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CONVEX  
OPTIMIZATION



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# Convex Optimization

†

# Euclidean Distance Geometry



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BY SZE WAN

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*for Jennie Columba*



*Antonio*



*& Sze Wan*

$$\texttt{EDM} = \mathbb{S}_h \cap \left( \mathbb{S}_c^\perp - \mathbb{S}_+ \right)$$

# Prelude

*The constant demands of my department and university and the ever increasing work needed to obtain funding have stolen much of my precious thinking time, and I sometimes yearn for the halcyon days of Bell Labs.*

— Steven Chu, Nobel laureate [92]

Convex Analysis is an emerging calculus of inequalities while Convex Optimization is its application. Analysis is inherently the domain of a mathematician while Optimization belongs to the engineer. A convex optimization problem is conventionally regarded as minimization of a convex objective function subject to an artificial convex domain imposed upon it by the problem constraints. The constraints comprise equalities and inequalities of convex functions whose simultaneous solution set generally constitutes the imposed convex domain: called *feasible set*.

It is easy to minimize a convex function over any convex subset of its domain because any local minimum must be a global minimum. But it is difficult to find the maximum of a convex function over some convex domain because there can be many local maxima; although this has practical application (Eternity II §4.8, §C.5), it is not a convex problem. Tremendous benefit accrues when a mathematical problem can be transformed to an equivalent convex optimization, primarily because any locally optimal solution is then guaranteed globally optimal.<sup>0.1</sup> An *optimal* solution is a best solution to the problem posed; a certificate can be obtained guaranteeing that no better solution exists.

To provide a concrete example of what it meant by *optimal*, recall the ordinary *least squares* problem espoused by Gauss and Legendre over 200 years ago: (§E.0.1.0.1)

$$\underset{x}{\text{minimize}} \quad \|Ax - b\|_2^2$$

Suppose we were to pose this problem a bit differently by *constraining* variable vector  $x$  simultaneously with the minimization. In particular, let's suppose that each entry of  $x$  were bounded above by the same maximum allowable value:

$$\begin{aligned} &\underset{x}{\text{minimize}} \quad \|Ax - b\|_2^2 \\ &\text{subject to} \quad x \preceq x_{\max} \end{aligned}$$

Would a constrained solution, so obtained, be equivalent to an ordinary least squares solution whose entries (exceeding the prescribed bound) are simply clipped to the maximum value? The two solutions are, generally, different when clipping occurs. We argue that a constrained solution is better than a clipped solution; indeed, it is optimal.

<sup>0.1</sup>Solving a nonlinear system for example, by instead solving an equivalent convex optimization problem, is therefore highly preferable and what motivates *geometric programming*; a form of convex optimization invented in 1960s [67] [90] that has driven great advances in the electronic circuit design industry. [38, §4.7] [288] [459] [462] [117] [215] [224] [225] [226] [227] [228] [305] [306] [356]

Both of the foregoing ordinary and bounded least squares problems are convex. Recognizing a problem as convex is an acquired skill; that being, to know when an objective function is convex and when constraints specify a convex feasible set. The challenge, which is indeed an art, is how to express difficult problems in a convex way: perhaps, problems previously believed nonconvex. Practitioners in the art of Convex Optimization engage themselves with discovery of which hard problems can be transformed into convex equivalents; because, once convex form of a problem is found, then a globally optimal solution is close at hand - the hard work is finished: Finding convex expression of a problem is itself, in a very real sense, its solution.

Yet, that skill acquired by understanding the geometry and application of Convex Optimization will remain more an art for some time to come; the reason being, there is generally no unique transformation of a given problem to its convex equivalent. This means, two researchers pondering the same problem are likely to formulate a convex equivalent differently; hence, one solution is likely different from the other although any convex combination of those two solutions remains optimal. Any presumption of only one right or correct solution becomes nebulous. Study of equivalence & sameness, uniqueness, and duality therefore pervade study of Optimization.

It can be difficult for the engineer to apply convex theory without an understanding of Analysis. These pages comprise my journal over an eighteen year period bridging gaps between engineer and mathematician; they constitute a translation, unification, and cohering of about five hundred papers, books, and reports from several different fields of mathematics and engineering. Although beacons of historical accomplishment are cited throughout, much of what is written here will not be found elsewhere. Care to detail, clarity, accuracy, consistency, and typography accompanies removal of ambiguity and verbosity, out of respect for the reader. But the book is nonlinear in its presentation. Consequently there is much indexing, cross referencing, linkage to online sources, and background material provided in the text, footnotes, and appendices so as to be more self-contained and to provide understanding of fundamental concepts.

Looking toward the future, there remains much to be done in the area of machine computation if mathematical Optimization is to become fully embraced by the signal processing community. Wordlength of contemporary computers and numerical burdens upon them prohibit real time solution and accuracy sufficient to embed optimization problems within a recursive mathematical setting. When optimization problems constitute only intermediate solution to much larger problems, acquiring only a “few digits” accuracy can throw off subsequent dependent calculations. *Barrier* methods of solution are the principal obstacle to accuracy while *simplex* methods are the principal setback to speed. Novel, not hybrid, methods of solution are needed.

Audio distortion & noise analysis and measurement §8.1-§8.4 was begun 2016. Sinusoid tracking proved superior to Fourier and other filtering methods in 2017. Discerning harmonic and intermodulation distortion of device under test (DUT), from that produced by D/A→DUT→A/D signal chain, was discovered then. By 2018, the preferred “analyzer” had become discrete D/A and A/D converter because commercial analyzers (test gear) could not accept 32-bit inputs required for antidistortion injection. So it was discovered how D/A and A/D could themselves become DUT, opening up analysis to converter chip designers; submeasurable capability never prior had. The term *submeasurable* was introduced in 2019 to define levels below what was then measurable by very best commercial analyzers. Entrepreneurship prohibited publication in those years.

– Jon Dattorro  
Stanford, California  
2019

# Convex Optimization

## Euclidean Distance Geometry<sup>2ε</sup>

<b>1</b>	<b>Overview</b>	<b>19</b>
<b>2</b>	<b>Convex Geometry</b>	<b>31</b>
2.1	Convex set . . . . .	31
2.2	Vectorized-matrix inner product . . . . .	42
2.3	Hulls . . . . .	50
2.4	Halfspace, Hyperplane . . . . .	58
2.5	Subspace representations . . . . .	69
2.6	Extreme, Exposed . . . . .	74
2.7	Cones . . . . .	77
2.8	Cone boundary . . . . .	85
2.9	Positive semidefinite (PSD) cone . . . . .	90
2.10	Conic independence (c.i.) . . . . .	111
2.11	When extreme means exposed . . . . .	115
2.12	Convex polyhedra . . . . .	116
2.13	Dual cone & generalized inequality . . . . .	122
<b>3</b>	<b>Geometry of Convex Functions</b>	<b>171</b>
3.1	Convex real and vector-valued function . . . . .	171
3.2	Practical norm functions, absolute value . . . . .	175
3.3	Powers, roots, and inverted functions . . . . .	183
3.4	Affine function . . . . .	186
3.5	Epigraph, Sublevel set . . . . .	189
3.6	Gradient . . . . .	195
3.7	First-order convexity condition, real function . . . . .	201
3.8	First-order convexity condition, vector-valued . . . . .	204
3.9	Second-order convexity condition, real function . . . . .	205
3.10	Second-order convexity condition, vector-valued . . . . .	205
3.11	Convex matrix-valued function . . . . .	206
3.12	First-order convexity condition, matrix-valued . . . . .	208
3.13	Epigraph of matrix-valued function, sublevel sets . . . . .	209
3.14	Second-order convexity condition, matrix-valued . . . . .	209
3.15	Quasiconvex . . . . .	211
3.16	Salient properties . . . . .	215

<b>4 Semidefinite Programming</b>	<b>217</b>
4.1 Conic problem . . . . .	218
4.2 Framework . . . . .	224
4.3 Rank reduction . . . . .	234
4.4 Cardinality reduction . . . . .	240
4.5 Rank constraint by Convex Iteration . . . . .	243
4.6 Constraining cardinality . . . . .	263
4.7 Cardinality and rank constraint examples . . . . .	274
4.8 Eternity II . . . . .	296
4.9 Quantum optimization . . . . .	311
4.10 Constraining rank of indefinite matrices . . . . .	318
4.11 Convex Iteration rank-1 . . . . .	322
<b>5 Euclidean Distance Matrix</b>	<b>329</b>
5.1 EDM . . . . .	329
5.2 First metric properties . . . . .	330
5.3 $\exists$ fifth Euclidean metric property . . . . .	331
5.4 EDM definition . . . . .	334
5.5 Invariance . . . . .	358
5.6 Injectivity of $\mathbf{D}$ & unique reconstruction . . . . .	361
5.7 Embedding in affine hull . . . . .	366
5.8 Euclidean metric <i>versus</i> matrix criteria . . . . .	370
5.9 Bridge: Convex polyhedra to EDMs . . . . .	375
5.10 EDM-entry composition . . . . .	380
5.11 EDM indefiniteness . . . . .	383
5.12 List reconstruction . . . . .	387
5.13 Reconstruction examples . . . . .	391
5.14 Fifth property of Euclidean metric . . . . .	395
<b>6 Cone of Distance Matrices</b>	<b>403</b>
6.1 Defining EDM cone . . . . .	404
6.2 Polyhedral bounds . . . . .	406
6.3 $\sqrt{\text{EDM}}$ cone is not convex . . . . .	407
6.4 EDM definition in $\mathbf{1}\mathbf{1}^T$ . . . . .	407
6.5 Correspondence to PSD cone $\mathbb{S}_+^{N-1}$ . . . . .	414
6.6 Vectorization & projection interpretation . . . . .	418
6.7 A geometry of completion . . . . .	424
6.8 Dual EDM cone . . . . .	429
6.9 Theorem of the alternative . . . . .	441
6.10 Postscript . . . . .	441
<b>7 Proximity Problems</b>	<b>443</b>
7.1 First prevalent problem: . . . . .	448
7.2 Second prevalent problem: . . . . .	456
7.3 Third prevalent problem: . . . . .	464
7.4 Conclusion . . . . .	471
<b>8 Audio Analysis</b>	<b>473</b>
8.1 Distortion & noise measurement . . . . .	473
8.2 Harmonic distortion measurement of Audio . . . . .	473
8.3 Intermodulation distortion measurement of Audio . . . . .	473
8.4 Distortion & noise test procedures . . . . .	473

8.5 Arbitrary magnitude analog filter design . . . . .	474
8.6 Signal dropout . . . . .	485
<b>A Linear Algebra</b>	<b>491</b>
A.1 Main-diagonal $\delta$ operator, $\lambda$ , tr, vec, $\circ$ , $\otimes$ . . . . .	491
A.2 Semidefiniteness: domain of test . . . . .	494
A.3 Proper statements of positive semidefiniteness . . . . .	497
A.4 Schur complement . . . . .	505
A.5 Eigenvalue decomposition . . . . .	509
A.6 Singular value decomposition, SVD . . . . .	512
A.7 Zeros . . . . .	516
<b>B Simple Matrices</b>	<b>523</b>
B.1 Rank-1 matrix (dyad) . . . . .	523
B.2 Doublet . . . . .	527
B.3 Elementary matrix . . . . .	528
B.4 Auxiliary $V$ -matrices . . . . .	530
B.5 Orthomatrices . . . . .	533
B.6 Arrow matrix . . . . .	538
<b>C Some Analytical Optimal Results</b>	<b>541</b>
C.1 Properties of infima . . . . .	541
C.2 Trace, singular and eigen values . . . . .	542
C.3 Orthogonal Procrustes problem . . . . .	547
C.4 Two-sided orthogonal Procrustes . . . . .	549
C.5 Quadratics . . . . .	553
<b>D Matrix Calculus</b>	<b>555</b>
D.1 Gradient, Directional derivative, Taylor series . . . . .	555
D.2 Tables of gradients and derivatives . . . . .	570
<b>E Projection</b>	<b>579</b>
E.1 Idempotent matrices . . . . .	583
E.2 $I - P$ , Projection on algebraic complement . . . . .	586
E.3 Symmetric idempotent matrices . . . . .	587
E.4 Algebra of projection on affine subsets . . . . .	591
E.5 Projection examples . . . . .	592
E.6 Vectorization interpretation . . . . .	599
E.7 Projection on matrix subspaces . . . . .	604
E.8 Range, Rowspace interpretation . . . . .	606
E.9 Projection on convex set . . . . .	607
E.10 Projection on intersection of subspaces . . . . .	618
E.11 Alternating projection . . . . .	619
<b>F Notation, Definitions, Glossary</b>	<b>633</b>
<b>Bibliography</b>	<b>649</b>
<b>Index</b>	<b>671</b>

# List of Tables

<b>2 Convex Geometry</b>	
Table 2.9.2.3.1, rank <i>versus</i> dimension of $\mathbb{S}_+^3$ faces	97
Table 2.10.0.0.1, maximum number of c.i. directions	111
Cone Table 1	151
Cone Table S	152
Cone Table A	153
Cone Table 1*	157
<b>4 Semidefinite Programming</b>	
Faces of $\mathbb{S}_+^3$ corresponding to faces of $\mathcal{S}_+^3$	222
Quantum impulse	315
Quantum step	316
Quantum AND function	317
<b>5 Euclidean Distance Matrix</b>	
Précis 5.7.2: affine dimension in terms of rank	369
<b>B Simple Matrices</b>	
Auxiliary V-matrix Table B.4.4	533
<b>D Matrix Calculus</b>	
Table D.2.1, algebraic gradients and derivatives	571
Table D.2.2, trace Kronecker gradients	572
Table D.2.3, trace gradients and derivatives	573
Table D.2.4, logarithmic determinant gradients, derivatives	575
Table D.2.5, determinant gradients and derivatives	576
Table D.2.6, logarithmic derivatives	576
Table D.2.7, exponential gradients and derivatives	577

# List of Figures

<b>1 Overview</b>	<b>19</b>
1 Sigma delta quantizer . . . . .	20
2 Room geometry estimation by first acoustic reflections . . . . .	20
3 <i>Orion nebula</i> . . . . .	21
4 Application of trilateration is localization . . . . .	22
5 Molecular conformation . . . . .	23
6 Facial recognition . . . . .	24
7 <i>Swiss roll</i> . . . . .	25
8 USA map reconstruction . . . . .	26
9 Honeycomb, Hexabenzocoronene molecule . . . . .	27
10 Robotic vehicles . . . . .	28
11 Reconstruction of David . . . . .	29
12 David by distance geometry . . . . .	29
<b>2 Convex Geometry</b>	<b>31</b>
13 Slab . . . . .	33
14 Open, closed, convex sets . . . . .	35
15 Intersection of line with boundary . . . . .	36
16 Tangentials . . . . .	38
17 Inverse image . . . . .	41
18 Inverse image under linear map . . . . .	41
19 <i>Tesseract</i> . . . . .	44
20 Linear injective mapping of Euclidean body . . . . .	45
21 Linear noninjective mapping of Euclidean body . . . . .	46
22 Convex hull of a random list of points . . . . .	50
23 Hulls . . . . .	52
24 Two Fantopes . . . . .	54
25 Nuclear Norm Ball . . . . .	55
26 Convex hull of rank-1 matrices . . . . .	56
27 A simplicial cone . . . . .	59
28 Hyperplane illustrated $\partial\mathcal{H}$ is a partially bounding line . . . . .	60
29 Hyperplanes in $\mathbb{R}^2$ . . . . .	62
30 Affine independence . . . . .	64
31 $\{z \in \mathcal{C} \mid a^T z = \kappa_i\}$ . . . . .	65
32 Hyperplane supporting closed set . . . . .	66
33 Minimizing hyperplane over affine subset in nonnegative orthant . . . . .	72
34 Maximizing hyperplane over convex set . . . . .	73
35 Closed convex set illustrating exposed and extreme points . . . . .	78

36	Two-dimensional nonconvex cone . . . . .	78
37	Nonconvex cone made from lines . . . . .	79
38	Nonconvex cone is convex cone boundary . . . . .	79
39	Union of convex cones is nonconvex cone . . . . .	79
40	Truncated nonconvex cone $\mathcal{X}$ . . . . .	80
41	Cone exterior is convex cone . . . . .	80
42	Not a cone . . . . .	81
43	Minimum element, Minimal element . . . . .	83
44	$\mathcal{K}$ is a pointed polyhedral cone not full-dimensional . . . . .	86
45	Exposed and extreme directions . . . . .	89
46	Positive semidefinite cone . . . . .	92
47	Convex Schur-form set . . . . .	93
48	Projection of truncated PSD cone . . . . .	95
49	Circular cone showing axis of revolution . . . . .	103
50	Circular section . . . . .	104
51	Polyhedral inscription . . . . .	106
52	Conically (in)dependent vectors . . . . .	112
53	Pointed six-faceted polyhedral cone and its dual . . . . .	113
54	Minimal set of generators for halfspace about origin . . . . .	115
55	Venn diagram for cones and polyhedra . . . . .	117
56	Range form polyhedron . . . . .	118
57	Simplex . . . . .	120
58	Two views of a simplicial cone and its dual . . . . .	121
59	Two equivalent constructions of dual cone . . . . .	123
60	Dual polyhedral cone construction by right angle . . . . .	124
61	Orthogonal cones . . . . .	126
62	Blades $\mathcal{K}$ and $\mathcal{K}^*$ . . . . .	127
63	$\mathcal{K}$ is a halfspace about the origin . . . . .	128
64	Iconic primal and dual objective functions . . . . .	129
65	Membership w.r.t $\mathcal{K}$ and orthant . . . . .	137
66	Shrouded polyhedral cone . . . . .	142
67	Simplicial cone $\mathcal{K}$ in $\mathbb{R}^2$ and its dual . . . . .	146
68	Monotone nonnegative cone $\mathcal{K}_{\mathcal{M}+}$ and its dual . . . . .	154
69	Monotone cone $\mathcal{K}_{\mathcal{M}}$ and its dual . . . . .	155
70	Two views of monotone cone $\mathcal{K}_{\mathcal{M}}$ and its dual . . . . .	156
71	First-order optimality condition . . . . .	159
72	Normal-cone progression . . . . .	160
73	Normal cone to ellotope . . . . .	161
<b>3</b>	<b>Geometry of Convex Functions</b> . . . . .	<b>171</b>
74	Convex functions having unique minimizer . . . . .	172
75	Minimum/Minimal element, dual cone characterization . . . . .	174
76	Norm balls . . . . .	175
77	1-norm ball $\mathcal{B}_1$ from compressed sensing/compressive sampling . . . . .	178
78	Cardinality minimization, phase transition, signed <i>versus</i> unsigned variable	179
79	1-norm variants . . . . .	179
80	Affine function . . . . .	187
81	Supremum of affine functions . . . . .	188
82	Epigraph . . . . .	188
83	Log function constraint . . . . .	195
84	Quadratic bowl and 1-norm gradients in $\mathbb{R}^2$ evaluated on grid . . . . .	196

85	Quadratic function convexity in terms of its gradient . . . . .	201
86	Contour plot of convex real function at selected levels . . . . .	202
87	Tangent hyperplane to nonconvex surface . . . . .	203
88	Taxicab distance on nonuniform rectangular grid . . . . .	208
89	Ionic quasiconvex function . . . . .	212
90	Quasiconcave monotonic function $xu$ . . . . .	214
91	Sum of convex functions . . . . .	216
<b>4</b>	<b>Semidefinite Programming</b>	<b>217</b>
92	Venn diagram of convex program types . . . . .	220
93	Visualizing positive semidefinite cone in high dimension . . . . .	221
94	Primal/Dual transformations . . . . .	228
95	Projection <i>versus</i> convex iteration . . . . .	246
96	Trace heuristic . . . . .	246
97	Sensor-network localization . . . . .	249
98	2-lattice of sensors and anchors for localization example . . . . .	251
99	3-lattice of sensors and anchors for localization example . . . . .	252
100	4-lattice of sensors and anchors for localization example . . . . .	253
101	5-lattice of sensors and anchors for localization example . . . . .	254
102	Uncertainty ellipsoids orientation and eccentricity . . . . .	255
103	2-lattice localization solution . . . . .	257
104	3-lattice localization solution . . . . .	258
105	4-lattice localization solution . . . . .	258
106	5-lattice localization solution . . . . .	259
107	10-lattice localization solution . . . . .	259
108	100 randomized noiseless sensors localization . . . . .	260
109	100 randomized sensors localization . . . . .	261
110	Regularization curve for convex iteration . . . . .	263
111	1-norm heuristic . . . . .	265
112	Sparse sampling theorem . . . . .	267
113	Simplex with intersecting line problem in compressed sensing . . . . .	269
114	Geometric interpretations of sparse-sampling constraints . . . . .	272
115	Permutation matrix column-norm and column-sum constraint . . . . .	277
116	MAX CUT problem . . . . .	283
117	Shepp-Logan phantom . . . . .	287
118	MRI radial sampling pattern in Fourier domain . . . . .	291
119	Aliased phantom . . . . .	292
120	Neighboring-pixel stencil on Cartesian grid . . . . .	294
121	Differentiable almost everywhere . . . . .	294
122	<i>Eternity II</i> . . . . .	297
123	<i>Eternity II</i> game-board grid . . . . .	298
124	<i>Eternity II</i> demo-game piece illustrating edge-color ordering . . . . .	299
125	<i>Eternity II</i> vectorized demo-game-board piece descriptions . . . . .	300
126	<i>Eternity II</i> difference $\Delta$ and boundary coefficient $\beta$ construction . . . . .	301
127	<i>Eternity II</i> composite variable matrix sparsity pattern . . . . .	303
128	<i>Eternity II</i> problem visualization in three dimensions . . . . .	308
129	<i>Eternity II</i> permutation polyhedron vertices visualization on sphere . . . . .	310
130	<i>Chimera</i> topology for D:Wave 1152-qubit chip . . . . .	312
131	D:Wave <i>Chimera</i> chip layout . . . . .	313
132	MIT logo . . . . .	319
133	One-pixel camera . . . . .	319

134	One-pixel camera - compression estimates . . . . .	320
135	Convergence of Singular Value Decomposition by Convex Iteration . . . . .	325
136	Straight line through three direction vectors by midpoint fit . . . . .	327
<b>5</b>	<b>Euclidean Distance Matrix</b>	<b>329</b>
137	Convex hull of three points . . . . .	330
138	Complete dimensionless <i>EDM graph</i> . . . . .	332
139	Fifth Euclidean metric property . . . . .	333
140	<i>Fermat point</i> . . . . .	340
141	Arbitrary hexagon in $\mathbb{R}^3$ . . . . .	341
142	Kissing number . . . . .	342
143	<i>Trilateration</i> . . . . .	346
144	This EDM graph provides unique isometric reconstruction . . . . .	349
145	Two sensors $\bullet$ and three anchors $\circ$ . . . . .	349
146	Two discrete linear trajectories of sensors . . . . .	350
147	Coverage in cellular telephone network . . . . .	353
148	Contours of equal signal power . . . . .	353
149	Depiction of molecular conformation . . . . .	354
150	Square diamond . . . . .	360
151	Orthogonal complements in $\mathbb{S}^N$ abstractly oriented . . . . .	362
152	Elliptope $\mathcal{E}^3$ . . . . .	376
153	Elliptope $\mathcal{E}^2$ interior to $\mathbb{S}_+^2$ . . . . .	377
154	Smallest eigenvalue of $-V_{\mathcal{N}}^T D V_{\mathcal{N}}$ . . . . .	381
155	Some entrywise EDM compositions . . . . .	381
156	Map of United States of America . . . . .	390
157	Largest ten eigenvalues of $-V_{\mathcal{N}}^T O V_{\mathcal{N}}$ . . . . .	392
158	<i>Relative-angle inequality tetrahedron</i> . . . . .	397
159	Nonsimplicial pyramid in $\mathbb{R}^3$ . . . . .	400
<b>6</b>	<b>Cone of Distance Matrices</b>	<b>403</b>
160	Relative boundary of cone of Euclidean distance matrices . . . . .	405
161	Example of $V_{\mathcal{X}}$ selection to make an EDM . . . . .	409
162	Vector $V_{\mathcal{X}}$ spirals . . . . .	411
163	Three views of translated negated elliptope . . . . .	417
164	Halfline $\mathcal{T}$ on PSD cone boundary . . . . .	420
165	Vectorization and projection interpretation example . . . . .	421
166	Intersection of EDM cone with hyperplane . . . . .	423
167	Neighborhood graph . . . . .	425
168	<i>Trefoil knot untied</i> . . . . .	426
169	<i>Trefoil ribbon</i> . . . . .	428
170	Orthogonal complement of geometric center subspace . . . . .	432
171	EDM cone construction by flipping PSD cone . . . . .	433
172	Decomposing member of polar EDM cone . . . . .	436
173	Ordinary dual EDM cone projected on $\mathbb{S}_h^3$ . . . . .	440
<b>7</b>	<b>Proximity Problems</b>	<b>443</b>
174	Pseudo-Venn diagram for <i>EDM</i> . . . . .	445
175	Elbow placed in path of projection . . . . .	445
176	Convex envelope . . . . .	459

<b>8 Audio Analysis</b>	<b>473</b>
177 Operational Amplifier implementation of third-order filter having a zero . . . . .	474
178 Mason flowgraph for operational amplifier arbitrary magnitude filter circuit . . . . .	475
179 Bisection method linearity . . . . .	478
180 Arbitrary magnitude analog filter design . . . . .	480
181 Nonnegative spectral factorization . . . . .	482
182 Signal dropout . . . . .	487
183 Signal dropout reconstruction . . . . .	488
<b>A Linear Algebra</b>	<b>491</b>
184 Geometrical interpretation of full SVD . . . . .	515
<b>B Simple Matrices</b>	<b>523</b>
185 Four fundamental subspaces for any dyad . . . . .	524
186 Four fundamental subspaces for doublet . . . . .	527
187 Four fundamental subspaces for elementary matrix . . . . .	528
188 Antireflection . . . . .	535
189 Gimbal . . . . .	537
190 Arrow matrix . . . . .	538
<b>D Matrix Calculus</b>	<b>555</b>
191 Convex quadratic bowl in $\mathbb{R}^2 \times \mathbb{R}$ . . . . .	563
<b>E Projection</b>	<b>579</b>
192 Action of pseudoinverse . . . . .	580
193 Nonorthogonal projection of $x \in \mathbb{R}^3$ on $\mathcal{R}(U) = \mathbb{R}^2$ . . . . .	585
194 Biorthogonal expansion of point $x \in \text{aff } \mathcal{K}$ . . . . .	593
195 Linear regression <i>versus</i> principal component analysis . . . . .	597
196 Dual interpretation of projection on convex set . . . . .	609
197 Projection on orthogonal complement . . . . .	611
198 Projection on dual cone . . . . .	613
199 Projection product on convex set in subspace . . . . .	617
200 von Neumann-style projection of point $b$ . . . . .	620
201 Alternating projection on two halfspaces . . . . .	621
202 Distance, feasibility, optimization . . . . .	622
203 Alternating projection on nonnegative orthant and hyperplane . . . . .	624
204 Geometric convergence of iterates in norm . . . . .	624
205 Distance between PSD cone and iterate in $\mathcal{A}$ . . . . .	628
206 Dykstra's alternating projection algorithm . . . . .	629
207 Polyhedral normal cones . . . . .	630



# Chapter 1

## Overview

### Convex Optimization Euclidean Distance Geometry

*People are so afraid of convex analysis.*

— Claude Lemaréchal, 2003

In layman's terms, the mathematical science of Optimization is a study of how to make good choices when confronted with conflicting requirements and demands. Optimization is a relatively new wisdom, historically, that can represent balance of real things. The qualifier *convex* means: when an optimal solution is found, then it is guaranteed to be a best solution; there is no better choice.

Any convex optimization problem has geometric interpretation. If a given optimization problem can be transformed to a convex equivalent, then this interpretive benefit is acquired. That is a powerful attraction: the ability to visualize geometry of an optimization problem. Conversely, recent advances in geometry and in graph theory hold convex optimization within their proofs' core. [471] [367]

This book is about convex optimization, convex geometry (with particular attention to distance geometry), and nonconvex, combinatorial, and geometrical problems that can be relaxed or transformed into convexity. A virtual flood of new applications follows by epiphany that many problems, presumed nonconvex, can be so transformed: [11] [12] [38, §4.3, p.316-322] [66] [106] [177] [180] [320] [345] [353] [413] [414] [467] [471] *e.g.*, sigma delta analog-to-digital audio converter (A/D) antialiasing (Figure 1).

Euclidean distance geometry is, fundamentally, a determination of point conformation (configuration, relative position or location) by inference from interpoint distance information. By *inference* we mean: *e.g.*, given only distance information, determine whether there corresponds a *realizable* conformation of points; a *list* of points in some dimension that attains the given interpoint distances. Each point may represent simply location or, abstractly, any entity expressible as a vector in finite-dimensional Euclidean space; *e.g.*, distance geometry of music [125].

It is a common misconception to presume that some desired point conformation cannot be recovered in absence of complete interpoint distance information. We might, for example, want to realize a constellation given only interstellar distance (or, equivalently, parsecs from our Sun and relative angular measurement; the Sun as vertex to two distant stars); called *stellar cartography*, an application evoked by Figure 3. At first it may seem

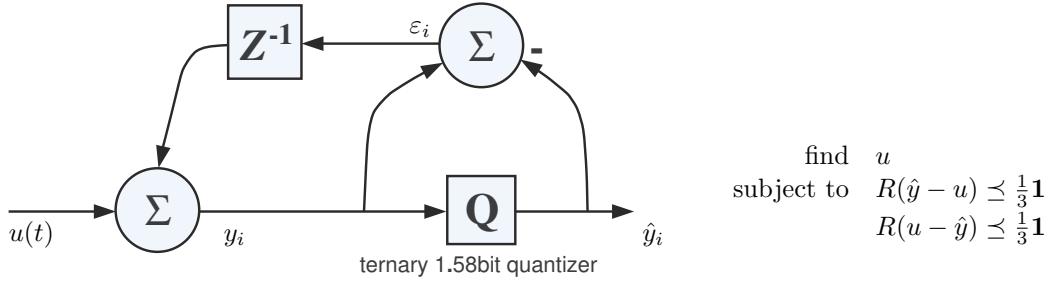


Figure 1: Multibit sigma delta quantization is predominant technology for analog to digital audio signal conversion. [2, p.6] Input signal  $u(t)$  is continuous. Delay  $z^{-1}$  here is analog, perhaps implemented by sample/hold circuit at MHz rate of  $\hat{y}_i$  samples. Observing vector  $\hat{y}$ , signal  $u$  can be reconstructed by finding a point feasible to the set of linear inequalities representing this coarse quantizer recursion.  $R$  is a lower triangular matrix of ones. [114]

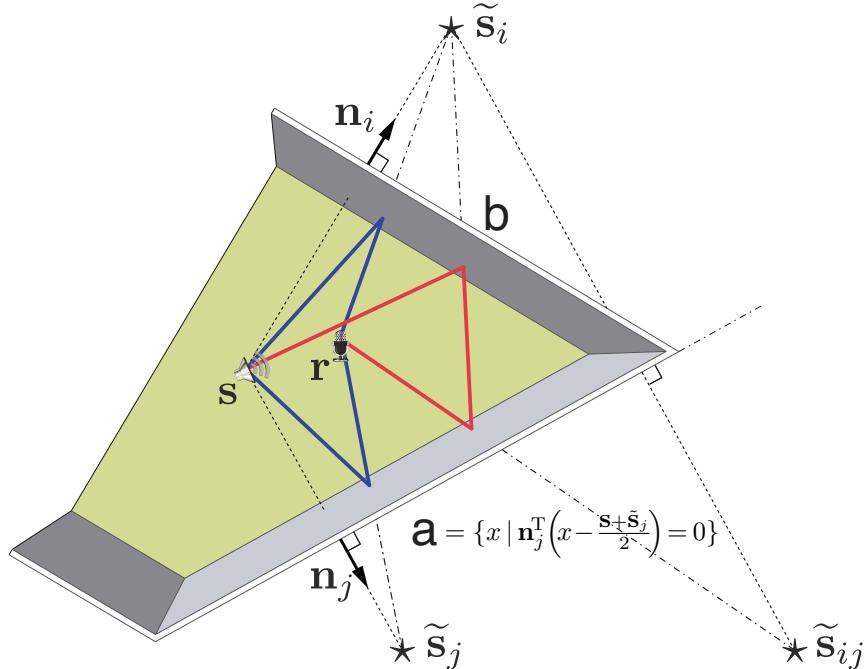


Figure 2: [137] [333] [134] Dokmanić & Parhizkar *et alii* discover an audio signal processing application of Euclidean distance matrices to room geometry estimation by discerning first acoustic reflections of stationary sound source  $s$ . Locations of source and phantom  $\star$  sources  $\tilde{s}_i$  and  $\tilde{s}_j$  are ascertained by measuring arrival times of first echoes (blue) at multiple microphone receivers. (Only one receiver  $r$  is illustrated. Second reflection (red) phantom  $\tilde{s}_{ij}$  ignored.) Phantom location is invariant to receiver position. All interpoint distances among receivers are known. Once source and phantoms are localized, normals  $\mathbf{n}_j$  and  $\mathbf{n}_i$  respectively identify truncated hyperplanes (walls)  $\mathbf{a}$  and  $\mathbf{b}$  bisecting perpendicular line segment connecting source  $s$  to a phantom.



Figure 3: *Orion nebula*. (Astrophotography by [Massimo Robberto](#).)

that  $O(N^2)$  data is required, yet there are many circumstances where this can be reduced to  $O(N)$ .

If we agree that a set of points may have a shape (three points can form a triangle and its interior, for example, four points a tetrahedron), then we can ascribe *shape* of a set of points to their convex hull. It should be apparent: from distance, these shapes can be determined only to within a *rigid transformation* (rotation, reflection, translation).

Absolute position information is generally lost, given only distance information, but we can determine the smallest possible dimension in which an unknown list of points can exist; that attribute is their *affine dimension* (a triangle in any ambient space has affine dimension 2, for example). In circumstances where stationary reference points are also provided, it becomes possible to determine absolute position or location; *e.g.* Figure 4.

Geometric problems involving distance between points can sometimes be reduced to convex optimization problems. Mathematics of this combined study of geometry and optimization is rich and deep. Its application has already proven invaluable discerning organic *molecular conformation* by measuring interatomic distance along covalent bonds; *e.g.* Figure 5. [100] [403] [164] [52] Many disciplines have already benefitted and simplified consequent to this theory; *e.g.*, distance based *pattern recognition* (Figure 6), *localization* in wireless sensor networks [53] [465] [51] by measurement of intersensor distance along channels of communication, *wireless location* of a radio-signal source such as cell phone by multiple measurements of signal strength, the *global positioning system* (GPS), *multidimensional scaling* (§5.12) which is a numerical representation of qualitative data by finding a low-dimensional scale, and audio signal processing: ultrasound tomography, room geometry estimation (Figure 2), and perhaps dereverberation by localization of phantom sound sources [135] [134] [137]. [136]

Euclidean distance geometry provides some foundation for *artificial intelligence*. Together with convex optimization, distance geometry has found application to:

- *machine learning* by discerning naturally occurring manifolds in:
  - Euclidean bodies (Figure 7, §6.7.0.0.1)
  - Fourier spectra of kindred utterances [248]
  - photographic image sequences [448]

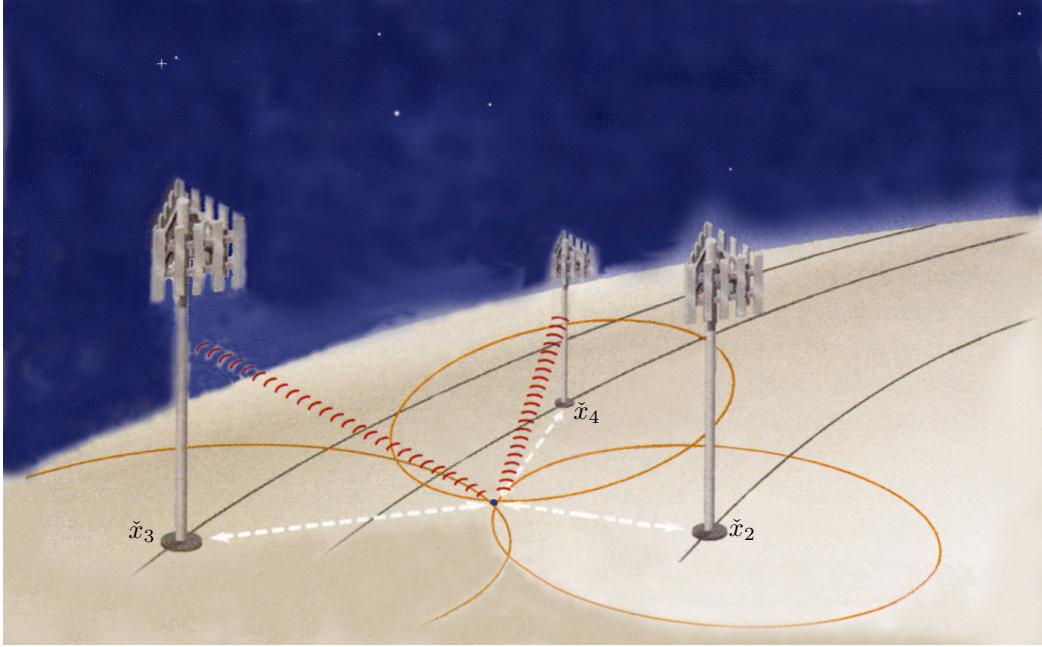


Figure 4: Application of trilateration (§5.4.2.2.8) is localization (determining position) of a radio signal source in 2 dimensions; more commonly known by radio engineers as the process “triangulation”. In this scenario, anchors  $\check{x}_2, \check{x}_3, \check{x}_4$  are illustrated as fixed antennae. [244] The radio signal source (a sensor  $\bullet x_1$ ) anywhere in affine hull of three antenna bases can be uniquely localized by measuring distance to each (dashed white arrowed line segments). Ambiguity of lone distance measurement to sensor is represented by circle about each antenna. Trilateration is expressible as a semidefinite program; hence, a convex optimization problem. [368]

- *robotics*; e.g., automated manufacturing, and autonomous navigation of vehicles maneuvering in formation (Figure 10).

## by chapter

We study the many manifestations and representations of pervasive convex Euclidean bodies. In particular, we make convex polyhedra, cones, and dual cones visceral through illustration in **Chapter 2 Convex Geometry** where geometric relationship of polyhedral cones to nonorthogonal bases (biorthogonal expansion) is examined. It is shown that coordinates are unique in any conic system whose basis cardinality equals or exceeds spatial dimension; for high cardinality, a new definition of *conic coordinate* is provided in Theorem 2.13.13.0.1. Conic analogue to linear independence, called *conic independence*, is introduced as a tool for study, analysis, and manipulation of cones; a natural extension and next logical step in progression: linear, affine, conic. We explain conversion between halfspace- and vertex-description of convex cone, we motivate dual cone and provide formulae for finding it, and we show how first-order optimality conditions or alternative systems of linear inequality or *linear matrix inequality* can be explained by *dual generalized inequalities* with respect to convex cones. Arcane theorems of alternative generalized inequality are, in fact, simply derived from cone *membership relations*; generalizations of algebraic *Farkas' lemma* translated to geometry of convex cones.

Any convex optimization problem can be visualized geometrically. Desire to visualize

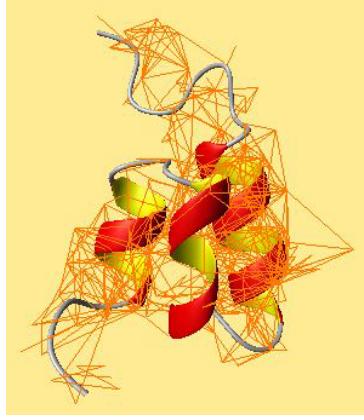


Figure 5: [223] [139] Distance data collected via nuclear magnetic resonance (NMR) helped render this three-dimensional depiction of a [protein molecule](#). At the beginning of the 1980s, Kurt Wüthrich [Nobel laureate] developed an idea about how NMR could be extended to cover biological molecules such as proteins. He invented a systematic method of pairing each NMR signal with the right hydrogen nucleus (proton) in the macromolecule. The method is called sequential assignment and is today a cornerstone of all NMR structural investigations. He also showed how it was subsequently possible to determine pairwise distances between a large number of hydrogen nuclei and use this information with a mathematical method based on distance-geometry to calculate a three-dimensional structure for the molecule. [454] [218] –[324]

in high dimension [[Sagan, \*Cosmos – The Edge of Forever\*, 22:55'](#)] is deeply embedded in the [mathematical psyche](#). [1] Chapter 2 provides tools to make visualization easier, and we teach how to visualize in high dimension. The concepts of face, extreme point, and extreme direction of a convex Euclidean body are explained here; crucial to understanding convex optimization. How to find the smallest face of any closed convex cone, containing convex set  $\mathcal{C}$ , is divulged; later shown to have practical application to presolving convex programs. The convex cone of positive semidefinite matrices, in particular, is studied in depth:

- We interpret, for example, inverse image of the positive semidefinite cone under affine transformation. ([Example 2.9.1.0.2](#))
- Subsets of the positive semidefinite cone, discriminated by rank exceeding some lower bound, are convex. In other words, high-rank subsets of the positive semidefinite cone boundary united with its interior are convex. ([Theorem 2.9.2.9.3](#)) There is a closed form for projection on those convex subsets.
- The positive semidefinite cone is a circular cone in low dimension; *Gershgorin discs* specify inscription of a polyhedral cone into it. ([Figure 51](#))

**Chapter 3 Geometry of Convex Functions** observes Fenchel's analogy between convex sets and functions: We explain, for example, how the real affine function relates to convex functions as the hyperplane relates to convex sets. A toolbox of practical useful convex functions and a cookbook for optimization problems, methods are drawn from the appendices about matrix calculus for determining convexity and discerning geometry.

**Chapter 4. Semidefinite Programming** has recently emerged to prominence because it admits a new problem type previously unsolvable by convex optimization techniques and because it theoretically subsumes other convex types: linear programming, quadratic programming, second-order cone programming. –[p.219](#) Semidefinite programming is

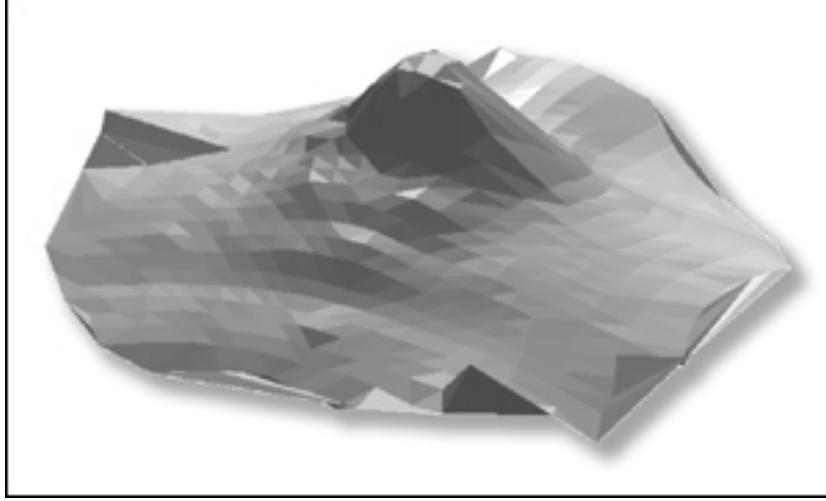


Figure 6: This coarsely discretized triangulated algorithmically flattened human face (made by Kimmel & the Bronsteins [263]) represents a stage in machine recognition of human identity; called *facial recognition*. Distance geometry is applied to determine discriminating-features.

reviewed with particular attention to optimality conditions for prototypical primal and dual problems, their interplay, and a perturbation method for rank reduction of optimal solutions (extant but not well known). *Positive definite Farkas' lemma* is derived, and we also show how to determine if a feasible set belongs exclusively to a positive semidefinite cone boundary. An arguably good three-dimensional polyhedral analogue to the positive semidefinite cone of  $3 \times 3$  symmetric matrices is introduced: a new tool for visualizing coexistence of low- and high-rank optimal solutions in six isomorphic dimensions and a mnemonic aid for understanding semidefinite programs. We find a minimal cardinality Boolean solution to an instance of  $Ax = b$ :

$$\begin{aligned} & \underset{x}{\text{minimize}} && \|x\|_0 \\ & \text{subject to} && Ax = b \\ & && x_i \in \{0, 1\}, \quad i=1 \dots n \end{aligned} \tag{715}$$

The *sensor-network localization* problem is solved in any dimension in this chapter. We introduce a method of *convex iteration* for constraining rank in the form  $\text{rank } G \leq \rho$  and cardinality in the form  $\text{card } x \leq k$ . Cardinality minimization is applied to a discrete image-gradient of the Shepp-Logan phantom, from Magnetic Resonance Imaging (MRI) in the field of medical imaging, for which we find a new lower bound of 1.9% cardinality. We show how to handle polynomial constraints, and how to transform a rank-constrained problem to a rank-1 problem.

The EDM is studied in **Chapter 5 Euclidean Distance Matrix**; its properties and relationship to both positive semidefinite and Gram matrices. We relate the EDM to the four classical properties of Euclidean metric; thereby, observing existence of an infinity of properties of the Euclidean metric beyond triangle inequality. We proceed by deriving the fifth Euclidean metric property and then explain why furthering this endeavor is inefficient because the ensuing criteria (while describing polyhedra in angle or area, volume, content, and so on *ad infinitum*) grow linearly in complexity and number with problem size.

Reconstruction methods are explained and applied to a map of the United States; *e.g.*, Figure 8. We also experimentally test a conjecture of Borg & Groenen by reconstructing

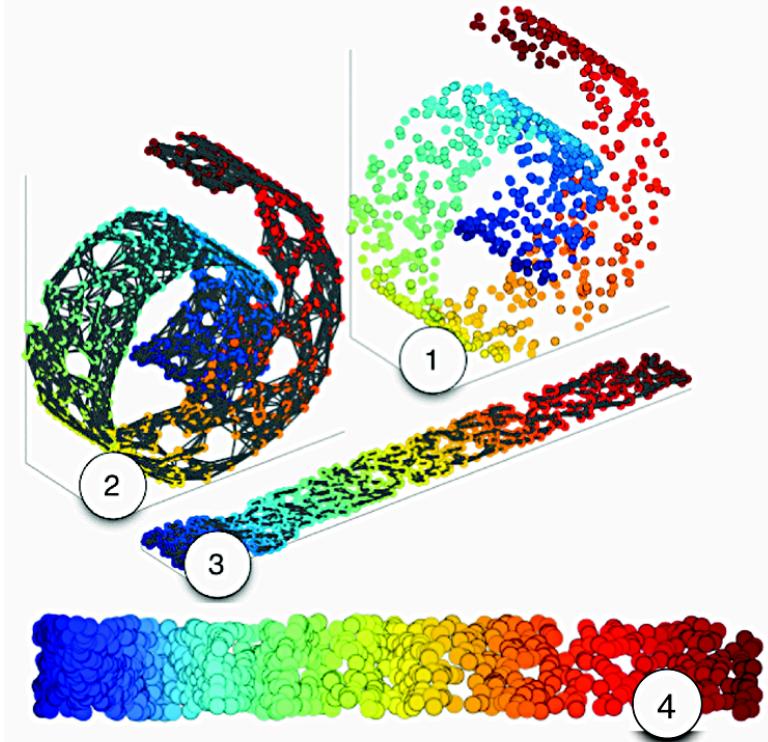


Figure 7: *Swiss roll*, Weinberger & Saul [448]. The problem of manifold learning, illustrated for  $N = 800$  data points sampled from a “Swiss roll” ①. A discretized manifold is revealed by connecting each data point and its  $k=6$  nearest neighbors ②. An unsupervised learning algorithm unfolds the Swiss roll while preserving the local geometry of nearby data points ③. Finally, the data points are projected onto the two-dimensional subspace that maximizes their variance, yielding a faithful embedding of the original manifold ④.

a distorted but recognizable isotonic map of the USA using only ordinal (comparative) distance data: Figure 156e-f. We demonstrate an elegant method for including dihedral (or *torsion*) angle constraints into a molecular conformation problem. We explain why *trilateration* (a.k.a *localization*) is a convex optimization problem. We show how to recover relative position given incomplete interpoint distance information, and how to pose EDM problems or transform geometrical problems to convex optimizations; *e.g.*, *kissing number* of packed spheres about a central sphere (solved in  $\mathbb{R}^3$  by Isaac Newton).

The set of all Euclidean distance matrices forms a pointed closed convex cone called the *EDM cone*:  $\text{EDM}^N$ . We offer a new proof of Schoenberg’s seminal characterization of EDMs:

$$D \in \text{EDM}^N \Leftrightarrow \begin{cases} -V_{\mathcal{N}}^T D V_{\mathcal{N}} \succeq 0 \\ D \in \mathbb{S}_h^N \end{cases} \quad (1025)$$

Our proof relies on fundamental geometry; assuming, any EDM must correspond to a list of points contained in some polyhedron (possibly at its vertices) and *vice versa*. It is known, but not obvious, this *Schoenberg criterion* implies nonnegativity of the EDM entries; proved herein.

We characterize eigenvalue spectrum of an EDM, then devise a polyhedral spectral cone for determining membership of a given matrix (in Cayley-Menger form) to the convex cone of Euclidean distance matrices; *id est*, a matrix is an EDM if and only if its nonincreasingly

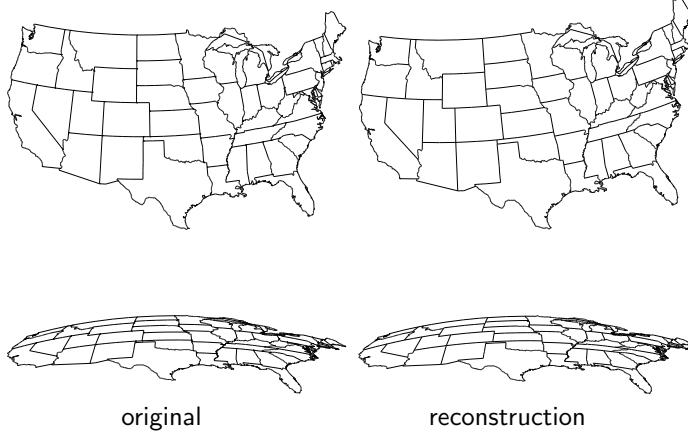


Figure 8: (*confer Figure 156*) About five thousand points along borders constituting United States were used to create an exhaustive matrix of interpoint distance for each and every pair of points in an ordered set (a *list*); called *Euclidean distance matrix*. From that noiseless distance information, it is easy to reconstruct this nonconvex map exactly via Schoenberg criterion (1025). (§5.13.1.0.1) Map reconstruction is exact (to within a rigid transformation) given any number of interpoint distances; the greater the number of distances, the greater the detail (as it is for all conventional map preparation).

ordered vector of eigenvalues belongs to a polyhedral spectral cone for  $\text{EDM}^N$

$$D \in \text{EDM}^N \Leftrightarrow \begin{cases} \lambda\left(\begin{bmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & -D \end{bmatrix}\right) \in \left[\begin{array}{c} \mathbb{R}_+^N \\ \mathbb{R}_- \end{array}\right] \cap \partial\mathcal{H} \\ D \in \mathbb{S}_h^N \end{cases} \quad (1243)$$

We will see: spectral cones are not unique.

In **Chapter 6 Cone of Distance Matrices** we explain a geometric relationship between the cone of Euclidean distance matrices, two positive semidefinite cones, and the ellipope. We illustrate geometric requirements, in particular, for projection of a given matrix on a positive semidefinite cone that establish its membership to the EDM cone. The faces of the EDM cone are described, but still open is the question whether all its faces are exposed as they are for the positive semidefinite cone.

The *Schoenberg criterion*,

$$D \in \text{EDM}^N \Leftrightarrow \begin{cases} -V_N^T D V_N \in \mathbb{S}_+^{N-1} \\ D \in \mathbb{S}_h^N \end{cases} \quad (1025)$$

for identifying a Euclidean distance matrix, is revealed to be a discretized *membership relation* (*dual generalized inequalities*, a new Farkas'-like lemma) between the EDM cone and its ordinary dual:  $\text{EDM}^{N^*}$ . A matrix criterion for membership to the dual EDM cone is derived that is simpler than the Schoenberg criterion:

$$D^* \in \text{EDM}^{N^*} \Leftrightarrow \delta(D^* \mathbf{1}) - D^* \succeq 0 \quad (1393)$$

There is a concise equality, relating the convex cone of Euclidean distance matrices to the positive semidefinite cone, apparently overlooked in the literature; an equality between two large convex Euclidean bodies:

$$\text{EDM}^N = \mathbb{S}_h^N \cap \left( \mathbb{S}_c^{N\perp} - \mathbb{S}_+^N \right) \quad (1387)$$

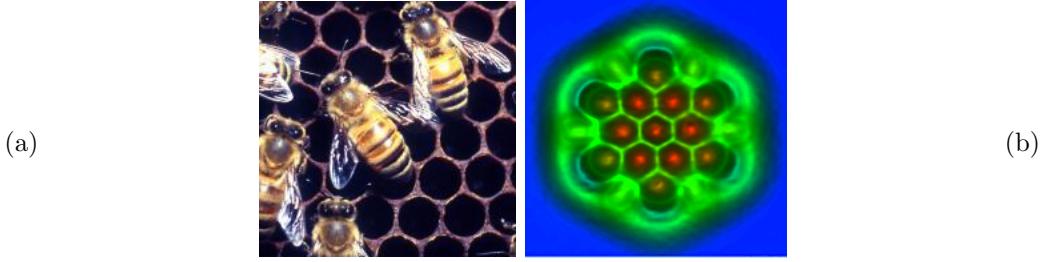


Figure 9: **(a)** These bees construct a honeycomb by solving a convex optimization problem (§5.4.2.2.6). The most dense packing of identical spheres about a central sphere in 2 dimensions is 6. Sphere centers describe a regular lattice. **(b)** A hexabenzocoronene molecule (diameter: 1.4nm) imaged by noncontact atomic force microscopy using a microscope tip terminated with a single carbon monoxide molecule. The carbon-carbon bonds in the imaged molecule appear with different contrast and apparent lengths. Based on these disparities, the bond orders and lengths of the individual bonds can be distinguished. (Image by Leo Gross.)

Seemingly innocuous problems in terms of point position  $x_i \in \mathbb{R}^n$  like

$$\underset{\{x_i\}}{\text{minimize}} \sum_{i, j \in \mathcal{I}} (\|x_i - x_j\| - h_{ij})^2 \quad (1427)$$

$$\underset{\{x_i\}}{\text{minimize}} \sum_{i, j \in \mathcal{I}} (\|x_i - x_j\|^2 - h_{ij})^2 \quad (1428)$$

are difficult to solve. So, in **Chapter 7 Proximity Problems**, we instead explore methods of their solution by transformation to a few fundamental and prevalent Euclidean distance matrix proximity problems; the problem of finding that distance matrix closest, in some sense, to a given matrix  $H = [h_{ij}]$ :

$$\begin{array}{ll} \underset{D}{\text{minimize}} & \| -V(D - H)V \|^2_F \\ \text{subject to} & \text{rank } VDV \leq \rho \\ & D \in \text{EDM}^N \end{array} \quad \begin{array}{ll} \underset{\sqrt[3]{D}}{\text{minimize}} & \| \sqrt[3]{D} - H \|^2_F \\ \text{subject to} & \text{rank } VDV \leq \rho \\ & \sqrt[3]{D} \in \sqrt{\text{EDM}^N} \end{array} \quad (1429)$$

$$\begin{array}{ll} \underset{D}{\text{minimize}} & \| D - H \|^2_F \\ \text{subject to} & \text{rank } VDV \leq \rho \\ & D \in \text{EDM}^N \end{array} \quad \begin{array}{ll} \underset{\sqrt[3]{D}}{\text{minimize}} & \| -V(\sqrt[3]{D} - H)V \|^2_F \\ \text{subject to} & \text{rank } VDV \leq \rho \\ & \sqrt[3]{D} \in \sqrt{\text{EDM}^N} \end{array}$$

We apply a convex iteration method for constraining rank. Known heuristics for rank minimization are also explained. We offer new geometrical proof, in §7.1.4.0.1, of a famous discovery by Eckart & Young in 1936 [153]: Euclidean projection on that generally nonconvex subset of the positive semidefinite cone boundary comprising all semidefinite matrices having rank not exceeding a prescribed bound  $\rho$ . We explain how this problem is transformed to a convex optimization for any rank  $\rho$ .

**Chapter 8 Audio Analysis** constitutes the latest edition: §8.1) Discernment of sinusoids at the same frequency, emanating from distinct sources, with application to harmonic and intermodulation distortion measurement. §8.5) Arbitrary magnitude analog filter design by quasiconvex optimization with application to parametric equalizer implementation having zeros of transfer.

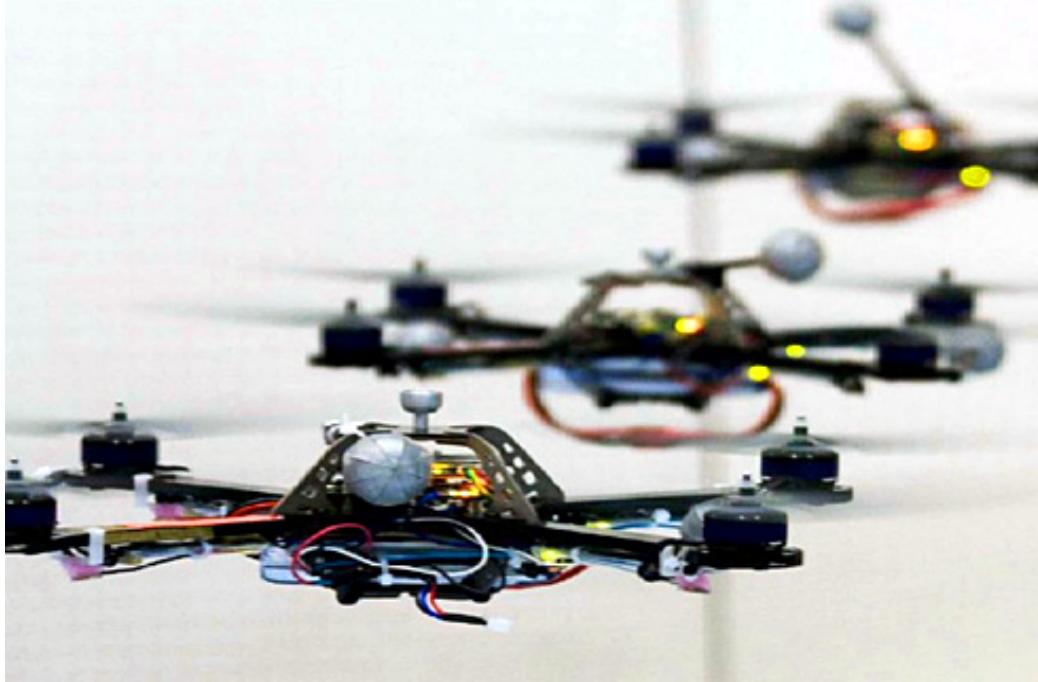


Figure 10: [Nanocopter swarm](#). Robotic vehicles in concert can move larger objects or localize a plume of gas, liquid, or radio waves. [163]

## appendices

We presume a reader already comfortable with elementary vector operations; [15, §3] formally known as *analytic geometry*. [456] Toolboxes are provided, in the form of appendices and code, so as to be more self-contained:

- **linear algebra** (Appendix A is primarily concerned with proper statements of semidefiniteness for square matrices)
- **simple matrices** (dyad, doublet, elementary, Householder, Schoenberg, orthogonal, *etcetera*, in Appendix B)
- collection of known **analytical solutions** to some important optimization problems (Appendix C)
- **matrix calculus** remains somewhat unsystematized when compared to ordinary calculus (Appendix D concerns matrix-valued functions, matrix differentiation and directional derivatives, Taylor series, and tables of first- and second-order gradients and matrix derivatives)
- elaborate exposition offering insight into orthogonal and nonorthogonal **projection** on convex sets (the connection between projection and positive semidefiniteness, for example, or between projection and a linear objective function in Appendix E)
- MATLAB **code** on [Wikimization](#) [436] to discriminate EDMs, to determine conic independence, to reduce or constrain rank of an optimal solution to a semidefinite program, to compress digital image and audio signals by compressive sampling (compressed sensing), and to reconstruct a map of the United States by two distinct methods: one given only distance data, the other given only comparative distance.



Figure 11: Three-dimensional reconstruction of David from distance data.

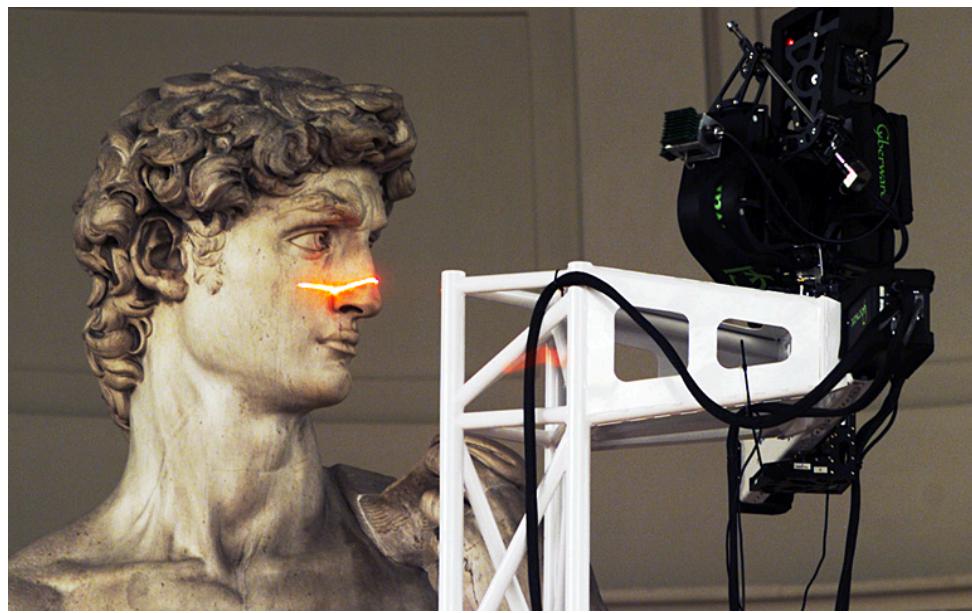


Figure 12: *Digital Michelangelo Project*, Stanford University. Measuring distance to David by laser rangefinder. (Spatial resolution: 0.29mm.) *Crystalix* commercialized a 3D image rendering laser by refining a stunning technique for interior engraving of cubic *photocrystal*.



# Chapter 2

# Convex Geometry

*Convexity has an immensely rich structure and numerous applications. On the other hand, almost every “convex” idea can be explained by a two-dimensional picture.*

— Alexander Barvinok [28, p.vii]

We study convex geometry because it is the easiest of geometries. For that reason, much of a practitioner’s energy is expended seeking invertible transformation of problematic sets to convex ones.

As convex geometry and linear algebra are inextricably bonded by linear inequality (*asymmetry*), we provide much background material on linear algebra (especially in the appendices) although a reader is assumed comfortable with [379] [381] [237] or any other intermediate-level text. The essential references to convex analysis are [234] [354]. The reader is referred to [377] [28] [447] [46] [68] [351] [410] for a comprehensive treatment of convexity. There is relatively less published pertaining to convex matrix-valued functions. [251] [238, §6.6] [340]

## 2.1 Convex set

A set  $\mathcal{C}$  is convex iff for all  $Y, Z \in \mathcal{C}$  and  $0 \leq \mu \leq 1$

$$\mu Y + (1 - \mu)Z \in \mathcal{C} \quad (1)$$

Under that defining condition on  $\mu$ , the linear sum in (1) is called a *convex combination* of  $Y$  and  $Z$ . If  $Y$  and  $Z$  are points in real finite-dimensional Euclidean *vector space* [264] [456]  $\mathbb{R}^n$  or  $\mathbb{R}^{m \times n}$  (matrices), then (1) represents the closed line segment joining them. Line segments are thereby convex sets;  $\mathcal{C}$  is convex iff the line segment connecting any two points in  $\mathcal{C}$  is itself in  $\mathcal{C}$ . Apparent from this definition: a convex set is a connected set. [299, §3.4, §3.5] [46, p.2] A convex set can, but does not necessarily, contain the *origin*  $\mathbf{0}$ .

An *ellipsoid* centered at  $x = a$  (Figure 15 p.36), given matrix  $C \in \mathbb{R}^{m \times n}$  and scalar  $\gamma$

$$\mathcal{B}_{\mathcal{E}} = \{x \in \mathbb{R}^n \mid \|C(x - a)\|^2 = (x - a)^T C^T C (x - a) \leq \gamma^2\} \quad (2)$$

(an *ellipsoidal ball*) is a good icon for a convex set. [2.1](#)

---

[2.1](#) Ellipsoid semiaxes are eigenvectors of  $C^T C$  whose lengths are reciprocal square root eigenvalues. This particular definition is slablike (Figure 13) in  $\mathbb{R}^n$  when  $C$  has nontrivial nullspace.

### 2.1.1 subspace

A nonempty subset  $\mathcal{R}$  of real Euclidean vector space  $\mathbb{R}^n$  is called a *subspace* (§2.5) if every vector<sup>2.2</sup> of the form  $\alpha x + \beta y$ , for  $\alpha, \beta \in \mathbb{R}$ , is in  $\mathcal{R}$  whenever vectors  $x$  and  $y$  are. [290, §2.3] A subspace is a convex set containing the origin, by definition. [354, p.4] Any subspace is therefore open in the sense that it contains no boundary, but closed in the sense [299, §2]

$$\mathcal{R} + \mathcal{R} = \mathcal{R} \quad (3)$$

It is not difficult to show

$$\mathcal{R} = -\mathcal{R} \quad (4)$$

as is true for any subspace  $\mathcal{R}$ , because  $x \in \mathcal{R} \Leftrightarrow -x \in \mathcal{R}$ . Given any  $x \in \mathcal{R}$

$$\mathcal{R} = x + \mathcal{R} \quad (5)$$

Intersection of an arbitrary collection of subspaces remains a subspace. Any subspace, not constituting the entire *ambient vector space*  $\mathbb{R}^n$ , is a *proper subspace*; e.g.,<sup>2.3</sup> any line (of infinite extent) through the origin in two-dimensional Euclidean space  $\mathbb{R}^2$ . Subspace  $\{\mathbf{0}\}$ , comprising only the origin, is proper though *trivial*. The vector space  $\mathbb{R}^n$  is itself a conventional subspace, inclusively, [264, §2.1] although not proper.

### 2.1.2 linear independence

Arbitrary given vectors in Euclidean space  $\{\Gamma_i \in \mathbb{R}^n, i=1 \dots N\}$  are *linearly independent* (l.i.) if and only if, for all  $\zeta \in \mathbb{R}^N$  ( $\zeta_i \in \mathbb{R}$ )

$$\Gamma_1 \zeta_1 + \cdots + \Gamma_{N-1} \zeta_{N-1} - \Gamma_N \zeta_N = \mathbf{0} \quad (6)$$

has only the *trivial solution*  $\zeta = \mathbf{0}$ ; in other words, iff no vector from the given set can be expressed as a linear combination of those remaining.

Geometrically, two nontrivial vector subspaces are linearly independent iff they intersect only at the origin.

#### 2.1.2.1 preservation of linear independence

(confer §2.4.2.4, §2.10.1) Linear transformation preserves linear dependence. [264, p.86] Conversely, linear independence can be preserved under linear transformation. Given  $Y = [y_1 \ y_2 \ \cdots \ y_N] \in \mathbb{R}^{N \times N}$ , consider the mapping

$$T(\Gamma) : \mathbb{R}^{n \times N} \rightarrow \mathbb{R}^{n \times N} \triangleq \Gamma Y \quad (7)$$

whose domain is the set of all matrices  $\Gamma \in \mathbb{R}^{n \times N}$  holding a linearly independent set columnar. Linear independence of  $\{\Gamma y_i \in \mathbb{R}^n, i=1 \dots N\}$  demands, by definition, there exist no nontrivial solution  $\zeta \in \mathbb{R}^N$  to

$$\Gamma y_1 \zeta_1 + \cdots + \Gamma y_{N-1} \zeta_{N-1} - \Gamma y_N \zeta_N = \mathbf{0} \quad (8)$$

By factoring out  $\Gamma$ , we see that triviality is ensured by linear independence of  $\{y_i \in \mathbb{R}^N\}$ .

---

<sup>2.2</sup>A *vector* is assumed, throughout, to be a column vector.

<sup>2.3</sup>We substitute abbreviation *e.g.* in place of the Latin *exempli gratia*; meaning, *for sake of example*.

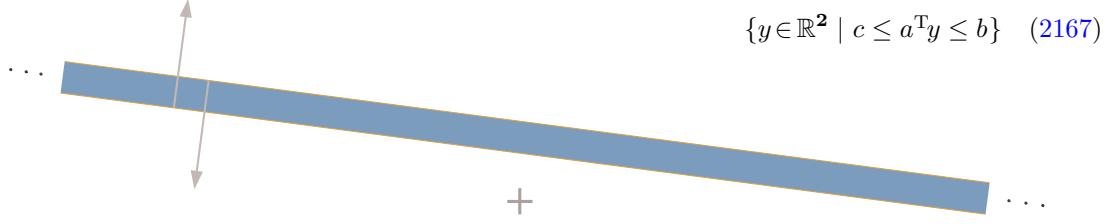


Figure 13: A *slab* is a convex Euclidean body infinite in extent but not affine. Illustrated in  $\mathbb{R}^2$ , it may be constructed by intersecting two opposing halfspaces whose bounding hyperplanes are parallel but not coincident. Because number of halfspaces used in its construction is finite, slab is a *polyhedron* (§2.12). (Cartesian axes + and vector inward-normal, to each halfspace-boundary, are drawn for reference.)

### 2.1.3 Orthant:

name given to a closed convex set that is the higher-dimensional generalization of *quadrant* from the classical Cartesian partition of  $\mathbb{R}^2$ ; a *Cartesian cone*. The most common is the nonnegative orthant  $\mathbb{R}_+^n$  or  $\mathbb{R}_+^{n \times n}$  (analogue to quadrant I) to which membership denotes nonnegative vector- or matrix-entries respectively; *e.g.*,

$$\mathbb{R}_+^n \triangleq \{x \in \mathbb{R}^n \mid x_i \geq 0 \ \forall i\} \quad (9)$$

The nonpositive orthant  $\mathbb{R}_-^n$  or  $\mathbb{R}_-^{n \times n}$  (analogue to quadrant III) denotes negative and 0 entries. Orthant convexity<sup>2.4</sup> is easily verified by definition (1).

### 2.1.4 affine set

A nonempty *affine set* (from the word *affinity*) is any subset of  $\mathbb{R}^n$  that is a translation of some subspace. Any affine set is convex, and open in the sense that it contains no boundary: *e.g.*, empty set  $\emptyset$ , point, line, plane, *hyperplane* (§2.4.2), subspace, *etcetera*. The intersection of an arbitrary collection of affine sets remains affine.

#### 2.1.4.0.1 Definition. Affine subset.

We analogize *affine subset* to subspace,<sup>2.5</sup> defining it to be any nonempty affine set of vectors; an affine subset of  $\mathbb{R}^n$ .  $\triangle$

For some *parallel*<sup>2.6</sup> subspace  $\mathcal{R}$  and any point  $x \in \mathcal{A}$

$$\begin{aligned} \mathcal{A} \text{ is affine} &\Leftrightarrow \mathcal{A} = x + \mathcal{R} \\ &= \{y \mid y - x \in \mathcal{R}\} \end{aligned} \quad (10)$$

*Affine hull* of a set  $\mathcal{C} \subseteq \mathbb{R}^n$  (§2.3.1) is the smallest affine set containing it.

### 2.1.5 dimension

*Dimension* of an arbitrary set  $\mathcal{Z}$  is Euclidean dimension of its affine hull; [447, p.14]

$$\dim \mathcal{Z} \triangleq \dim \text{aff } \mathcal{Z} = \dim \text{aff}(\mathcal{Z} - s), \quad s \in \mathcal{Z} \quad (11)$$

<sup>2.4</sup>All orthants are selfdual simplicial cones. (§2.13.6.1, §2.12.3.1.1)

<sup>2.5</sup>The popular term *affine subspace* is an oxymoron.

<sup>2.6</sup>Two affine sets are *parallel* when one is a translation of the other. [354, p.4]

the same as dimension of the subspace parallel to that affine set  $\text{aff } \mathcal{Z}$  when nonempty. Hence dimension (of a set) is synonymous with *affine dimension*. [234, A.2.1]

### 2.1.6 empty set *versus* empty interior

*Emptiness*  $\emptyset$  of a set is handled differently than *interior* in the classical literature. It is common for a nonempty convex set to have empty interior; *e.g.*, paper in the real world:

- An ordinary flat sheet of paper is a nonempty convex set having empty interior in  $\mathbb{R}^3$  but nonempty interior relative to its affine hull.

#### 2.1.6.1 relative interior

Although it is always possible to pass to a smaller ambient Euclidean space where a nonempty set acquires an interior [28, §II.2.3], we prefer the qualifier *relative* which is the conventional fix to this ambiguous terminology.<sup>2.7</sup> So we distinguish *interior* from *relative interior* throughout: [377] [447] [410]

- Classical interior  $\text{intr } \mathcal{C}$  is defined as a union of points:  $x$  is an interior point of  $\mathcal{C} \subseteq \mathbb{R}^n$  if there exists an open Euclidean ball

$$\mathcal{B} \triangleq \{y \in \mathbb{R}^n \mid \|y - x\| < \gamma\} \quad (12)$$

of dimension  $n$  and nonzero radius  $\gamma$  centered at  $x$  that is contained in  $\mathcal{C}$ .

- Relative interior  $\text{rel intr } \mathcal{C}$  of a convex set  $\mathcal{C} \subseteq \mathbb{R}^n$  is interior relative to its affine hull.<sup>2.8</sup>

Thus defined, it is common (though confusing) for  $\text{intr } \mathcal{C}$  the interior of  $\mathcal{C}$  to be empty while its relative interior is not: this happens whenever dimension of its affine hull is less than dimension of the ambient space ( $\dim \text{aff } \mathcal{C} < n$ ; *e.g.*, were  $\mathcal{C}$  paper) or in the exception when  $\mathcal{C}$  is a single point; [299, §2.2.1]

$$\text{rel intr}\{x\} \triangleq \text{aff}\{x\} = \{x\}, \quad \text{intr}\{x\} = \emptyset, \quad x \in \mathbb{R}^n \quad (13)$$

In any case, *closure* of the relative interior of a convex set  $\mathcal{C}$  always yields closure of the set itself;

$$\overline{\text{rel intr } \mathcal{C}} = \overline{\mathcal{C}} \quad (14)$$

Closure is invariant to translation. If  $\mathcal{C}$  is convex then  $\text{rel intr } \mathcal{C}$  and  $\overline{\mathcal{C}}$  are convex. [234, p.24] If  $\mathcal{C}$  has nonempty interior, then

$$\text{rel intr } \mathcal{C} = \text{intr } \mathcal{C} \quad (15)$$

Given the intersection of convex set  $\mathcal{C}$  with affine set  $\mathcal{A}$

$$\text{rel intr}(\mathcal{C} \cap \mathcal{A}) = \text{rel intr}(\mathcal{C}) \cap \mathcal{A} \iff \text{rel intr}(\mathcal{C}) \cap \mathcal{A} \neq \emptyset \quad (16)$$

Because an affine set  $\mathcal{A}$  is open

$$\text{rel intr } \mathcal{A} = \mathcal{A} \quad (17)$$

---

<sup>2.7</sup>Superfluous mingling of terms as in *relatively nonempty set* would be an unfortunate consequence. From the opposite perspective, some authors use the term *full* or *full-dimensional* to describe a set having nonempty interior.

<sup>2.8</sup>Likewise for *relative boundary* (§2.1.7.2), although *relative closure* is superfluous. [234, §A.2.1]

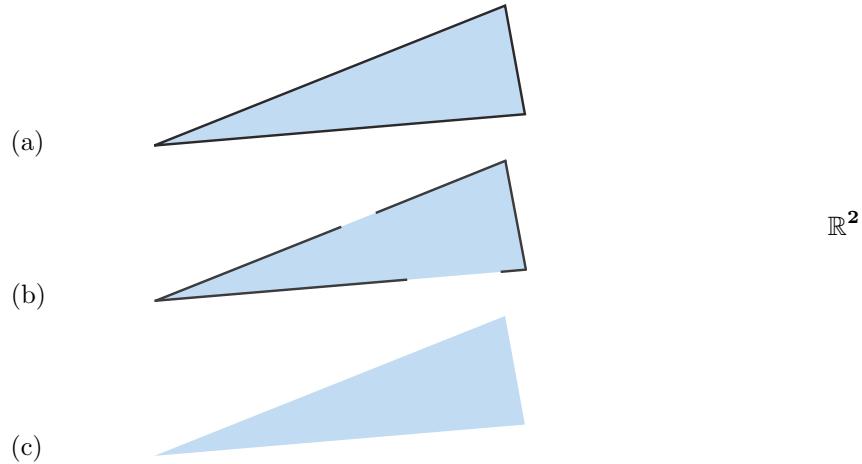


Figure 14: (a) Closed convex set. (b) Neither open, closed, or convex. Yet PSD cone can remain convex in absence of certain boundary components (§2.9.2.9.3). Nonnegative orthant with origin excluded (§2.6) and positive orthant with origin adjoined [354, p.49] are convex. (c) Open convex set.

### 2.1.7 classical boundary

(confer §2.1.7.2) *Boundary* of a set  $\mathcal{C}$  is the closure of  $\mathcal{C}$  less its interior;

$$\partial\mathcal{C} = \overline{\mathcal{C}} \setminus \text{intr } \mathcal{C} \quad (18)$$

[61, §1.1] which follows from the fact

$$\overline{\text{intr } \mathcal{C}} = \overline{\mathcal{C}} \quad \Leftrightarrow \quad \partial \text{intr } \mathcal{C} = \partial \mathcal{C} \quad (19)$$

and presumption of nonempty interior.<sup>2.9</sup> Implications are:

- $\text{intr } \mathcal{C} = \overline{\mathcal{C}} \setminus \partial \mathcal{C}$
- a bounded open set has *boundary* defined but not contained in the set
- interior of an open set is equivalent to the set itself;

from which an open set is defined: [299, p.109]

$$\mathcal{C} \text{ is open} \Leftrightarrow \text{intr } \mathcal{C} = \mathcal{C} \quad (20)$$

$$\mathcal{C} \text{ is closed} \Leftrightarrow \overline{\text{intr } \mathcal{C}} = \mathcal{C} \quad (21)$$

The set illustrated in Figure 14b is not open because it is not equivalent to its interior, for example, it is not closed because it does not contain its boundary, and it is not convex because it does not contain all convex combinations of its boundary points.

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<sup>2.9</sup>Otherwise, for  $x \in \mathbb{R}^n$  as in (13), [299, §2.1-§2.3]

$$\overline{\text{intr}\{x\}} = \overline{\emptyset} = \emptyset$$

the empty set is both open and closed.