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Convex Optimization † Euclidean Distance Geometry 2ε

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Convex Optimization
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Euclidean Distance Geometry

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& *Sze Wan*

$$\mathbb{EDM} = \mathbb{S}_h \cap \left(\mathbb{S}_c^{\perp} - \mathbb{S}_+\right)$$

Prelude

The constant demands of my department and university and the ever increasing work needed to obtain funding have stolen much of my precious thinking time, and I sometimes yearn for the halcyon days of Bell Labs.

−Steven Chu, Nobel laureate [92]

Convex Analysis is an emerging calculus of inequalities while Convex Optimization is its application. Analysis is inherently the domain of a mathematician while Optimization belongs to the engineer. A convex optimization problem is conventionally regarded as minimization of a convex objective function subject to an artificial convex domain imposed upon it by the problem constraints. The constraints comprise equalities and inequalities of convex functions whose simultaneous solution set generally constitutes the imposed convex domain: called *feasible set.*

It is easy to minimize a convex function over any convex subset of its domain because any local minimum must be a global minimum. But it is difficult to find the maximum of a convex function over some convex domain because there can be many local maxima; although this has practical application (Eternity II §4.8, §C.5), it is not a convex problem. Tremendous benefit accrues when a mathematical problem can be transformed to an equivalent convex optimization, primarily because any locally optimal solution is then guaranteed globally optimal.[0.1] An *optimal* solution is a best solution to the problem posed; a certificate can be obtained guaranteeing that no better solution exists.

To provide a concrete example of what it meant by *optimal,* recall the ordinary *least squares* problem espoused by Gauss and Legendre over 200 years ago: (§E.0.1.0.1)

$$\underset{x}{\text{minimize}}\ \ \|Ax-b\|_2^2$$

Suppose we were to pose this problem a bit differently by *constraining* variable vector x simultaneously with the minimization. In particular, let's suppose that each entry of x were bounded above by the same maximum allowable value:

$$\begin{array}{cl}\underset{x}{\text{minimize}} & \|Ax-b\|_2^2\\ \text{subject to} & x\preceq x_{\max}\end{array}$$

Would a constrained solution, so obtained, be equivalent to an ordinary least squares solution whose entries (exceeding the prescribed bound) are simply clipped to the maximum value? The two solutions are, generally, different when clipping occurs. We argue that a constrained solution is better than a clipped solution; indeed, it is optimal.

[0.1] Solving a nonlinear system for example, by instead solving an equivalent convex optimization problem, is therefore highly preferable and what motivates *geometric programming*; a form of convex optimization invented in 1960s [67] [90] that has driven great advances in the electronic circuit design industry. [38, §4.7] [288] [459] [462] [117] [215] [224] [225] [226] [227] [228] [305] [306] [356]

Both of the foregoing ordinary and bounded least squares problems are convex. Recognizing a problem as convex is an acquired skill; that being, to know when an objective function is convex and when constraints specify a convex feasible set. The challenge, which is indeed an art, is how to express difficult problems in a convex way: perhaps, problems previously believed nonconvex. Practitioners in the art of Convex Optimization engage themselves with discovery of which hard problems can be transformed into convex equivalents; because, once convex form of a problem is found, then a globally optimal solution is close at hand - the hard work is finished: Finding convex expression of a problem is itself, in a very real sense, its solution.

Yet, that skill acquired by understanding the geometry and application of Convex Optimization will remain more an art for some time to come; the reason being, there is generally no unique transformation of a given problem to its convex equivalent. This means, two researchers pondering the same problem are likely to formulate a convex equivalent differently; hence, one solution is likely different from the other although any convex combination of those two solutions remains optimal. Any presumption of only one right or correct solution becomes nebulous. Study of equivalence & sameness, uniqueness, and duality therefore pervade study of Optimization.

It can be difficult for the engineer to apply convex theory without an understanding of Analysis. These pages comprise my journal over an eighteen year period bridging gaps between engineer and mathematician; they constitute a translation, unification, and cohering of about five hundred papers, books, and reports from several different fields of mathematics and engineering. Although beacons of historical accomplishment are cited throughout, much of what is written here will not be found elsewhere. Care to detail, clarity, accuracy, consistency, and typography accompanies removal of ambiguity and verbosity, out of respect for the reader. But the book is nonlinear in its presentation. Consequently there is much indexing, cross referencing, linkage to online sources, and background material provided in the text, footnotes, and appendices so as to be more self-contained and to provide understanding of fundamental concepts.

Looking toward the future, there remains much to be done in the area of machine computation if mathematical Optimization is to become fully embraced by the signal processing community. Wordlength of contemporary computers and numerical burdens upon them prohibit real time solution and accuracy sufficient to embed optimization problems within a recursive mathematical setting. When optimization problems constitute only intermediate solution to much larger problems, acquiring only a "few digits" accuracy can throw off subsequent dependent calculations. *Barrier* methods of solution are the principal obstacle to accuracy while *simplex* methods are the principal setback to speed. Novel, not hybrid, methods of solution are needed.

Audio distortion & noise analysis and measurement §8.1-§8.4 was begun 2016. Sinusoid tracking proved superior to Fourier and other filtering methods in 2017. Discerning harmonic and intermodulation distortion of device under test (DUT), from that produced by D/A→DUT→A/D signal chain, was discovered then. By 2018, the preferred "analyzer" had become discrete D/A and A/D converter because commercial analyzers (test gear) could not accept 32-bit inputs required for antidistortion injection. So it was discovered how D/A and A/D could themselves become DUT, opening up analysis to converter chip designers; submeasurable capability never prior had. The term *submeasurable* was introduced in 2019 to define levels below what was then measurable by very best commercial analyzers. Entrepreneurship prohibited publication in those years.

−Jon Dattorro
Stanford, California
2019

Convex Optimization
Euclidean Distance Geometry $^{2\varepsilon}$

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Chapter 1

Overview

Convex Optimization
Euclidean Distance Geometry

People are so afraid of convex analysis.

−Claude Lemaréchal, 2003

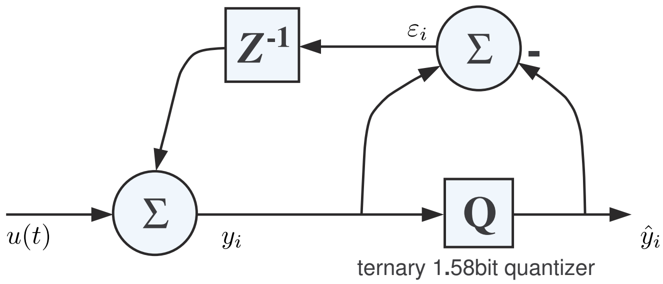
In layman's terms, the mathematical science of Optimization is a study of how to make good choices when confronted with conflicting requirements and demands. Optimization is a relatively new wisdom, historically, that can represent balance of real things. The qualifier *convex* means: when an optimal solution is found, then it is guaranteed to be a best solution; there is no better choice.

Any convex optimization problem has geometric interpretation. If a given optimization problem can be transformed to a convex equivalent, then this interpretive benefit is acquired. That is a powerful attraction: the ability to visualize geometry of an optimization problem. Conversely, recent advances in geometry and in graph theory hold convex optimization within their proofs' core. [471] [367]

This book is about convex optimization, convex geometry (with particular attention to distance geometry), and nonconvex, combinatorial, and geometrical problems that can be relaxed or transformed into convexity. A virtual flood of new applications follows by epiphany that many problems, presumed nonconvex, can be so transformed: [11] [12] [38, §4.3, p.316-322] [66] [106] [177] [180] [320] [345] [353] [413] [414] [467] [471] *e.g*, sigma delta analog-to-digital audio converter (A/D) antialiasing (Figure **1**).

Euclidean distance geometry is, fundamentally, a determination of point conformation (configuration, relative position or location) by inference from interpoint distance information. By *inference* we mean: *e.g*, given only distance information, determine whether there corresponds a *realizable* conformation of points; a *list* of points in some dimension that attains the given interpoint distances. Each point may represent simply location or, abstractly, any entity expressible as a vector in finite-dimensional Euclidean space; *e.g*, distance geometry of music [125].

It is a common misconception to presume that some desired point conformation cannot be recovered in absence of complete interpoint distance information. We might, for example, want to realize a constellation given only interstellar distance (or, equivalently, parsecs from our Sun and relative angular measurement; the Sun as vertex to two distant stars); called *stellar cartography,* an application evoked by Figure **3**. At first it may seem



$$\begin{array}{rl} \text{find} & u \\ \text{subject to} & R(\hat{y}-u) \preceq \frac{1}{3}\mathbf{1} \\ & R(u-\hat{y}) \preceq \frac{1}{3}\mathbf{1} \end{array}$$

Figure 1: Multibit sigma delta quantization is predominant technology for analog to digital audio signal conversion. [2, p.6] Input signal $u(t)$ is continuous. Delay z^{-1} here is analog, perhaps implemented by sample/hold circuit at MHz rate of $\hat{y}_i$ samples. Observing vector $\hat{y}$, signal u can be reconstructed by finding a point feasible to the set of linear inequalities representing this coarse quantizer recursion. R is a lower triangular matrix of ones. [114]

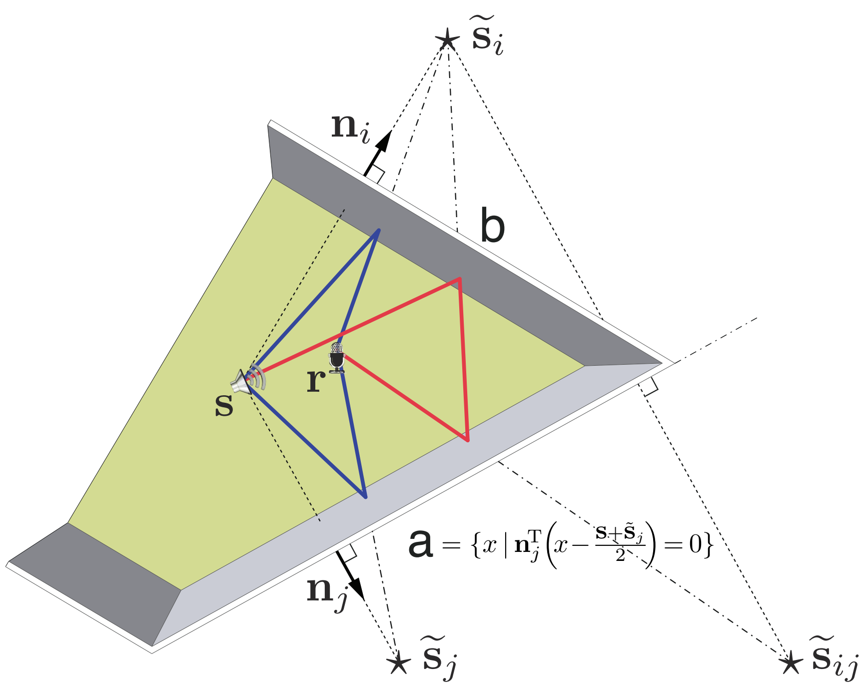


Figure 2: [137] [333] [134] Dokmanić & Parhizkar *et alii* discover an audio signal processing application of Euclidean distance matrices to room geometry estimation by discerning first acoustic reflections of stationary sound source $\mathbf{s}$. Locations of source and phantom $\star$ sources $\tilde{\mathbf{s}}_i$ and $\tilde{\mathbf{s}}_j$ are ascertained by measuring arrival times of first echoes (blue) at multiple microphone receivers. (Only one receiver $\mathbf{r}$ is illustrated. Second reflection (red) phantom $\tilde{\mathbf{s}}_{ij}$ ignored.) Phantom location is invariant to receiver position. All interpoint distances among receivers are known. Once source and phantoms are localized, normals $\mathbf{n}_j$ and $\mathbf{n}_i$ respectively identify truncated hyperplanes (walls) $\mathbf{a}$ and $\mathbf{b}$ bisecting perpendicular line segment connecting source $\mathbf{s}$ to a phantom.

Figure 3: *Orion nebula*. (Astrophotography by Massimo Robberto.)

that $\text{O}(N^2)$ data is required, yet there are many circumstances where this can be reduced to $\text{O}(N)$.

If we agree that a set of points may have a shape (three points can form a triangle and its interior, for example, four points a tetrahedron), then we can ascribe *shape* of a set of points to their convex hull. It should be apparent: from distance, these shapes can be determined only to within a *rigid transformation* (rotation, reflection, translation).

Absolute position information is generally lost, given only distance information, but we can determine the smallest possible dimension in which an unknown list of points can exist; that attribute is their *affine dimension* (a triangle in any ambient space has affine dimension 2, for example). In circumstances where stationary reference points are also provided, it becomes possible to determine absolute position or location; *e.g*, Figure **4**.

Geometric problems involving distance between points can sometimes be reduced to convex optimization problems. Mathematics of this combined study of geometry and optimization is rich and deep. Its application has already proven invaluable discerning organic *molecular conformation* by measuring interatomic distance along covalent bonds; *e.g*, Figure **5**. [100] [403] [164] [52] Many disciplines have already benefitted and simplified consequent to this theory; *e.g*, distance based *pattern recognition* (Figure **6**), *localization* in wireless sensor networks [53] [465] [51] by measurement of intersensor distance along channels of communication, *wireless location* of a radio-signal source such as cell phone by multiple measurements of signal strength, the *global positioning system* (GPS), *multidimensional scaling* (§5.12) which is a numerical representation of qualitative data by finding a low-dimensional scale, and audio signal processing: ultrasound tomography, room geometry estimation (Figure **2**), and perhaps dereverberation by localization of phantom sound sources [135] [134] [137]. [136]

Euclidean distance geometry provides some foundation for *artificial intelligence.* Together with convex optimization, distance geometry has found application to:

- *machine learning* by discerning naturally occurring manifolds in:
  - Euclidean bodies (Figure **7**, §6.7.0.0.1)
  - Fourier spectra of kindred utterances [248]
  - photographic image sequences [448]

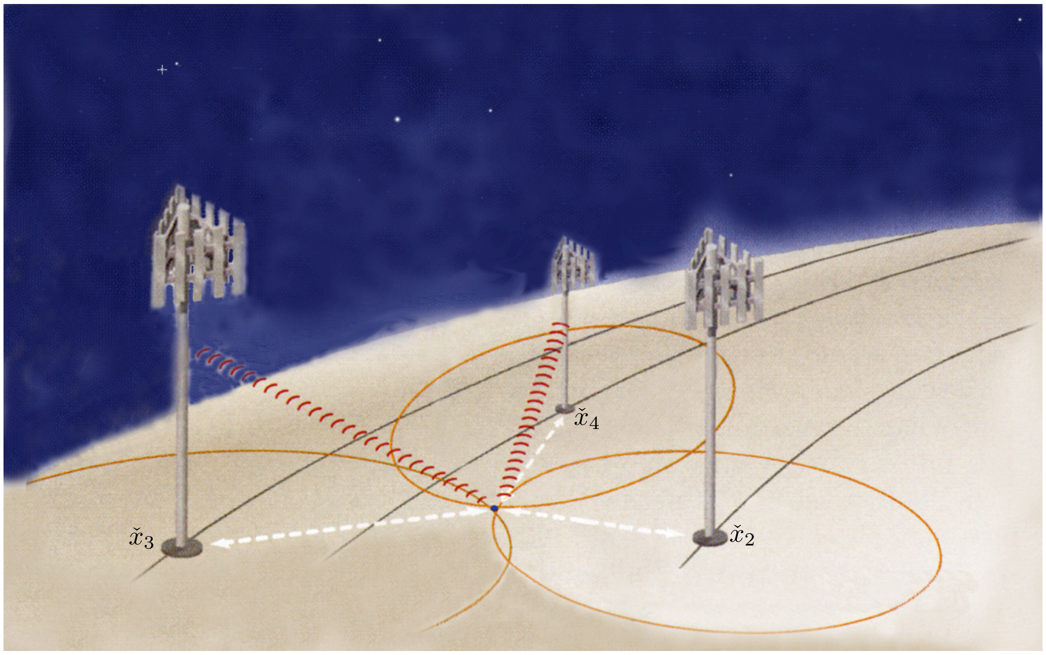


Figure 4: Application of trilateration (§5.4.2.2.8) is localization (determining position) of a radio signal source in 2 dimensions; more commonly known by radio engineers as the process "triangulation". In this scenario, anchors $\check{x}_2$, $\check{x}_3$, $\check{x}_4$ are illustrated as fixed antennae. [244] The radio signal source (a sensor • x_1) anywhere in affine hull of three antenna bases can be uniquely localized by measuring distance to each (dashed white arrowed line segments). Ambiguity of lone distance measurement to sensor is represented by circle about each antenna. Trilateration is expressible as a semidefinite program; hence, a convex optimization problem. [368]

- *robotics*; *e.g*, automated manufacturing, and autonomous navigation of vehicles maneuvering in formation (Figure **10**).

by chapter

We study the many manifestations and representations of pervasive convex Euclidean bodies. In particular, we make convex polyhedra, cones, and dual cones visceral through illustration in **Chapter 2 Convex Geometry** where geometric relationship of polyhedral cones to nonorthogonal bases (biorthogonal expansion) is examined. It is shown that coordinates are unique in any conic system whose basis cardinality equals or exceeds spatial dimension; for high cardinality, a new definition of *conic coordinate* is provided in Theorem 2.13.13.0.1. Conic analogue to linear independence, called *conic independence,* is introduced as a tool for study, analysis, and manipulation of cones; a natural extension and next logical step in progression: linear, affine, conic. We explain conversion between halfspace- and vertex-description of convex cone, we motivate dual cone and provide formulae for finding it, and we show how first-order optimality conditions or alternative systems of linear inequality or *linear matrix inequality* can be explained by *dual generalized inequalities* with respect to convex cones. Arcane theorems of alternative generalized inequality are, in fact, simply derived from cone *membership relations*; generalizations of algebraic *Farkas' lemma* translated to geometry of convex cones.

Any convex optimization problem can be visualized geometrically. Desire to visualize

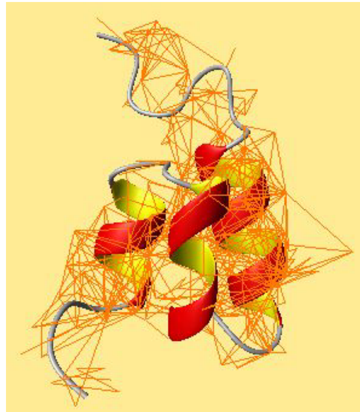


Figure 5: [223] [139] Distance data collected via nuclear magnetic resonance (NMR) helped render this three-dimensional depiction of a protein molecule. *At the beginning of the 1980s, Kurt Wüthrich* [Nobel laureate] *developed an idea about how NMR could be extended to cover biological molecules such as proteins. He invented a systematic method of pairing each NMR signal with the right hydrogen nucleus (proton) in the macromolecule. The method is called* sequential assignment *and is today a cornerstone of all NMR structural investigations. He also showed how it was subsequently possible to determine pairwise distances between a large number of hydrogen nuclei and use this information with a mathematical method based on distance-geometry to calculate a three-dimensional structure for the molecule.* [454] [218] −[324]

in high dimension [Sagan, *Cosmos − The Edge of Forever,* 22:55′] is deeply embedded in the mathematical psyche. [1] Chapter 2 provides tools to make visualization easier, and we teach how to visualize in high dimension. The concepts of face, extreme point, and extreme direction of a convex Euclidean body are explained here; crucial to understanding convex optimization. How to find the smallest face of any closed convex cone, containing convex set $\mathcal{C}$, is divulged; later shown to have practical application to presolving convex programs. The convex cone of positive semidefinite matrices, in particular, is studied in depth:

- We interpret, for example, inverse image of the positive semidefinite cone under affine transformation. (Example 2.9.1.0.2)
- Subsets of the positive semidefinite cone, discriminated by rank exceeding some lower bound, are convex. In other words, high-rank subsets of the positive semidefinite cone boundary united with its interior are convex. (Theorem 2.9.2.9.3) There is a closed form for projection on those convex subsets.
- The positive semidefinite cone is a circular cone in low dimension; *Geršgorin discs* specify inscription of a polyhedral cone into it. (Figure **51**)

Chapter 3 Geometry of Convex Functions observes Fenchel's analogy between convex sets and functions: We explain, for example, how the real affine function relates to convex functions as the hyperplane relates to convex sets. A toolbox of practical useful convex functions and a cookbook for optimization problems, methods are drawn from the appendices about matrix calculus for determining convexity and discerning geometry.

Chapter 4. **Semidefinite Programming** *has recently emerged to prominence because it admits a new problem type previously unsolvable by convex optimization techniques and because it theoretically subsumes other convex types: linear programming, quadratic programming, second-order cone programming*. −p.219 Semidefinite programming is

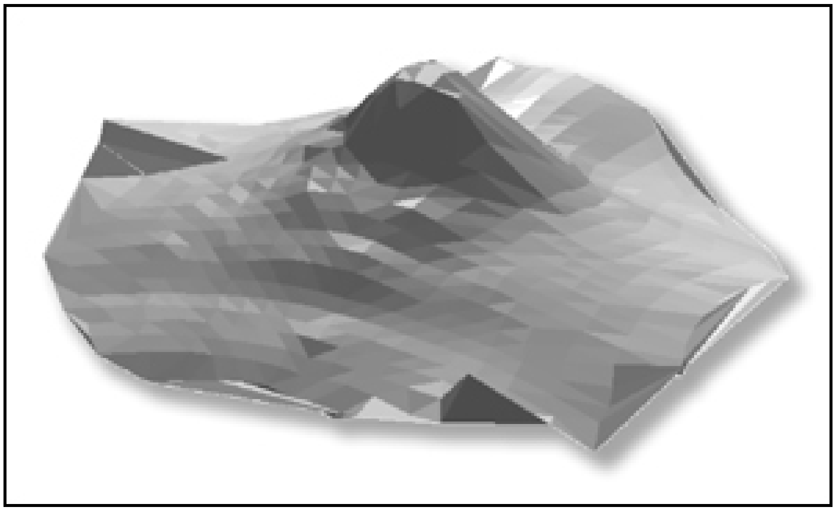


Figure 6: This coarsely discretized triangulated algorithmically flattened human face (made by Kimmel & the Bronsteins [263]) represents a stage in machine recognition of human identity; called *facial recognition.* Distance geometry is applied to determine discriminating-features.

reviewed with particular attention to optimality conditions for prototypical primal and dual problems, their interplay, and a perturbation method for rank reduction of optimal solutions (extant but not well known). *Positive definite Farkas' lemma* is derived, and we also show how to determine if a feasible set belongs exclusively to a positive semidefinite cone boundary. An arguably good three-dimensional polyhedral analogue to the positive semidefinite cone of 3×3 symmetric matrices is introduced: a new tool for visualizing coexistence of low- and high-rank optimal solutions in six isomorphic dimensions and a mnemonic aid for understanding semidefinite programs. We find a minimal cardinality Boolean solution to an instance of $Ax=b$:

$$\begin{array}{cl}\underset{x}{\text{minimize}} & \|x\|_0\\ \text{subject to} & Ax=b\\ & x_i\in\{0,1\}\ ,\quad i=1\ldots n\end{array}\tag{715}$$

The *sensor-network localization* problem is solved in any dimension in this chapter. We introduce a method of *convex iteration* for constraining rank in the form $\operatorname{rank}G\leq\rho$ and cardinality in the form $\operatorname{card}x\leq k$. Cardinality minimization is applied to a discrete image-gradient of the Shepp-Logan phantom, from Magnetic Resonance Imaging (MRI) in the field of medical imaging, for which we find a new lower bound of 1.9% cardinality. We show how to handle polynomial constraints, and how to transform a rank-constrained problem to a rank-1 problem.

The EDM is studied in **Chapter 5 Euclidean Distance Matrix**; its properties and relationship to both positive semidefinite and Gram matrices. We relate the EDM to the four classical properties of Euclidean metric; thereby, observing existence of an infinity of properties of the Euclidean metric beyond triangle inequality. We proceed by deriving the fifth Euclidean metric property and then explain why furthering this endeavor is inefficient because the ensuing criteria (while describing polyhedra in angle or area, volume, content, and so on *ad infinitum*) grow linearly in complexity and number with problem size.

Reconstruction methods are explained and applied to a map of the United States; *e.g*, Figure **8**. We also experimentally test a conjecture of Borg & Groenen by reconstructing

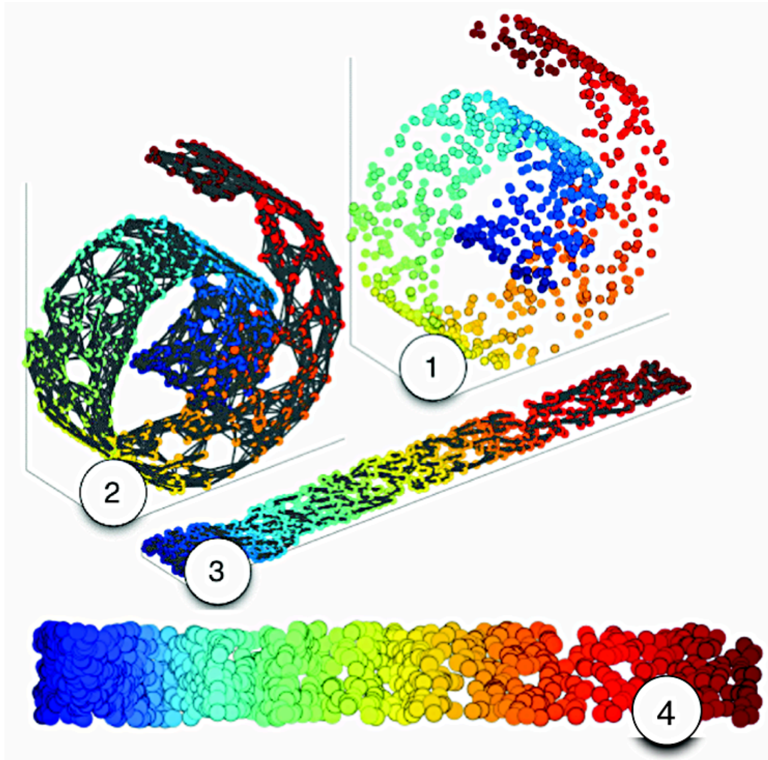


Figure 7: *Swiss roll,* Weinberger & Saul [448]. *The problem of manifold learning, illustrated for $N=800$ data points sampled from a "Swiss roll"* ①. *A discretized manifold is revealed by connecting each data point and its $k=6$ nearest neighbors* ②. *An unsupervised learning algorithm unfolds the Swiss roll while preserving the local geometry of nearby data points* ③. *Finally, the data points are projected onto the two-dimensional subspace that maximizes their variance, yielding a faithful embedding of the original manifold* ④.

a distorted but recognizable isotonic map of the USA using only ordinal (comparative) distance data: Figure **156**e-f. We demonstrate an elegant method for including dihedral (or *torsion*) angle constraints into a molecular conformation problem. We explain why *trilateration* (`a.k.a` *localization*) is a convex optimization problem. We show how to recover relative position given incomplete interpoint distance information, and how to pose EDM problems or transform geometrical problems to convex optimizations; *e.g*, *kissing number* of packed spheres about a central sphere (solved in $\mathbb{R}^{\mathbf{3}}$ by Isaac Newton).

The set of all Euclidean distance matrices forms a pointed closed convex cone called the *EDM cone*: $\mathbb{EDM}^N$. We offer a new proof of Schoenberg's seminal characterization of EDMs:

$$D \in \mathbb{EDM}^N \quad \Leftrightarrow \quad \left\{ \begin{array}{c} -V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} \succeq 0 \\ D \in \mathbb{S}_h^N \end{array} \right. \tag{1025}$$

Our proof relies on fundamental geometry; assuming, any EDM must correspond to a list of points contained in some polyhedron (possibly at its vertices) and *vice versa*. It is known, but not obvious, this *Schoenberg criterion* implies nonnegativity of the EDM entries; proved herein.

We characterize eigenvalue spectrum of an EDM, then devise a polyhedral spectral cone for determining membership of a given matrix (in Cayley-Menger form) to the convex cone of Euclidean distance matrices; *id est*, a matrix is an EDM if and only if its nonincreasingly

Figure 8: (*confer* Figure **156**) About five thousand points along borders constituting United States were used to create an exhaustive matrix of interpoint distance for each and every pair of points in an ordered set (a *list*); called *Euclidean distance matrix.* From that noiseless distance information, it is easy to reconstruct this nonconvex map exactly via Schoenberg criterion (1025). (§5.13.1.0.1) Map reconstruction is exact (to within a rigid transformation) given any number of interpoint distances; the greater the number of distances, the greater the detail (as it is for all conventional map preparation).

ordered vector of eigenvalues belongs to a polyhedral spectral cone for $\mathbb{EDM}^N$

$$D \in \mathbb{EDM}^N \;\Leftrightarrow\; \left\{\begin{array}{l} \lambda\left(\left[\begin{array}{cc} 0 & \mathbf{1}^{\mathrm{T}} \\ \mathbf{1} & -D \end{array}\right]\right) \in \left[\begin{array}{c} \mathbb{R}_+^N \\ \mathbb{R}_- \end{array}\right] \cap \partial\mathcal{H} \\ D \in \mathbb{S}_h^N \end{array}\right. \tag{1243}$$

We will see: spectral cones are not unique.

In **Chapter 6 Cone of Distance Matrices** we explain a geometric relationship between the cone of Euclidean distance matrices, two positive semidefinite cones, and the elliptope. We illustrate geometric requirements, in particular, for projection of a given matrix on a positive semidefinite cone that establish its membership to the EDM cone. The faces of the EDM cone are described, but still open is the question whether all its faces are exposed as they are for the positive semidefinite cone.

The *Schoenberg criterion,*

$$D \in \mathbb{EDM}^N \;\Leftrightarrow\; \left\{\begin{array}{c} -V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} \in \mathbb{S}_+^{N-1} \\ D \in \mathbb{S}_h^N \end{array}\right. \tag{1025}$$

for identifying a Euclidean distance matrix, is revealed to be a discretized *membership relation* (*dual generalized inequalities,* a new Farkas'-like lemma) between the EDM cone and its ordinary dual: $\mathbb{EDM}^{N^*}$. A matrix criterion for membership to the dual EDM cone is derived that is simpler than the Schoenberg criterion:

$$D^* \in \mathbb{EDM}^{N^*} \Leftrightarrow\; \delta(D^*\mathbf{1}) - D^* \succeq 0 \tag{1393}$$

There is a concise equality, relating the convex cone of Euclidean distance matrices to the positive semidefinite cone, apparently overlooked in the literature; an equality between two large convex Euclidean bodies:

$$\mathbb{EDM}^N = \mathbb{S}_h^N \cap \left(\mathbb{S}_c^{N\perp} - \mathbb{S}_+^N\right) \tag{1387}$$

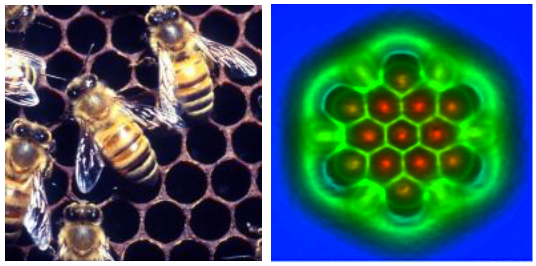
(a)  (b)

Figure 9: **(a)** These bees construct a honeycomb by solving a convex optimization problem (§5.4.2.2.6). The most dense packing of identical spheres about a central sphere in 2 dimensions is 6. Sphere centers describe a regular lattice. **(b)** *A hexabenzocoronene molecule (diameter: 1.4nm) imaged by noncontact atomic force microscopy using a microscope tip terminated with a single carbon monoxide molecule. The carbon-carbon bonds in the imaged molecule appear with different contrast and apparent lengths. Based on these disparities, the bond orders and lengths of the individual bonds can be distinguished.* (Image by Leo Gross.)

Seemingly innocuous problems in terms of point position $x_i \in \mathbb{R}^n$ like

$$\underset{\{x_i\}}{\text{minimize}} \sum_{i\,,\,j\,\in\,\mathcal{I}} (\|x_i - x_j\| - h_{ij})^2 \qquad (1427)$$

$$\underset{\{x_i\}}{\text{minimize}} \sum_{i\,,\,j\,\in\,\mathcal{I}} (\|x_i - x_j\|^2 - h_{ij})^2 \qquad (1428)$$

are difficult to solve. So, in **Chapter 7 Proximity Problems**, we instead explore methods of their solution by transformation to a few fundamental and prevalent Euclidean distance matrix proximity problems; the problem of finding that distance matrix closest, in some sense, to a given matrix $H = [h_{ij}]$:

$$\begin{array}{llll}
\underset{D}{\text{minimize}} & \|-V(D-H)V\|_{\mathrm{F}}^2 & \underset{\sqrt[\circ]{D}}{\text{minimize}} & \|\sqrt[\circ]{D} - H\|_{\mathrm{F}}^2 \\
\text{subject to} & \operatorname{rank} VDV \leq \rho & \text{subject to} & \operatorname{rank} VDV \leq \rho \\
 & D \in \mathbb{EDM}^N & & \sqrt[\circ]{D} \in \sqrt{\mathbb{EDM}^N} \\
 & & & \\
\underset{D}{\text{minimize}} & \|D-H\|_{\mathrm{F}}^2 & \underset{\sqrt[\circ]{D}}{\text{minimize}} & \|-V(\sqrt[\circ]{D} - H)V\|_{\mathrm{F}}^2 \\
\text{subject to} & \operatorname{rank} VDV \leq \rho & \text{subject to} & \operatorname{rank} VDV \leq \rho \\
 & D \in \mathbb{EDM}^N & & \sqrt[\circ]{D} \in \sqrt{\mathbb{EDM}^N}
\end{array} \qquad (1429)$$

We apply a convex iteration method for constraining rank. Known heuristics for rank minimization are also explained. We offer new geometrical proof, in §7.1.4.0.1, of a famous discovery by Eckart & Young in 1936 [153]: Euclidean projection on that generally nonconvex subset of the positive semidefinite cone boundary comprising all semidefinite matrices having rank not exceeding a prescribed bound ρ. We explain how this problem is transformed to a convex optimization for any rank ρ.

Chapter 8 Audio Analysis constitutes the latest edition: §8.1) Discernment of sinusoids at the same frequency, emanating from distinct sources, with application to harmonic and intermodulation distortion measurement. §8.5) Arbitrary magnitude analog filter design by quasiconvex optimization with application to parametric equalizer implementation having zeros of transfer.

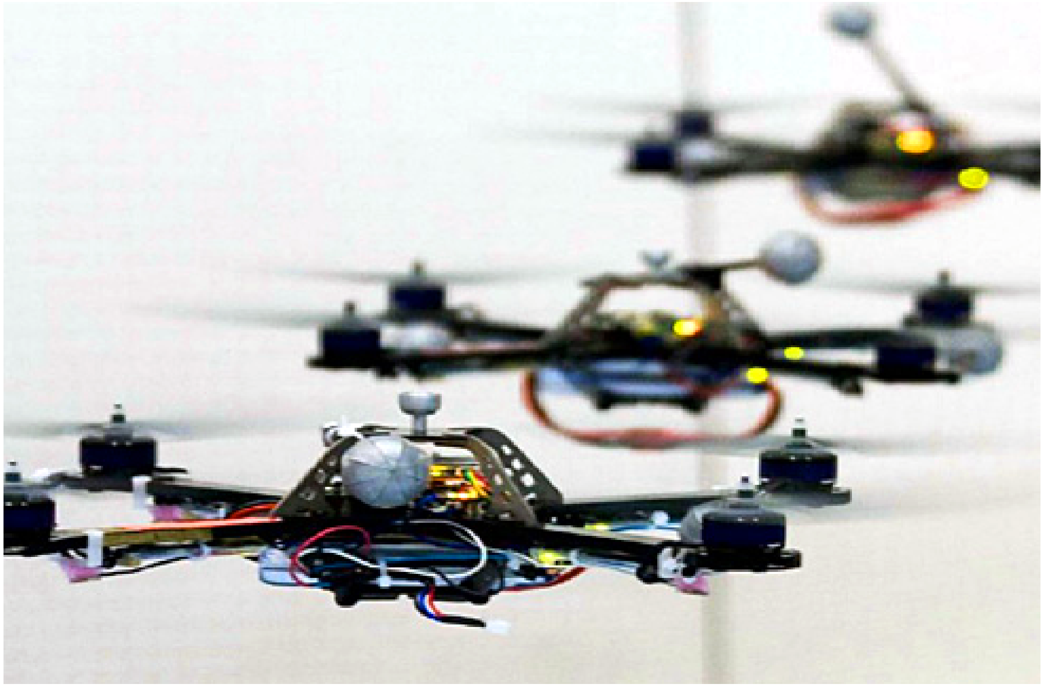


Figure 10: Nanocopter swarm. Robotic vehicles in concert can move larger objects or localize a plume of gas, liquid, or radio waves. [163]

appendices

We presume a reader already comfortable with elementary vector operations; [15, §3] formally known as *analytic geometry*. [456] Toolboxes are provided, in the form of appendices and code, so as to be more self-contained:

- **linear algebra** (Appendix A is primarily concerned with proper statements of semidefiniteness for square matrices)
- **simple matrices** (dyad, doublet, elementary, Householder, Schoenberg, orthogonal, *etcetera*, in Appendix B)
- collection of known **analytical solutions** to some important optimization problems (Appendix C)
- **matrix calculus** remains somewhat unsystematized when compared to ordinary calculus (Appendix D concerns matrix-valued functions, matrix differentiation and directional derivatives, Taylor series, and tables of first- and second-order gradients and matrix derivatives)
- elaborate exposition offering insight into orthogonal and nonorthogonal **projection** on convex sets (the connection between projection and positive semidefiniteness, for example, or between projection and a linear objective function in Appendix E)
- MATLAB **code** on *Wıκımization* [436] to discriminate EDMs, to determine conic independence, to reduce or constrain rank of an optimal solution to a semidefinite program, to compress digital image and audio signals by compressive sampling (compressed sensing), and to reconstruct a map of the United States by two distinct methods: one given only distance data, the other given only comparative distance.

Figure 11: Three-dimensional reconstruction of David from distance data.

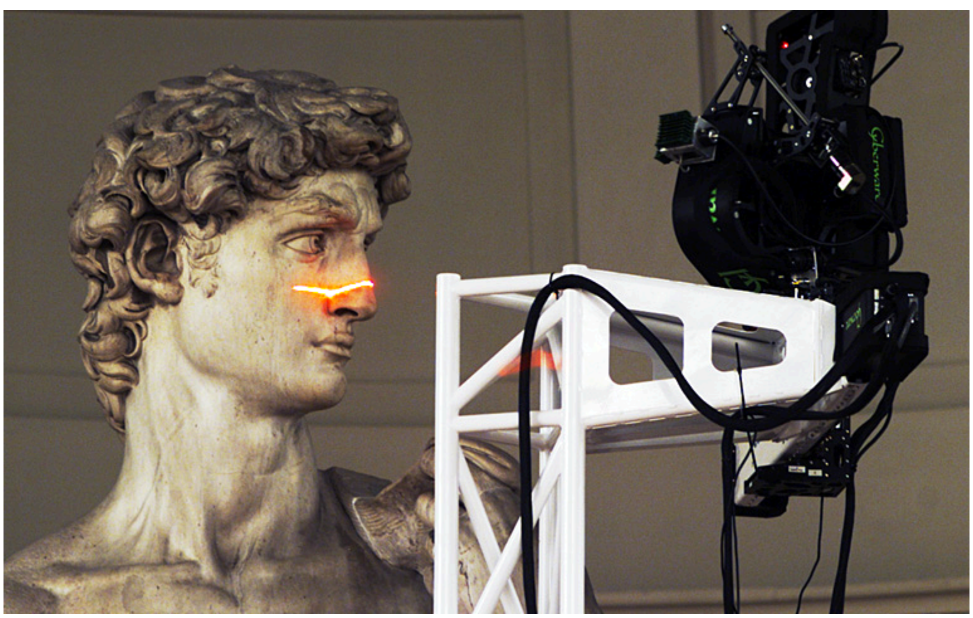


Figure 12: *Digital Michelangelo Project*, Stanford University. Measuring distance to David by laser rangefinder. (Spatial resolution: 0.29mm.) *Crystalix* commercialized a 3D image rendering laser by refining a stunning technique for interior engraving of cubic *photocrystal*.

Chapter 2

Convex Geometry

Convexity has an immensely rich structure and numerous applications. On the other hand, almost every "convex" idea can be explained by a two-dimensional picture.

−Alexander Barvinok [28, p.vii]

We study convex geometry because it is the easiest of geometries. For that reason, much of a practitioner's energy is expended seeking invertible transformation of problematic sets to convex ones.

As convex geometry and linear algebra are inextricably bonded by linear inequality (*asymmetry*), we provide much background material on linear algebra (especially in the appendices) although a reader is assumed comfortable with [379] [381] [237] or any other intermediate-level text. The essential references to convex analysis are [234] [354]. The reader is referred to [377] [28] [447] [46] [68] [351] [410] for a comprehensive treatment of convexity. There is relatively less published pertaining to convex matrix-valued functions. [251] [238, §6.6] [340]

2.1 Convex set

A set $\mathcal{C}$ is convex iff for all $Y, Z \in \mathcal{C}$ and $0 \leq \mu \leq 1$

$$\mu Y + (1-\mu)Z \in \mathcal{C} \tag{1}$$

Under that defining condition on μ, the linear sum in (1) is called a *convex combination* of Y and Z. If Y and Z are points in real finite-dimensional Euclidean *vector space* [264] [456] $\mathbb{R}^n$ or $\mathbb{R}^{m\times n}$ (matrices), then (1) represents the closed line segment joining them. Line segments are thereby convex sets; $\mathcal{C}$ is convex iff the line segment connecting any two points in $\mathcal{C}$ is itself in $\mathcal{C}$. Apparent from this definition: a convex set is a connected set. [299, §3.4, §3.5] [46, p.2] A convex set can, but does not necessarily, contain the *origin* $\mathbf{0}$.

An *ellipsoid* centered at $x=a$ (Figure **15** p.36), given matrix $C \in \mathbb{R}^{m\times n}$ and scalar γ

$$\mathcal{B}_{\mathcal{E}} = \{x \in \mathbb{R}^n \mid \|C(x-a)\|^2 = (x-a)^{\mathrm{T}} C^{\mathrm{T}} C(x-a) \leq \gamma^2\} \tag{2}$$

(an *ellipsoidal ball*) is a good icon for a convex set.[2.1]

[2.1] Ellipsoid semiaxes are eigenvectors of $C^{\mathrm{T}}C$ whose lengths are reciprocal square root eigenvalues. This particular definition is slablike (Figure **13**) in $\mathbb{R}^n$ when C has nontrivial nullspace.

2.1.1 subspace

A nonempty subset $\mathcal{R}$ of real Euclidean vector space $\mathbb{R}^n$ is called a *subspace* (§2.5) if every vector[2.2] of the form $\alpha x + \beta y$, for $\alpha, \beta \in \mathbb{R}$, is in $\mathcal{R}$ whenever vectors x and y are. [290, §2.3] A subspace is a convex set containing the origin, by definition. [354, p.4] Any subspace is therefore open in the sense that it contains no boundary, but closed in the sense [299, §2]

$$\mathcal{R} + \mathcal{R} = \mathcal{R} \tag{3}$$

It is not difficult to show

$$\mathcal{R} = -\mathcal{R} \tag{4}$$

as is true for any subspace $\mathcal{R}$, because $x \in \mathcal{R} \Leftrightarrow -x \in \mathcal{R}$. Given any $x \in \mathcal{R}$

$$\mathcal{R} = x + \mathcal{R} \tag{5}$$

Intersection of an arbitrary collection of subspaces remains a subspace. Any subspace, not constituting the entire *ambient vector space* $\mathbb{R}^n$, is a *proper subspace*; *e.g*,[2.3] any line (of infinite extent) through the origin in two-dimensional Euclidean space $\mathbb{R}^2$. Subspace $\{\mathbf{0}\}$, comprising only the origin, is proper though *trivial.* The vector space $\mathbb{R}^n$ is itself a conventional subspace, inclusively, [264, §2.1] although not proper.

2.1.2 linear independence

Arbitrary given vectors in Euclidean space $\{\Gamma_i \in \mathbb{R}^n ,\; i = 1 \ldots N\}$ are *linearly independent* (l.i.) if and only if, for all $\zeta \in \mathbb{R}^N$ $(\zeta_i \in \mathbb{R})$

$$\Gamma_1\, \zeta_1 + \cdots + \Gamma_{N-1}\, \zeta_{N-1} - \Gamma_N\, \zeta_N = \mathbf{0} \tag{6}$$

has only the *trivial solution* $\zeta = \mathbf{0}$; in other words, iff no vector from the given set can be expressed as a linear combination of those remaining.

Geometrically, two nontrivial vector subspaces are linearly independent iff they intersect only at the origin.

2.1.2.1 preservation of linear independence

(*confer* §2.4.2.4, §2.10.1) Linear transformation preserves linear dependence. [264, p.86] Conversely, linear independence can be preserved under linear transformation. Given $Y = [\, y_1 \;\; y_2 \; \cdots \; y_N \,] \in \mathbb{R}^{N \times N}$, consider the mapping

$$T(\Gamma) : \mathbb{R}^{n \times N} \to \mathbb{R}^{n \times N} \;\triangleq\; \Gamma\, Y \tag{7}$$

whose domain is the set of all matrices $\Gamma \in \mathbb{R}^{n \times N}$ holding a linearly independent set columnar. Linear independence of $\{\Gamma y_i \in \mathbb{R}^n ,\; i = 1 \ldots N\}$ demands, by definition, there exist no nontrivial solution $\zeta \in \mathbb{R}^N$ to

$$\Gamma y_1\, \zeta_i + \cdots + \Gamma y_{N-1}\, \zeta_{N-1} - \Gamma y_N\, \zeta_N = \mathbf{0} \tag{8}$$

By factoring out Γ , we see that triviality is ensured by linear independence of $\{y_i \in \mathbb{R}^N\}$.

[2.2] A *vector* is assumed, throughout, to be a column vector.

[2.3] We substitute abbreviation *e.g* in place of the Latin *exempli gratia*; meaning, *for sake of example.*

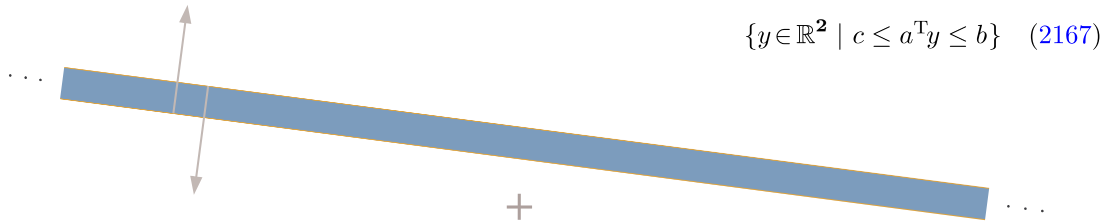


Figure 13: A *slab* is a convex Euclidean body infinite in extent but not affine. Illustrated in $\mathbb{R}^2$, it may be constructed by intersecting two opposing halfspaces whose bounding hyperplanes are parallel but not coincident. Because number of halfspaces used in its construction is finite, slab is a *polyhedron* (§2.12). (Cartesian axes $+$ and vector inward-normal, to each halfspace-boundary, are drawn for reference.)

2.1.3 Orthant:

name given to a closed convex set that is the higher-dimensional generalization of *quadrant* from the classical Cartesian partition of $\mathbb{R}^2$; a *Cartesian cone*. The most common is the nonnegative orthant $\mathbb{R}^n_+$ or $\mathbb{R}^{n\times n}_+$ (analogue to quadrant I) to which membership denotes nonnegative vector- or matrix-entries respectively; *e.g*,

$$\mathbb{R}^n_+ \triangleq \{x \in \mathbb{R}^n \mid x_i \geq 0 \;\; \forall\, i\} \tag{9}$$

The nonpositive orthant $\mathbb{R}^n_-$ or $\mathbb{R}^{n\times n}_-$ (analogue to quadrant III) denotes negative and 0 entries. Orthant convexity[2.4] is easily verified by definition (1).

2.1.4 affine set

A nonempty *affine set* (from the word *affinity*) is any subset of $\mathbb{R}^n$ that is a translation of some subspace. Any affine set is convex, and open in the sense that it contains no boundary: *e.g*, empty set $\emptyset$, point, line, plane, *hyperplane* (§2.4.2), subspace, *etcetera.* The intersection of an arbitrary collection of affine sets remains affine.

2.1.4.0.1 Definition. *Affine subset.*
We analogize *affine subset* to subspace,[2.5] defining it to be any nonempty affine set of vectors; an affine subset of $\mathbb{R}^n$. $\triangle$

For some *parallel*[2.6] subspace $\mathcal{R}$ and any point $x \in \mathcal{A}$

$$\begin{aligned} \mathcal{A} \text{ is affine} \;\Leftrightarrow\;\; \mathcal{A} &= x + \mathcal{R} \\ &= \{y \mid y - x \in \mathcal{R}\} \end{aligned} \tag{10}$$

Affine hull of a set $\mathcal{C} \subseteq \mathbb{R}^n$ (§2.3.1) is the smallest affine set containing it.

2.1.5 dimension

Dimension of an arbitrary set $\mathcal{Z}$ is Euclidean dimension of its affine hull; [447, p.14]

$$\dim \mathcal{Z} \triangleq \dim \operatorname{aff} \mathcal{Z} \;=\; \dim \operatorname{aff}(\mathcal{Z} - s)\,, \quad s \in \mathcal{Z} \tag{11}$$

[2.4] All orthants are selfdual simplicial cones. (§2.13.6.1, §2.12.3.1.1)
[2.5] The popular term *affine subspace* is an oxymoron.
[2.6] Two affine sets are *parallel* when one is a translation of the other. [354, p.4]

the same as dimension of the subspace parallel to that affine set $\operatorname{aff}\mathcal{Z}$ when nonempty. Hence dimension (of a set) is synonymous with *affine dimension.* [234, A.2.1]

2.1.6 empty set *versus* empty interior

Emptiness $\emptyset$ of a set is handled differently than *interior* in the classical literature. It is common for a nonempty convex set to have empty interior; *e.g*, paper in the real world:

- An ordinary flat sheet of paper is a nonempty convex set having empty interior in $\mathbb{R}^3$ but nonempty interior relative to its affine hull.

2.1.6.1 relative interior

Although it is always possible to pass to a smaller ambient Euclidean space where a nonempty set acquires an interior [28, §II.2.3], we prefer the qualifier *relative* which is the conventional fix to this ambiguous terminology.[2.7] So we distinguish *interior* from *relative interior* throughout: [377] [447] [410]

- Classical interior $\operatorname{intr}\mathcal{C}$ is defined as a union of points: x is an interior point of $\mathcal{C}\subseteq\mathbb{R}^n$ if there exists an open Euclidean ball

$$\mathcal{B} \triangleq \{y\in\mathbb{R}^n \mid \|y-x\| < \gamma\} \tag{12}$$

  of dimension n and nonzero radius γ centered at x that is contained in $\mathcal{C}$.

- Relative interior $\operatorname{rel\,intr}\mathcal{C}$ of a convex set $\mathcal{C}\subseteq\mathbb{R}^n$ is interior relative to its affine hull.[2.8]

Thus defined, it is common (though confusing) for $\operatorname{intr}\mathcal{C}$ the interior of $\mathcal{C}$ to be empty while its relative interior is not: this happens whenever dimension of its affine hull is less than dimension of the ambient space ($\dim\operatorname{aff}\mathcal{C}<n$; *e.g*, were $\mathcal{C}$ paper) or in the exception when $\mathcal{C}$ is a single point; [299, §2.2.1]

$$\operatorname{rel\,intr}\{x\} \triangleq \operatorname{aff}\{x\} = \{x\}\ , \qquad \operatorname{intr}\{x\} = \emptyset\ , \qquad x\in\mathbb{R}^n \tag{13}$$

In any case, *closure* of the relative interior of a convex set $\mathcal{C}$ always yields closure of the set itself;

$$\overline{\operatorname{rel\,intr}\mathcal{C}} = \overline{\mathcal{C}} \tag{14}$$

Closure is invariant to translation. If $\mathcal{C}$ is convex then $\operatorname{rel\,intr}\mathcal{C}$ and $\overline{\mathcal{C}}$ are convex. [234, p.24] If $\mathcal{C}$ has nonempty interior, then

$$\operatorname{rel\,intr}\mathcal{C} = \operatorname{intr}\mathcal{C} \tag{15}$$

Given the intersection of convex set $\mathcal{C}$ with affine set $\mathcal{A}$

$$\operatorname{rel\,intr}(\mathcal{C}\cap\mathcal{A}) = \operatorname{rel\,intr}(\mathcal{C})\cap\mathcal{A} \quad\Leftarrow\quad \operatorname{rel\,intr}(\mathcal{C})\cap\mathcal{A}\neq\emptyset \tag{16}$$

Because an affine set $\mathcal{A}$ is open

$$\operatorname{rel\,intr}\mathcal{A} = \mathcal{A} \tag{17}$$

[2.7] Superfluous mingling of terms as in *relatively nonempty set* would be an unfortunate consequence. From the opposite perspective, some authors use the term *full* or *full-dimensional* to describe a set having nonempty interior.

[2.8] Likewise for *relative boundary* (§2.1.7.2), although *relative closure* is superfluous. [234, §A.2.1]

(a)

(b)

(c)

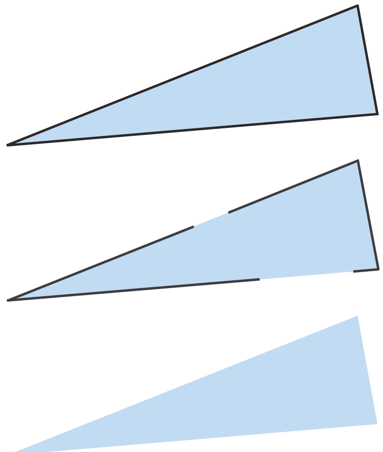


Figure 14: **(a)** Closed convex set. **(b)** Neither open, closed, or convex. Yet PSD cone can remain convex in absence of certain boundary components (§2.9.2.9.3). Nonnegative orthant with origin excluded (§2.6) and positive orthant with origin adjoined [354, p.49] are convex. **(c)** Open convex set.

2.1.7 classical boundary

(*confer* §2.1.7.2) *Boundary* of a set $\mathcal{C}$ is the closure of $\mathcal{C}$ less its interior;

$$\partial\mathcal{C} = \overline{\mathcal{C}} \setminus \operatorname{intr}\mathcal{C} \tag{18}$$

[61, §1.1] which follows from the fact

$$\overline{\operatorname{intr}\mathcal{C}} = \overline{\mathcal{C}} \quad \Leftrightarrow \quad \partial\operatorname{intr}\mathcal{C} = \partial\mathcal{C} \tag{19}$$

and presumption of nonempty interior.[2.9] Implications are:

- $\operatorname{intr}\mathcal{C} = \overline{\mathcal{C}} \setminus \partial\mathcal{C}$
- a bounded open set has *boundary* defined but not contained in the set
- interior of an open set is equivalent to the set itself;

from which an open set is defined: [299, p.109]

$$\mathcal{C} \text{ is } \textit{open} \;\Leftrightarrow\; \operatorname{intr}\mathcal{C} = \mathcal{C} \tag{20}$$
$$\mathcal{C} \text{ is } \textit{closed} \;\Leftrightarrow\; \overline{\operatorname{intr}\mathcal{C}} = \mathcal{C} \tag{21}$$

The set illustrated in Figure 14b is not open because it is not equivalent to its interior, for example, it is not closed because it does not contain its boundary, and it is not convex because it does not contain all convex combinations of its boundary points.

[2.9] Otherwise, for $x \in \mathbb{R}^n$ as in (13), [299, §2.1-§2.3]

$$\overline{\operatorname{intr}\{x\}} = \overline{\emptyset} = \emptyset$$

the empty set is both open and closed.

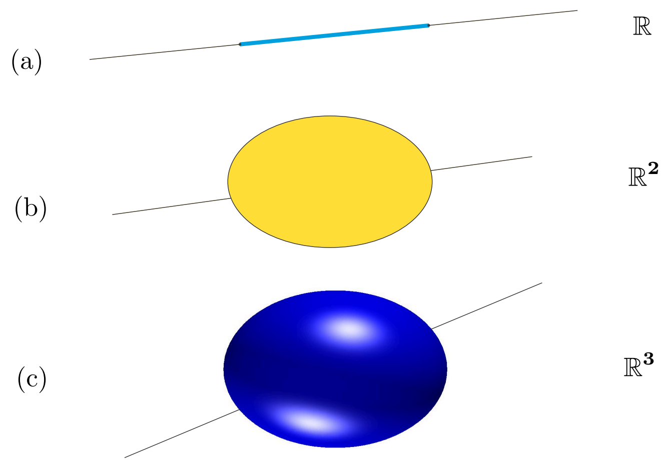


Figure 15: **(a)** Ellipsoid in $\mathbb{R}$ is a line segment whose boundary comprises two points. Intersection of line with ellipsoid in $\mathbb{R}$, **(b)** in $\mathbb{R}^2$, **(c)** in $\mathbb{R}^3$. Each ellipsoid illustrated has entire boundary constituted by zero-dimensional faces; in fact, by *vertices* (§2.6.1.0.1). Intersection of line with boundary is a point at entry to interior. These same facts hold in higher dimension.

2.1.7.1 Line intersection with boundary

A line can intersect the boundary of a convex set in any dimension at a point demarcating the line's entry to the set interior. On one side of that entry-point along the line is the exterior of the set, on the other side is the set interior. In other words,

- starting from any point of a convex set, a move toward the interior is an immediate entry into the interior. [28, §II.2]

When a line intersects the interior of a convex body in any dimension, the boundary appears to the line to be as thin as a point. This is intuitively plausible because, for example, a line intersects the boundary of the ellipsoids in Figure **15** at a point in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$. Such thinness is a remarkable fact when pondering visualization of convex *polyhedra* (§2.12, §5.14.3) in four Euclidean dimensions, for example, having boundaries constructed from other three-dimensional convex polyhedra called *faces.*

We formally define *face* in (§2.6). For now, we observe the boundary of a convex body to be entirely constituted by all its faces of dimension lower than the body itself. Any face of a convex set is convex. For example: The ellipsoids in Figure **15** have boundaries composed only of zero-dimensional faces. The two-dimensional slab in Figure **13** is an unbounded *polyhedron* having one-dimensional faces making its boundary. The three-dimensional bounded polyhedron in Figure **22** has zero-, one-, and two-dimensional polygonal faces constituting its boundary.

2.1.7.1.1 Example. *Intersection of line with boundary in* $\mathbb{R}^6$.
The convex cone of positive semidefinite matrices $\mathbb{S}^3_+$ (§2.9), in the ambient subspace of symmetric matrices $\mathbb{S}^3$ (§2.2.2.0.1), is a six-dimensional Euclidean body in *isometrically isomorphic* $\mathbb{R}^6$ (§2.2.1). Boundary of the positive semidefinite cone, in this dimension, comprises faces having only the dimensions 0, 1, and 3; *id est*, $\{\rho(\rho+1)/2\,,\ \rho=0,1,2\}$.

Unique minimum-distance projection PX (§E.9) of any point $X\in\mathbb{S}^3$ on that cone $\mathbb{S}_+^3$ is known in closed form (§7.1.2). Given, for example, $\lambda\in\operatorname{intr}\mathbb{R}_+^3$ and *diagonalization* (§A.5.1) of exterior point

$$X = Q\Lambda Q^{\mathrm{T}} \in \mathbb{S}^3, \quad \Lambda \triangleq \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \lambda_2 & \\ \mathbf{0} & & -\lambda_3 \end{bmatrix} \tag{22}$$

where $Q\in\mathbb{R}^{3\times 3}$ is an *orthogonal matrix,* then the projection on $\mathbb{S}_+^3$ in $\mathbb{R}^6$ is

$$PX = Q\begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \lambda_2 & \\ \mathbf{0} & & 0 \end{bmatrix} Q^{\mathrm{T}} \in \mathbb{S}_+^3 \tag{23}$$

This positive semidefinite matrix PX nearest X thus has rank 2, found by discarding all negative eigenvalues in Λ. The line connecting these two points is $\{X + (PX - X)t \mid t\in\mathbb{R}\}$ where $t=0 \Leftrightarrow X$ and $t=1 \Leftrightarrow PX$. Because this line intersects the boundary of the *positive semidefinite cone* $\mathbb{S}_+^3$ at point PX and passes through its interior (by assumption), then the matrix corresponding to an infinitesimally positive perturbation of t there should reside interior to the cone (rank 3). Indeed, for ε an arbitrarily small positive constant,

$$X + (PX - X)t\big|_{t=1+\varepsilon} = Q\big(\Lambda + (P\Lambda - \Lambda)(1+\varepsilon)\big)Q^{\mathrm{T}} = Q\begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \lambda_2 & \\ \mathbf{0} & & \varepsilon\lambda_3 \end{bmatrix} Q^{\mathrm{T}} \in \operatorname{intr}\mathbb{S}_+^3 \tag{24}$$

□

2.1.7.1.2 Example. *Tangential line intersection with boundary.*
A higher-dimensional boundary $\partial\mathcal{C}$ of a convex Euclidean body $\mathcal{C}$ is simply a dimensionally larger set through which a line can pass when it does not intersect the body's interior. Still, for example, a line existing in five or more dimensions may pass *tangentially* (intersecting no point interior to $\mathcal{C}$ [398, §15.3]) through a single point relatively interior to a three-dimensional face on $\partial\mathcal{C}$. Let's understand why by inductive reasoning.

Figure **16**a shows a vertical line-segment whose boundary comprises its two endpoints. For a line to pass through the boundary tangentially (intersecting no point relatively interior to the line-segment), it must exist in an ambient space of at least two dimensions. Otherwise, the line is confined to the same one-dimensional space as the line-segment and must pass along the segment to reach the end points.

Figure **16**b illustrates a two-dimensional ellipsoid whose boundary is constituted entirely by zero-dimensional faces. Again, a line must exist in at least two dimensions to tangentially pass through any single arbitrarily chosen point on the boundary (without intersecting the ellipsoid interior).

Now let's move to an ambient space of three dimensions. Figure **16**c shows a polygon rotated into three dimensions. For a line to pass through its zero-dimensional boundary (one of its *vertices*) tangentially, it must exist in at least the two dimensions of the polygon. But for a line to pass tangentially through a single arbitrarily chosen point in the relative interior of a one-dimensional face on the boundary as illustrated, it must exist in at least three dimensions.

Figure **16**d illustrates a solid circular cone (drawn truncated) whose one-dimensional faces are halflines emanating from its pointed end (*vertex*). This cone's boundary is constituted solely by those one-dimensional halflines. A line may pass through the boundary tangentially, striking only one arbitrarily chosen point relatively interior to a one-dimensional face, if it exists in at least the three-dimensional ambient space of the cone.

From these few examples, we may deduce a general rule (without proof):

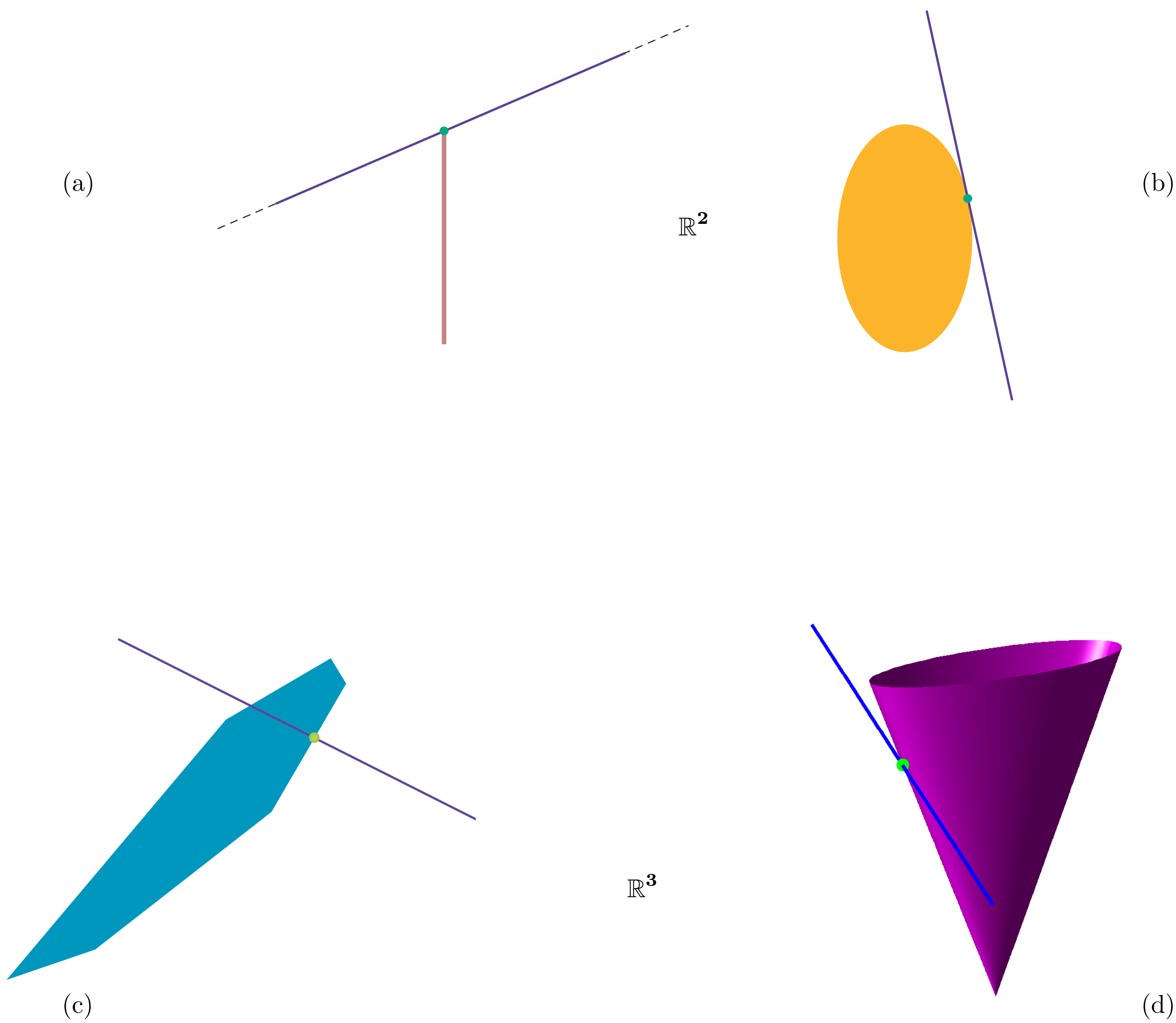


Figure 16: Line tangential: **(a) (b)** to relative interior of a zero-dimensional face in $\mathbb{R}^2$, **(c) (d)** to relative interior of a one-dimensional face in $\mathbb{R}^3$.

- A line may pass tangentially through a single arbitrarily chosen point relatively interior to a k-dimensional face on the boundary of a convex Euclidean body if the line exists in dimension at least equal to $k+2$.

Now the interesting part, with regard to Figure **22** showing a bounded polyhedron in $\mathbb{R}^3$; call it $\mathcal{P}$: A line existing in at least four dimensions is required in order to pass tangentially (without hitting $\operatorname{intr}\mathcal{P}$) through a single arbitrary point in the relative interior of any two-dimensional polygonal face on the boundary of polyhedron $\mathcal{P}$. Now imagine that polyhedron $\mathcal{P}$ is itself a three-dimensional face of some other polyhedron in $\mathbb{R}^4$. To pass a line tangentially through polyhedron $\mathcal{P}$ itself, striking only one point from its relative interior $\operatorname{rel\,intr}\mathcal{P}$ as claimed, requires a line existing in at least five dimensions.[2.10]

It is not too difficult to deduce:

- A line may pass through a single arbitrarily chosen point interior to a k-dimensional convex Euclidean body (hitting no other interior point) if that line exists in dimension at least equal to $k+1$.

In layman's terms, this means: a being capable of navigating four spatial dimensions (one Euclidean dimension beyond our physical reality) could see inside three-dimensional objects. □

2.1.7.2 Relative boundary

The classical definition of *boundary* of a set $\mathcal{C}$ presumes nonempty interior:

$$\partial\mathcal{C} = \overline{\mathcal{C}} \setminus \operatorname{intr}\mathcal{C} \tag{18}$$

More suitable to study of convex sets is the *relative boundary*; defined [234, §A.2.1.2]

$$\operatorname{rel}\partial\mathcal{C} \triangleq \overline{\mathcal{C}} \setminus \operatorname{rel\,intr}\mathcal{C} \tag{25}$$

boundary relative to affine hull of $\mathcal{C}$.

In the exception when $\mathcal{C}$ is a single point $\{x\}$, (13)

$$\operatorname{rel}\partial\{x\} = \overline{\{x\}}\setminus\{x\} = \emptyset\,, \qquad x\in\mathbb{R}^n \tag{26}$$

A bounded convex polyhedron (§2.3.2, §2.12.0.0.1) in subspace $\mathbb{R}$, for example, has boundary constructed from two points, in $\mathbb{R}^2$ from at least three line segments, in $\mathbb{R}^3$ from convex polygons, while a convex *polychoron* (a bounded polyhedron in $\mathbb{R}^4$ [449]) has boundary constructed from three-dimensional convex polyhedra. A halfspace is partially bounded by a hyperplane; its interior therefore excludes that hyperplane. An affine set has no relative boundary. Ellipsoid (2) has relative boundary

$$\partial\mathcal{B}_{\mathcal{E}} = \{x\in\mathbb{R}^n \mid \|C(x-a)\|^2 = (x-a)^{\mathrm T}C^{\mathrm T}C(x-a) = \gamma^2\} \tag{27}$$

Relative boundary of a convex set consisting of more than a single point is nonconvex.

2.1.8 intersection, sum, difference, product

2.1.8.0.1 Theorem. *Intersection.* [354, §2, thm.6.5]
Intersection of an arbitrary collection of convex sets $\{\mathcal{C}_i\}$ is convex. For a finite collection of N sets, a necessarily nonempty intersection of relative interior $\bigcap_{i=1}^N \operatorname{rel\,intr}\mathcal{C}_i = \operatorname{rel\,intr}\bigcap_{i=1}^N \mathcal{C}_i$ equals relative interior of intersection. And for a possibly infinite collection, $\bigcap\overline{\mathcal{C}_i} = \overline{\bigcap\mathcal{C}_i}$. ⋄

[2.10] This rule can help determine whether there exists unique solution to a convex optimization problem whose *feasible set* is an intersecting line; *e.g*, the *trilateration* problem (§5.4.2.2.8).

In converse this theorem is implicitly false insofar as a convex set can be formed by the intersection of sets that are not. Unions of convex sets are generally not convex. [234, p.22]

Vector sum of two convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$ is convex [234, p.24] (`a.k.a` *Minkowski sum*)

$$\mathcal{C}_1 + \mathcal{C}_2 = \{x + y \mid x \in \mathcal{C}_1 \;,\; y \in \mathcal{C}_2\} \tag{28}$$

but not necessarily closed unless at least one set is closed and bounded.

By additive inverse, we can similarly define *vector difference* of two convex sets

$$\mathcal{C}_1 - \mathcal{C}_2 = \{x - y \mid x \in \mathcal{C}_1 \;,\; y \in \mathcal{C}_2\} \tag{29}$$

which is convex. Applying this definition to nonempty convex set $\mathcal{C}_1$, its selfdifference $\mathcal{C}_1 - \mathcal{C}_1$ is generally nonempty, nontrivial, and convex; *e.g*, for any *convex cone* $\mathcal{K}$, (§2.7.2) the set $\mathcal{K} - \mathcal{K}$ constitutes its affine hull. [354, p.15]

Cartesian product of convex sets

$$\mathcal{C}_1 \times \mathcal{C}_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \in \mathcal{C}_1 \;,\; y \in \mathcal{C}_2 \right\} = \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix} \tag{30}$$

remains convex. The converse also holds; *id est*, a Cartesian product is convex iff each set is. [234, p.23]

Convex results are also obtained for scaling $\kappa\,\mathcal{C}$ of a convex set $\mathcal{C}$, rotation/reflection $Q\,\mathcal{C}$, or translation $\mathcal{C} + \alpha$; each similarly defined.

Given any operator T and convex set $\mathcal{C}$, we are prone to write $T(\mathcal{C})$ meaning

$$T(\mathcal{C}) \triangleq \{T(x) \mid x \in \mathcal{C}\} \tag{31}$$

Given linear operator T, it therefore follows from (28),

$$\begin{aligned} T(\mathcal{C}_1 + \mathcal{C}_2) &= \{T(x+y) \mid x \in \mathcal{C}_1 \;,\; y \in \mathcal{C}_2\} \\ &= \{T(x) + T(y) \mid x \in \mathcal{C}_1 \;,\; y \in \mathcal{C}_2\} \\ &= T(\mathcal{C}_1) + T(\mathcal{C}_2) \end{aligned} \tag{32}$$

2.1.9 inverse image

While *epigraph* (§3.5) of a convex function must be convex, it generally holds that inverse image (Figure **17**) of a convex function is not. The most prominent examples to the contrary are affine functions (§3.4):

2.1.9.0.1 Theorem. *Inverse image.* [354, §3]
Let f be a mapping from $\mathbb{R}^{p\times k}$ to $\mathbb{R}^{m\times n}$.

- The image of a convex set $\mathcal{C}$ under any affine function

$$f(\mathcal{C}) = \{f(X) \mid X \in \mathcal{C}\} \subseteq \mathbb{R}^{m\times n} \tag{33}$$

  is convex.

- Inverse image of a convex set $\mathcal{F}$,

$$f^{-1}(\mathcal{F}) = \{X \mid f(X) \in \mathcal{F}\} \subseteq \mathbb{R}^{p\times k} \tag{34}$$

  a single- or many-valued mapping, under any affine function f is convex. ⋄