# Vector Integrals

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## 1 Introduction

In this short paper, I am going to show a simple 2D integration technique using eigenspace of a matrix.

## 2 Problem

Our goal is to evalute the following integral:

$$\int_{\mathbb{R}^2} \exp\left(-\mathbf{u} \cdot A\mathbf{u}\right) d\mathbf{u} \tag{1}$$

Where:

$$\mathbf{u} \in \mathbb{R}^2 \text{ and } A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

First we need to find the eigenvalues of the matrix $A$ to diagonalize it. We start by finding the roots of the characteristic equation:

$$\chi = 0 = |A - \lambda I| = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 \tag{2}$$

Therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$ and the corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \text{ and } \mathbf{v}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Now we can write our similar matrix equation as follows:

$$A = J^{-1} D J \text{ where } J = [\,\hat{\mathbf{v}}_1 \;\; \hat{\mathbf{v}}_2\,] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3}$$

Namely:

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \;,\; J^{-1} = J \;,\; \text{ and } D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that $J$ is an orthonormal involutory matrix, i.e., $J = J^{-1} = J^T$ this orthonormal property will be handy for our solution.

## 3 Solution

Now we have all the tools to evaluate our integral.

To start we need to use the diagonal matrix instead of our original matrix. Refer to Gantmakher's book for a refresher on linear algebra. [1]

$$A = J^{-1}DJ = J^T DJ$$

And recall the inner (dot) product definition: $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b}$ to rewrite the original dot product into the following form:

$$\mathbf{u} \cdot A\mathbf{u} = \mathbf{u}^T A\mathbf{u} = \mathbf{u}^T J^T DJ\mathbf{u} = (J\mathbf{u})^T DJ\mathbf{u} \tag{4}$$

Notice that the product $J\mathbf{u}$ can be written as another vector $\mathbf{x}$ to get $\mathbf{x}^T D\mathbf{x}$. This is fascinating because we now have just arrived at a quadratic form that can be written in a one-dimensional expression of multiple variables. Observe:

$$\mathbf{x}^T D\mathbf{x} = \left\{ \begin{array}{cc} x_1 & x_2 \end{array} \right\} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = 3x_1^2 + x_2^2 \tag{5}$$

Back to our integral:

$$\int_{\mathbb{R}^2} \exp\left(-\mathbf{u} \cdot A\mathbf{u}\right) d\mathbf{u} = \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left(-\left(3x_1^2 + x_2^2\right)\right) dx_1 dx_2 \tag{6}$$

This particular integral can be written as the product of two integrals as follows:

$$\int_{\mathbb{R}} \exp\left(-3x_1^2\right) dx_1 \int_{\mathbb{R}} \exp\left(-x_2^2\right) dx_2 \tag{7}$$

Each of these two are Gaussian Integrals [2] and could be evaulated as follows:

$$\int_{-\infty}^{\infty} \exp\left(-3x_1^2\right) dx_1 \int_{-\infty}^{\infty} \exp\left(-x_2^2\right) dx_2 = \sqrt{\frac{\pi}{3}}\sqrt{\pi} \tag{8}$$

The final solution of our problem given this particular matrix $A$ becomes:

$$\boxed{\int_{\mathbb{R}^2} \exp\left(-\mathbf{u} \cdot A\mathbf{u}\right) d\mathbf{u} = \frac{\pi}{\sqrt{3}}} \tag{9}$$

Notice that our final result can be written as $\dfrac{\pi}{\sqrt{\det D}} = \dfrac{\pi}{\sqrt{\prod_i \lambda_i}}$ and this is NOT a coincidence.

## References

[1] Feliks Ruvimovich Gantmakher. *The theory of matrices*, volume 131. American Mathematical Soc., 2000.

[2] Eric W Weisstein. Gaussian integral. 2004.