1 Introduction and simplification.

This is the story of an infinite product integral that seems intimidating at first but is actually easier than it looks. Let's take a look:

$$\int \frac{1}{x} \prod_{n=1}^{\infty} \left(1 - \tan^2 \frac{x}{2^n}\right) dx \tag{1}$$

Recall some trig identities:

$$1 - \tan^2(a) = 2 - \sec^2(a) = \left(2\cos^2(a) - 1\right)\sec^2(a) = \cos(2a)\sec^2(a)$$

This simplifies our integral to:

$$\to \int \frac{1}{x} \prod_{n=1}^{\infty} \left(\cos\left(\frac{x}{2^{n-1}}\right)\sec^2\left(\frac{x}{2^n}\right)\right) dx \tag{2}$$

Use this property of products: $\prod_n a_n^k = [\,\prod_n a_n\,]^k$ to get:

$$\to \int \frac{1}{x} \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^{n-1}}\right) \left[\prod_{n=1}^{\infty} \sec\left(\frac{x}{2^n}\right)\right]^2 dx \tag{3}$$

2 Finding a way to simplify the infinite product.

We seek to find the convergence of $\cos\left(\frac{x}{2^{n-1}}\right)$ with respect to n:

$$\lim_{k\to\infty} \prod_{n=1}^{k} \cos\left(\frac{x}{2^{n-1}}\right) = \cos(x) \lim_{k\to\infty} \prod_{n=1}^{k} \cos\left(\frac{x}{2^n}\right) \tag{4}$$

Now we need to recall this simple trig identity:

$$\sin(2a) = 2\sin(a)\cos(a) \to \cos(a) = \tfrac{\sin(2a)}{2\sin(a)}$$

Redefine the limit to get:

$$\Rightarrow \cos(x) \lim_{k\to\infty} \left[\frac{\sin(x)}{2\sin\left(\frac{x}{2}\right)} \cdot \frac{\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2^2}\right)} \cdot \frac{\sin\left(\frac{x}{2^2}\right)}{2\sin\left(\frac{x}{2^3}\right)} \cdots \frac{\sin\left(\frac{x}{2^{k-1}}\right)}{2\sin\left(\frac{x}{2^k}\right)}\right] \tag{5}$$

Now this is interesting, because we are looking at a telescoping series of product:

$$\Rightarrow \cos(x) \lim_{k\to\infty} \left[\frac{\sin(x)}{2\cancel{\sin\left(\frac{x}{2}\right)}} \cdot \frac{\cancel{\sin\left(\frac{x}{2}\right)}}{2\cancel{\sin\left(\frac{x}{2^2}\right)}} \cdot \frac{\cancel{\sin\left(\frac{x}{2^2}\right)}}{2\cancel{\sin\left(\frac{x}{2^3}\right)}} \cdots \frac{\cancel{\sin\left(\frac{x}{2^{k-1}}\right)}}{2\sin\left(\frac{x}{2^k}\right)}\right] \tag{6}$$

So our product series ends with this:

$$\Rightarrow \cos(x) \lim_{k\to\infty} \frac{\sin(x)}{2^k \sin\frac{x}{2^k}} \tag{7}$$

Using L'Hospital's rule to evaluate the limit:

$$\Rightarrow \cos(x)\sin(x) \lim_{k\to\infty} \frac{\frac{d}{dk}2^{-k}}{\frac{d}{dk}\sin\frac{x}{2^k}} = \cos(x)\sin(x) \lim_{k\to\infty} \frac{\cancel{-2^{-k}\ln(2)}}{\cancel{(-2^{-k}\ln 2)}x\cos\frac{x}{2^k}} = \boxed{\frac{\cos(x)\sin(x)}{x}} \tag{8}$$

3 The final stretch.

Recall our integral from before:

$$\int \frac{1}{x} \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^{n-1}}\right) \left[\prod_{n=1}^{\infty} \sec\left(\frac{x}{2^{n}}\right)\right]^2 dx \tag{9}$$

Use the substitutions from before:

$$\prod_{n=1}^{\infty} \cos\left(\frac{x}{2^{n-1}}\right) = \frac{\sin(x)\cos(x)}{x}$$

And similarly:

$$\left[\prod_{n=1}^{\infty} \sec\left(\frac{x}{2^{n}}\right)\right]^2 = \frac{x^2}{\sin^2(x)}$$

Now it can be rewritten as:

$$\int \frac{1}{\cancel{x}} \left[\frac{\cos(x)\,\cancel{\sin(x)}}{\cancel{x}} \cdot \frac{\cancel{x^2}}{\sin^{\cancel{2}} x}\right] dx = \int \cot(x)\, dx = \ln|\sin x| + \mathrm{C} \tag{10}$$

Plot for $c = 0$: