# PROBABILITY

## Some Definitions:

**Random Experiments:** An experiment E is called a random experiment if i) all the possible outcomes of E are known in advance, ii) it is impossible to predict which outcome will occur at a particular performance of E, iii) E can be repeated, at least conceptually, under identical conditions for infinite number of times.

**Elementary Events:** If a random experiment is performed, then each of its outcomes is known as an elementary event.

**Sample Space:** The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.

**Example:** Let there be a bag containing 3 white and 2 black balls. Let the white balls be denoted by $W_1$, $W_2$, $W_3$ and black balls be denoted by $B_1$, $B_2$. If we draw two balls from the bag, then there are ${}^5C_2$ elementary events associated to this experiment. These elementary are $B_1W_1$, $B_1W_2$, $B_1W_3$, $B_2W_1$, $B_2W_2$ $B_2W_3$, $W_1W_2$, $W_1W_3$, $W_2W_3$, $B_1B_2$.The set of all these elementary events is the sample space associated to the experiment.

**Compound Event:** A subset of the sample space associated to a random experiment is defined as compound event if it is disjoint union of single element subsets of the sample space.

**Notes:** i) If there are n elementary events associated to a random experiment, then the sample space associated to it has n elements and so there are $2^n$ subsets of it. Out of these $2^n$ subsets, there are n single element subsets. These single element subsets define n elementary events and the remaining $2^n - (n + 1)$ subsets (excluding null set) define compound events. Some of these compound events can be described in words whereas for others there may not be general description.

ii) It is evident from the above discussion that every event associated to a random experiment is represented by subsets (say) of the sample space S associated to the experiment and conversely, every subsets of S represents an event associated to the random experiment. That follows, the event represented by the subset A of sample space Swill be denoted by A i.e., we will be using the same symbol to represent an event and the subset of the sample space representing it.

**Occurrence of an Event:** An event A associated to a random experiment is said to occur if any one of the elementary events associated to it is an outcome.

**Negation of an Event:** Corresponding to every event A associated with a random experiment we define an event "not A" which occurs when and only when A doesn't occur.

**Favourable Elementary Events:** Let S be the sample space associated with a random experiment and A be an event associated to the experiment. Then elementary events belonging to A are known as favourable elementary events to the event A.

## Probability:

If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A is denoted by P(A) and is defined as the ratio

$$P(A) = \frac{m}{n}$$

Clearly, $0 \leq m \leq n$. Therefore, $0 \leq \frac{m}{n} \leq 1 \Rightarrow 0 \leq P(A) \leq 1$

If P(A) = 1, then A is called the certain event and if P(A) = 0,then A is called an impossible event. Denoted by P(S) and $P(\phi)$ .

The number of elementary events which will ensure the non-occurrence of A i.e., which ensure the occurrence of $A^c$ is (n-m). Therefore,

$$P(A^c) = \frac{n-m}{n} = 1-\frac{m}{n} = 1-P(A) \Rightarrow P(A)+P(A^c) = 1$$

The odds in favour of occurrence of the event A are defined by m :(n-m) i.e. P(A): $P(A^c)$ and the odds against the occurrence of A are defined by n - m i.e., $P(A^c)$ : P(A).

**Types of Events:**

**Mutually Exclusive Events** Two or more events associated to a random experiment are mutually exclusive if the occurrence of one of them prevents or denies the occurrence of all others. Thus, if two events A and B are mutually exclusive, then $P(A \cap B) = 0$.Similarly, if A, B and C are mutually exclusive events, then $P(A \cap B \cap C) = 0$.

**Exhaustive Events** Two or more events associated to a random experiment are exhaustive, if their union is the sample space i.e., events $A_1, A_2, \ldots, A_n$ associated to a random experiment with sample space S are exhaustive, if $A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_n = S$.

**Independent Events:** Two events A and B associated to a random experiment are independent, if the probability of occurrence or non-occurrence of A is not affected by the occurrence or non-occurrence of B.

## Addition Theorems on Probability:

**Addition Theorem for two events:** If A and B are two events associated with a random experiment, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

**Addition Theorem for three events** If A, B, C are three events associated with a random experiment, then $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

**Generalized Addition Theorem:** If $A_1, A_2, \ldots, A_n$ are n events associated to a random experiment, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \leq i \leq j \leq n} P(A_i \cap A_j) + \sum\sum\sum_{1 \leq i \leq j \leq k \leq n} P(A_i \cap A_j \cap A_k) - \ldots\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots\ldots A_n)$$

Let A and B be two events associated to a random experiment. Then,

(a) $P(A^c \cap B) = P(B) - P(A \cap B)$

(b) $P(A \cap B^c) = P(A) - P(A \cap B)$

(c) $P((A^c \cap B) \cup (A \cap B^c)) = P(A) + P(B) - 2P(A \cap B)$

For any two events A and B, $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$.

For any two events A and B, the probability that exactly one of A, B occurs = $P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B)$.

If A, B, C are three events, then

(a) P (at least two of A, B, C occur) = $P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$

(b) P (Exactly two of A, B, C occur) = $P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$

(c) P (Exactly one of A, B, C occur)= $P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$.

## Conditional Probability

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$, is called the conditional probability and it is denoted by P(A/B). Thus, we have P(A/B)= Probability of occurrence of A given that B has already occurred. In fact, the meanings of symbols P(A/B) and P (B/A) depend on the nature of the events A and B and also on the nature of the random experiment. These two symbols have the following meanings also: P(A/B)= Probability of occurrence of A when B occurs or probability of occurrence of A with respect to B and, P (B/A) = probability of occurrence of B when A occurs or probability of occurrence of B with respect to A.

**Multiplication Theorems on Probability**: If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A)P(B/A), \text{ if } P(A) \neq 0$$

$$\text{or} \quad P(A \cap B) = P(B)P(A/B), \text{ if } P(B) \neq 0.$$

**Extension of multiplication theorem** If $A_1, A_2, \ldots, A_n$ are n events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)P(A_2/A_1)P(A_3/ A_1 \cap A_2) \ldots P(A_n / A_1 \cap A_2 \cap \ldots \cap A_{n-1}),$$

Where $P(A_j / A_1 \cap A_2 \cap \ldots \cap A_{j-1})$ represents the conditional probability of the occurrence of event $A_j$, given that the events $A_1, A_2 \ldots A_{j-1}$ have already occurred.

**More on Independent Events**

**Pairwise Independent Events:** Let $A_1, A_2, \ldots, A_n$ be n events associated to a random experiment. These events are said to be pairwise independent, if $P(A_i \cap A_j) = P(A_i)P(A_j)$ for $i \neq j$; i, j = 1, 2,...., n

**Mutually Independent Events:** Let $A_1, A_2, \ldots, A_n$ be n events associated to a random experiment. These events are said to be mutually independent, if the probability of the simultaneous occurrence of any finite number of them is equal to the product of their separate probabilities i.e.

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2) \ldots P(A_n)$$

- If $A_1, A_2, \ldots, A_n$ are pairwise independent events, then the total number of conditions for their pairwise independence is ${}^nC_2$ whereas for their mutually independences there must be ${}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - n - 1$ conditions
- It follows from the above definitions that mutually independent events are always pairwise independent but the converse need not be true.
- In case of two events only associated to a random experiment, there is no distinction between their mutual independence and pairwise independence.
- If A and B are independent events associated with a random experiment, then

i) $A^c$ and B are independent events (ii) A and $B^c$ are independent events (iii) $A^c$ and $B^c$ are also independent events.

- The term independent events will mean mutually independent events.

- If A and B are independent events associated to a random experiment, then
  Probability of occurrence of at least one
  $= P(A \cup B) = P(A)+P(B)-P(A \cap B) = P(A)+P(B)-P(A)P(B)$
  $= 1 - [1 - P(A) - P(B) + P(A)P(B)] = 1-(1-P(A))(1-P(B)) = 1-P(A^c)P(B^c)$
- If $A_1, A_2, \ldots, A_n$ are independent events associated with a random experiment, then
  Probability of occurrence of at least one
  $= P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = 1 - P((A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)^c)$
  $= 1 - P(A^c_1 \cap A^c_2 \cap A^c_3 \cap \ldots \cap A^c_n) = 1-P(A^c_1)P(A^c_2)\ldots P(A^c_n)$.

**The Law of Total Probability:** Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with $E_1$ and $E_2$ or.... or $E_n$, then $P(A)=P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \ldots + P(E_n)P(A/E_n)=\sum_{r=1}^{n} P(E_r)P(A/E_r)$.

**Baye's Rule:** Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with $E_1$ or $E_2$ or.... or $E_n$, then

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{r=1}^{n} P(E_r)P(A/E_r)}.$$

**Random Variable and Its Probability Distribution**

**Random variable** Let S be the sample space associated with a given random experiment. Then, a real valued function X which assigns to each events $w \in S$ to a unique real number X(w) is called a random variable.

**Probability distribution** If a random variable X takes values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, then

| | | | |
|---|---|---|---|
| X : | $x_1$ | $x_2$ ………… | $x_n$ |
| P(X) : | $p_1$ | $p_2$ ………… | $p_n$ |

is known as the probability distribution of X.

This, a tabular description giving the values of the random variable along with the corresponding probabilities is called its probability distribution. The probability distribution of a random variable X is defined only when we have the various values of the random variable, $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ satisfying $\sum_{r=1}^{n} p_r = 1$

**Mean and Variance of a Random Variable**

**Mean** If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$, then the mean $\bar{X}$ of X is defined as

$$\bar{X} = \frac{x_1 + x_2 + \ldots + x_n}{n}, \text{ or } \quad \boxed{\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

• The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by E(X).

• In case of a frequency distribution the mean $\overline{X}$ is given by $(f_1x_1 + f_2x_2 + .... + f_nx_n)/N$, where $N = f_1 + f_2 + .... + f_n$. Thus, if we replace $f_i/N$ by $p_i$, in the definition of mean, we obtain the mean of a discrete random variable. Consequently, the term 'mean' is appropriate for the sum $\sum_{i=1}^{n} p_i x_i$ .

**Variance:** If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with the respective probabilities $p_1, p_2, ...., p_n$, then variance of X is defined as

$$\mathrm{Var}(X) = p_1(x_1 - \overline{X})^2 + p_2(x_1 - \overline{X})^2 + ... + p_n(x_1 - \overline{X})^2 = \sum_{i=1}^{n} p_i (x_i - \overline{X})^2$$

. Where, $\overline{X}$ is the mean of X.

Now $\mathrm{Var}(X) = \sum_{i=1}^{n} p_i (x_i - \overline{X})^2 = \sum_{i=1}^{n} p_i (x_i^2 - 2x_i\overline{X} + \overline{X}^2) = \sum_{i=1}^{n} p_i x^2_i - 2\overline{X} \sum_{i=1}^{n} p_i x_i + \overline{X}^2 (\sum_{i=1}^{n} p_i)$

$= \sum_{i=1}^{n} p_i x^2_i - 2\overline{X} . \overline{X} + \overline{X}^2 = \sum_{i=1}^{n} p_i x^2_i - \overline{X}^2$ . So $\boxed{\mathrm{Var}(X) = \sum_{i=1}^{n} p_i x^2_i - \left( \sum_{i=1}^{n} p_i x_i \right)^2}$

**Exercise:** 1. The probability that out of 10 persons, all born in June, at least two have same birthday is

a) ${}^{30}C_{10}/(30)^{10}$ b) ${}^{30}C_{10}/(30)!$ c) $(30^{10} - {}^{30}C_{10})/(30)^{10}$ d) None of these

2. There are four machines and it is known to them that exactly two of them are faulty. They are tested one by one, in a random order till both the faulty machine re identified. Then the probability that only two test are needed is

a) 1/3 b) 1/6 c) 1/2 d) 1/4

3. An elevator start with m passengers and stops at n floors $(m \leq n)$.the probability that no two passenger alight at same floor is

a) $\frac{{}^{n}P_{m}}{m^{n}}$ b) $\frac{{}^{n}P_{m}}{n^{m}}$ c) $\frac{{}^{n}P_{m}}{m^{n}}$ d) $\frac{{}^{n}C_{m}}{n^{m}}$

4. The numbers 1,2….,n are arranged in a random order. The probability that the digits 1,2,…,k (k<n) appear as neighbor in that order is

a) $\frac{1}{n!}$ b) $\frac{k!}{n!}$ c) $\frac{(n-k)!}{n!}$ d) None of these

5. In a convex hexagon two diagonals are drawn at random. The probability that two diagonals intersect at an interior point of hexagon is

a) 5/12 b) 2/5 c) 7/12 d) None of these

6. 15 persons, among whom A and B, sit at random at a round table. The probability that there are 4 persons between them is

a) 9! /14! b) 10! /14! c) 9! /15! d) None of these

7. A team of 8 couples (husbands and wives) attend a lucky draw in which 4 persons picked for a prize. Then the probability that there is at least one couple, is

a) 11/39 b) 12/39 c) 14/39 d) 15/39

8. A number is chosen from the set A = $\{33^n : n \in \mathbf{N}\}$. Then what is the probability that it has 3 in unit place?

a) 1/2 b) 1/3 c) 1/4 d) None of these

9. A fair coin is tossed 99 times. If X is the number of times heads occur, then P(X=r) is maximum when r is

a) 49, 50 b) 50, 51 c) 51, 52 d) None of these

10. Four digit numbers are formed with each of the digits 1,2,….,8 only once. One number form them is picked up at random. The probability that selected number contains unity is

a) 1/8 b) 1/4 c) 1/2 d) None of these

11. The probability that $\sin^{-1}(\sin x)+\cos^{-1}(\cos y)$ is an integer x, y $\in$ {1, 2, 3, 4} is

a) 1/16 b) 3/16 c)15/16 d) None of these

12. If $0 \le a \le 20$, then the probability that the equation $16x^2+8(a+5)x-7a-5=0$ has imaginary roots, is

a) 13/20 b) 20/13 c)13/24 d) 15/20

13. If P(A/B) = P(B/A). A and B are two non-mutually exclusive events, then

a) A and B are necessarily same events   b) P(A) = P(B)   c) $P(A\cap B) = P(A).P(B)$   d) All of the above

14. 2 positive real number x & y satisfying $x \le 1$ & $y \le 1$ are chosen at random. The probability that $x + y \le 1$, given that $x^2 + y^2 \ge 1/4$, is

a) $\frac{8-\pi}{16-\pi}$   b) $\frac{4-\pi}{16-\pi}$   c) $\frac{4-\pi}{8-\pi}$   d) None of these

15. Suppose $f(x) = x^3 + ax^2 + bx + c$, where a, b, c are chosen respectively by throwing a die three times. Then the probability that f(x) is an increasing function, is

a) 4/9   b) 3/8   c)2/5   d) 16/34

16. Three die are thrown simultaneously, the probability of getting a sum of 15, is

a) 1/72   b) 5/36   c) 5/72   d) None of these

17. There are 4 urns. The first contains 1 white and 1 black ball, the 2nd urn contains 2 white and 3 black balls, the 3rd urn contains 3 white and 5 black balls, and 4th urn contains 4 white and 7 black balls. The selection of each urn is not equally likely. The probability of selecting $i^{th}$ urn is $\frac{i^2+1}{34}$(i=1, 2, 3, 4). If we randomly select one urns and draw a ball, the probability of ball being white, is

a) 569/1496   b) 498/569   c) 494/569   d) None of these

18. A 5 digit number is formed by using the digit 0, 1, 2, 3, 4 and 5 without repetition. The probability that the number is divisible by 6, is

a) 0.08   b) 0.17   c) 0.18   d) 0.36

19. If three normal die are thrown together, the probability that the sum of numbers appearing on the dice is k $(9 \le k \le 14)$, is

a) $\frac{21k\text{-}k^2-83}{216}$   b) $\frac{k^2-21k}{216}$   c) $\frac{3k^2-16k}{216}$   d) None of these

20. Let A, B are two events such that P(A) =1/4 and $P(A \cup B) = 1/2$ , then the value of $P(B/A^c)$ is

a) 2/3 b) 1/3 c) 5/6 d) ½

A box contains n coins. Let $P(E_i)$be the probability that exactly i out of n coins are biased. If $P(E_i)$ is directly proportional to $i(i+1); 1 \le i \le n$.

21. Proportionality constant is equal to

a) $\frac{3}{n\left(n^2+1\right)}$ b) $\frac{1}{(n+2)\left(n^2+1\right)}$ c) $\frac{1}{n(n+1)(n+2)}$ d) $\frac{1}{(n+1)(n+2)(n+3)}$

22. If P be the probability that a coin is selected randomly is biased, then $\lim_{n\to\infty} P$ is

a) 1/4 b) 3/4 c) 3/5 d 7/8)

23. If a coin is selected at random is found to be biased, then the probability that it is the only biased coin in the box, is

a) $\frac{1}{(n+1)(n+2)(n+3)(n+4)}$ b) $\frac{1}{n(n+2)(n+3)(3n+1)}$ c) $\frac{1}{n(n+2)(n+3)(2n+1)}$ d) None of these

24. A bag contains some white and some black balls out of 10 balls. If 3 balls are drawn at random without replacement and all them are found to be black, then the probability that it contains 1 white and 9 black balls, is

a)14/55 b)12/55 c) 2/11 d) 8/55

25. Out of 21 tickets marked with 1,2,……..,21 three are drawn at random. T he probability that they are in A.P is

a) 10/133 b) 9/15 c) 14/261 d) 13/261