \beta_i x_i + \sum_{j=1}^{k-1} \gamma_j y_j$ for some $\beta_i, \gamma_j \in F$

Consider $S_1 = \{x_1, x_2 \dots x_m, y_1, \dots y_{k-1}, y_{k+1} \dots y_n\}$

Let $y \in V$

then $y = \sum_{l=1}^{n} s_l y_l$, $s_l \in F$, $1 \le l \le n$

($\because B$ is basis of V)

$= \sum_{\substack{l=1 \\ l \neq k}}^{n} s_l y_l + s_k y_k$

$= \sum_{\substack{l=1 \\ l \neq k}}^{n} s_l y_l + s_k \left[\sum_{i=1}^{m} \beta_i x_i + \sum_{j=1}^{k-1} \gamma_j y_j \right]$

$= s_k \sum_{i=1}^{m} \beta_i x_i + \sum_{l=1}^{k-1} (s_l + s_k \gamma_l) y_l + \sum_{l=k+1}^{m} s_l y_l$

$\in L(S_1)$

$\Rightarrow V = \langle S_1 \rangle$

If S_1 is L.I, the S_1 is req basis containing

But if S_1 is not L.I., then we go on repeating the above process to get new set S_k such that S_k is l.I, $\langle S_k \rangle = V$ & S_k is extension of S.

At most by repeating above process, we shall get the set $\{x_1 \dots x_m\}$ which is given to be L.I set & hence it becomes the basis of V.

Ans 4)

let $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

The characteristic equation of A is

$|A - \lambda I| = 0$

$\begin{vmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{vmatrix} = 0$

$\Rightarrow (\cos(\theta) - \lambda)^2 + \sin^2(\theta) = 0$

$= \cos^2(\theta) + \lambda^2 - 2\lambda\cos(\theta) + \sin^2(\theta) = 0$

$= \lambda^2 - 2\lambda\cos(\theta) + 1 = 0$

$= \lambda = \dfrac{-(-2\cos(\theta)) \pm \sqrt{(-2\cos(\theta))^2 - 4(1)(1)}}{2(1)}$

$\Rightarrow \lambda = \dfrac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2}$

$\Rightarrow \lambda = \dfrac{2\cos(\theta) \pm 2\sqrt{\cos^2(\theta) - 1}}{2}$

$\Rightarrow \lambda = \dfrac{2\cos(\theta) \pm 2\sqrt{-\sin^2(\theta)}}{2}$

$\lambda = \dfrac{2\cos(\theta) \pm 2i\sin(\theta)}{2}$

$\lambda = \cos(\theta) \pm i\sin(\theta)$

$\lambda = e^{\pm i\theta}$

Distinct eigen values are $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$

Now write the augmented matrix for $\lambda_1 = e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$[A - \lambda I : 0] = \left(\begin{array}{cc|c} -i\sin(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & -i\sin(\theta) & 0 \end{array}\right)$$

$$R_1 \to -\frac{1}{i\sin(\theta)} R_1 \to \left(\begin{array}{cc|c} 1 & -i & 0 \\ \sin(\theta) & -i\sin(\theta) & 0 \end{array}\right)$$

$$R_2 \to R_2 - \sin(\theta) R_1 \to \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The system of equation is $x - iy = 0$

The rank of the ~~eq~~ coefficient matrix is 1 and number of variables is 2.

$\therefore$ There is 1 free variable.

Let y be the free variables, let $y = t \Rightarrow x = iy = it$.

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda_1 = e^{i\theta}$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now write augmentated matrix for $\lambda_2 = e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

$$[A - \lambda I \; 0] = \left(\begin{array}{cc|c} i\sin(\theta) & -i\sin(\theta) & 0 \\ \sin(\theta) & i\sin(\theta) & 0 \end{array}\right)$$

$$R_1 \to \frac{1}{i\sin(\theta)} R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & i & 0 \\ \sin(\theta) & i\sin(\theta) & 0 \end{array}\right]$$

$$R_2 \to R_2 - \sin(\theta) R_1 \to \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The system of equation is $x + iy = 0$.
The rank of coefficient matrix is 1 & the number of variables is 2.
$\therefore$ There is one free variable.
Let y be the free variable, & let $y = t \Rightarrow x = -iy = -it$.

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Eigen vectors corresponding to $\lambda_2 = e^{-i\theta}$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

The eigenspace is $= \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$$

$$P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

$P^{-1}AP = D$

$\Rightarrow AP = PD$

$= A(AP) = A(PD)$

$= (AA)P = (AP)D$

$= A^2P = (PD)D$

$= A^2P = P(DD)$

$= A^2P = PD^2$

Repeating this upto n

$A^nP = PD^n$

$\Rightarrow A^n = PD^nP^{-1}$

$A^n = PD^nP^{-1}$

$$= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}^n \tfrac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

$$= \tfrac{1}{2} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{bmatrix} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

$$= \tfrac{1}{2} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(n\theta) + i\sin(n\theta) & 0 \\ 0 & \cos(n\theta) - i\sin(n\theta) \end{bmatrix} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

$$= \tfrac{1}{2} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i\cos(n\theta) + \sin(n\theta) & \cos(n\theta) + i\sin(n\theta) \\ i\cos(n\theta) + \sin(n\theta) & \cos(n\theta) - i\sin(n\theta) \end{bmatrix}$$

$$\begin{bmatrix} \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta) & i\cos(n\theta) - \sin(n\theta) - i\cos(n\theta) - \sin(n\theta) \\ -i\cos(n\theta) + \sin(n\theta) + i\cos(n\theta) + \sin(n\theta) & \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2\cos(n\theta) & -2\sin(n\theta) \\ 2\sin(n\theta) & 2\cos(n\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

$$\therefore \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

Ans 5)

$$\begin{bmatrix} 2 & 1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & & & \\ \vdots & & \ddots & 2 & -1 & 0 \\ 0 & & & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

$L =$ $D =$ $U =$

$A = L + D + U$

$$L = \begin{bmatrix} 2 & & \text{rest all zeros} & & \\ -1 & \ddots & & & \\ 0 & \cdots & \ddots & & \\ \vdots & \cdots & & & \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Jacobi $\boxed{A = G - H}$ for splitting

If $G = D$ & $H = -(L+U)$

Then $x_{k+1} = -D^{-1}(L+U)x_k + D^{-1}b$

$x_k + D^{-1}(b - Ax_k)$ (formula)

& $x_{(k+1)i} = \frac{1}{A_{i,i}}\left(b_i - \sum_{\substack{j=1 \\ j\neq 1}}^{n} A_{ij}(x_k)_j\right)$ $i=1,2,3,4$ $\because (n=4)$

& Gauss Seidel

$$G = (D+L) \quad \& \quad H = -U$$

$$x_{k+1} = (D+L)(-U)x_k + (D+L)^{-1}b =$$

$$x_k + (D+L)^{-1}(b - Ax_k)$$

$$\underline{N = D+L}$$

Derivation.

$$x_{k+1} = \frac{1}{A_{ii}}\left(b_i - \sum_{j=1}^{i-1} A_{ij}(x_{k+1})_j - \sum_{j=i+1}^{n} A_{ij}(x_k)_i\right)$$

$$= (x_k)_i + \frac{1}{A_{ii}}\left(b_i - \sum_{j=1}^{i-1} A_{ij}(x_{k+1})_j - \sum_{j=1}^{n} A_{ij}(x_k)_j\right)$$