1
Real Analysis

1.1 Elementary Calculus

Problem 1.1.1 (Fa87) *Prove that* $(\cos\theta)^p \leqslant \cos(p\theta)$ *for* $0 \leqslant \theta \leqslant \pi/2$ *and* $0 < p < 1$.

Problem 1.1.2 (Fa77) *Let* $f : [0,1] \to \mathbb{R}$ *be continuously differentiable, with* $f(0) = 0$. *Prove that*

$$\sup_{0 \leqslant x \leqslant 1} |f(x)| \leqslant \sqrt{\int_0^1 (f'(x))^2\, dx}.$$

Problem 1.1.3 (Sp81) *Let* $f(x)$ *be a real valued function defined for all* $x \geqslant 1$, *satisfying* $f(1) = 1$ *and*

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that

$$\lim_{x \to \infty} f(x)$$

exists and is less than $1 + \dfrac{\pi}{4}$.

Problem 1.1.4 (Sp95) *Let* $f, g : [0,1] \to [0, \infty)$ *be continuous functions satisfying*

$$\sup_{0 \leqslant x \leqslant 1} f(x) = \sup_{0 \leqslant x \leqslant 1} g(x).$$

Prove that there exists $t \in [0,1]$ *with* $f(t)^2 + 3f(t) = g(t)^2 + 3g(t)$.

Problem 1.1.5 (Fa86) *For f a real valued function on the real line, define the function Δf by $\Delta f(x) = f(x+1) - f(x)$. For $n \geqslant 2$, define $\Delta^n f$ recursively by $\Delta^n f = \Delta(\Delta^{n-1} f)$. Prove that $\Delta^n f = 0$ if and only if f has the form $f(x) = a_0(x) + a_1(x)x + \cdots + a_{n-1}(x)x^{n-1}$ where $a_0, a_1, \ldots, a_{n-1}$ are periodic functions of period* 1.

Problem 1.1.6 (Fa00) *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nonconstant function such that $f(x) \leqslant f(y)$ whenever $x \leqslant y$. Prove that there exist $a \in \mathbb{R}$ and $c > 0$ such that $f(a+x) - f(a-x) \geqslant cx$ for all $x \in [0, 1]$.*

Problem 1.1.7 (Fa81) *Either prove or disprove (by a counterexample) each of the following statements:*

1. *Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be such that*
$$\lim_{t \to a} g(t) = b \ \text{ and } \ \lim_{t \to b} f(t) = c.$$
Then
$$\lim_{t \to a} f(g(t)) = c.$$
2. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and U is an open set in $\mathbb{R}$, then $f(U)$ is an open set in $\mathbb{R}$.*
3. *Let f be of class C^∞ on the interval $(-1, 1)$. Suppose that $|f^{(n)}(x)| \leqslant 1$ for all $n \geqslant 1$ and all x in the interval. Then f is* real analytic; *that is, it has a convergent power series expansion in a neighborhood of each point of the interval.*

Problem 1.1.8 (Fa97) *Prove that for all $x > 0$, $\sin x > x - \dfrac{x^3}{6}$.*

Problem 1.1.9 (Su81) *Let*
$$y(h) = 1 - 2\sin^2(2\pi h), \qquad f(y) = \frac{2}{1 + \sqrt{1-y}}.$$
Justify the statement
$$f(y(h)) = 2 - 4\sqrt{2}\pi\,|h| + O(h^2)$$
where
$$\limsup_{h \to 0} \frac{O(h^2)}{h^2} < \infty.$$

Problem 1.1.10 (Fa82) 1. *Prove that there is no continuous map from the closed interval* $[0, 1]$ *onto the open interval* $(0, 1)$.

2. *Find a continuous surjective map from the open interval* $(0, 1)$ *onto the closed interval* $[0, 1]$.

3. *Prove that no map in Part 2 can be bijective.*

Problem 1.1.11 (Sp99) *Suppose that f is a twice differentiable real function such that $f''(x) > 0$ for all $x \in [a, b]$. Find all numbers $c \in [a, b]$ at which the area between the graph $y = f(x)$, the tangent to the graph at $(c, f(c))$, and the lines $x = a$, $x = b$, attains its minimum value.*

Problem 1.1.12 (Fa94, Sp98) *Find the maximum area of all triangles that can be inscribed in an ellipse with semiaxes a and b, and describe the triangles that have maximum area.*
Note: See also Problem 2.2.2.

Problem 1.1.13 (Fa93) *Let f be a continuous real valued function on $[0, \infty)$. Let A be the set of real numbers a that can be expressed as $a = \lim_{n\to\infty} f(x_n)$ for some sequence (x_n) in $[0, \infty)$ such that $\lim_{n\to\infty} x_n = \infty$. Prove that if A contains the two numbers a and b, then contains the entire interval with endpoints a and b.*

Problem 1.1.14 (Su81) *Show that the equation*

$$x\left(1+\log\left(\frac{1}{\varepsilon\sqrt{x}}\right)\right)=1, \quad x>0, \quad \varepsilon>0\,,$$

has, for each sufficiently small $\varepsilon > 0$, exactly two solutions. Let $x(\varepsilon)$ be the smaller one. Show that

1. *$x(\varepsilon) \to 0$ as $\varepsilon \to 0+$;*

yet for any $s > 0$,

2. *$\varepsilon^{-s}x(\varepsilon) \to \infty$ as $\varepsilon \to 0+$.*

Problem 1.1.15 (Sp82) *Suppose that $f(x)$ is a polynomial with real coefficients and a is a real number with $f(a) \neq 0$. Show that there exists a real polynomial $g(x)$ such that if we define p by $p(x) = f(x)g(x)$, we have $p(a) = 1$, $p'(a) = 0$, and $p''(a) = 0$.*

Problem 1.1.16 (Su84) *Let $p(z)$ be a nonconstant polynomial with real coefficients such that for some real number a, $p(a) \neq 0$ but $p'(a) = p''(a) = 0$. Prove that the equation $p(z) = 0$ has a nonreal root.*

Problem 1.1.17 (Fa84, Fa97) *Let f be a C^2 function on the real line. Assume f is bounded with bounded second derivative. Let*

$$A = \sup_{x\in\mathbb{R}} |f(x)|, \quad B = \sup_{x\in\mathbb{R}} |f''(x)|.$$

Prove that

$$\sup_{x\in\mathbb{R}} |f'(x)| \leqslant 2\sqrt{AB}.$$

Problem 1.1.18 (Fa90) *Find all pairs of integers a and b satisfying* $0 < a < b$ *and* $a^b = b^a$.

Problem 1.1.19 (Sp92) *For which positive numbers a and b, with* $a > 1$, *does the equation* $\log_a x = x^b$ *have a positive solution for x?*

Problem 1.1.20 (Sp84) *Which number is larger,* π^3 *or* 3^π*?*

Problem 1.1.21 (Sp94) *For which numbers a in* $(1, \infty)$ *is it true that* $x^a \leqslant a^x$ *for all x in* $(1, \infty)$*?*

Problem 1.1.22 (Sp96) *Show that a positive constant t can satisfy*

$$e^x > x^t \quad \textit{for all} \quad x > 0$$

if and only if $t < e$.

Problem 1.1.23 (Su77) *Suppose that* $f(x)$ *is defined on* $[-1, 1]$, *and that* $f'''(x)$ *is continuous. Show that the series*

$$\sum_{n=1}^{\infty} \left(n \left(f\left(\frac{1}{n}\right) - f\left(-\frac{1}{n}\right) \right) - 2f'(0) \right)$$

converges.

Problem 1.1.24 (Fa96) *If f is a* C^2 *function on an open interval, prove that*

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) .$$

Problem 1.1.25 (Su85) *1. For* $0 \leqslant \theta \leqslant \frac{\pi}{2}$, *show that*

$$\sin\theta \geqslant \frac{2}{\pi}\theta .$$

2. *By using Part 1, or by any other method, show that if* $\lambda < 1$, *then*

$$\lim_{R \to \infty} R^\lambda \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta = 0.$$

Problem 1.1.26 (Su78) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuous. Suppose that* $\mathbb{R}$ *contains a countably infinite subset S such that*

$$\int_p^q f(x)\, dx = 0$$

if p and q are not in S. Prove that f is identically 0.

Problem 1.1.27 (Fa89) *Let the function f from* $[0, 1]$ *to* $[0, 1]$ *have the following properties:*

- *f is of class C^1;*
- *$f(0) = f(1) = 0$;*
- *f' is nonincreasing (i.e., f is concave).*

Prove that the arclength of the graph of f does not exceed 3.

Problem 1.1.28 (Sp93) *Let f be a real valued C^1 function on $[0, \infty)$ such that the improper integral $\int_1^\infty |f'(x)|dx$ converges. Prove that the infinite series $\sum_{n=1}^\infty f(n)$ converges if and only if the integral $\int_1^\infty f(x)dx$ converges.*

Problem 1.1.29 (Su82) *Let E be the set of all continuous real valued functions $u : [0, 1] \to \mathbb{R}$ satisfying*

$$|u(x) - u(y)| \leqslant |x - y|, \quad 0 \leqslant x, y \leqslant 1, \quad u(0) = 0.$$

Let $\varphi : E \to \mathbb{R}$ be defined by

$$\varphi(u) = \int_0^1 \left(u(x)^2 - u(x)\right) dx .$$

Show that φ achieves its maximum value at some element of E.

Problem 1.1.30 (Fa87) *Let S be the set of all real C^1 functions f on $[0, 1]$ such that $f(0) = 0$ and*

$$\int_0^1 f'(x)^2 \, dx \leqslant 1.$$

Define

$$J(f) = \int_0^1 f(x) \, dx .$$

Show that the function J is bounded on S, and compute its supremum. Is there a function $f_0 \in S$ at which J attains its maximum value? If so, what is f_0?

Problem 1.1.31 (Fa82, Fa96) *Let f be a real valued continuous nonnegative function on $[0, 1]$ such that*

$$f(t)^2 \leqslant 1 + 2\int_0^t f(s) \, ds$$

for $t \in [0, 1]$. Show that $f(t) \leqslant 1 + t$ for $t \in [0, 1]$.

Problem 1.1.32 (Sp96) *Suppose φ is a C^1 function on $\mathbb{R}$ such that*

$$\varphi(x) \to a \quad \text{and} \quad \varphi'(x) \to b \quad \text{as} \quad x \to \infty.$$

Prove or give a counterexample: b must be zero.

Problem 1.1.33 (Su77) *Show that*

$$F(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\cos^2 x}}$$

$0 \leqslant k < 1$, *is an increasing function of* k.

Problem 1.1.34 (Fa79) *Given that*

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,,$$

find $f'(t)$ *explicitly, where*

$$f(t) = \int_{-\infty}^{\infty} e^{-tx^2}\, dx, \qquad t > 0.$$

Problem 1.1.35 (Fa80) *Define*

$$F(x) = \int_{\sin x}^{\cos x} e^{(t^2 + xt)}\, dt.$$

Compute $F'(0)$.

Problem 1.1.36 (Fa95) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a nonzero* C^∞ *function such that* $f(x)f(y) = f\left(\sqrt{x^2 + y^2}\right)$ *for all* x *and* y *and that* $f(x) \to 0$ *as* $|x| \to \infty$.

1. *Prove that* f *is an even function and that* $f(0)$ *is* 1.
2. *Prove that* f *satisfies the differential equation* $f'(x) = f''(0)xf(x)$, *and find the most general function satisfying the given conditions.*

Problem 1.1.37 (Fa01) *Let* S *be the set of continuous real-valued functions on* $[0, 1]$ *such that* $f(x)$ *is rational whenever* x *is rational. Prove that* S *is uncountable.*

1.2 Limits and Continuity

Problem 1.2.1 (Sp02) *Let* f *be a bounded, continuous, real-valued function on* $\mathbb{R}^2$. *Define the function* g *on* $\mathbb{R}$ *by*

$$g(x) = \int_{-\infty}^{\infty} \frac{f(x, t)}{1 + t^2}\, dt\,.$$

Prove that g *is continuous.*

Problem 1.2.2 (Fa90) *Suppose that f maps the compact interval I into itself and that*

$$|f(x) - f(y)| < |x - y|$$

for all $x, y \in I$, $x \neq y$. Can one conclude that there is some constant $M < 1$ such that, for all $x, y \in I$,

$$|f(x) - f(y)| \leqslant M|x - y|\,?$$

Problem 1.2.3 (Sp90) *Let the real valued function f on $[0, 1]$ have the following two properties:*

- *If $[a, b] \subset [0, 1]$, then $f([a, b])$ contains the interval with endpoints $f(a)$ and $f(b)$ (i.e., f has the Intermediate Value Property).*
- *For each $c \in \mathbb{R}$, the set $f^{-1}(c)$ is closed.*

Prove that f is continuous.

Problem 1.2.4 (Fa00) *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $f(0) = 0$. Prove: there exists a positive number B such that $|f(x)| \leqslant 1 + B|x|$, for all x.*

Problem 1.2.5 (Sp83, Sp01) *Suppose that f is a continuous function on $\mathbb{R}$ which is periodic with period 1, i.e., $f(x + 1) = f(x)$. Show:*

1. *The function f is bounded above and below and achieves its maximum and minimum.*
2. *The function f is uniformly continuous on $\mathbb{R}$.*
3. *There exists a real number x_0 such that*

$$f(x_0 + \pi) = f(x_0).$$

Problem 1.2.6 (Sp77) *Let $h : [0, 1) \to \mathbb{R}$ be a function defined on the half-open interval $[0, 1)$. Prove that if h is uniformly continuous, there exists a unique continuous function $g : [0, 1] \to \mathbb{R}$ such that $g(x) = h(x)$ for all $x \in [0, 1)$.*

Problem 1.2.7 (Fa99) *Let f be a continuous real valued function on $[0, \infty)$ such that $\lim\limits_{x\to\infty} f(x)$ exists (finitely). Prove that f is uniformly continuous.*

Problem 1.2.8 (Sp84) *Prove or supply a counterexample: If the function f from $\mathbb{R}$ to $\mathbb{R}$ has both a left limit and a right limit at each point of $\mathbb{R}$, then the set of discontinuities of f is, at most, countable.*

Problem 1.2.9 (Fa78) *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \leqslant f(y)$ for $x \leqslant y$. Prove that the set where f is not continuous is finite or countably infinite.*

Problem 1.2.10 (Su85, Fa96) *A function $f : [0,1] \to \mathbb{R}$ is said to be upper semicontinuous if given $x \in [0,1]$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|y-x| < \delta$, then $f(y) < f(x) + \varepsilon$. Prove that an upper semicontinuous function f on $[0,1]$ is bounded above and attains its maximum value at some point $p \in [0,1]$.*

Problem 1.2.11 (Su83) *Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets must be monotonic.*

Problem 1.2.12 (Fa91) *Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \geqslant |x-y|$ for all x and y. Prove that the range of f is all of $\mathbb{R}$.*
Note: See also Problem 2.1.8.

Problem 1.2.13 (Fa81) *Let f be a continuous function on $[0,1]$. Evaluate the following limits.*

1.
$$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx\,.$$

2.
$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx\,.$$

Problem 1.2.14 (Fa88, Sp97) *Let f be a function from $[0,1]$ into itself whose graph*
$$G_f = \{(x, f(x)) \mid x \in [0,1]\}$$
is a closed subset of the unit square. Prove that f is continuous.
Note: See also Problem 2.1.2.

Problem 1.2.15 (Sp89) *Let f be a continuous real valued function defined on $[0,1] \times [0,1]$. Let the function g on $[0,1]$ be defined by*
$$g(x) = \max\{f(x,y) \mid y \in [0,1]\}\,.$$
Prove that g is continuous.

Problem 1.2.16 (Fa01) *Let the function $f : \mathbb{R} \to \mathbb{R}$ be bounded on bounded sets and have the property that $f^{-1}(K)$ is closed whenever K is compact. Prove f is continuous.*

1.3 Sequences, Series, and Products

Problem 1.3.1 (Su85) *Let $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_k \geqslant 0$. Evaluate*
$$\lim_{n\to\infty} \left(A_1^n + A_2^n + \cdots + A_k^n\right)^{1/n}.$$
Note: See also Problem 5.1.11.

Problem 1.3.2 (Sp96) *Compute*

$$L = \lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n}.$$

Problem 1.3.3 (Sp92) *Let* $x_0 = 1$ *and*

$$x_{n+1} = \frac{3+2x_n}{3+x_n}, \qquad n \geqslant 0.$$

Prove that $x_\infty = \lim_{n\to\infty} x_n$ *exists, and find its value.*

Problem 1.3.4 (Fa97) *Define a sequence of real numbers* (x_n) *by*

$$x_0 = 1, \qquad x_{n+1} = \frac{1}{2+x_n} \quad \text{for} \quad n \geqslant 0.$$

Show that (x_n) *converges, and evaluate its limit.*

Problem 1.3.5 (Fa89, Sp94) *Let* α *be a number in* $(0, 1)$. *Prove that any sequence* (x_n) *of real numbers satisfying the recurrence relation*

$$x_{n+1} = \alpha x_n + (1-\alpha)x_{n-1}$$

has a limit, and find an expression for the limit in terms of α, x_0 *and* x_1.

Problem 1.3.6 (Fa92) *Let* k *be a positive integer. Determine those real numbers* c *for which every sequence* (x_n) *of real numbers satisfying the recurrence relation*

$$\frac{1}{2}(x_{n+1} + x_{n-1}) = cx_n$$

has period k *(i.e.,* $x_{n+k} = x_n$ *for all* n*).*

Problem 1.3.7 (Sp84) *Let* a *be a positive real number. Define a sequence* (x_n) *by*

$$x_0 = 0, \quad x_{n+1} = a + x_n^2, \quad n \geqslant 0.$$

Find a necessary and sufficient condition on a *in order that a finite limit* $\lim_{n\to\infty} x_n$ *should exist.*

Problem 1.3.8 (Sp03) *Let* x_n *be a sequence of real numbers so that*

$$\lim_{n\to\infty} (2x_{n+1} - x_n) = x.$$

Show that $\lim_{n\to\infty} x_n = x$.

Problem 1.3.9 (Sp00) *Let* a *and* x_0 *be positive numbers, and define the sequence* $(x_n)_{n=1}^{\infty}$ *recursively by*

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right).$$

Prove that this sequence converges, and find its limit.

Problem 1.3.10 (Fa95) *Let x_1 be a real number, $0 < x_1 < 1$, and define a sequence by $x_{n+1} = x_n - x_n^{n+1}$. Show that $\liminf_{n\to\infty} x_n > 0$.*

Problem 1.3.11 (Fa80) *Let $f(x) = \frac{1}{4} + x - x^2$. For any real number x, define a sequence (x_n) by $x_0 = x$ and $x_{n+1} = f(x_n)$. If the sequence converges, let x_∞ denote the limit.*

1. *For $x = 0$, show that the sequence is bounded and nondecreasing and find $x_\infty = \lambda$.*
2. *Find all $y \in \mathbb{R}$ such that $y_\infty = \lambda$.*

Problem 1.3.12 (Fa81) *The Fibonacci numbers $f_1, f_2, \ldots$ are defined recursively by $f_1 = 1$, $f_2 = 2$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geqslant 2$. Show that*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$$

exists, and evaluate the limit.
Note: See also Problem 7.5.20.

Problem 1.3.13 (Fa79) *Prove that*

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \log 2.$$

Problem 1.3.14 (Sp02) *For n a positive integer, let H_n denote the n^{th} partial sum of the harmonic series*

$$H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

Let $k > 1$ be an integer. Prove that

$$\log k - \frac{C}{n} < H_{nk} - H_n < \log k \qquad (n = 1, 2, \ldots),$$

where $\log k$ is the natural logarithm of k, and C is a constant.

Problem 1.3.15 (Sp90) *Suppose $x_1, x_2, x_3, \ldots$ is a sequence of nonnegative real numbers satisfying*

$$x_{n+1} \leqslant x_n + \frac{1}{n^2}$$

for all $n \geqslant 1$. Prove that $\lim_{n\to\infty} x_n$ exists.

Problem 1.3.16 (Sp93) *Let (a_n) and (ε_n) be sequences of positive numbers. Assume that $\lim_{n\to\infty} \varepsilon_n = 0$ and that there is a number k in $(0, 1)$ such that $a_{n+1} \leqslant k a_n + \varepsilon_n$ for every n. Prove that $\lim_{n\to\infty} a_n = 0$.*

Problem 1.3.17 (Fa83) *Prove or disprove (by giving a counterexample), the following assertion: Every infinite sequence $x_1, x_2, \ldots$ of real numbers has either a nondecreasing subsequence or a nonincreasing subsequence.*

Problem 1.3.18 (Su83) *Let $b_1, b_2, \ldots$ be positive real numbers with*

$$\lim_{n\to\infty} b_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} (b_n/b_{n+1}) = 1.$$

Assume also that $b_1 < b_2 < b_3 < \cdots$. Show that the set of quotients $(b_m/b_n)_{1 \leqslant n < m}$ is dense in $(1, \infty)$.

Problem 1.3.19 (Sp81) *Which of the following series converges?*

1.

$$\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}.$$

2.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}.$$

Problem 1.3.20 (Fa91) *Let $a_1, a_2, a_3, \ldots$ be positive numbers.*

1. *Prove that $\sum a_n < \infty$ implies $\sum \sqrt{a_n a_{n+1}} < \infty$.*
2. *Prove that the converse of the above statement is false.*

Problem 1.3.21 (Su80, Sp97) *For each $(a, b, c) \in \mathbb{R}^3$, consider the series*

$$\sum_{n=3}^{\infty} \frac{a^n}{n^b (\log n)^c}.$$

Determine the values of (a, b, c) for which the series

1. *converges absolutely;*
2. *converges but not absolutely;*
3. *diverges.*

Problem 1.3.22 (Sp91) *For which real numbers x does the infinite series*

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^x}$$

converge?

Problem 1.3.23 (Fa94) *For which values of the real number a does the series*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\frac{1}{n}\right)^{a}$$

converge?

Problem 1.3.24 (Sp91) *Let A be the set of positive integers that do not contain the digit 9 in their decimal expansions. Prove that*

$$\sum_{a\in A}\frac{1}{a}<\infty;$$

that is, A defines a convergent subseries of the harmonic series.

Problem 1.3.25 (Sp89) *Let $a_1, a_2, \ldots$ be positive numbers such that*

$$\sum_{n=1}^{\infty}a_n<\infty\,.$$

Prove that there are positive numbers $c_1, c_2, \ldots$ such that

$$\lim_{n\to\infty}c_n=\infty \quad \textit{and} \quad \sum_{n=1}^{\infty}c_n a_n<\infty\,.$$

Problem 1.3.26 (Fa90) *Evaluate the limit*

$$\lim_{n\to\infty}\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^n}.$$

1.4 Differential Calculus

Problem 1.4.1 (Su83) *Outline a proof, starting from basic properties of the real numbers, of the following theorem: Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that $f'(x)=0$ for all $x\in(a,b)$. Then $f(b)=f(a)$.*

Problem 1.4.2 (Sp84) *Let $f(x)=x\log\left(1+x^{-1}\right)$, $0<x<\infty$.*

1. *Show that f is strictly monotonically increasing.*
2. *Compute* $\lim f(x)$ *as $x\to 0$ and $x\to\infty$.*

Problem 1.4.3 (Sp85) *Let $f(x)$, $0\leqslant x<\infty$, be continuous, differentiable, with $f(0)=0$, and that $f'(x)$ is an increasing function of x for $x\geqslant 0$. Prove that*

$$g(x)=\begin{cases} f(x)/x, & x>0\\ f'(0), & x=0\end{cases}$$

is an increasing function of x.

Problem 1.4.4 (Su79, Fa97) *1. Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative f' is not continuous.*

2. *Let f be as in Part 1. If $f'(0) < 2 < f'(1)$, prove that $f'(x) = 2$ for some $x \in [0, 1]$.*

Problem 1.4.5 (Sp90) *Let $y : \mathbb{R} \to \mathbb{R}$ be a C^∞ function that satisfies the differential equation*

$$y'' + y' - y = 0$$

for $x \in [0, L]$, where L is a positive real number. Suppose that $y(0) = y(L) = 0$. Prove that $y \equiv 0$ on $[0, L]$.

Problem 1.4.6 (Su85) *Let $u(x)$, $0 \leqslant x \leqslant 1$, be a real valued C^2 function which satisfies the differential equation*

$$u''(x) = e^x u(x).$$

1. *Show that if $0 < x_0 < 1$, then u cannot have a positive local maximum at x_0. Similarly, show that u cannot have a negative local minimum at x_0.*
2. *Now suppose that $u(0) = u(1) = 0$. Prove that $u(x) \equiv 0$, $0 \leqslant x \leqslant 1$.*

Problem 1.4.7 (Sp98) *Let K be a real constant. Suppose that $y(t)$ is a positive differentiable function satisfying $y'(t) \leqslant Ky(t)$ for $t \geqslant 0$. Prove that $y(t) \leqslant e^{Kt}y(0)$ for $t \geqslant 0$.*

Problem 1.4.8 (Sp77, Su82) *Suppose f is a differentiable function from the reals into the reals. Suppose $f'(x) > f(x)$ for all $x \in \mathbb{R}$, and $f(x_0) = 0$. Prove that $f(x) > 0$ for all $x > x_0$.*

Problem 1.4.9 (Sp99) *Suppose that f is a twice differentiable real-valued function on $\mathbb{R}$ such that $f(0) = 0$, $f'(0) > 0$, and $f''(x) \geqslant f(x)$ for all $x \geqslant 0$. Prove that $f(x) > 0$ for all $x > 0$.*

Problem 1.4.10 (Sp87) *Show that the equation $ae^x = 1 + x + x^2/2$, where a is a positive constant, has exactly one real root.*

Problem 1.4.11 (Su78, Fa89) *Suppose $f : [0, 1] \to \mathbb{R}$ is continuous with $f(0) = 0$, and for $0 < x < 1$, f is differentiable and $0 \leqslant f'(x) \leqslant 2f(x)$. Prove that f is identically 0.*

Problem 1.4.12 (Sp84) *Let $f : [0, 1] \to \mathbb{R}$ be continuous function, with $f(0) = f(1) = 0$. Assume that f'' exists on $0 < x < 1$, with $f'' + 2f' + f \geqslant 0$. Show that $f(x) \leqslant 0$ for all $0 \leqslant x \leqslant 1$.*

Problem 1.4.13 (Sp85) *Let v_1 and v_2 be two real valued continuous functions on $\mathbb{R}$ such that $v_1(x) < v_2(x)$ for all $x \in \mathbb{R}$. Let $\varphi_1(t)$ and $\varphi_2(t)$ be, respectively, solutions of the differential equations*

$$\frac{dx}{dt} = v_1(x) \quad \text{and} \quad \frac{dx}{dt} = v_2(x)$$

for $a < t < b$. *If* $\varphi_1(t_0) = \varphi_2(t_0)$ *for some* $t_0 \in (a, b)$, *show that* $\varphi_1(t) \leqslant \varphi_2(t)$ *for all* $t \in (t_0, b)$.

Problem 1.4.14 (Fa83, Fa84) *Prove or supply a counterexample: If* f *and* g *are* C^1 *real valued functions on* $(0, 1)$, *if*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0,$$

if g *and* g' *never vanish, and if*

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = c,$$

then

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = c.$$

Problem 1.4.15 (Fa00) *Let* f *be a real-valued differentiable function on* $(-1, 1)$ *such that* $f(x)/x^2$ *has a finite limit as* $x \to 0$. *Does it follow that* $f''(0)$ *exists? Give a proof or a counterexample.*

Problem 1.4.16 (Sp90, Fa91) *Let* f *be an infinitely differentiable function from* $\mathbb{R}$ *to* $\mathbb{R}$. *Suppose that, for some positive integer* n,

$$f(1) = f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0.$$

Prove that $f^{(n+1)}(x) = 0$ *for some* x *in* $(0, 1)$.

Problem 1.4.17 (Sp00) *Let* f *be a positive function of class* C^2 *on* $(0, \infty)$ *such that* $f' \leqslant 0$ *and* f'' *is bounded. Prove that* $\lim_{t\to\infty} f'(t) = 0$.

Problem 1.4.18 (Sp86) *Let* f *be a positive differentiable function on* $(0, \infty)$. *Prove that*

$$\lim_{\delta\to 0} \left(\frac{f(x + \delta x)}{f(x)} \right)^{1/\delta}$$

exists (finitely) and is nonzero for each x.

Problem 1.4.19 (Sp88) *Suppose that* $f(x)$, $-\infty < x < \infty$, *is a continuous real valued function, that* $f'(x)$ *exists for* $x \neq 0$, *and that* $\lim_{x\to 0} f'(x)$ *exists. Prove that* $f'(0)$ *exists.*

Problem 1.4.20 (Sp88) *For each real value of the parameter* t, *determine the number of real roots, counting multiplicities, of the cubic polynomial* $p_t(x) = (1 + t^2)x^3 - 3t^3x + t^4$.

Problem 1.4.21 (Sp91) *Let the real valued function* f *be defined in an open interval about the point* a *on the real line and be differentiable at* a. *Prove that if*

(x_n) is an increasing sequence and (y_n) is a decreasing sequence in the domain of f, and both sequences converge to a, then

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$

Problem 1.4.22 (Fa86) *Let f be a continuous real valued function on $[0, 1]$ such that, for each $x_0 \in [0, 1)$,*

$$\limsup_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geqslant 0.$$

Prove that f is nondecreasing.

Problem 1.4.23 (Sp84) *Let I be an open interval in $\mathbb{R}$ containing zero. Assume that f' exists on a neighborhood of zero and $f''(0)$ exists. Show that*

$$f(x) = f(0) + f'(0)\sin x + \frac{1}{2} f''(0)\sin^2 x + o(x^2)$$

($o(x^2)$ denotes a quantity such that $\frac{o(x^2)}{x^2} \to 0$ as $x \to 0$).

Problem 1.4.24 (Sp84) *Prove that the Taylor coefficients at the origin of the function*

$$f(z) = \frac{z}{e^z - 1}$$

are rational numbers.

Problem 1.4.25 (Sp79) *Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ having all three of the following properties:*

- *$f(x) = 0$ for $x < 0$ and $x > 2$,*
- *$f'(1) = 1$,*
- *f has derivatives of all orders.*

Problem 1.4.26 (Sp99) *Prove that if n is a positive integer and α, ε are real numbers with $\varepsilon > 0$, then there is a real function f with derivatives of all orders such that*

1. *$|f^{(k)}(x)| \leqslant \varepsilon$ for $k = 0, 1, \ldots, n-1$ and all $x \in \mathbb{R}$,*
2. *$f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n-1$,*
3. *$f^{(n)}(0) = \alpha$.*

Problem 1.4.27 (Su83) *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, periodic of period* 1, *and nonnegative. Show that*

$$\frac{d}{dx}\left(\frac{f(x)}{1+cf(x)}\right) \to 0 \quad (as\ c \to \infty)$$

uniformly in x.

Problem 1.4.28 (Su81) *Let $I \subset \mathbb{R}$ be the open interval from* 0 *to* 1. *Let $f : I \to \mathbb{C}$ be C^1 (i.e., the real and imaginary parts are continuously differentiable). Suppose that $f(t) \to 0$, $f'(t) \to C \neq 0$ as $t \to 0+$. Show that the function $g(t) = |f(t)|$ is C^1 for sufficiently small $t > 0$ and that $\lim_{t\to 0+} g'(t)$ exists, and evaluate the limit.*

Problem 1.4.29 (Fa95) *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function. Assume that $f(x)$ has a local minimum at $x = 0$. Prove there is a disc centered on the y axis which lies above the graph of f and touches the graph at $(0, f(0))$.*

1.5 Integral Calculus

Problem 1.5.1 (Fa98) *Let f be a real function on $[a, b]$. Assume that f is differentiable and that f' is Riemann integrable. Prove that*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Problem 1.5.2 (Sp98) *Using the properties of the Riemann integral, show that if f is a non-negative continuous function on $[0, 1]$, and $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for all $x \in [0, 1]$.*

Problem 1.5.3 (Fa90) *Suppose f is a continuous real valued function. Show that*

$$\int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi)$$

for some $\xi \in [0, 1]$.

Problem 1.5.4 (Sp77) *Suppose that f is a real valued function of one real variable such that*

$$\lim_{x\to c} f(x)$$

exists for all $c \in [a, b]$. Show that f is Riemann integrable on $[a, b]$.

Problem 1.5.5 (Sp78) *Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable over $[b, 1]$ for all b such that $0 < b \leqslant 1$.*

1. *If f is bounded, prove that f is Riemann integrable over $[0, 1]$.*

2. *What if f is not bounded?*

Problem 1.5.6 (Su81) *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with*

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Show that there is a sequence (x_n) such that $x_n \to \infty$, $x_n f(x_n) \to 0$, and $x_n f(-x_n) \to 0$ as $n \to \infty$.

Problem 1.5.7 (Su85) *Let*

$$f(x) = e^{x^2/2} \int_x^{\infty} e^{-t^2/2}\, dt$$

for $x > 0$.

1. *Show that $0 < f(x) < \dfrac{1}{x}$.*
2. *Show that $f(x)$ is strictly decreasing for $x > 0$.*

Problem 1.5.8 (Su84) *Let $\varphi(s)$ be a C^2 function on $[1, 2]$ with φ and φ' vanishing at $s = 1, 2$. Prove that there is a constant $C > 0$ such that for any $\lambda > 1$,*

$$\left| \int_1^2 e^{i\lambda x} \varphi(x)\, dx \right| \leqslant \frac{C}{\lambda^2}.$$

Problem 1.5.9 (Fa85) *Let $0 \leqslant a \leqslant 1$ be given. Determine all nonnegative continuous functions f on $[0, 1]$ which satisfy the following three conditions:*

$$\int_0^1 f(x)\, dx = 1,$$

$$\int_0^1 x f(x)\, dx = a,$$

$$\int_0^1 x^2 f(x)\, dx = a^2.$$

Problem 1.5.10 (Fa85, Sp90) *Let f be a differentiable function on $[0, 1]$ and let*

$$\sup_{0<x<1} |f'(x)| = M < \infty.$$

Let n be a positive integer. Prove that

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)\, dx \right| \leqslant \frac{M}{2n}.$$

Problem 1.5.11 (Fa83) *Let $f : [0, \infty) \to \mathbb{R}$ be a uniformly continuous function with the property that*

$$\lim_{b\to\infty}\int_0^b f(x)\,dx$$

exists (as a finite limit). Show that

$$\lim_{x\to\infty} f(x) = 0.$$

Problem 1.5.12 (Fa86) *Let f be a real valued continuous function on $[0, \infty)$ such that*

$$\lim_{x\to\infty}\left(f(x) + \int_0^x f(t)\,dt\right)$$

exists. Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

Problem 1.5.13 (Sp83) *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone decreasing function, defined on the positive real numbers with*

$$\int_0^\infty f(x)\,dx < \infty.$$

Show that

$$\lim_{x\to\infty} xf(x) = 0.$$

Problem 1.5.14 (Fa90, Sp97) *Let f be a continuous real valued function satisfying $f(x) \geqslant 0$, for all x, and*

$$\int_0^\infty f(x)\,dx < \infty.$$

Prove that

$$\frac{1}{n}\int_0^n xf(x)\,dx \to 0$$

as $n \to \infty$.

Problem 1.5.15 (Sp87) *Evaluate the integral*

$$I = \int_0^{1/2} \frac{\sin x}{x}\,dx$$

to an accuracy of two decimal places; that is, find a number I^* such that $|I - I^*| < 0.005$.

Problem 1.5.16 (Fa87) *Show that the following limit exists and is finite:*

$$\lim_{t\to 0^+}\left(\int_0^1 \frac{dx}{\left(x^4 + t^4\right)^{1/4}} + \log t\right).$$

Problem 1.5.17 (Fa95) *Let f and f' be continuous on $[0, \infty)$ and $f(x) = 0$ for $x \geqslant 10^{10}$. Show that*

$$\int_0^\infty f(x)^2 dx \leqslant 2\sqrt{\int_0^\infty x^2 f(x)^2 dx}\,\sqrt{\int_0^\infty f'(x)^2 dx}\ .$$

Problem 1.5.18 (Fa88) *Let f be a continuous, strictly increasing function from $[0, \infty)$ onto $[0, \infty)$ and let $g = f^{-1}$. Prove that*

$$\int_0^a f(x)\,dx + \int_0^b g(y)\,dy \geqslant ab$$

for all positive numbers a and b, and determine the condition for equality.

Problem 1.5.19 (Sp94) *Let f be a continuous real valued function on $\mathbb{R}$ such that the improper Riemann integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges. Define the function g on $\mathbb{R}$ by*

$$g(y) = \int_{-\infty}^{\infty} f(x)\cos(xy)\,dx\,.$$

Prove that g is continuous.

Problem 1.5.20 (Fa99) *Let f and g be continuous real valued functions on $\mathbb{R}$ such that $\lim_{|x|\to\infty} f(x) = 0$ and $\int_{-\infty}^{\infty} |g(x)|dx < \infty$. Define the function h on $\mathbb{R}$ by*

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy\,.$$

Prove that $\lim\limits_{|x|\to\infty} h(x) = 0$.

Problem 1.5.21 (Sp88) *Prove that the integrals*

$$\int_0^\infty \cos x^2\,dx \quad \text{and} \quad \int_0^\infty \sin x^2\,dx$$

converge.

Problem 1.5.22 (Fa85) *Let $f(x)$, $0 \leqslant x \leqslant 1$, be a real valued continuous function. Show that*

$$\lim_{n\to\infty} (n+1)\int_0^1 x^n f(x)\,dx = f(1).$$

Problem 1.5.23 (Su83, Sp84, Fa89) *Compute*

$$\int_0^\infty \frac{\log x}{x^2 + a^2}\,dx$$

where $a > 0$ is a constant.

Problem 1.5.24 (Sp85) *Show that*

$$I = \int_0^\pi \log(\sin x)\,dx$$

converges as an improper Riemann integral. Evaluate I.

Problem 1.5.25 (Sp02) *Prove that* $\displaystyle\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx$ *converges as an improper Riemann integral, but that* $\displaystyle\int_0^\infty \frac{|\sin x|}{\sqrt{x}}\,dx = \infty$.

1.6 Sequences of Functions

Problem 1.6.1 (Fa84) *Prove or supply a counterexample: If f is a nondecreasing real valued function on $[0, 1]$, then there is a sequence of continuous functions on $[0, 1]$, $\{f_n\}$, such that for each $x \in [0, 1]$,*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Problem 1.6.2 (Fa77, Sp80) *Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, \ldots$ with $|f_n'(x)| \leqslant 1$ for all n and x. Assume*

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

Problem 1.6.3 (Fa87) *Suppose that $\{f_n\}$ is a sequence of nondecreasing functions which map the unit interval into itself. Suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

pointwise and that f is a continuous function. Prove that $f_n(x) \to f(x)$ uniformly as $n \to \infty$, $0 \leqslant x \leqslant 1$. Note that the functions f_n are not necessarily continuous.

Problem 1.6.4 (Fa85) *Let f and f_n, $n = 1, 2, \ldots$, be functions from $\mathbb{R}$ to $\mathbb{R}$. Assume that $f_n(x_n) \to f(x)$ as $n \to \infty$ whenever $x_n \to x$. Show that f is continuous. Note: The functions f_n are not assumed to be continuous.*

Problem 1.6.5 (Sp99) *Suppose that a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly on $\mathbb{R}$ to a function $f : \mathbb{R} \to \mathbb{R}$, and that $c_n = \lim_{x\to\infty} f_n(x)$ exists for each positive integer n. Prove that $\lim_{n\to\infty} c_n$ and $\lim_{x\to\infty} f(x)$ both exist and are equal.*

Problem 1.6.6 (Sp81) *1. Give an example of a sequence of C^1 functions*

$$f_k : [0, \infty) \to \mathbb{R}, \quad k = 0, 1, 2, \dots$$

such that $f_k(0) = 0$ for all k, and $f_k'(x) \to f_0'(x)$ for all x as $k \to \infty$, but $f_k(x)$ does not converge to $f_0(x)$ for all x as $k \to \infty$.

2. *State an extra condition which would imply that $f_k(x) \to f_0(x)$ for all x as $k \to \infty$.*

Problem 1.6.7 (Fa84) *Show that if f is a homeomorphism of $[0, 1]$ onto itself, then there is a sequence $\{p_n\}$, $n = 1, 2, 3, \dots$ of polynomials such that $p_n \to f$ uniformly on $[0, 1]$ and each p_n is a homeomorphism of $[0, 1]$ onto itself.*

Problem 1.6.8 (Sp95) *Let $f_n : [0, 1] \to [0, \infty)$ be a continuous function, for $n = 1, 2, \dots$. Suppose that one has*

$$(*) \qquad f_1(x) \geqslant f_2(x) \geqslant f_3(x) \geqslant \cdots \quad \textit{for all } x \in [0, 1].$$

Let $f(x) = \lim_{n\to\infty} f_n(x)$ and $M = \sup_{0\leqslant x\leqslant 1} f(x)$.

1. *Prove that there exists $t \in [0, 1]$ with $f(t) = M$.*
2. *Show by example that the conclusion of Part 1 need not hold if instead of $(*)$ we merely know that for each $x \in [0, 1]$ there exists n_x such that for all $n \geqslant n_x$ one has $f_n(x) \geqslant f_{n+1}(x)$.*

Problem 1.6.9 (Fa82) *Let $f_1, f_2, \dots$ be continuous functions on $[0, 1]$ satisfying $f_1 \geqslant f_2 \geqslant \cdots$ and such that $\lim_{n\to\infty} f_n(x) = 0$ for each x. Must the sequence $\{f_n\}$ converge to 0 uniformly on $[0, 1]$?*

Problem 1.6.10 (Sp78) *Let $k \geqslant 0$ be an integer and define a sequence of maps*

$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \frac{x^k}{x^2 + n}, \qquad n = 1, 2, \dots.$$

For which values of k does the sequence converge uniformly on $\mathbb{R}$? On every bounded subset of $\mathbb{R}$?

Problem 1.6.11 (Sp81) *Let $f : [0, 1] \to \mathbb{R}$ be continuous and $k \in \mathbb{N}$. Prove that there is a real polynomial $P(x)$ of degree $\leqslant k$ which minimizes (for all such polynomials)*

$$\sup_{0\leqslant x\leqslant 1} |f(x) - P(x)|.$$

Problem 1.6.12 (Fa79, Fa80) *Let $\{P_n\}$ be a sequence of real polynomials of degree $\leqslant D$, a fixed integer. Suppose that $P_n(x) \to 0$ pointwise for $0 \leqslant x \leqslant 1$. Prove that $P_n \to 0$ uniformly on $[0, 1]$.*

Problem 1.6.13 (Su85) *Let f be a real valued continuous function on a compact interval $[a, b]$. Given $\varepsilon > 0$, show that there is a polynomial p such that $p(a) = f(a)$, $p'(a) = 0$, and $|p(x) - f(x)| < \varepsilon$ for $x \in [a, b]$.*

Problem 1.6.14 (Sp95) *For each positive integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \cos nx$. Prove that the sequence of functions $\{f_n\}$ has no uniformly convergent subsequence.*

Problem 1.6.15 (Fa03) *Let $C_{[0,1]}$ denote the space of continuous functions on $[0, 1]$. Define*

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

1. *Show that d is a metric on $C_{[0,1]}$.*
2. *Show that $(C_{[0,1]}, d)$ is not a complete metric space.*

Problem 1.6.16 (Fa86) *The Arzelà–Ascoli Theorem asserts that the sequence $\{f_n\}$ of continuous real valued functions on a metric space Ω is precompact (i.e., has a uniformly convergent subsequence) if*

(i) *Ω is compact,*

(ii) $\sup \|f_n\| < \infty$ *(where* $\|f_n\| = \sup\{|f_n(x)| \mid x \in \Omega\}$*),*

(iii) *the sequence is equicontinuous.*

Give examples of sequences which are not precompact such that: (i) and (ii) hold but (iii) fails; (i) and (iii) hold but (ii) fails; (ii) and (iii) hold but (i) fails. Take Ω to be a subset of the real line. Sketch the graph of a typical member of the sequence in each case.

Problem 1.6.17 (Sp01) *Let the functions $f_n : [0, 1] \to [0, 1]$ $(n = 1, 2, \ldots)$ satisfy $|f_n(x) - f_n(y)| \leqslant |x - y|$ whenever $|x - y| \geqslant 1/n$. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.*

Problem 1.6.18 (Fa92) *Let $\{f_n\}$ be a sequence of real valued C^1 functions on $[0, 1]$ such that, for all n,*

$$|f_n'(x)| \leqslant \frac{1}{\sqrt{x}} \quad (0 < x \leqslant 1),$$

$$\int_0^1 f_n(x)\, dx = 0.$$

Prove that the sequence has a subsequence that converges uniformly on $[0, 1]$.

Problem 1.6.19 (Fa96) *Let M be the set of real valued continuous functions f on $[0, 1]$ such that f' is continuous on $[0, 1]$, with the norm*

$$\|f\| = \sup_{0 \leqslant x \leqslant 1} |f(x)| + \sup_{0 \leqslant x \leqslant 1} |f'(x)|.$$

Which subsets of M are compact?

Problem 1.6.20 (Su80) *Let (a_n) be a sequence of nonzero real numbers. Prove that the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$*

$$f_n(x) = \frac{1}{a_n}\sin(a_n x) + \cos(x + a_n)$$

has a subsequence converging to a continuous function.

Problem 1.6.21 (Sp82) *Let $\{f_n\}$ be a sequence of continuous functions from $[0, 1]$ to $\mathbb{R}$. Suppose that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in [0, 1]$ and also that, for some constant K, we have*

$$\left|\int_0^1 f_n(x)\,dx\right| \leqslant K < \infty$$

for all n. Does

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 0\,?$$

Problem 1.6.22 (Sp82, Sp93) *Let $\{g_n\}$ be a sequence of twice differentiable functions on $[0, 1]$ such that $g_n(0) = g_n'(0) = 0$ for all n. Suppose also that $|g_n''(x)| \leqslant 1$ for all n and all $x \in [0, 1]$. Prove that there is a subsequence of $\{g_n\}$ which converges uniformly on $[0, 1]$.*

Problem 1.6.23 (Fa93) *Let K be a continuous real valued function defined on $[0, 1] \times [0, 1]$. Let F be the family of functions f on $[0, 1]$ of the form*

$$f(x) = \int_0^1 g(y)K(x, y)\,dy$$

with g a real valued continuous function on $[0, 1]$ satisfying $|g| \leqslant 1$ everywhere. Prove that the family F is equicontinuous.

Problem 1.6.24 (Fa78) *Let $\{g_n\}$ be a sequence of Riemann integrable functions from $[0, 1]$ into $\mathbb{R}$ such that $|g_n(x)| \leqslant 1$ for all n, x. Define*

$$G_n(x) = \int_0^x g_n(t)\,dt\,.$$

Prove that a subsequence of $\{G_n\}$ converges uniformly.

Problem 1.6.25 (Su79) *Let $\{f_n\}$ be a sequence of continuous real functions defined $[0, 1]$ such that*

$$\int_0^1 (f_n(y))^2\,dy \leqslant 5$$

for all n. Define $g_n : [0, 1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^1 \sqrt{x + y}\,f_n(y)\,dy.$$

1. *Find a constant $K \geqslant 0$ such that $|g_n(x)| \leqslant K$ for all n.*
2. *Prove that a subsequence of the sequence $\{g_n\}$ converges uniformly.*

Problem 1.6.26 (Su81) *Let $\{f_n\}$ be a sequence of continuous functions defined from $[0, 1] \to \mathbb{R}$ such that*

$$\int_0^1 (f_n(x) - f_m(x))^2\, dx \to 0 \quad as \quad n, m \to \infty.$$

Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Define $g_n : [0, 1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^1 K(x, y) f_n(y)\, dy.$$

Prove that the sequence $\{g_n\}$ converges uniformly.

Problem 1.6.27 (Fa82) *Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be nonnegative continuous functions on $[0, 1]$ such that the limit*

$$\lim_{n\to\infty} \int_0^1 x^k \varphi_n(x)\, dx$$

exists for every $k = 0, 1, \ldots$. Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(x)\varphi_n(x)\, dx$$

exists for every continuous function f on $[0, 1]$.

Problem 1.6.28 (Sp83) *Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ be real numbers. Show that the infinite series*

$$\sum_{n=1}^{\infty} \frac{e^{i\lambda_n x}}{n^2}$$

converges uniformly over $\mathbb{R}$ to a continuous limit function $f : \mathbb{R} \to \mathbb{C}$. Show, further, that the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx$$

exists.

Problem 1.6.29 (Sp85) *Define the function ζ by*

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Prove that $\zeta(x)$ is defined and has continuous derivatives of all orders in the interval $1 < x < \infty$.

$$f_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right).$$

Prove that $f_n(x)$ converges uniformly to a limit on every finite interval $[a, b]$.

Problem 1.6.31 (Sp87) *Let f be a continuous real valued function on $\mathbb{R}$ satisfying*

$$|f(x)| \leqslant \frac{C}{1+x^2},$$

where C is a positive constant. Define the function F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

1. *Prove that F is continuous and periodic with period 1.*

2. *Prove that if G is continuous and periodic with period 1, then*

$$\int_0^1 F(x)G(x)\,dx = \int_{-\infty}^{\infty} f(x)G(x)\,dx .$$

Problem 1.6.32 (Sp79) *Show that for any continuous function $f : [0, 1] \to \mathbb{R}$ and $\varepsilon > 0$, there is a function of the form*

$$g(x) = \sum_{k=0}^{n} C_k x^{4k}$$

for some $n \in \mathbb{Z}$, where $C_0, \ldots, C_n \in \mathbb{Q}$ and $|g(x) - f(x)| < \varepsilon$ for all x in $[0, 1]$.

1.7 Fourier Series

Problem 1.7.1 (Sp80) *Let $f : \mathbb{R} \to \mathbb{R}$ be the unique function such that $f(x) = x$ if $-\pi \leqslant x < \pi$ and $f(x + 2n\pi) = f(x)$ for all $n \in \mathbb{Z}$.*

1. *Prove that the Fourier series of f is*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\sin nx}{n}.$$

2. *Prove that the series does not converge uniformly.*

3. *For each $x \in \mathbb{R}$, find the sum of the series.*

Problem 1.7.2 (Su81) *Let $f : \mathbb{R} \to \mathbb{R}$ be the function of period 2π such that $f(x) = x^3$ for $-\pi \leqslant x < \pi$.*

1. *Prove that the Fourier series for f has the form $\sum_1^\infty b_n \sin nx$ and write an integral formula for b_n (do not evaluate it).*
2. *Prove that the Fourier series converges for all x.*
3. *Prove*
$$\sum_{n=1}^{\infty} b_n^2 = \frac{2\pi^6}{7}.$$

Problem 1.7.3 (Su82) *Let $f : [0, \pi] \to \mathbb{R}$ be continuous and such that*

$$\int_0^\pi f(x) \sin nx \, dx = 0$$

for all integers $n \geqslant 1$. Is f identically 0?

Problem 1.7.4 (Sp86) *Let f be a continuous real valued function on $\mathbb{R}$ such that*

$$f(x) = f(x+1) = f\left(x + \sqrt{2}\right)$$

for all x. Prove that f is constant.

Problem 1.7.5 (Sp88) *Does there exist a continuous real valued function $f(x)$, $0 \leqslant x \leqslant 1$, such that*

$$\int_0^1 x f(x)\, dx = 1 \quad \text{and} \quad \int_0^1 x^n f(x)\, dx = 0$$

for $n = 0, 2, 3, 4, \ldots$? Give an example or a proof that no such f exists.

Problem 1.7.6 (Fa80) *Let g be 2π-periodic, continuous on $[-\pi, \pi]$ and have Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Let f be 2π-periodic and satisfy the differential equation

$$f''(x) + kf(x) = g(x)$$

where $k \neq n^2, n = 1, 2, 3, \ldots$. Find the Fourier series of f and prove that it converges everywhere.

Problem 1.7.7 (Su83) *Let f be a twice differentiable real valued function on $[0, 2\pi]$, with $\int_0^{2\pi} f(x)dx = 0 = f(2\pi) - f(0)$. Show that*

$$\int_0^{2\pi} (f(x))^2\, dx \leqslant \int_0^{2\pi} \left(f'(x)\right)^2 dx.$$

Problem 1.7.8 (Fa81) *Let f and g be continuous functions on $\mathbb{R}$ such that $f(x+1)=f(x)$, $g(x+1)=g(x)$, for all $x \in \mathbb{R}$. Prove that*

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)\,dx = \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx\,.$$

1.8 Convex Functions

Problem 1.8.1 (Sp81) *Let $f:[0,1]\to\mathbb{R}$ be continuous with $f(0)=0$. Show there is a continuous concave function $g:[0,1]\to\mathbb{R}$ such that $g(0)=0$ and $g(x)\geqslant f(x)$ for all $x\in[0,1]$.*
Note: A function $g: I\to\mathbb{R}$ is concave *if*

$$g(tx+(1-t)y)\geqslant tg(x)+(1-t)g(y)$$

for all x and y in I and $0\leqslant t\leqslant 1$.

Problem 1.8.2 (Sp82) *Let $f: I\to\mathbb{R}$ (where I is an interval of $\mathbb{R}$) be such that $f(x)>0$, $x\in I$. Suppose that $e^{cx}f(x)$ is convex in I for every real number c. Show that $\log f(x)$ is convex in I.*
Note: A function $g: I\to\mathbb{R}$ is convex *if*

$$g(tx+(1-t)y)\leqslant tg(x)+(1-t)g(y)$$

for all x and y in I and $0\leqslant t\leqslant 1$.

Problem 1.8.3 (Sp86) *Let f be a real valued continuous function on $\mathbb{R}$ satisfying the* mean value inequality *below:*

$$f(x)\leqslant \frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy, \qquad x\in\mathbb{R}, \qquad h>0.$$

Prove:

1. *The maximum of f on any closed interval is assumed at one of the endpoints.*

2. *f is convex.*

2
Multivariable Calculus

2.1 Limits and Continuity

Problem 2.1.1 (Fa94) *Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following two conditions:*

(i) $f(K)$ is compact whenever K is a compact subset of $\mathbb{R}^n$.

(ii) If $\{K_n\}$ is a decreasing sequence of compact subsets of $\mathbb{R}^n$, then

$$f\left(\bigcap_1^\infty K_n\right) = \bigcap_1^\infty f(K_n).$$

Prove that f is continuous.

Problem 2.1.2 (Sp78) *Prove that a map $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous only if its graph is closed in $\mathbb{R}^n \times \mathbb{R}^n$. Is the converse true?*
Note: See also Problem 1.2.14.

Problem 2.1.3 (Su79) *Let $U \subset \mathbb{R}^n$ be an open set. Suppose that the map $h : U \to \mathbb{R}^n$ is a homeomorphism from U onto $\mathbb{R}^n$, which is uniformly continuous. Prove $U = \mathbb{R}^n$.*

Problem 2.1.4 (Sp89) *Let f be a real valued function on $\mathbb{R}^2$ with the following properties:*

1. *For each y_0 in $\mathbb{R}$, the function $x \mapsto f(x, y_0)$ is continuous.*

2. *For each x_0 in $\mathbb{R}$, the function $y \mapsto f(x_0, y)$ is continuous.*

3. *$f(K)$ is compact whenever K is a compact subset of $\mathbb{R}^2$.*

Prove that f is continuous.

Problem 2.1.5 (Sp91) *Let f be a continuous function from the ball $B_n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ into itself. (Here, $\|\cdot\|$ denotes the Euclidean norm.) Assume $\|f(x)\| < \|x\|$ for all nonzero $x \in B_n$. Let x_0 be a nonzero point of B_n, and define the sequence (x_k) by setting $x_k = f(x_{k-1})$. Prove that $\lim x_k = 0$.*

Problem 2.1.6 (Su78) *Let N be a* norm *on the vector space $\mathbb{R}^n$; that is, $N : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\begin{aligned}
&N(x) \geqslant 0 \ \ \textit{and} \ \ N(x) = 0 \ \ \textit{only if} \ \ x = 0, \\
&N(x+y) \leqslant N(x) + N(y), \\
&N(\lambda x) = |\lambda| N(x)
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

1. *Prove that N is bounded on the unit sphere.*

2. *Prove that N is continuous.*

3. *Prove that there exist constants $A > 0$ and $B > 0$, such that for all $x \in \mathbb{R}^n$, $A\|x\| \leqslant N(x) \leqslant B\|x\|$.*

Problem 2.1.7 (Fa97) *A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is* proper *if it is continuous and $f^{-1}(B)$ is compact for each compact subset B of $\mathbb{R}^n$; f is* closed *if it is continuous and $f(A)$ is closed for each closed subset A of $\mathbb{R}^m$.*

1. *Prove that every proper map $f : \mathbb{R}^m \to \mathbb{R}^n$ is closed.*

2. *Prove that every one-to-one closed map $f : \mathbb{R}^m \to \mathbb{R}^n$ is proper.*

Problem 2.1.8 (Sp83) *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies*

$$\|F(x) - F(y)\| \geqslant \lambda \|x - y\|$$

for all $x, y \in \mathbb{R}^n$ and some $\lambda > 0$. Prove that F is one-to-one, onto, and has a continuous inverse.
Note: See also Problem 1.2.12.

2.2 Differential Calculus

Problem 2.2.1 (Sp93) *Prove that $\dfrac{x^2 + y^2}{4} \leqslant e^{x+y-2}$ for $x \geqslant 0,\ y \geqslant 0$.*

Problem 2.2.2 (Fa98) *Find the minimal value of the areas of hexagons circumscribing the unit circle in* $\mathbb{R}^2$.
Note: See also Problem 1.1.12.

Problem 2.2.3 (Sp03) *Define* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $f(x, 0) = 0$ *and*

$$f(x, y) = \left(1 - \cos \frac{x^2}{y}\right) \sqrt{x^2 + y^2}$$

for $y \neq 0$.

1. *Show that* f *is continuous at* $(0, 0)$.
2. *Calculate all the directional derivatives of* f *at* $(0, 0)$.
3. *Show that* f *is not differentiable at* $(0, 0)$.

Problem 2.2.4 (Fa86) *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by:*

$$f(x, y) = \begin{cases} x^{4/3} \sin(y/x) & if \quad x \neq 0 \\ 0 & if \quad x = 0 . \end{cases}$$

Determine all points at which f *is differentiable.*

Problem 2.2.5 (Sp00) *Let* F, *with components* $F_1, \dots, F_n$, *be a differentiable map of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$ *such that* $F(0) = 0$. *Assume that*

$$\sum_{j,k=1}^{n} \left| \frac{\partial F_j(0)}{\partial x_k} \right|^2 = c < 1 .$$

Prove that there is a ball B *in* $\mathbb{R}^n$ *with center* 0 *such that* $F(B) \subset B$.

Problem 2.2.6 (Fa02) *Let* p *be a polynomial over* $\mathbb{R}$ *of positive degree. Define the function* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $f(x, y) = (p(x + y), p(x - y))$. *Prove that the derivative* $Df(x, y)$ *is invertible for an open dense set of points* (x, y) *in* $\mathbb{R}^2$.

Problem 2.2.7 (Fa02) *Find the most general continuously differentiable function* $g : \mathbb{R} \to (0, \infty)$ *such that the function* $h(x, y) = g(x)g(y)$ *on* $\mathbb{R}^2$ *is constant on each circle with center* $(0, 0)$.

Problem 2.2.8 (Sp80, Fa92) *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be continuously differentiable. Assume the Jacobian matrix* $\left(\partial f_i / \partial x_j\right)$ *has rank* n *everywhere. Suppose* f *is* proper; *that is,* $f^{-1}(K)$ *is compact whenever* K *is compact. Prove* $f(\mathbb{R}^n) = \mathbb{R}^n$.

Problem 2.2.9 (Sp89) *Suppose* f *is a continuously differentiable map of* $\mathbb{R}^2$ *into* $\mathbb{R}^2$. *Assume that* f *has only finitely many singular points, and that for each positive number* M, *the set* $\{z \in \mathbb{R}^2 \mid |f(z)| \leqslant M\}$ *is bounded. Prove that* f *maps* $\mathbb{R}^2$ *onto* $\mathbb{R}^2$.

Problem 2.2.10 (Fa81) *Let f be a real valued function on $\mathbb{R}^n$ of class C^2. A point $x \in \mathbb{R}^n$ is a critical point of f if all the partial derivatives of f vanish at x; a critical point is* nondegenerate *if the $n \times n$ matrix*

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$$

is nonsingular.

Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e., the nondegenerate critical points are isolated).

Problem 2.2.11 (Su80) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function whose partial derivatives of order $\leqslant 2$ are everywhere defined and continuous.*

1. *Let $a \in \mathbb{R}^n$ be a critical point of f (i.e., $\frac{\partial f}{\partial x_j}(a) = 0,\ i = 1, \ldots, n$). Prove that a is a local minimum provided the Hessian matrix*

   $$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

   is positive definite at $x = a$.

2. *Assume the Hessian matrix is positive definite at all x. Prove that f has, at most, one critical point.*

Problem 2.2.12 (Fa88) *Prove that a real valued C^3 function f on $\mathbb{R}^2$ whose Laplacian,*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

is everywhere positive cannot have a local maximum.

Problem 2.2.13 (Fa01) *Let the function u on $\mathbb{R}^2$ be harmonic, not identically 0, and homogeneous of degree d, where $d > 0$. (The homogeneity condition means that $u(tx, ty) = t^d u(x, y)$ for $t > 0$.) Prove that d is an integer.*

Problem 2.2.14 (Su82) *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ and assume that 0 is a* regular value *of f (i.e., the differential of f has rank 2 at each point of $f^{-1}(0)$). Prove that $\mathbb{R}^3 \setminus f^{-1}(0)$ is arcwise connected.*

Problem 2.2.15 (Sp87) *Let the transformation T from the subset $U = \{(u, v) \mid u > v\}$ of $\mathbb{R}^2$ into $\mathbb{R}^2$ be defined by $T(u, v) = (u + v, u^2 + v^2)$.*

1. *Prove that T is locally one-to-one.*

2. *Determine the range of T, and show that T is globally one-to-one.*

Problem 2.2.16 (Fa91) *Let f be a C^1 function from the interval $(-1, 1)$ into $\mathbb{R}^2$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that there is a number ε in $(0, 1)$ such that $\|f(t)\|$ is an increasing function of t on $(0, \varepsilon)$.*

Problem 2.2.17 (Fa80) *For a real 2×2 matrix*

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

let $\|X\| = x^2 + y^2 + z^2 + t^2$, and define a metric by $d(X, Y) = \|X - Y\|$. Let $\Sigma = \{X \mid \det(X) = 0\}$. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Find the minimum distance from A to Σ and exhibit an $S \in \Sigma$ that achieves this minimum.

Problem 2.2.18 (Su80) *Let $S \subset \mathbb{R}^3$ denote the ellipsoidal surface defined by*

$$2x^2 + (y-1)^2 + (z-10)^2 = 1.$$

Let $T \subset \mathbb{R}^3$ be the surface defined by

$$z = \frac{1}{x^2 + y^2 + 1}.$$

Prove that there exist points $p \in S$, $q \in T$, such that the line $\overline{pq}$ is perpendicular to S at p and to T at q.

Problem 2.2.19 (Sp80) *Let P_2 denote the set of real polynomials of degree $\leqslant 2$. Define the map $J : P_2 \to \mathbb{R}$ by*

$$J(f) = \int_0^1 f(x)^2\, dx.$$

Let $Q = \{f \in P_2 \mid f(1) = 1\}$. Show that J attains a minimum value on Q and determine where the minimum occurs.

Problem 2.2.20 (Su79) *Let X be the space of orthogonal real $n \times n$ matrices. Let $v_0 \in \mathbb{R}^n$. Locate and describe the elements of X, where the map*

$$f : X \to \mathbb{R}, \qquad f(A) = \langle v_0, Av_0 \rangle$$

takes its maximum and minimum values.

Problem 2.2.21 (Fa78) *Let $W \subset \mathbb{R}^n$ be an open connected set and f a real valued function on W such that all partial derivatives of f are 0. Prove that f is constant.*

Problem 2.2.22 (Sp77) *In $\mathbb{R}^2$, consider the region $\mathcal{A}$ defined by $x^2 + y^2 > 1$. Find differentiable real valued functions f and g on $\mathcal{A}$ such that $\partial f/\partial x = \partial g/\partial y$ but there is no real valued function h on $\mathcal{A}$ such that $f = \partial h/\partial y$ and $g = \partial h/\partial x$.*

Problem 2.2.23 (Sp77) *Suppose that $u(x, t)$ is a continuous function of the real variables x and t with continuous second partial derivatives. Suppose that u and its first partial derivatives are periodic in x with period 1, and that*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

Prove that

$$E(t) = \frac{1}{2}\int_0^1 \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) dx$$

is a constant independent of t.

Problem 2.2.24 (Su77) *Let $f(x, t)$ be a C^1 function such that $\partial f/\partial x = \partial f/\partial t$. Suppose that $f(x, 0) > 0$ for all x. Prove that $f(x, t) > 0$ for all x and t.*

Problem 2.2.25 (Fa77) *Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives and satisfy*

$$\left|\frac{\partial f}{\partial x_j}(x)\right| \leqslant K$$

for all $x = (x_1, \ldots, x_n)$, $j = 1, \ldots, n$. Prove that

$$|f(x) - f(y)| \leqslant \sqrt{n}K\|x - y\|$$

(where $\|u\| = \sqrt{u_1^2 + \cdots + u_n^2}$).

Problem 2.2.26 (Fa83, Sp87) *Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a function which is continuously differentiable and whose partial derivatives are uniformly bounded:*

$$\left|\frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)\right| \leqslant M$$

for all $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$. Show that if $n \geqslant 2$, then f can be extended to a continuous function defined on all of $\mathbb{R}^n$. Show that this is false if $n = 1$ by giving a counterexample.

Problem 2.2.27 (Sp79) *Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be differentiable. Suppose*

$$\lim_{x \to 0} \frac{\partial f}{\partial x_j}(x)$$

exists for each $j = 1, \ldots, n$.

1. *Can f be extended to a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$?*

2. *Assuming continuity at the origin, is f differentiable from $\mathbb{R}^n$ to $\mathbb{R}$?*

Problem 2.2.28 (Sp82) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ have directional derivatives in all directions at the origin. Is f differentiable at the origin? Prove or give a counterexample.*

Problem 2.2.29 (Fa78) *Let $f : \mathbb{R}^n \to \mathbb{R}$ have the following properties: f is differentiable on $\mathbb{R}^n \setminus \{0\}$, f is continuous at 0, and*

$$\lim_{p \to 0} \frac{\partial f}{\partial x_i}(p) = 0$$

for $i = 1, \ldots, n$. Prove that f is differentiable at 0.

Problem 2.2.30 (Su78) *Let $U \subset \mathbb{R}^n$ be a convex open set and $f : U \to \mathbb{R}^n$ a differentiable function whose partial derivatives are uniformly bounded but not necessarily continuous. Prove that f has a unique continuous extension to the closure of U.*

Problem 2.2.31 (Fa78)

1. *Show that if $u, v : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable and $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$, then $u = \dfrac{\partial f}{\partial x}$, $v = \dfrac{\partial f}{\partial y}$ for some $f : \mathbb{R}^2 \to \mathbb{R}$.*

2. *Prove there is no $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that*

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2 + y^2} \quad \textit{and} \quad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}.$$

Problem 2.2.32 (Su79) *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be such that*

$$f^{-1}(0) = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}.$$

Suppose f has continuous partial derivatives of orders $\leqslant 2$. Is there a $p \in \mathbb{R}^3$ with $\|p\| \leqslant 1$ such that

$$\frac{\partial^2 f}{\partial x^2}(p) + \frac{\partial^2 f}{\partial y^2}(p) + \frac{\partial^2 f}{\partial z^2}(p) \geqslant 0 \ ?$$

Problem 2.2.33 (Sp92) *Let f be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that there is a differentiable function g from $\mathbb{R}^n$ to $\mathbb{R}$ having no critical points such that $g \circ f$ vanishes identically. Prove that the Jacobian determinant of f vanishes identically.*

Problem 2.2.34 (Fa83) *Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions with $f(0) = 0$ and $f'(0) \neq 0$. Consider the equation $f(x) = tg(x)$, $t \in \mathbb{R}$.*

1. *Show that in a suitably small interval $|t| < \delta$, there is a unique continuous function $x(t)$ which solves the equation and satisfies $x(0) = 0$.*

2. *Derive the first order Taylor expansion of $x(t)$ about $t = 0$.*

Problem 2.2.35 (Sp78) *Consider the system of equations*

$$\begin{aligned} 3x + y - z + u^4 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

1. *Prove that for some $\varepsilon > 0$, the system can be solved for (x, y, u) as a function of $z \in [-\varepsilon, \varepsilon]$, with $x(0) = y(0) = u(0) = 0$. Are such functions $x(z)$, $y(z)$ and $u(z)$ continuous? Differentiable? Unique?*

2. *Show that the system cannot be solved for (x, y, z) as a function of $u \in [-\delta, \delta]$, for all $\delta > 0$.*

Problem 2.2.36 (Sp81) *Describe the two regions in (a, b)-space for which the function*

$$f_{a,b}(x, y) = ay^2 + bx$$

restricted to the circle $x^2 + y^2 = 1$, has exactly two, and exactly four critical points, respectively.

Problem 2.2.37 (Fa87) *Let u and v be two real valued C^1 functions on $\mathbb{R}^2$ such that the gradient ∇u is never 0, and such that, at each point, ∇v and ∇u are linearly dependent vectors. Given $p_0 = (x_0, y_0) \in \mathbb{R}^2$, show that there is a C^1 function F of one variable such that $v(x, y) = F(u(x, y))$ in some neighborhood of p_0.*

Problem 2.2.38 (Fa94) *Let f be a continuously differentiable function from $\mathbb{R}^2$ into $\mathbb{R}$. Prove that there is a continuous one-to-one function g from $[0, 1]$ into $\mathbb{R}^2$ such that the composite function $f \circ g$ is constant.*

Problem 2.2.39 (Su84) *Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and let*

$$\begin{aligned} u &= f(x) \\ v &= -y + xf(x). \end{aligned}$$

If $f'(x_0) \neq 0$, show that this transformation is locally invertible near (x_0, y_0) and the inverse has the form

$$\begin{aligned} x &= g(u) \\ y &= -v + ug(u). \end{aligned}$$

Problem 2.2.40 (Su78, Fa99) *Let $M_{n\times n}$ denote the vector space of real $n\times n$ matrices. Define a map $f: M_{n\times n} \to M_{n\times n}$ by $f(X) = X^2$. Find the derivative of f.*

Problem 2.2.41 (Su82) *Let $M_{2\times 2}$ be the four-dimensional vector space of all 2×2 real matrices and define $f : M_{2\times 2} \to M_{2\times 2}$ by $f(X) = X^2$.*

1. *Show that f has a local inverse near the point*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. *Show that f does not have a local inverse near the point*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 2.2.42 (Fa80) *Show that there is an $\varepsilon > 0$ such that if A is any real 2×2 matrix satisfying $|a_{ij}| \leqslant \varepsilon$ for all entries a_{ij} of A, then there is a real 2×2 matrix X such that $X^2 + X^t = A$, where X^t is the transpose of X. Is X unique?*

Problem 2.2.43 (Sp96) *Let $M_{2\times 2}$ be the space of 2×2 matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^4$. Let the function F from $M_{2\times 2}$ into $M_{2\times 2}$ be defined by*

$$F(X) = X + X^2.$$

Prove that the range of F contains a neighborhood of the origin.

Problem 2.2.44 (Fa78) *Let $M_{n\times n}$ denote the vector space of $n \times n$ real matrices. Prove that there are neighborhoods U and V in $M_{n\times n}$ of the identity matrix such that for every A in U, there is a unique X in V such that $X^4 = A$.*

Problem 2.2.45 (Sp79, Fa93) *Let $M_{n\times n}$ denote the vector space of $n \times n$ real matrices for $n \geqslant 2$. Let* det : $M_{n\times n} \to \mathbb{R}$ *be the determinant map.*

1. *Show that* det *is C^∞.*

2. *Show that the derivative of* det *at $A \in M_{n\times n}$ is zero if and only if A has rank $\leqslant n-2$.*

Problem 2.2.46 (Fa83) *Let $F(t) = \left(f_{ij}(t)\right)$ be an $n\times n$ matrix of continuously differentiable functions $f_{ij} : \mathbb{R} \to \mathbb{R}$, and let*

$$u(t) = \operatorname{tr}\left(F(t)^3\right).$$

Show that u is differentiable and

$$u'(t) = 3\operatorname{tr}\left(F(t)^2 F'(t)\right).$$

Problem 2.2.47 (Fa81) *Let $A = \left(a_{ij}\right)$ be an $n \times n$ matrix whose entries a_{ij} are real valued differentiable functions defined on $\mathbb{R}$. Assume that the determinant* $\det(A)$ *of A is everywhere positive. Let $B = \left(b_{ij}\right)$ be the inverse matrix of A. Prove the formula*

$$\frac{d}{dt}\log(\det(A)) = \sum_{i,j=1}^{n} \frac{da_{ij}}{dt} b_{ji}.$$

Problem 2.2.48 (Sp03)

1. *Prove that there is no continuously differentiable, measure-preserving bijective function $f : \mathbb{R} \to \mathbb{R}_+$.*

2. *Find an example of a continuously differentiable, measure-preserving bijective function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}_+$.*

2.3 Integral Calculus

Problem 2.3.1 (Sp78) *What is the volume enclosed by the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1?$$

Problem 2.3.2 (Sp78) *Evaluate*

$$\int\!\!\int_{\mathcal{A}} e^{-x^2-y^2}\,dxdy\,,$$

where $\mathcal{A}=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leqslant 1\}$.

Problem 2.3.3 (Sp98) *Given the fact that* $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$, *evaluate the integral*

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+(y-x)^2+y^2)}dx\,dy\,.$$

Problem 2.3.4 (Fa86) *Evaluate*

$$\int\!\!\int_{\mathcal{R}}(x^3-3xy^2)\,dxdy\,,$$

where

$$\mathcal{R}=\{(x,y)\in\mathbb{R}^2 \mid (x+1)^2+y^2\leqslant 9,\quad (x-1)^2+y^2\geqslant 1\}.$$

Problem 2.3.5 (Fa98) *Let* $\varphi(x,y)$ *be a function with continuous second order partial derivatives such that*

1. $\varphi_{xx}+\varphi_{yy}+\varphi_x=0$ *in the punctured plane* $\mathbb{R}^2\setminus\{0\}$,
2. $r\varphi_x\to\dfrac{x}{2\pi r}$ *and* $r\varphi_y\to\dfrac{y}{2\pi r}$ *as* $r=\sqrt{x^2+y^2}\to 0$.

Let C_R *be the circle* $x^2+y^2=R^2$. *Show that the line integral*

$$\int_{C_R} e^x(-\varphi_y\,dx+\varphi_x\,dy)$$

is independent of R, and evaluate it.

Problem 2.3.6 (Sp80) *Let* $\mathcal{S}=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ *denote the unit sphere in* $\mathbb{R}^3$. *Evaluate the surface integral over* $\mathcal{S}$:

$$\int\!\!\int_{\mathcal{S}}(x^2+y+z)\,dA\,.$$

Problem 2.3.7 (Sp81) *Let $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ be the usual unit vectors in $\mathbb{R}^3$. Let $\vec{F}$ denote the vector field*

$$(x^2 + y - 4)\vec{\imath} + 3xy\vec{\jmath} + (2xz + z^2)\vec{k}.$$

1. *Compute $\nabla \times \vec{F}$ (the curl of $\vec{F}$).*
2. *Compute the integral of $\nabla \times \vec{F}$ over the surface $x^2 + y^2 + z^2 = 16$, $z \geqslant 0$.*

Problem 2.3.8 (Sp91) *Let the vector field F in $\mathbb{R}^3$ have the form*

$$F(r) = g(\|r\|)r \qquad\qquad (r \neq (0, 0, 0)),$$

where g is a real valued smooth function on $(0, \infty)$ and $\|\cdot\|$ denotes the Euclidean norm. (F is undefined at $(0, 0, 0)$.) Prove that

$$\int_C F \cdot ds = 0$$

for any smooth closed path C in $\mathbb{R}^3$ that does not pass through the origin.

Problem 2.3.9 (Fa91) *Let $\mathcal{B}$ denote the unit ball of $\mathbb{R}^3$, $\mathcal{B} = \{r \in \mathbb{R}^3 \mid \|r\| \leqslant 1\}$. Let $J = (J_1, J_2, J_3)$ be a smooth vector field on $\mathbb{R}^3$ that vanishes outside of $\mathcal{B}$ and satisfies $\nabla \cdot \vec{J} = 0$.*

1. *For f a smooth, scalar-valued function defined on a neighborhood of $\mathcal{B}$, prove that*
$$\int_{\mathcal{B}} (\nabla f) \cdot \vec{J}\, dxdydz = 0.$$
2. *Prove that*
$$\int_{\mathcal{B}} J_1\, dxdydz = 0.$$

Problem 2.3.10 (Fa94) *Let $\mathcal{D}$ denote the open unit disc in $\mathbb{R}^2$. Let u be an* eigenfunction for the Laplacian *in $\mathcal{D}$; that is, a real valued function of class C^2 defined in $\overline{\mathcal{D}}$, zero on the boundary of $\mathcal{D}$ but not identically zero, and satisfying the differential equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u\,,$$

where λ is a constant. Prove that

$$(*) \qquad \iint_{\mathcal{D}} |\text{grad}\, u|^2\, dxdy + \lambda \iint_{\mathcal{D}} u^2 dxdy = 0\,,$$

and hence that $\lambda < 0$.

Problem 2.3.11 (Fa03) *Let $\lambda, a \in \mathbb{R}$, with $a < 0$. Let $u(x, y)$ be an infinitely differentiable function defined on an open neighborhood of closed unit disc $\mathcal{D}$ such that*

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \lambda u && \text{in } \operatorname{int}(\mathcal{D}) \\ D_n u &= au && \text{in } \partial\mathcal{D}. \end{aligned}$$

Here $D_n u$ denotes the directional derivative of u in the direction of the outward unit normal. Prove that if u is not identically zero in the interior of $\mathcal{D}$ then $\lambda < 0$.

Problem 2.3.12 (Sp92) *Let f be a one-to-one C^1 map of $\mathbb{R}^3$ into $\mathbb{R}^3$, and let J denote its Jacobian determinant. Prove that if x_0 is any point of $\mathbb{R}^3$ and $Q_r(x_0)$ denotes the cube with center x_0, side length r, and edges parallel to the coordinate axes, then*

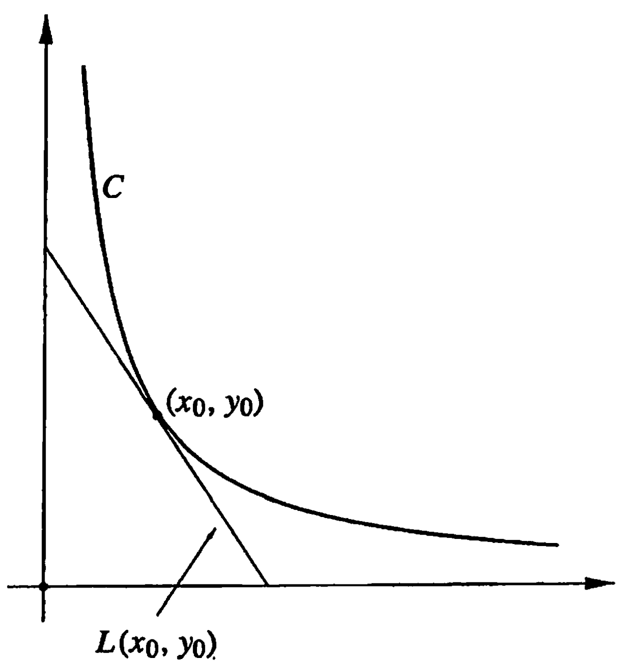
$$|J(x_0)| = \lim_{r\to 0} r^{-3} \operatorname{vol}(f(Q_r(x_0))) \leqslant \limsup_{x\to x_0} \frac{\|f(x) - f(x_0)\|^3}{\|x - x_0\|^3}.$$

Here, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$.

3
Differential Equations

3.1 First Order Equations

Problem 3.1.1 (Fa87) *Find a curve C in $\mathbb{R}^2$, passing through the point* $(3, 2)$, *with the following property: Let $L(x_0, y_0)$ be the segment of the tangent line to C at (x_0, y_0) which lies in the first quadrant. Then each point (x_0, y_0) of C is the midpoint of $L(x_0, y_0)$.*



Problem 3.1.2 (Su78) *Solve the differential equation $g' = 2g$, $g(0) = a$, where a is a real constant.*

Problem 3.1.3 (Fa93, Fa77) *Let n be an integer larger than 1. Is there a differentiable function on $[0, \infty)$ whose derivative equals its n^{th} power and whose value at the origin is positive?*

Problem 3.1.4 (Sp78)

1. *For which real numbers $\alpha > 0$ does the differential equation*

$$\frac{dx}{dt} = x^{\alpha}, \qquad x(0) = 0,$$

have a solution on some interval $[0, b]$, $b > 0$?

2. *For which values of α are there intervals on which two solutions are defined?*

Problem 3.1.5 (Sp78) *Consider the differential equation*

$$\frac{dx}{dt} = x^2 + t^2, \quad x(0) = 1.$$

1. *Prove that for some $b > 0$, there is a solution defined for $t \in [0, b]$.*
2. *Find an explicit value of b having the property in Part 1.*
3. *Find a $c > 0$ such that there is no solution on $[0, c]$.*

Problem 3.1.6 (Fa78) *Solve the differential equation*

$$\frac{dy}{dx} = x^2 y - 3x^2, \qquad y(0) = 1.$$

Problem 3.1.7 (Sp93) *Prove that every solution $x(t)$ $(t \geqslant 0)$ of the differential equation*

$$\frac{dx}{dt} = x^2 - x^6$$

with $x(0) > 0$ satisfies $\lim_{t\to\infty} x(t) = 1$.

Problem 3.1.8 (Sp80) *Consider the differential equation*

$$x' = \frac{x^3 - x}{1 + e^x}.$$

1. *Find all its constant solutions.*
2. *Discuss $\lim_{t\to\infty} x(t)$, where $x(t)$ is the solution such that $x(0) = \frac{1}{2}$.*

Problem 3.1.9 (Su77, Su80, Sp82, Sp83) *Prove that the initial value problem*

$$\frac{dx}{dt} = 3x + 85\cos x, \qquad x(0) = 77\,,$$

has a solution $x(t)$ defined for all $t \in \mathbb{R}$.

Problem 3.1.10 (Fa82) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nowhere vanishing function, and consider the differential equation*

$$\frac{dy}{dx} = f(y).$$

1. *For each real number c, show that this equation has a unique, continuously differentiable solution $y = y(x)$ on a neighborhood of 0 which satisfies the initial condition $y(0) = c$.*
2. *Deduce the conditions on f under which the solution y exists for all $x \in \mathbb{R}$, for every initial value c.*

Problem 3.1.11 (Sp79) *Find all solutions to the differential equation*

$$\frac{dy}{dx} = \sqrt{y}, \qquad y(0) = 0\,.$$

Problem 3.1.12 (Sp83) *Find all solutions $y : \mathbb{R} \to \mathbb{R}$ to*

$$\frac{dy}{dx} = \sqrt{y\,(y-2)}, \qquad y(0) = 0.$$

Problem 3.1.13 (Su83) *Find all real valued C^1 solutions $y(x)$ of the differential equation*

$$x\frac{dy}{dx} + y = x \qquad (-1 < x < 1).$$

Problem 3.1.14 (Sp84) *Consider the equation*

$$\frac{dy}{dx} = y - \sin y.$$

Show that there is an $\varepsilon > 0$ such that if $|y_0| < \varepsilon$, then the solution $y = f(x)$ with $f(0) = y_0$ satisfies

$$\lim_{x \to -\infty} f(x) = 0.$$

Problem 3.1.15 (Fa84) *Consider the differential equation*

$$\frac{dy}{dx} = 3xy + \frac{y}{1+y^2}.$$

Prove

1. *For each* $n = 1, 2, \ldots$, *there is a unique solution* $y = f_n(x)$ *defined for* $0 \leqslant x \leqslant 1$ *such that* $f_n(0) = 1/n$.

2. $\lim_{n\to\infty} f_n(1) = 0$.

Problem 3.1.16 (Fa85) *Let* $y(t)$ *be a real valued solution, defined for* $0 < t < \infty$, *of the differential equation*

$$\frac{dy}{dt} = e^{-y} - e^{-3y} + e^{-5y}.$$

Show that $y(t) \to +\infty$ *as* $t \to +\infty$.

Problem 3.1.17 (Sp01) *Consider the* differential-delay equation *given by* $y'(t) = -y(t - t_0)$. *Here, the independent variable* t *is a real variable, the function* y *is allowed to be complex valued, and* t_0 *is a positive constant. Prove that if* $0 < t_0 < \pi/2$ *then every solution of the form* $y(t) = e^{\lambda t}$, *with* λ *complex, tends to 0 as* $t \to +\infty$.

Problem 3.1.18 (Fa86) *Prove the following theorem, or find a counterexample: If* p *and* q *are continuous real valued functions on* $\mathbb{R}$ *such that* $|q(x)| \leqslant |p(x)|$ *for all* x, *and if every solution* f *of the differential equation*

$$f' + qf = 0$$

satisfies $\lim_{x\to+\infty} f(x) = 0$, *then every solution* f *of the differential equation*

$$f' + pf = 0$$

satisfies $\lim_{x\to+\infty} f(x) = 0$.

Problem 3.1.19 (Fa86) *Discuss the solvability of the differential equation*

$$(e^x \sin y)(y')^3 + (e^x \cos y)y' + e^y \tan x = 0$$

with the initial condition $y(0) = 0$. *Does a solution exist in some interval about 0? If so, is it unique?*

Problem 3.1.20 (Fa92) *Let* f *and* g *be positive continuous functions on* $\mathbb{R}$, *with* $g \leqslant f$ *everywhere. Assume the initial value problem*

$$\frac{dx}{dt} = f(x), \quad x(0) = 0,$$

has a solution defined on all of $\mathbb{R}$. *Prove that the initial value problem*

$$\frac{dx}{dt} = g(x), \quad x(0) = 0,$$

also has a solution defined on all of $\mathbb{R}$.

Problem 3.1.21 (Sp95) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a bounded continuously differentiable function. Show that every solution of* $y'(x) = f(y(x))$ *is monotone.*

3.2 Second Order Equations

Problem 3.2.1 (Sp97) *Suppose that $f''(x) = (x^2 - 1)f(x)$ for all $x \in \mathbb{R}$, and that $f(0) = 1$, $f'(0) = 0$. Show that $f(x) \to 0$ as $x \to \infty$.*

Problem 3.2.2 (Sp77) *Find the solution of the differential equation*

$$y'' - 2y' + y = 0,$$

subject to the conditions

$$y(0) = 1, \qquad y'(0) = 1.$$

Problem 3.2.3 (Fa77) *Find all solutions of the differential equation*

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = \sin t$$

subject to the condition $x(0) = 1$ and $x'(0) = 0$.

Problem 3.2.4 (Su79) *Let $x : \mathbb{R} \to \mathbb{R}$ be a solution to the differential equation*

$$5x'' + 10x' + 6x = 0.$$

Prove that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = \frac{x(t)^2}{1 + x(t)^4}$$

attains a maximum value.

Problem 3.2.5 (Su84) *Let $x(t)$ be the solution of the differential equation*

$$x''(t) + 8x'(t) + 25x(t) = 2\cos t$$

with initial conditions $x(0) = 0$ and $x'(0) = 0$. Show that for suitable constants α and δ,

$$\lim_{t\to\infty} (x(t) - \alpha\cos(t - \delta)) = 0.$$

Problem 3.2.6 (Fa79, Su81, Fa92) *Let $y = y(x)$ be a solution of the differential equation $y'' = -|y|$ with $-\infty < x < \infty$, $y(0) = 1$ and $y'(0) = 0$.*

1. *Show that y is an even function.*
2. *Show that y has exactly one zero on the positive real axis.*

Problem 3.2.7 (Fa95) *Determine all real numbers $L > 1$ so that the boundary value problem*

$$x^2y''(x) + y(x) = 0, \qquad 1 \leqslant x \leqslant L$$
$$y(1) = y(L) = 0$$

has a nonzero solution.

Problem 3.2.8 (Fa83) *For which real values of p does the differential equation*

$$y'' + 2py' + y = 3$$

admit solutions $y = f(x)$ *with infinitely many critical points?*

Problem 3.2.9 (Sp87) *Let p, q and r be continuous real valued functions on* $\mathbb{R}$, *with* $p > 0$. *Prove that the differential equation*

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$

is equivalent to (i.e., has exactly the same solutions as) a differential equation of the form

$$\big(a(t)x'(t)\big)' + b(t)x(t) = 0,$$

where a is continuously differentiable and b is continuous.

Problem 3.2.10 (Fa93) *Let the function* $x(t)$ $(-\infty < t < \infty)$ *be a solution of the differential equation*

$$\frac{d^2x}{dt^2} - 2b\frac{dx}{dt} + cx = 0$$

such that $x(0) = x(1) = 0$. *(Here, b and c are real constants.) Prove that* $x(n) = 0$ *for every integer n.*

Problem 3.2.11 (Sp93) *Let k be a positive integer. For which values of the real number c does the differential equation*

$$\frac{d^2x}{dt^2} - 2c\frac{dx}{dt} + x = 0$$

have a solution satisfying $x(0) = x(2\pi k) = 0$?

Problem 3.2.12 (Fa01) *Consider the second-order linear differential equation*

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0\,. \qquad (*)$$

Here, the independent variable t varies over $\mathbb{R}$, *the unknown function x is assumed to be real valued, and p and q are continuous functions on* $\mathbb{R}$. *Assume that the solutions of* $(*)$ *are defined for all t (which is actually guaranteed by the theory), and that the solution set is translation invariant: if f is a solution and s is a real number, then the function* $g(t) = f(t+s)$ *is also a solution. Prove that p and q are constant.*

Problem 3.2.13 (Sp85) *Let* $h > 0$ *be given. Consider the linear difference equation*

$$\frac{y((n+2)h) - 2y((n+1)h) + y(nh)}{h^2} = -y(nh), \quad n = 0, 1, 2, \ldots.$$

(Note the analogy with the differential equation $y'' = -y$.*)*

1. *Find the general solution of the equation by trying suitable exponential substitutions.*

2. *Find the solution with $y(0) = 0$ and $y(h) = h$. Denote it by $S_h(nh)$, $n = 1, 2, \ldots$.*

3. *Let x be fixed and $h = \frac{x}{n}$. Show that*

$$\lim_{n\to\infty} S_{\frac{x}{n}}\left(\frac{nx}{n}\right) = \sin x\,.$$

3.3 Higher Order Equations

Problem 3.3.1 (Su78) *Let E be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are solutions to the differential equation $f''' + f'' - 2f = 0$.*

1. *Prove that E is a vector space and find its dimension.*

2. *Let $E_0 \subset E$ be the subspace of solutions g such that $\lim_{t\to\infty} g(t) = 0$. Find $g \in E_0$ such that $g(0) = 0$ and $g'(0) = 2$.*

Problem 3.3.2 (Fa98) *Find a function $y(x)$ such that $y^{(4)} + y = 0$ for $x \geqslant 0$, $y(0) = 0$, $y'(0) = 1$ and $\lim_{x\to\infty} y(x) = \lim_{x\to\infty} y'(x) = 0$.*

Problem 3.3.3 (Sp87) *Let V be a finite-dimensional linear subspace of $C^\infty(\mathbb{R})$ (the space of complex valued, infinitely differentiable functions). Assume that V is closed under D, the operator of differentiation (i.e., $f \in V \Rightarrow Df \in V$). Prove that there is a constant coefficient differential operator*

$$L = \sum_{k=0}^{n} a_k D^k$$

such that V consists of all solutions of the differential equation $Lf = 0$.

Problem 3.3.4 (Fa94) 1. *Find a basis for the space of real solutions of the differential equation*

$$(*) \qquad \sum_{n=0}^{7} \frac{d^n x}{dt^n} = 0.$$

2. *Find a basis for the subspace of real solutions of $(*)$ that satisfy*

$$\lim_{t\to+\infty} x(t) = 0.$$

Problem 3.3.5 (Sp94) *1. Suppose the functions* $\sin t$ *and* $\sin 2t$ *are both solutions of the differential equation*

$$\sum_{k=0}^{n} c_k \frac{d^k x}{dt^k} = 0,$$

where $c_0, \ldots, c_n$ *are real constants. What is the smallest possible order of the equation? Write down an equation of minimum order having the given functions as solutions.*

2. *Will the answers to Part 1 be different if the constants* $c_0, \ldots, c_n$ *are allowed to be complex?*

Problem 3.3.6 (Sp95) *Let* $y : \mathbb{R} \to \mathbb{R}$ *be a three times differentiable function satisfying the differential equation* $y''' - y = 0$*. Suppose that* $\lim_{x\to\infty} y(x) = 0$*. Find real numbers a, b, c, and d, not all zero, such that* $ay(0)+y'(0)+cy''(0) = d$*.*

3.4 Systems of Differential Equations

Problem 3.4.1 (Sp79) *Consider the system of differential equations:*

$$\begin{aligned}\frac{dx}{dt} &= y + tz\\ \frac{dy}{dt} &= z + t^2 x\\ \frac{dz}{dt} &= x + e^t y.\end{aligned}$$

Prove there exists a solution defined for all $t \in [0, 1]$*, such that*

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\\ z(0)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

and also

$$\int_0^1 \left(x(t)^2 + y(t)^2 + z(t)^2\right) dt = 1.$$

Problem 3.4.2 (Su79, Fa79, Fa82, Su85) *Find all pairs of* C^∞ *functions* $x(t)$ *and* $y(t)$ *on* $\mathbb{R}$ *satisfying*

$$x'(t) = 2x(t) - y(t), \qquad y'(t) = x(t).$$

Problem 3.4.3 (Su80) *Consider the differential equations*

$$\frac{dx}{dt} = -x + y, \qquad \frac{dy}{dt} = \log(20 + x) - y.$$

Let $x(t)$ *and* $y(t)$ *be a solution defined for all* $t \geqslant 0$ *with* $x(0) > 0$ *and* $y(0) > 0$*. Prove that* $x(t)$ *and* $y(t)$ *are bounded.*

Problem 3.4.4 (Sp81) *Consider the system of differential equations*

$$\frac{dx}{dt} = y + x(1 - x^2 - y^2)$$
$$\frac{dy}{dt} = -x + y(1 - x^2 - y^2).$$

1. *Show that for any x_0 and y_0, there is a unique solution $(x(t), y(t))$ defined for all $t \in \mathbb{R}$ such that $x(0) = x_0$, $y(0) = y_0$.*
2. *Show that if $x_0 \neq 0$ and $y_0 \neq 0$, the solution referred to in Part 1 approaches the circle $x^2 + y^2 = 1$ as $t \to \infty$.*

Problem 3.4.5 (Sp84) *Show that the system of differential equations*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

has a solution which tends to ∞ as $t \to -\infty$ and tends to the origin as $t \to +\infty$.

Problem 3.4.6 (Sp91) *Let $x(t)$ be a nontrivial solution to the system*

$$\frac{dx}{dt} = Ax,$$

where

$$A = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}.$$

Prove that $\|x(t)\|$ is an increasing function of t. (Here, $\|\cdot\|$ denotes the Euclidean norm.)

Problem 3.4.7 (Su84) *Consider the solution curve $(x(t), y(t))$ to the equations*

$$\frac{dx}{dt} = 1 + \frac{1}{2}x^2 \sin y$$
$$\frac{dy}{dt} = 3 - x^2$$

with initial conditions $x(0) = 0$ and $y(0) = 0$. Prove that the solution must cross the line $x = 1$ in the xy plane by the time $t = 2$.

Problem 3.4.8 (Fa80) *Consider the differential equation $x'' + x' + x^3 = 0$ and the function $f(x, x') = (x + x')^2 + (x')^2 + x^4$.*

1. *Show that f decreases along trajectories of the differential equation.*
2. *Show that if $x(t)$ is any solution, then $(x(t), x'(t))$ tends to $(0, 0)$ as $t \to \infty$.*

Problem 3.4.9 (Fa84) *Consider the differential equation*

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -ay - x^3 - x^5, \quad \text{where } a > 0.$$

1. *Show that*

$$F(x, y) = \frac{y^2}{2} + \frac{x^4}{4} + \frac{x^6}{6}$$

decreases along solutions.

2. *Show that for any* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that whenever* $\| (x(0), y(0)) \| < \delta$, *there is a unique solution* $(x(t), y(t))$ *of the given equations with the initial condition* $(x(0), y(0))$ *which is defined for all* $t \geqslant 0$ *and satisfies* $\| (x(t), y(t)) \| < \varepsilon$.

Problem 3.4.10 (Fa83) 1. *Let* $u(t)$ *be a real valued differentiable function of a real variable* t *which satisfies an inequality of the form*

$$u'(t) \leqslant au(t), \quad t \geqslant 0, \quad u(0) \leqslant b,$$

where a *and* b *are positive constants. Starting from first principles, derive an upper bound for* $u(t)$ *for* $t > 0$.

2. *Let* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *be a differentiable function from* $\mathbb{R}$ *to* $\mathbb{R}^n$ *which satisfies a differential equation of the form*

$$x'(t) = f(x(t)),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function. Assuming that* f *satisfies the condition*

$$\langle f(y), y \rangle \leqslant \|y\|^2, \quad y \in \mathbb{R}^n$$

derive an inequality showing that the norm $\|x(t)\|$ *grows, at most, exponentially.*

Problem 3.4.11 (Fa81) *Consider an autonomous system of differential equations*

$$\frac{dx_i}{dt} = F_i(x_1, \ldots, x_n),$$

where $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ *is a* C^1 *vector field.*

1. *Let* U *and* V *be two solutions on* $a < t < b$. *Assuming that*

$$\langle DF(x)z, z \rangle \leqslant 0$$

for all x, z *in* $\mathbb{R}^n$, *show that* $\|U(t) - V(t)\|^2$ *is a decreasing function of* t.

2. *Let $W(t)$ be a solution defined for $t > 0$. Assuming that*

$$\langle DF(x)z, z\rangle \leqslant -\|z\|^2,$$

show that there exists $C \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} W(t) = C.$$

Problem 3.4.12 (Fa81) *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and consider the system of second order differential equations*

$$x_i''(t) = f_i(x(t)), \qquad 1 \leqslant i \leqslant n,$$

where

$$f_i = -\frac{\partial V}{\partial x_i}.$$

Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a solution of this system on a finite interval $a < t < b$.

1. *Show that the function*

$$H(t) = \frac{1}{2}\langle x'(t), x'(t)\rangle + V(x(t))$$

is constant for $a < t < b$.

2. *Assuming that $V(x) \geqslant M > -\infty$ for all $x \in \mathbb{R}^n$, show that $x(t)$, $x'(t)$, and $x''(t)$ are bounded on $a < t < b$, and then prove all three limits*

$$\lim_{t\to b} x(t), \quad \lim_{t\to b} x'(t), \quad \lim_{t\to b} x''(t)$$

exist.

Problem 3.4.13 (Sp86) *For λ a real number, find all solutions of the integral equations*

$$\varphi(x) = e^x + \lambda\int_0^x e^{(x-y)}\varphi(y)\,dy, \qquad 0 \leqslant x \leqslant 1,$$

$$\psi(x) = e^x + \lambda\int_0^1 e^{(x-y)}\psi(y)\,dy, \qquad 0 \leqslant x \leqslant 1.$$

Problem 3.4.14 (Sp86) *Let V be a finite-dimensional vector space (over $\mathbb{C}$) of C^∞ complex valued functions on $\mathbb{R}$ (the linear operations being defined pointwise). Prove that if V is closed under differentiation (i.e., $f'(x)$ belongs to V whenever $f(x)$ does), then V is closed under translations (i.e., $f(x+a)$ belongs to V whenever $f(x)$ does, for all real numbers a).*

Problem 3.4.15 (Fa99) *Describe all three dimensional vector spaces V of C^∞ complex valued functions on $\mathbb{R}$ that are invariant under the operator of differentiation.*

Problem 3.4.16 (Fa88) *Let the real valued functions $f_1, \dots, f_{n+1}$ on $\mathbb{R}$ satisfy the system of differential equations*

$$\begin{aligned} f'_{k+1} + f'_k &= (k+1)f_{k+1} - kf_k, \quad k = 1, \dots, n \\ f'_{n+1} &= -(n+1)f_{n+1}. \end{aligned}$$

Prove that for each k,

$$\lim_{t\to\infty} f_k(t) = 0.$$

Problem 3.4.17 (Fa91) *Consider the vector differential equation*

$$\frac{dx(t)}{dt} = A(t)x(t)$$

where A is a smooth $n \times n$ function on $\mathbb{R}$. Assume A has the property that $\langle A(t)y, y\rangle \leqslant c\|y\|^2$ for all y in $\mathbb{R}^n$ and all t, where c is a fixed real number. Prove that any solution $x(t)$ of the equation satisfies $\|x(t)\| \leqslant e^{ct}\|x(0)\|$ for all $t > 0$.

Problem 3.4.18 (Sp94) *Let W be a real 3×3 antisymmetric matrix, i.e., $W^t = -W$. Let the function*

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

be a real solution of the vector differential equation $\dfrac{dX}{dt} = WX$.

1. *Prove that $\|X(t)\|$, the Euclidean norm of $X(t)$, is independent of t.*
2. *Prove that if v is a vector in the null space of W, then $X(t)\cdot v$ is independent of t.*
3. *Prove that the values $X(t)$ all lie on a fixed circle in $\mathbb{R}^3$.*

Problem 3.4.19 (Sp80) *For each $t \in \mathbb{R}$, let $P(t)$ be a symmetric real $n\times n$ matrix whose entries are continuous functions of t. Suppose for all t that the eigenvalues of $P(t)$ are all $\leqslant -1$. Let $x(t) = (x_1(t), \dots, x_n(t))$ be a solution of the vector differential equation*

$$\frac{dx}{dt} = P(t)x.$$

Prove that

$$\lim_{t\to\infty} x(t) = 0.$$

Problem 3.4.20 (Sp89) *Let*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Find the general solution of the matrix differential equation $\dfrac{dX}{dt} = AXB$ *for the unknown* 4×4 *matrix function* $X(t)$.

4

Metric Spaces

4.1 Topology of $\mathbb{R}^n$

Problem 4.1.1 (Fa02) *Let S be a subset of $\mathbb{R}$. Let C be the set of points x in $\mathbb{R}$ with the property that $S \cap (x - \delta, x + \delta)$ is uncountable for every $\delta > 0$. Prove that $S \setminus C$ is finite or countable.*

Problem 4.1.2 (Sp03) *Let $A \subseteq \mathbb{R}$ be uncountable.*

1. *Show that A has at least one accumulation point.*

2. *Show that A has uncountably many accumulation points.*

Problem 4.1.3 (Fa02) *Let $\{x(i, j) \mid i, j \in \mathbb{N}\}$ be a doubly indexed set in a complete metric space (X, ρ) such that*

$$\rho(x(i, j), x(k, \ell)) \leqslant \min\left\{\max\left\{\frac{1}{i}, \frac{1}{k}\right\}, \max\left\{\frac{1}{j}, \frac{1}{\ell}\right\}\right\}$$

Prove that the iterated limits $\lim\limits_{i\to\infty}\lim\limits_{j\to\infty} x(i, j)$, $\lim\limits_{j\to\infty}\lim\limits_{i\to\infty} x(i, j)$ exist and are equal.

Problem 4.1.4 (Sp01) *Let T_0 be the interior of a triangle in $\mathbb{R}^2$ with vertices A, B, C. Let T_1 be the interior of the triangle whose vertices are the midpoints of the sides of T_0, T_2 the interior of the triangle whose vertices are the midpoints of the sides of T_1, and so on. Describe the set $\bigcap_{n=0}^{\infty} T_n$.*

Problem 4.1.5 (Sp86, Sp94, Sp96, Fa98) *Let K be a compact subset of $\mathbb{R}^n$ and $\{B_j\}$ a sequence of open balls that covers K. Prove that there is a positive number*

ε such that each ε-ball centered at a point of K is contained in one of the balls B_j.

Problem 4.1.6 (Su81) *Prove or disprove: The set $\mathbb{Q}$ of rational numbers is the intersection of a countable family of open subsets of $\mathbb{R}$.*

Problem 4.1.7 (Fa77) *Let $X \subset \mathbb{R}$ be a nonempty connected set of real numbers. If every element of X is rational, prove X has only one element.*

Problem 4.1.8 (Su80) *Give an example of a subset of $\mathbb{R}$ having uncountably many connected components. Can such a subset be open? Closed?*

Problem 4.1.9 (Fa00) *Let $f_n : \mathbb{R}^k \to \mathbb{R}^m$ be continuous ($n = 1, 2, \ldots$). Let K be a compact subset of $\mathbb{R}^k$. Suppose $f_n \to f$ uniformly on K. Prove that $S = f(K) \cup \bigcup_{n=1}^{\infty} f_n(K)$ is compact.*

Problem 4.1.10 (Sp83) *Show that the interval $[0, 1]$ cannot be written as a countably infinite disjoint union of closed subintervals of $[0, 1]$.*

Problem 4.1.11 (Su78, Sp99, Sp03) *Let X and Y be nonempty subsets of a metric space M. Define*

$$d(X, Y) = \inf\{d(x, y) \mid x \in X, y \in Y\}.$$

1. *Suppose X contains only one point x, and Y is closed. Prove*

   $$d(X, Y) = d(x, y)$$

   for some $y \in Y$.

2. *Suppose X is compact and Y is closed. Prove*

   $$d(X, Y) = d(x, y)$$

   for some $x \in X$, $y \in Y$.

3. *Show by example that the conclusion of Part 2 can be false if X and Y are closed but not compact.*

Problem 4.1.12 (Sp82) *Let $S \subset \mathbb{R}^n$ be a subset which is uncountable. Prove that there is a sequence of distinct points in S converging to a point of S.*

Problem 4.1.13 (Fa89) *Let $X \subset \mathbb{R}^n$ be a closed set and r a fixed positive real number. Let $Y = \{y \in \mathbb{R}^n \mid |x - y| = r \text{ for some } x \in X\}$. Show that Y is closed.*

Problem 4.1.14 (Sp92, Fa99) *Show that every infinite closed subset of $\mathbb{R}^n$ is the closure of a countable set.*

Problem 4.1.15 (Fa86) *Let $\{U_1, U_2, \ldots\}$ be a cover of $\mathbb{R}^n$ by open sets. Prove that there is a cover $\{V_1, V_2, \ldots\}$ such that*

1. *$V_j \subset U_j$ for each j;*
2. *each compact subset of $\mathbb{R}^n$ is disjoint from all but finitely many of the V_j.*

Problem 4.1.16 (Sp87) *A standard theorem states that a continuous real valued function on a compact set is bounded. Prove the converse: If K is a subset of $\mathbb{R}^n$ and if every continuous real valued function on K is bounded, then K is compact.*

Problem 4.1.17 (Su77) *Let $A \subset \mathbb{R}^n$ be compact, $x \in A$; let (x_i) be a sequence in A such that every convergent subsequence of (x_i) converges to x.*

1. *Prove that the entire sequence (x_i) converges.*
2. *Give an example to show that if A is not compact, the result in Part 1 is not necessarily true.*

Problem 4.1.18 (Fa89) *Let $X \subset \mathbb{R}^n$ be compact and let $f : X \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$, show there is an M such that for all $x, y \in X$,*

$$|f(x) - f(y)| \leqslant M|x - y| + \varepsilon.$$

Problem 4.1.19 (Su78) *Let $\{S_\alpha\}$ be a family of connected subsets of $\mathbb{R}^2$ all containing the origin. Prove that $\bigcup_\alpha S_\alpha$ is connected.*

Problem 4.1.20 (Fa79) *Consider the following properties of a function $f : \mathbb{R}^n \to \mathbb{R}$:*

1. *f is continuous.*
2. *The graph of f is connected in $\mathbb{R}^n \times \mathbb{R}$.*

Prove or disprove the implications $1 \Rightarrow 2$, $2 \Rightarrow 1$.

Problem 4.1.21 (Sp01) *Let U be a nonempty, proper, open subset of $\mathbb{R}^n$. Construct a function $f : \mathbb{R}^n \to \mathbb{R}$ that is discontinuous at each point of U and continuous at each point of $\mathbb{R}^n \setminus U$.*

Problem 4.1.22 (Sp82) *Prove or give a counterexample: Every connected, locally pathwise connected set in $\mathbb{R}^n$ is pathwise connected.*

Problem 4.1.23 (Sp81) *The set of real 3×3 symmetric matrices is a real, finite-dimensional vector space isomorphic to $\mathbb{R}^6$. Show that the subset of such matrices of signature $(2, 1)$ is an open connected subspace in the usual topology on $\mathbb{R}^6$.*

Problem 4.1.24 (Fa78) *Let $M_{n \times n}$ be the vector space of real $n \times n$ matrices, identified with $\mathbb{R}^{n^2}$. Let $X \subset M_{n \times n}$ be a compact set. Let $S \subset \mathbb{C}$ be the set of all numbers that are eigenvalues of at least one element of X. Prove that S is compact.*

Problem 4.1.25 (Su81) *Let $\mathbb{SO}(3)$ denote the group of orthogonal transformations of $\mathbb{R}^3$ of determinant 1. Let $Q \subset \mathbb{SO}(3)$ be the subset of symmetric transformations $\neq I$. Let P^2 denote the space of lines through the origin in $\mathbb{R}^3$.*

1. *Show that P^2 and $\mathbb{SO}(3)$ are compact metric spaces (in their usual topologies).*
2. *Show that P^2 and Q are homeomorphic.*

Problem 4.1.26 (Fa83) *Let m and n be positive integers, with $m < n$. Let $M_{m\times n}$ be the space of linear transformations of $\mathbb{R}^m$ into $\mathbb{R}^n$ (considered as $n \times m$ matrices) and let L be the set of transformations in $M_{m\times n}$ which have rank m.*

1. *Show that L is an open subset of $M_{m\times n}$.*
2. *Show that there is a continuous function $T : L \to M_{m\times n}$ such that $T(A)A = I_m$ for all A, where I_m is the identity on $\mathbb{R}^m$.*

Problem 4.1.27 (Fa91) *Let $M_{n\times n}$ be the space of real $n\times n$ matrices. Regard it as a metric space with the distance function*

$$d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}| \qquad (A = (a_{ij}), B = (b_{ij})) .$$

Prove that the set of nilpotent matrices in $M_{n\times n}$ is a closed set.

Problem 4.1.28 (Sp00) *Let S be an uncountable subset of $\mathbb{R}$. Prove that there exists a real number t such that both sets $S \cap (-\infty, t)$ and $S \cap (t, \infty)$ are uncountable.*

4.2 General Theory

Problem 4.2.1 (Sp02) *Let (X, ρ) and (Y, σ) be metric spaces. Assume that:*

1. *$f, f_1, f_2, \ldots$ are bijective functions of X onto Y with inverses $g, g_1, g_2, \ldots$*
2. *g is uniformly continuous*
3. *$f_n \to f$ uniformly as $n \to \infty$.*

Prove that $g_n \to g$ uniformly as $n \to \infty$.

Problem 4.2.2 (Fa99) *Let $E_1, E_2, \ldots$ be nonempty closed subsets of a complete metric space (X, d) with $E_{n+1} \subset E_n$ for all positive integers n, and such that $\lim\limits_{n\to\infty} \operatorname{diam}(E_n) = 0$, where $\operatorname{diam}(E)$ is defined to be*

$$\sup\{d(x, y) \mid x, y \in E\}.$$

Prove that $\bigcap\limits_{n=1}^{\infty} E_n \neq \emptyset$.

Problem 4.2.3 (Fa00) *Let A be a subset of a compact metric space (X, d). Assume that, for every continuous function $f : X \to \mathbb{R}$, the restriction of f to A attains a maximum on A. Prove that A is compact.*

Problem 4.2.4 (Fa93) *Let X be a metric space and (x_n) a convergent sequence in X with limit x_0. Prove that the set $C = \{x_0, x_1, x_2, \dots\}$ is compact.*

Problem 4.2.5 (Sp79) *Prove that every compact metric space has a countable dense subset.*

Problem 4.2.6 (Fa80) *Let X be a compact metric space and $f : X \to X$ an isometry. Show that $f(X) = X$.*

Problem 4.2.7 (Sp97) *Let M be a metric space with metric d. Let C be a nonempty closed subset of M. Define $f : M \to \mathbb{R}$ by*

$$f(x) = \inf\{d(x, y) \mid y \in C\}.$$

Show that f is continuous, and that $f(x) = 0$ if and only if $x \in C$.

Problem 4.2.8 (Su84) *Let $C^{1/3}$ be the set of real valued functions f on the closed interval $[0, 1]$ such that*

1. *$f(0) = 0$;*

2. *$\|f\|$ is finite, where by definition*

$$\|f\| = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|^{1/3}} \mid x \neq y\right\}.$$

Verify that $\|\cdot\|$ is a norm for the space $C^{1/3}$, and prove that $C^{1/3}$ is complete with respect to this norm.

Problem 4.2.9 (Sp00) *Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded equicontinuous sequence of real-valued functions on the compact metric space (X, d). Define the functions $g_n : X \to \mathbb{R}$, for $n \in \mathbb{N}$ by*

$$g_n(x) = \max\{f_1(x), \dots, f_n(x)\}.$$

Prove that the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly.

Problem 4.2.10 (Sp87) *Let $\mathcal{F}$ be a uniformly bounded, equicontinuous family of real valued functions on the metric space (X, d). Prove that the function*

$$g(x) = \sup\{f(x) \mid f \in \mathcal{F}\}$$

is continuous.

Problem 4.2.11 (Fa91) *Let X and Y be metric spaces and f a continuous map of X into Y. Let $K_1, K_2, \dots$ be nonempty compact subsets of X such that $K_{n+1} \subset K_n$ for all n, and let $K = \bigcap K_n$. Prove that $f(K) = \bigcap f(K_n)$.*

Problem 4.2.12 (Fa92) *Let* (X_1, d_1) *and* (X_2, d_2) *be metric spaces and* $f : X_1 \to X_2$ *a continuous surjective map such that* $d_1(p, q) \leqslant d_2(f(p), f(q))$ *for every pair of points* p, q *in* X_1.

1. *If* X_1 *is complete, must* X_2 *be complete? Give a proof or a counterexample.*
2. *If* X_2 *is complete, must* X_1 *be complete? Give a proof or a counterexample.*

4.3 Fixed Point Theorem

Problem 4.3.1 (Fa79) *An accurate map of California is spread out flat on a table in Evans Hall, in Berkeley. Prove that there is exactly one point on the map lying directly over the point it represents.*

Problem 4.3.2 (Fa87) *Define a sequence of positive numbers as follows. Let* $x_0 > 0$ *be any positive number, and let* $x_{n+1} = (1 + x_n)^{-1}$. *Prove that this sequence converges, and find its limit.*

Problem 4.3.3 (Su80) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be monotonically increasing (perhaps discontinuous). Suppose* $0 < f(0)$ *and* $f(100) < 100$. *Prove* $f(x) = x$ *for some* x.

Problem 4.3.4 (Su82, Sp95) *Let* K *be a nonempty compact set in a metric space with distance function* d. *Suppose that* φ: $K \to K$ *satisfies*

$$d(\varphi(x), \varphi(y)) < d(x, y)$$

for all $x \neq y$ *in* K. *Show there exists precisely one point* $x \in K$ *such that* $x = \varphi(x)$.

Problem 4.3.5 (Fa82) *Let* K *be a continuous function on the unit square* $0 \leqslant x, y \leqslant 1$ *satisfying* $|K(x, y)| < 1$ *for all* x *and* y. *Show that there is a continuous function* $f(x)$ *on* $[0, 1]$ *such that we have*

$$f(x) + \int_0^1 K(x, y) f(y)\, dy = e^{x^2}.$$

Can there be more than one such function f?

Problem 4.3.6 (Fa88) *Let* g *be a continuous real valued function on* $[0, 1]$. *Prove that there exists a continuous real valued function* f *on* $[0, 1]$ *satisfying the equation*

$$f(x) - \int_0^x f(x - t) e^{-t^2}\, dt = g(x).$$

Problem 4.3.7 (Su84) *Show there is a unique continuous real valued function* $f : [0, 1] \to \mathbb{R}$ *such that*

$$f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}}\, dy.$$

Problem 4.3.8 (Fa85, Sp98) *Let (M, d) be a nonempty complete metric space. Let S map M into M, and write S^2 for $S \circ S$; that is, $S^2(x) = S(S(x))$. Suppose that S^2 is a* strict contraction*; that is, there is a constant $\lambda < 1$ such that for all points $x, y \in M$, $d\left(S^2(x), S^2(y)\right) \leqslant \lambda d(x, y)$. Show that S has a unique fixed point in M.*

5

Complex Analysis

5.1 Complex Numbers

Problem 5.1.1 (Fa77) *If a and b are complex numbers and $a \neq 0$, the set a^b consists of those complex numbers c having a logarithm of the form $b\alpha$, for some logarithm α of a. (That is, $e^{b\alpha} = c$ and $e^{\alpha} = a$ for some complex number α.) Describe set a^b when $a = 1$ and $b = 1/3 + i$.*

Problem 5.1.2 (Su77) *Write all values of i^i in the form $a + bi$.*

Problem 5.1.3 (Sp85) *Show that a necessary and sufficient condition for three points a, b, and c in the complex plane to form an equilateral triangle is that*

$$a^2 + b^2 + c^2 = bc + ca + ab.$$

Problem 5.1.4 (Fa86) *Let the points a, b, and c lie on the unit circle of the complex plane and satisfy $a + b + c = 0$. Prove that a, b, and c form the vertices of an equilateral triangle.*

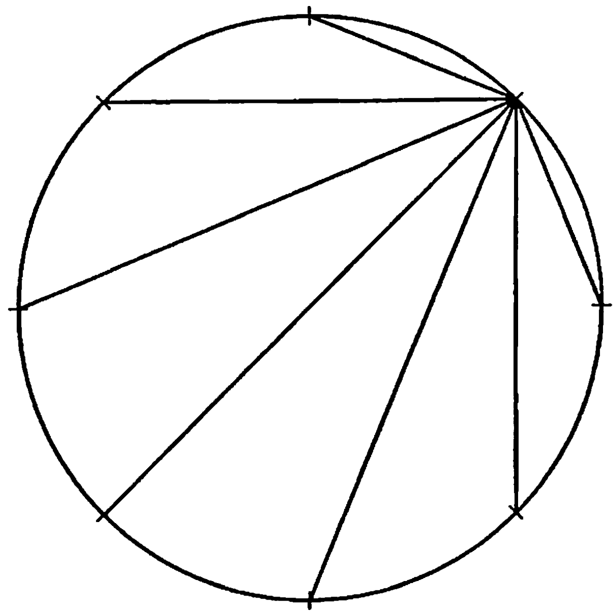
Problem 5.1.5 (Sp77) *1. Evaluate $P_{n-1}(1)$, where $P_{n-1}(x)$ is the polynomial*

$$P_{n-1}(x) = \frac{x^n - 1}{x - 1}.$$

2. *Consider a circle of radius 1, and let $Q_1, Q_2, \ldots, Q_n$ be the vertices of a regular n-gon inscribed in the circle. Join Q_1 to $Q_2, Q_3, \ldots, Q_n$ by segments of a straight line. You obtain $(n-1)$ segments of lengths $\lambda_2, \lambda_3, \ldots, \lambda_n$.*

Show that

$$\prod_{i=2}^{n} \lambda_i = n.$$



Problem 5.1.6 (Sp03) *Prove that for each integer $n \geqslant 0$ there is a polynomial $T_n(x)$ with integer coefficients such that the identity*

$$2\cos nz = T_n(2\cos z)$$

holds for all z.

Problem 5.1.7 (Sp90) *Let $z_1, z_2, \ldots, z_n$ be complex numbers. Prove that there exists a subset $J \subset \{1, 2, \ldots, n\}$ such that*

$$\left|\sum_{j\in J} z_j\right| \geqslant \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j|.$$

Problem 5.1.8 (Sp94) *Let $a_1, a_2, \ldots, a_n$ be complex numbers. Prove that there is a point x in $[0, 1]$ such that*

$$\left|1 - \sum_{k=1}^{n} a_k e^{2\pi i k x}\right| \geqslant 1.$$

Problem 5.1.9 (Fa82) *Let a and b be complex numbers whose real parts are negative or 0. Prove the inequality $|e^a - e^b| \leqslant |a - b|$.*

Problem 5.1.10 (Fa95) *Let A be a finite subset of the unit disc in the plane, and let $N(A, r)$ be the set of points at distance $\leqslant r$ from A, where $0 < r < 1$. Show* *that the length of the boundary $N(A, r)$ is, at most, C/r for some constant C independent of A.*

Problem 5.1.11 (Su82) *For complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, prove*

$$\limsup_n \left|\sum_{j=1}^{k} \alpha_j^n\right|^{1/n} = \sup_j |\alpha_j|.$$

Note: See also Problem 1.3.1.

5.2 Series and Sequences of Functions

Problem 5.2.1 (Fa95) *Show that*

$$(1+z+z^2+\cdots+z^9)(1+z^{10}+z^{20}+\cdots+z^{90})(1+z^{100}+z^{200}+\cdots+z^{900})\cdots=\frac{1}{1-z}$$

for $|z| < 1$.

Problem 5.2.2 (Fa94) *Suppose the coefficients of the power series*

$$\sum_{n=0}^{\infty} a_n z^n$$

are given by the recurrence relation

$$a_0 = 1,\ a_1 = -1,\ 3a_n + 4a_{n-1} - a_{n-2} = 0,\ \ n = 2, 3, \ldots.$$

Find the radius of convergence of the series and the function to which it converges in its disc of convergence.

Problem 5.2.3 (Fa03) *Show that the differential equation*

$$f''(z) = zf(z), \qquad f(0) = 1, \qquad f'(0) = 1$$

has an unique entire solution in the complex plane.

Problem 5.2.4 (Fa93) *Describe the region in the complex plane where the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(\frac{nz}{z-2}\right)$$

converges. Draw a sketch of the region.

Problem 5.2.5 (Su77) *Let f be an analytic function such that*

$$f(z) = 1 + 2z + 3z^2 + \cdots \quad \text{for } |z| < 1.$$

Define a sequence of real numbers $a_0, a_1, a_2, \ldots$ by

$$f(z) = \sum_{n=0}^{\infty} a_n (z+2)^n.$$

What is the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n ?$$

Problem 5.2.6 (Sp77) *Let the sequence $a_0, a_1, \ldots$ be defined by the equation*

$$1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} a_n (x-3)^n \quad (0 < x < 1).$$

Find

$$\limsup_{n\to\infty} \left(|a_n|^{\frac{1}{n}} \right).$$

Problem 5.2.7 (Su78) *Suppose the power series*

$$\sum_{n=0}^{\infty} a_n z^n$$

converges for $|z| < R$ where z and the a_n are complex numbers. If $b_n \in \mathbb{C}$ are such that $|b_n| < n^2 |a_n|$ for all n, prove that

$$\sum_{n=0}^{\infty} b_n z^n$$

converges for $|z| < R$.

Problem 5.2.8 (Sp99) *Let $b_1, b_2, \ldots$ be a sequence of real numbers such that $b_k \geqslant b_{k+1}$ for all k and $\lim_{k\to\infty} b_k = 0$. Prove that the power series $\sum_{k=1}^{\infty} b_k z^k$ converges for all complex numbers z such that $|z| \leqslant 1$ and $z \neq 1$.*

Problem 5.2.9 (Sp79) *For which $z \in \mathbb{C}$ does*

$$\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n} \right)$$

converge?

Problem 5.2.10 (Su79) *Show that*

$$\sum_{n=0}^{\infty} \frac{z}{(1+z^2)^n}$$

converges for all complex numbers z exterior to the lemniscate

$$\left|1+z^2\right| = 1.$$

Problem 5.2.11 (Su82) *Determine the complex numbers z for which the power series*

$$\sum_{n=1}^{\infty} \frac{z^n}{n^{\log n}}$$

and its term by term derivatives of all orders converge absolutely.

Problem 5.2.12 (Su84) *Suppose*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$. Show that

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

is entire and that for $0 < r < R$, there is a constant M such that

$$|h(z)| \leqslant M e^{|z|/r}.$$

Problem 5.2.13 (Sp01) *Let the power series $\sum_{n=0}^{\infty} c_n z^n$, with positive radius of convergence R, represent the function f in the disk $|z| < R$. For $k = 0, 1, \ldots$ let s_k be the k-th partial sum of the series $s_k(z) = \sum_{n=0}^{k} c_n z^n$. Prove that*

$$\sum_{k=0}^{\infty} |f(z) - s_k(z)| < \infty$$

for each z in the disk $|z| < R$.

Problem 5.2.14 (Sp85) *Let $R > 1$ and let f be analytic on $|z| < R$ except at $z = 1$, where f has a simple pole. If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1)$$

is the Maclaurin series for f, show that $\lim_{n\to\infty} a_n$ exists.

Problem 5.2.15 (Fa95) *Find the radius of convergence R of the Taylor series about $z = 1$ of the function $f(z) = 1/(1 + z^2 + z^4 + z^6 + z^8 + z^{10})$. Express the answer in terms of real numbers and square roots only.*

Problem 5.2.16 (Sp78) *Prove that the uniform limit of a sequence of complex analytic functions is complex analytic. Is the analogous theorem true for real analytic functions?*

Problem 5.2.17 (Su79) *Let $g_n(z)$ be an entire function having only real zeros, $n = 1, 2, \ldots$. Suppose*

$$\lim_{n\to\infty} g_n(z) = g(z)$$

uniformly on compact sets in $\mathbb{C}$, with g not identically zero. Prove that $g(z)$ has only real zeros.

Problem 5.2.18 (Sp86) *Let $f, g_1, g_2, \ldots$ be entire functions. Assume that*

1. $|g_n^{(k)}(0)| \leqslant |f^{(k)}(0)|$ *for all n and k;*
2. $\lim_{n\to\infty} g_n^{(k)}(0)$ *exists for all k.*

Prove that the sequence $\{g_n\}$ converges uniformly on compact sets and that its limit is an entire function.

Problem 5.2.19 (Sp92) *Find a Laurent series that converges in the annulus $1 < |z| < 2$ to a branch of the function $\log\left(\dfrac{z(2-z)}{1-z}\right)$.*

5.3 Conformal Mappings

Problem 5.3.1 (Fa77) *Consider the following four types of transformations:*

$$z \mapsto z + b, \quad z \mapsto 1/z, \quad z \mapsto kz \quad (\textit{where } k \neq 0),$$

$$z \mapsto \frac{az+b}{cz+d} \quad (\textit{where } ad - bc \neq 0).$$

Here, z is a variable complex number and the other letters denote constant complex numbers. Show that each transformation takes circles to either circles or straight lines.

Problem 5.3.2 (Fa78) *Give examples of conformal maps as follows:*

1. *from $\{z \mid |z| < 1\}$ onto $\{z \mid \Re z < 0\}$,*
2. *from $\{z \mid |z| < 1\}$ onto itself, with $f(0) = 0$ and $f(1/2) = i/2$,*
3. *from $\{z \mid z \neq 0, 0 < \arg z < \frac{3\pi}{2}\}$ onto $\{z \mid z \neq 0, 0 < \arg z < \frac{\pi}{2}\}$.*

Problem 5.3.3 (Sp83) *A fractional linear transformation maps the annulus $r < |z| < 1$ (where $r > 0$) onto the domain bounded by the two circles $|z-\frac{1}{4}| = \frac{1}{4}$ and $|z| = 1$. Find r.*

Problem 5.3.4 (Sp80) *Does there exist an analytic function mapping the annulus*

$$A = \{z \mid 1 \leqslant |z| \leqslant 4\}$$

onto the annulus

$$B = \{z \mid 1 \leqslant |z| \leqslant 2\}$$

and taking $C_1 \to C_1$, $C_4 \to C_2$, where C_r is the circle of radius r?

Problem 5.3.5 (Su80) *Exhibit a conformal map from $\{z \in \mathbb{C} \mid |z| < 1, \Re z > 0\}$ onto $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.*

Problem 5.3.6 (Sp90) *Find a one-to-one conformal map of the semidisc*

$$\left\{z \in \mathbb{C} \mid \Im z > 0, \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}$$

onto the upper half-plane.

Problem 5.3.7 (Fa97) *Conformally map the region inside the disc given by $\{z \in \mathbb{C} \mid |z - 1| \leqslant 1\}$ and outside the disc $\{z \in \mathbb{C} \mid |z - \frac{1}{2}| \leqslant \frac{1}{2}\}$ onto the upper half-plane.*

Problem 5.3.8 (Sp95) *Prove that there is no one-to-one conformal map of the punctured disc $G = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ onto the annulus $A = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$.*

Problem 5.3.9 (Fa02) *Let n be a positive integer. Find a group of linear fractional transformations of $\mathbb{C}$ that fixes the two points $z = 1$ and $z = -1$ and has order n.*

5.4 Functions on the Unit Disc

Problem 5.4.1 (Fa02) *Let α be a number in $(0, \pi/2)$. Prove that the function $f(z) = e^{-1/z}$ is uniformly continuous in $S_\alpha = \{z \mid 0 < |z| \leqslant 1, |\operatorname{Arg} z| \leqslant \alpha\}$, a sector of the complex plane.*

Problem 5.4.2 (Fa82) *Let a and b be nonzero complex numbers and $f(z) = az + bz^{-1}$. Determine the image under f of the unit circle $\{z \mid |z| = 1\}$.*

Problem 5.4.3 (Su83, Fa96) *Let f be analytic on and inside the unit circle $C = \{z \mid |z| = 1\}$. Let L be the length of the image of C under f. Show that $L \geqslant 2\pi|f'(0)|$.*

Problem 5.4.4 (Sp80) *Let*

$$f(z)=\sum_{n=0}^{\infty} c_n z^n$$

be analytic in the disc $\mathbb{D}=\{z\in\mathbb{C} \mid |z|<1\}$. *Assume* f *maps* $\mathbb{D}$ *one-to-one onto a domain* G *having area* A. *Prove*

$$A=\pi\sum_{n=1}^{\infty} n|c_n|^2.$$

Problem 5.4.5 (Su83) *Compute the area of the image of the unit disc* $\{z \mid |z|<1\}$ *under the map* $f(z)=z+z^2/2$.

Problem 5.4.6 (Sp80) *Let*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

be an analytic function in the open unit disc $\mathbb{D}$. *Assume that*

$$\sum_{n=2}^{\infty} n|a_n| \leqslant |a_1| \quad \textit{with} \quad a_1\neq 0.$$

Prove that f *is injective.*

Problem 5.4.7 (Sp03) *Let* $f(z)$ *be a function that is analytic in the unit disc* $\mathbb{D}=\{|z|<1\}$. *Suppose that* $|f(z)|\leqslant 1$ *in* $\mathbb{D}$. *Prove that if* $f(z)$ *has at least two fixed points* z_1 *and* z_2, *then* $f(z)=z$ *for all* $z\in\mathbb{D}$.

Problem 5.4.8 (Su85) *For each* $k>0$, *let* X_k *be the set of analytic functions* $f(z)$ *on the open unit disc* $\mathbb{D}$ *such that*

$$\sup_{z\in\mathbb{D}}\left\{(1-|z|)^k\,|f(z)|\right\}$$

is finite. Show that $f\in X_k$ *if and only if* $f'\in X_{k+1}$.

Problem 5.4.9 (Sp88) *Let the function* f *be analytic in the open unit disc of the complex plane and real valued on the radii* $[0,1)$ *and* $[0,e^{i\pi\sqrt{2}})$. *Prove that* f *is constant.*

Problem 5.4.10 (Fa91) *Let the function* f *be analytic in the disc* $|z|<1$ *of the complex plane. Assume that there is a positive constant* M *such that*

$$\int_0^{2\pi}|f'(re^{i\theta})|\,d\theta\leqslant M, \qquad (0\leqslant r<1).$$

Prove that

$$\int_{[0,1)}|f(x)|\,dx<\infty.$$

Problem 5.4.11 (Fa78) *Suppose $h(z)$ is analytic in the whole plane, $h(0) = 3 + 4i$, and $|h(z)| \leqslant 5$ if $|z| < 1$. What is $h'(0)$?*

Problem 5.4.12 (Fa98) *Let f be analytic in the closed unit disc, with $f(-\log 2) = 0$ and $|f(z)| \leqslant |e^z|$ for all z with $|z| = 1$. How large can $|f(\log 2)|$ be?*

Problem 5.4.13 (Fa79, Fa90) *Suppose that f is analytic on the open upper half-plane and satisfies $|f(z)| \leqslant 1$ for all z, $f(i) = 0$. How large can $|f(2i)|$ be under these conditions?*

Problem 5.4.14 (Fa85) *Let $f(z)$ be analytic on the right half-plane $H = \{z \mid \Re z > 0\}$ and suppose $|f(z)| \leqslant 1$ for $z \in H$. Suppose also that $f(1) = 0$. What is the largest possible value of $|f'(1)|$?*

Problem 5.4.15 (Su82) *Let $f(z)$ be analytic on the open unit disc $\mathbb{D}=\{z \mid |z|<1\}$. Prove that there is a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$ and $(f(z_n))$ is bounded.*

Problem 5.4.16 (Sp93) *Let f be an analytic function in the unit disc, $|z| < 1$.*

1. *Prove that there is a sequence (z_n) in the unit disc with $\lim_{n\to\infty} |z_n| = 1$ and $\lim_{n\to\infty} f(z_n)$ exists (finitely).*
2. *Assume f nonconstant. Prove that there are two sequences (z_n) and (w_n) in the disc such that $\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |w_n| = 1$, and such that both limits $\lim_{n\to\infty} f(z_n)$ and $\lim_{n\to\infty} f(w_n)$ exist (finitely) and are not equal.*

Problem 5.4.17 (Fa81, Sp89, Fa97) *Let f be a holomorphic map of the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$ into itself, which is not the identity map $f(z) = z$. Show that f can have, at most, one fixed point.*

Problem 5.4.18 (Fa87) *If $f(z)$ is analytic in the open disc $|z| < 1$, and $|f(z)| \leqslant 1/(1 - |z|)$, show that*

$$|a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leqslant (n+1)\left(1+\frac{1}{n}\right)^n < e(n+1).$$

Problem 5.4.19 (Sp88)

1. *Let f be an analytic function that maps the open unit disc, $\mathbb{D}$, into itself and vanishes at the origin. Prove that $|f(z) + f(-z)| \leqslant 2|z|^2$ in $\mathbb{D}$.*
2. *Prove that the inequality in Part 1 is strict, except at the origin, unless f has the form $f(z) = \lambda z^2$ with λ a constant of absolute value one.*

Problem 5.4.20 (Sp91) *Let the function f be analytic in the unit disc, with $|f(z)| \leqslant 1$ and $f(0) = 0$. Assume that there is a number r in $(0, 1)$ such that $f(r) = f(-r) = 0$. Prove that*

$$|f(z)| \leqslant |z|\left|\frac{z^2 - r^2}{1 - r^2 z^2}\right|.$$

Problem 5.4.21 (Sp85) *Let $f(z)$ be an analytic function that maps the open disc $|z| < 1$ into itself. Show that $|f'(z)| \leqslant 1/(1-|z|^2)$.*

Problem 5.4.22 (Sp87, Fa89) *Let f be an analytic function in the open unit disc of the complex plane such that $|f(z)| \leqslant C/(1-|z|)$ for all z in the disc, where C is a positive constant. Prove that $|f'(z)| \leqslant 4C/(1-|z|)^2$.*

5.5 Growth Conditions

Problem 5.5.1 (Sp03) *Let f be an entire function such that $\Re f(z) \geqslant -2$ for all $z \in \mathbb{C}$. Show that f is constant.*

Problem 5.5.2 (Fa90) *Let the function f be analytic in the entire complex plane, and suppose that $f(z)/z \to 0$ as $|z| \to \infty$. Prove that f is constant.*

Problem 5.5.3 (Fa99) *Let the rational function f in the complex plane have no poles for $\Im z \geqslant 0$. Prove that*

$$\sup\{|f(z)| \mid \Im z \geqslant 0\} = \sup\{|f(z)| \mid \Im z = 0\}.$$

Problem 5.5.4 (Fa97) *Let f be an entire function such that, for all z, $|f(z)| = |\sin z|$. Prove that there is a constant C of modulus 1 such that $f(z) = C \sin z$.*

Problem 5.5.5 (Fa79, Su81) *Suppose f and g are entire functions with $|f(z)| \leqslant |g(z)|$ for all z. Prove that $f(z) = cg(z)$ for some constant c.*

Problem 5.5.6 (Sp02) *Let p and q be nonconstant complex polynomials of the same degree whose zeros lie in the open disk $|z| < 1$. Prove that if $|p(z)| = |q(z)|$ for $|z| = 1$ then $q = \lambda p$ for a unimodular constant λ.*

Problem 5.5.7 (Fa98) *Let f be an entire function. Define $\Omega = \mathbb{C} \setminus (-\infty, 0]$, the complex plane with the ray $(-\infty, 0]$ removed. Suppose that for all $z \in \Omega$, $|f(z)| \leqslant |\log z|$, where $\log z$ is the principal branch of the logarithm. What can one conclude about the function f?*

Problem 5.5.8 (Sp97) *Let f and g be two entire functions such that, for all $z \in \mathbb{C}$, $\Re f(z) \leqslant k \Re g(z)$ for some real constant k (independent of z). Show that there are constants a, b such that*

$$f(z) = ag(z) + b.$$

Problem 5.5.9 (Sp02) *Let the continuous real-valued function φ on the complex plane satisfy the Sub-Mean-Value Property: for any point z_0,*

$$\varphi(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0 + re^{i\theta})\,d\theta \qquad (0 < r < 1).$$

Prove that φ obeys the Maximum Modulus Principle: the maximum of φ over any compact set K is attained on the boundary of K.

Problem 5.5.10 (Su78) *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $a > 0$ and $b > 0$ be constants.*

1. *If $|f(z)| \leqslant a\sqrt{|z|} + b$ for all z, prove that f is a constant.*
2. *What can one prove about f if*

$$|f(z)| \leqslant a|z|^{5/2} + b$$

for all z?

Problem 5.5.11 (Fa90) *Let the function f be analytic in the entire complex plane and satisfy*

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leqslant r^{17/3}$$

for all $r > 0$. Prove that f is the zero function.

Problem 5.5.12 (Fa96) *Does there exist a function f, analytic in the punctured plane $\mathbb{C} \setminus \{0\}$, such that*

$$|f(z)| \geqslant \frac{1}{\sqrt{|z|}}$$

for all nonzero z?

Problem 5.5.13 (Fa91) *Let the function f be analytic in the entire complex plane and satisfy the inequality $|f(z)| \leqslant |\Re z|^{-1/2}$ off the imaginary axis. Prove that f is constant.*

Problem 5.5.14 (Fa01) *Let the nonconstant entire function f satisfy the conditions*

1. *$f(0) = 0$;*
2. *for each $M > 0$, the set $\{z \mid |f(z)| < M\}$ is connected.*

Prove that $f(z) = cz^n$ for some constant c and positive integer n.

5.6 Analytic and Meromorphic Functions

Problem 5.6.1 (Sp96) *Let $f = u + iv$ be analytic in a connected open set D, where u and v are real valued. Suppose there are real constants a, b and c such that $a^2 + b^2 \neq 0$ and*

$$au + bv = c$$

in D. Show that f is constant in D.

Problem 5.6.2 (Sp88) *True or false: A function $f(z)$ analytic on $|z - a| < r$ and continuous on $|z - a| \leqslant r$ extends, for some $\delta > 0$, to a function analytic on $|z - a| < r + \delta$? Give a proof or a counterexample.*

Problem 5.6.3 (Fa80) *Do there exist functions $f(z)$ and $g(z)$ that are analytic at $z = 0$ and that satisfy*

1. $f(1/n) = f(-1/n) = 1/n^2$, $n = 1, 2, \dots$,
2. $g(1/n) = g(-1/n) = 1/n^3$, $n = 1, 2, \dots$?

Problem 5.6.4 (Fa02) *Let k be an integer larger than 1. Find all entire functions f that satisfy $f(z^k) = (f(z))^k$.*

Problem 5.6.5 (Fa01) *Let F be a polynomial over $\mathbb{C}$ of positive degree d, and let S be its zero set. Prove that every rational function R whose finite poles lie in S can be written uniquely as*

$$R = \sum_{k=m}^{n} a_k F^k ,$$

where m and n are integers, $m \leq n$, and the a_k are polynomials whose degrees are less than d.

Problem 5.6.6 (Fa99) *Let $A = \{0\} \cup \{1/n \mid n \in \mathbb{Z}, n > 1\}$, and let $\mathbb{D}$ be the open unit disc in the complex plane. Prove that every bounded holomorphic function on $\mathbb{D} \setminus A$ extends to a holomorphic function on $\mathbb{D}$.*

Problem 5.6.7 (Fa00) *Let U be a connected and simply connected open subset of the complex plane, and let f be a holomorphic function on U. Suppose a is a point of U such that the Taylor series of f at a converges on an open disc D that intersects the complement of U. Does it follow that f extends to a holomorphic function on $U \cup D$? Give a proof or a counterexample.*

Problem 5.6.8 (Su78)

1. *Suppose f is analytic on a connected open set $U \subset \mathbb{C}$ and f takes only real values. Prove that f is constant.*
2. *Suppose $W \subset \mathbb{C}$ is open, g is analytic on W, and $g'(z) \neq 0$ for all $z \in W$. Show that*

$$\{\Re g(z) + \Im g(z) \mid z \in W\} \subset \mathbb{R}$$

is an open subset of $\mathbb{R}$.

Problem 5.6.9 (Sp78) *Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.*

Problem 5.6.10 (Fa03) *Let L be a line in $\mathbb{C}$, and let f be an entire function such that $f(\mathbb{C}) \cap L = \emptyset$. Prove that f is constant. (Do not use the theorem of Picard that the image of a nonconstant entire function omits at most one complex number.)*

Problem 5.6.11 (Sp01)

1. *Prove that an entire function with a positive real part is constant.*

2. *Prove the analogous result for* 2×2 *matrix functions: If* $F(z) = (f_{jk}(z))$ *is a matrix function in the complex plane, each entry* f_{jk} *being entire, and if* $F(z) + F(z)^*$ *is positive definite for each* z*, then* F *is constant. (Here,* $F(z)^*$ *is the conjugate transpose of* $F(z)$*.)*

Problem 5.6.12 (Su82) *Let* $s(y)$ *and* $t(y)$ *be real differentiable functions of* y*,* $-\infty < y < \infty$*, such that the complex function*

$$f(x+iy) = e^x\,(s(y) + it(y))$$

is complex analytic with $s(0) = 1$ *and* $t(0) = 0$*. Determine* $s(y)$ *and* $t(y)$*.*

Problem 5.6.13 (Sp83) *Determine all the complex analytic functions* f *defined on the unit disc* $\mathbb{D}$ *which satisfy*

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0$$

for $n = 2, 3, 4, \ldots$.

Problem 5.6.14 (Fa00) *Assume the nonconstant entire function* f *takes real values on two intersecting lines in the complex plane. Prove that the measure of either angle formed by the lines is a rational multiple of* π*.*

Problem 5.6.15 (Su83) *Let* Ω *be an open subset of* $\mathbb{R}^2$*, and let* $f : \Omega \to \mathbb{R}^2$ *be a smooth map. Assume that* f *preserves orientation and maps any pair of orthogonal curves to a pair of orthogonal curves. Show that* f *is holomorphic. Note: Here we identify* $\mathbb{R}^2$ *with* $\mathbb{C}$*.*

Problem 5.6.16 (Sp87) *Let* f *be a complex valued function in the open unit disc,* $\mathbb{D}$*, of the complex plane such that the functions* $g = f^2$ *and* $h = f^3$ *are both analytic. Prove that* f *is analytic in* $\mathbb{D}$*.*

Problem 5.6.17 (Fa84) *Prove or supply a counterexample: If* f *is a continuous complex valued function defined on a connected open subset of the complex plane and if* f^2 *is analytic, then* f *is analytic.*

Problem 5.6.18 (Sp88) 1. *Let* G *be an open connected subset of the complex plane,* f *an analytic function in* G*, not identically* 0*, and* n *a positive integer. Assume that* f *has an analytic* n^{th} *root in* G*; that is, there is an analytic function* g *in* G *such that* $g^n = f$*. Prove that* f *has exactly* n *analytic* n^{th} *roots in* G*.*

2. *Give an example of a continuous real valued function on* $[0, 1]$ *that has more than two continuous square roots on* $[0, 1]$*.*

Problem 5.6.19 (Sp99) 1. *Prove that if* f *is holomorphic on the unit disc* $\mathbb{D}$ *and* $f(z) \neq 0$ *for all* $z \in \mathbb{D}$*, then there is a holomorphic function* g *on* $\mathbb{D}$ *such that* $f(z) = e^{g(z)}$ *for all* $z \in \mathbb{D}$*.*

2. *Does the conclusion of Part 1 remain true if $\mathbb{D}$ is replaced by an arbitrary connected open set in $\mathbb{C}$?*

Problem 5.6.20 (Fa92) *Let the function f be analytic in the region $|z| > 1$ of the complex plane. Prove that if f is real valued on the interval $(1, \infty)$ of the real axis, then f is also real valued on the interval $(-\infty, -1)$.*

Problem 5.6.21 (Fa94) *Let the function f be analytic in the complex plane, real on the real axis, 0 at the origin, and not identically 0. Prove that if f maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.*

Problem 5.6.22 (Fa01) *Let the entire function f be real valued on the lines $\Im z = 0$ and $\Im z = \pi$. Prove that f has $2\pi i$ as a period: $f(z + 2\pi i) = f(z)$ for all z.*

Problem 5.6.23 (Fa87) *Let $f(z)$ be analytic for $z \neq 0$, and suppose that $f(1/z) = f(z)$. Suppose also that $f(z)$ is real for all z on the unit circle $|z| = 1$. Prove that $f(z)$ is real for all real $z \neq 0$.*

Problem 5.6.24 (Fa91) *Let p be a nonconstant complex polynomial whose zeros are all in the half-plane $\Im z > 0$.*

1. *Prove that $\Im(p'/p) > 0$ on the real axis.*

2. *Find a relation between* deg p *and*

$$\int_{-\infty}^{\infty} \Im \frac{p'(x)}{p(x)}\, dx .$$

Problem 5.6.25 (Sp92) *Let f be an analytic function in the connected open subset G of the complex plane. Assume that for each point z in G, there is a positive integer n such that the n^{th} derivative of f vanishes at z. Prove that f is a polynomial.*

Problem 5.6.26 (Sp92) *Let the function f be analytic in the entire complex plane, real valued on the real axis, and of positive imaginary part in the upper half-plane. Prove $f'(x) > 0$ for x real.*

Problem 5.6.27 (Sp94)

1. *Let U and V be open connected subsets of the complex plane, and let f be an analytic function in U such that $f(U) \subset V$. Assume $f^{-1}(K)$ is compact whenever K is a compact subset of V. Prove that $f(U) = V$.*

2. *Prove that the last equality can fail if* analytic *is replaced by* continuous *in the preceding statement.*

Problem 5.6.28 (Sp94) *Let $f = u+iv$ and $g = p+iq$ be analytic functions defined in a neighborhood of the origin in the complex plane. Assume $|g'(0)| < |f'(0)|$. Prove that there is a neighborhood of the origin in which the function $h = f + \overline{g}$ is one-to-one.*

Problem 5.6.29 (Sp87) *Prove or disprove: If the function f is analytic in the entire complex plane, and if f maps every unbounded sequence to an unbounded sequence, then f is a polynomial.*

Problem 5.6.30 (Fa88, Sp97) *Determine the group* $\mathrm{Aut}(\mathbb{C})$ *of all one-to-one analytic maps of $\mathbb{C}$ onto $\mathbb{C}$.*

Problem 5.6.31 (Sp77) *Let $f(z)$ be a nonconstant meromorphic function. A complex number w is called a period of f if $f(z+w) = f(z)$ for all z.*

1. *Show that if w_1 and w_2 are periods, so are $n_1w_1 + n_2w_2$ for all integers n_1 and n_2.*

2. *Show that there are, at most, a finite number of periods of f in any bounded region of the complex plane.*

Problem 5.6.32 (Sp91) *Let the function f be analytic in the punctured disc $0 < |z| < r_0$, with Laurent series*

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n.$$

Assume there is a positive number M such that

$$r^4 \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < M, \qquad 0 < r < r_0.$$

Prove that $c_n = 0$ for $n < -2$.

Problem 5.6.33 (Sp98) *Let $a > 0$. Show that the complex function*

$$f(z) = \frac{1 + z + az^2}{1 - z + az^2}$$

satisfies $|f(z)| < 1$ for all z in the open left half-plane $\Re z < 0$.

Problem 5.6.34 (Fa93) *Let f be a continuous real valued function on $[0, 1]$, and let the function h in the complex plane be defined by*

$$h(z) = \int_0^1 f(t) \cos zt \, dt.$$

1. *Prove that h is analytic in the entire plane.*

2. *Prove that h is the zero function only if f is the zero function.*

Problem 5.6.35 (Su79, Sp82, Sp91, Sp96) *Let f be a continuous complex valued function on $[0, 1]$, and define the function g by*

$$g(z) = \int_0^1 f(t)e^{tz}\,dt \qquad (z \in \mathbb{C}).$$

Prove that g is analytic in the entire complex plane.

Problem 5.6.36 (Fa84, Fa95) *Let f and g be analytic functions in the open unit disc, and let C_r denote the circle with center 0 and radius r, oriented counterclockwise.*

1. *Prove that the integral*

$$\frac{1}{2\pi i}\int_{C_r} \frac{1}{w} f(w) g\left(\frac{z}{w}\right) dw$$

is independent of r as long as $|z| < r < 1$ and that it defines an analytic function $h(z)$, $|z| < 1$.

2. *Prove or supply a counterexample: If $f \not\equiv 0$ and $g \not\equiv 0$, then $h \not\equiv 0$.*

Problem 5.6.37 (Sp84) *Let F be a continuous complex valued function on the interval $[0, 1]$. Let*

$$f(z) = \int_0^1 \frac{F(t)}{t - z}\,dt,$$

for z a complex number not in $[0, 1]$.

1. *Prove that f is an analytic function.*

2. *Express the coefficients of the Laurent series of f about ∞ in terms of F. Use the result to show that F is uniquely determined by f.*

Problem 5.6.38 (Fa03) *Let $f(z)$ be a meromorphic function on the complex plane. Suppose that for every polynomial $p(z) \in \mathbb{C}[z]$ and every closed contour Γ avoiding the poles of f, we have*

$$\int_\Gamma p(z)^2 f(z)\,dz = 0.$$

Prove that $f(z)$ is entire.

5.7 Cauchy's Theorem

Problem 5.7.1 (Fa85) *Evaluate*

$$\int_0^{2\pi} e^{e^{i\theta}}\,d\theta\,.$$

Problem 5.7.2 (Su78) *Evaluate*

$$\int_0^{2\pi} e^{(e^{i\theta}-i\theta)}\, d\theta\,.$$

Problem 5.7.3 (Sp98) *Let a be a complex number with $|a| < 1$. Evaluate the integral*

$$\int_{|z|=1} \frac{|dz|}{|z-a|^2}$$

Problem 5.7.4 (Fa99) *For $0 < a < b$, evaluate the integral*

$$I = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|ae^{i\theta}-b|^4}\,d\theta\,\cdot$$

Problem 5.7.5 (Sp77, Sp82) *Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial with complex coefficients has a complex root.*

Problem 5.7.6 (Su77) *Let f be continuous on $\mathbb{C}$ and analytic on $\{z \mid \Im z \neq 0\}$. Prove that f must be analytic on $\mathbb{C}$.*

Problem 5.7.7 (Fa00) *Let $f(z)$ be the rational function $p(z)/q(z)$, where $p(z)$ and $q(z)$ are nonzero polynomials with complex coefficients, such that the degree of $p(z)$ is less than the degree of $q(z)$, and such that $q(z)$ has no complex zeros with nonnegative imaginary part. Prove that if z_0 is a complex number with positive imaginary part, then*

$$f(z_0) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{t-z_0}\,dt.$$

Problem 5.7.8 (Fa78, Su79) *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree $n > 0$. Prove*

$$\frac{1}{2\pi i}\int_{|z|=R} z^{n-1}|f(z)|^2\,dz = a_0\bar{a}_n R^{2n}.$$

Problem 5.7.9 (Sp01) *Let f be an entire function such that*

$$\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta \leqslant Ar^{2k} \qquad (0 < r < \infty)\,,$$

where k is a positive integer and A is a positive constant. Prove that f is a constant multiple of the function z^k.

Problem 5.7.10 (Fa95) *Let $f(z) = u(z) + iv(z)$ be holomorphic on $|z| < 1$, u and v real. Show that*

$$\int_0^{2\pi} u(re^{i\theta})^2 d\theta = \int_0^{2\pi} v(re^{i\theta})^2 d\theta$$

for $0 < r < 1$ if $u(0)^2 = v(0)^2$.

Problem 5.7.11 (Su83) *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that*

$$\left(1 + |z|^k\right)^{-1} \frac{d^m f}{dz^m}$$

is bounded for some k and m. Prove that $d^n f/dz^n$ is identically zero for sufficiently large n. How large must n be, in terms of k and m?

Problem 5.7.12 (Su83) *Suppose Ω is a bounded domain in $\mathbb{C}$ with a boundary consisting of a smooth Jordan curve γ. Let f be holomorphic on a neighborhood of the closure of Ω, and suppose that $f(z) \neq 0$ for $z \in \gamma$. Let $z_1, \ldots, z_k$ be the zeros of f in Ω, and let n_j be the order of the zero of f at z_j (for $j = 1, \ldots, k$).*

1. *Use Cauchy's integral formula to show that*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{k} n_j .$$

2. *Suppose that f has only one zero z_1 in Ω with multiplicity $n_1 = 1$. Find a boundary integral involving f whose value is the point z_1.*

Problem 5.7.13 (Fa88) *Let f be an analytic function on a disc D whose center is the point z_0. Assume that $|f'(z) - f'(z_0)| < |f'(z_0)|$ on D. Prove that f is one-to-one on D.*

Problem 5.7.14 (Fa89) *Let $f(z)$ be analytic in the annulus $\Omega = \{1 < |z| < 2\}$. Assume that f has no zeros in Ω. Show that there exists an integer n and an analytic function g in Ω such that, for all $z \in \Omega$, $f(z) = z^n e^{g(z)}$.*

Problem 5.7.15 (Sp02) *Define the function f on $\mathbb{C} \setminus [0, 1]$ by*

$$f(z) = \int_0^1 \frac{\sqrt{t}}{t - z}\, dt .$$

Prove that f is analytic, and find its Laurent series about ∞.

Problem 5.7.16 (Sp90) *Let the function f be analytic and bounded in the complex half-plane $\Re z > 0$. Prove that for any positive real number c, the function f is uniformly continuous in the half-plane $\Re z > c$.*

5.8 Zeros and Singularities

Problem 5.8.1 (Fa77, Fa96) *Let $\mathbb{C}^3$ denote the set of ordered triples of complex numbers. Define a map $F : \mathbb{C}^3 \to \mathbb{C}^3$ by*

$$F(u, v, w) = (u + v + w, uv + vw + wu, uvw).$$

Prove that F is onto but not one-to-one.

Problem 5.8.2 (Fa79, Fa89) *Prove that the polynomial*

$$p(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$$

has at least one root in the disc $|z| < 1$.

Problem 5.8.3 (Fa80) *Suppose that* f *is analytic inside and on the unit circle* $|z| = 1$ *and satisfies* $|f(z)| < 1$ *for* $|z| = 1$. *Show that the equation* $f(z) = z^3$ *has exactly three solutions (counting multiplicities) inside the unit circle.*

Problem 5.8.4 (Fa81)

1. *How many zeros does the function* $f(z) = 3z^{100} - e^z$ *have inside the unit circle (counting multiplicities)?*
2. *Are the zeros distinct?*

Problem 5.8.5 (Fa92)

1. *How many roots does the polynomial defined by* $p(z) = 2z^5 + 4z^2 + 1$ *have in the disc* $|z| < 1$?
2. *How many roots does the same polynomial have on the real axis?*

Problem 5.8.6 (Su80) *How many zeros does the complex polynomial*

$$3z^9 + 8z^6 + z^5 + 2z^3 + 1$$

have in the annulus $1 < |z| < 2$?

Problem 5.8.7 (Fa83) *Consider the polynomial*

$$p(z) = z^5 + z^3 + 5z^2 + 2.$$

How many zeros (counting multiplicities) does p *have in the annular region* $1 < |z| < 2$?

Problem 5.8.8 (Sp84, Fa87, Fa96) *Find the number of roots of*

$$z^7 - 4z^3 - 11 = 0$$

which lie between the two circles $|z| = 1$ *and* $|z| = 2$.

Problem 5.8.9 (Fa01) *Let* ε *be a positive number. Prove that the polynomial* $p(z) = \varepsilon z^3 - z^2 - 1$ *has exactly two roots in the half-plane* $\Re z < 0$.

Problem 5.8.10 (Sp96) *Let* $r < 1 < R$. *Show that for all sufficiently small* $\varepsilon > 0$, *the polynomial*

$$p(z) = \varepsilon z^7 + z^2 + 1$$

has exactly five roots (counted with their multiplicities) inside the annulus

$$r\varepsilon^{-1/5} < |z| < R\varepsilon^{-1/5}.$$

Problem 5.8.11 (Sp86) *Let the 3×3 matrix function A be defined on the complex plane by*

$$A(z) = \begin{pmatrix} 4z^2 & 1 & -1 \\ -1 & 2z^2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

How many distinct values of z are there such that $|z| < 1$ and $A(z)$ is not invertible?

Problem 5.8.12 (Fa85) *How many roots does the polynomial $z^4 + 3z^2 + z + 1$ have in the right half z-plane?*

Problem 5.8.13 (Sp87) *Prove that if the nonconstant polynomial $p(z)$, with complex coefficients, has all of its roots in the half-plane $\Re z > 0$, then all of the roots of its derivative are in the same half-plane.*

Problem 5.8.14 (Sp00) *Let f be a nonconstant entire function whose values on the real axis are real and nonnegative. Prove that all real zeros of f have even order.*

Problem 5.8.15 (Sp92) *Let p be a nonconstant polynomial with real coefficients and only real roots. Prove that for each real number r, the polynomial $p - rp'$ has only real roots.*

Problem 5.8.16 (Sp79, Su85, Sp89) *Prove that if $1 < \lambda < \infty$, the function*

$$f_\lambda(z) = z + \lambda - e^z$$

has only one zero in the half-plane $\Re z < 0$, and that this zero is real.

Problem 5.8.17 (Fa85) *Prove that for every $\lambda > 1$, the equation $ze^{\lambda - z} = 1$ has exactly one root in the disc $|z| < 1$ and that this root is real.*

Problem 5.8.18 (Sp85) *Prove that for any $a \in \mathbb{C}$ and any integer $n \geqslant 2$, the equation $1 + z + az^n = 0$ has at least one root in the disc $|z| \leqslant 2$.*

Problem 5.8.19 (Sp98) *Prove that the polynomial $z^4 + z^3 + 1$ has exactly one root in the quadrant $\{z = x + iy \mid x, y > 0\}$.*

Problem 5.8.20 (Sp98) *Let f be analytic in an open set containing the closed unit disc. Suppose that $|f(z)| > m$ for $|z| = 1$ and $|f(0)| < m$. Prove that $f(z)$ has at least one zero in the open unit disc $|z| < 1$.*

Problem 5.8.21 (Su82) *Let $0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$. Prove that the equation*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

has no roots in the disc $|z| < 1$.

Problem 5.8.22 (Fa86) *Show that the polynomial $p(z) = z^5 - 6z + 3$ has five distinct complex roots, of which exactly three (and not five) are real.*

Problem 5.8.23 (Sp90) *Let $c_0, c_1, \ldots, c_{n-1}$ be complex numbers. Prove that all the zeros of the polynomial*

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$$

lie in the open disc with center 0 *and radius*

$$\sqrt{1 + |c_{n-1}|^2 + \cdots + |c_1|^2 + |c_0|^2}\,.$$

Problem 5.8.24 (Sp95) *Let $P(x)$ be a polynomial with real coefficients and with leading coefficient* 1. *Suppose that $P(0) = -1$ and that $P(x)$ has no complex zeros inside the unit circle. Prove that $P(1) = 0$.*

Problem 5.8.25 (Su81) *Prove that the number of roots of the equation $z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$ (n a natural number, α and β real, nonzero) that have positive real part is*

1. *n if n is even, and*
2. *$n - 1$ if n is odd.*

Problem 5.8.26 (Su84) *Let $\rho > 0$. Show that for n large enough, all the zeros of*

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$$

lie in the circle $|z| < \rho$.

Problem 5.8.27 (Fa88) *Do the functions $f(z) = e^z + z$ and $g(z) = ze^z + 1$ have the same number of zeros in the strip $-\frac{\pi}{2} < \Im z < \frac{\pi}{2}$?*

Problem 5.8.28 (Sp93) *Let a be a complex number and ε a positive number. Prove that the function $f(z) = \sin z + \frac{1}{z-a}$ has infinitely many zeros in the strip $|\Im z| < \varepsilon$.*

Problem 5.8.29 (Sp99) *Suppose that f is holomorphic on some neighborhood of a in the complex plane. Prove that either f is constant on some neighborhood of a, or there exist an integer $n > 0$ and real numbers $\delta, \varepsilon > 0$ such that for each complex number b satisfying $0 < |b - f(a)| < \varepsilon$, the equation $f(z) = b$ has exactly n roots in $\{z \in \mathbb{C} \mid |z - a| < \delta\}$.*

Problem 5.8.30 (Su77, Sp81) *Let $\hat{a}_0 + \hat{a}_1 z + \cdots + \hat{a}_n z^n$ be a polynomial having $\hat{z}$ as a simple root. Show that there is a continuous function $r : U \to \mathbb{C}$, where U is a neighborhood of $(\hat{a}_0, \ldots, \hat{a}_n)$ in $\mathbb{C}^{n+1}$, such that $r(a_0, \ldots, a_n)$ is always a root of $a_0 + a_1 z + \cdots + a_n z^n$, and $r(\hat{a}_0, \ldots, \hat{a}_n) = \hat{z}$.*

Problem 5.8.31 (Su85) *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where all the a_n are nonnegative reals, and the series has radius of convergence 1. *Prove that $f(z)$ cannot be analytically continued to a function analytic in a neighborhood of $z = 1$.*

Problem 5.8.32 (Su80) *Let f be a meromorphic function on $\mathbb{C}$ which is analytic in a neighborhood of* 0. *Let its Maclaurin series be*

$$\sum_{k=0}^{\infty} a_k z^k$$

with all $a_k \geqslant 0$. Suppose there is a pole of modulus $r > 0$ and no pole has modulus $< r$. Prove there is a pole at $z = r$.

Problem 5.8.33 (Sp82) *Decide, without too much computation, whether a finite limit*

$$\lim_{z \to 0} \left((\tan z)^{-2} - z^{-2} \right)$$

exists, where z is a complex variable, and if yes, compute the limit.

Problem 5.8.34 (Sp89, Sp00) *Let f and g be entire functions such that $\lim_{z \to \infty} f(g(z)) = \infty$. Prove that f and g are polynomials.*

Problem 5.8.35 (Fa98) *Let $z_1, \ldots, z_n$ be distinct complex numbers, and let $a_1, \ldots, a_n$ be nonzero complex numbers such that $S_p = \sum_{j=1}^{n} a_j z_j^p = 0$ for $p = 0, 1, \ldots, m-1$ but $S_m \neq 0$. Here $1 \leqslant m \leqslant n-1$. How many zeros does the rational function $f(z) = \sum_{j=1}^{n} \frac{a_j}{z - z_j}$ have in $\mathbb{C}$? Why is $m \geqslant n$ impossible.*

5.9 Harmonic Functions

Problem 5.9.1 (Fa77, Fa81) *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $u(x, y) = x^3 - 3xy^2$. Show that u is harmonic and find $v : \mathbb{R}^2 \to \mathbb{R}$ such that the function $f : \mathbb{C} \to \mathbb{C}$ defined by*

$$f(x + iy) = u(x, y) + iv(x, y)$$

is analytic.

Problem 5.9.2 (Fa80) *Let $f(z)$ be an analytic function defined for $|z| \leqslant 1$ and let*

$$u(x, y) = \Re f(z), \qquad z = x + iy.$$

Prove that

$$\int_C \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0$$

where C is the unit circle, $x^2 + y^2 = 1$.

Problem 5.9.3 (Fa83) *1. Let f be a complex function which is analytic on an open set containing the disc $|z| \leqslant 1$, and which is real valued on the unit circle. Prove that f is constant.*

2. *Find a nonconstant function which is analytic at every point of the complex plane except for a single point on the unit circle $|z| = 1$, and which is real valued at every other point of the unit circle.*

Problem 5.9.4 (Fa92) *Let s be a real number, and let the function u be defined in $\mathbb{C}\backslash(-\infty, 0]$ by*

$$u(re^{i\theta}) = r^s \cos s\theta \qquad (r > 0, \quad -\pi < \theta < \pi).$$

Prove that u is a harmonic function.

Problem 5.9.5 (Fa87) *Let u be a positive* harmonic *function on $\mathbb{R}^2$; that is,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Show that u is constant.

Problem 5.9.6 (Sp94) *Let u be a real valued harmonic function in the complex plane such that*

$$u(z) \leqslant a\,|\log|z|| + b$$

for all z, where a and b are positive constants. Prove that u is constant.

Problem 5.9.7 (Fa03) *Let $D = \{z \in \mathbb{C} : |z| \leq 1\} \setminus \{1, -1\}$. Find an explicit continuous function $f : D \to \mathbb{R}$ satisfying all the following conditions:*

- *f is harmonic on the interior of D (the open unit disk),*
- *$f(z) = 1$ when $|z| = 1$ and $\Im z > 0$, and*
- *$f(z) = -1$ when $|z| = 1$ and $\Im z < 0$.*

5.10 Residue Theory

Problem 5.10.1 (Fa83) *Let $r_1, r_2, \ldots, r_n$ be distinct complex numbers. Show that a rational function of the form*

$$f(z) = \frac{b_0 + b_1 z + \cdots + b_{n-2} z^{n-2} + b_{n-1} z^{n-1}}{(z - r_1)(z - r_2)\cdots(z - r_n)}$$

can be written as a sum

$$f(z) = \frac{A_1}{z - r_1} + \frac{A_2}{z - r_2} + \cdots + \frac{A_n}{z - r_n}$$

for suitable constants $A_1, \ldots, A_n$.

Problem 5.10.2 (Fa82) *Let*

$$\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be the Laurent expansion for $\cot(\pi z)$ *on the annulus* $1 < |z| < 2$. *Compute the* a_n *for* $n < 0$.

Problem 5.10.3 (Sp78) *Show that there is a complex analytic function defined on the set* $U = \{z \in \mathbb{C} \mid |z| > 4\}$ *whose derivative is*

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Is there a complex analytic function on U *whose derivative is*

$$\frac{z^2}{(z-1)(z-2)(z-3)}\ ?$$

Problem 5.10.4 (Fa88) *Let* n *be a positive integer. Prove that the polynomial*

$$f(x) = \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

in $\mathbb{R}[x]$ *has* n *distinct complex zeros,* $z_1, z_2, \ldots, z_n$, *and that they satisfy*

$$\sum_{i=1}^{n} z_i^{-j} = 0 \quad for \quad 2 \leqslant j \leqslant n.$$

Problem 5.10.5 (Sp79, Sp83) *Let* P *and* Q *be complex polynomials with the degree of* Q *at least two more than the degree of* P. *Prove there is an* $r > 0$ *such that if* C *is a closed curve outside* $|z| = r$, *then*

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

Problem 5.10.6 (Sp93) *Prove that for any fixed complex number* ζ,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{2\zeta \cos\theta}\, d\theta = \sum_{n=0}^{\infty} \left(\frac{\zeta^n}{n!}\right)^2.$$

Problem 5.10.7 (Fa99) *Evaluate the integral*

$$I = \frac{1}{2\pi i} \int_{|z|=1} \frac{(z+2)^2}{z^2(2z-1)}\, dz,$$

where the direction of integration is counterclockwise.

Problem 5.10.8 (Sp80) *Let $a > 0$ be a constant $\neq 2$. Let C_a denote the positively oriented circle of radius a centered at the origin. Evaluate*

$$\int_{C_a} \frac{z^2 + e^z}{z^2(z-2)}\, dz\,.$$

Problem 5.10.9 (Sp02) *Let r be a positive number and a a complex number such that $|a| \neq r$.*
Evaluate the integral

$$I(a,r) = \int_{|z|=r} \frac{1}{a - \bar{z}}\, dz$$

where the circle $|z| = r$ has the counterclockwise orientation.

Problem 5.10.10 (Su80) *Let C denote the positively oriented circle $|z| = 2$, $z \in \mathbb{C}$. Evaluate the integral*

$$\int_C \sqrt{z^2 - 1}\, dz$$

where the branch of the square root is chosen so that $\sqrt{2^2 - 1} > 0$.

Problem 5.10.11 (Fa00) *Evaluate the integral*

$$I = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{\sin 4z}$$

where the direction of integration is counterclockwise.

Problem 5.10.12 (Fa02) *Evaluate the integrals*

$$I_n = \int_{C_n} \frac{1}{z^3 \sin z}\, dz\,, \qquad n = 0, 1, \ldots,$$

where C_n is the circle $|z| = \left(n + \frac{1}{2}\right)\pi$, with the counterclockwise orientation.

Problem 5.10.13 (Su81) *Compute*

$$\frac{1}{2\pi i} \int_C \frac{dz}{\sin \frac{1}{z}},$$

where C is the circle $|z| = \dfrac{1}{5}$, positively oriented.

Problem 5.10.14 (Su84) *1. Show that there is a unique analytic branch outside the unit circle of the function $f(z) = \sqrt{z^2 + z + 1}$ such that $f(t)$ is positive when $t > 1$.*

2. *Using the branch determined in Part 1, calculate the integral*

$$\frac{1}{2\pi i}\int_{C_r}\frac{dz}{\sqrt{z^2+z+1}}$$

where C_r is the positively oriented circle $|z| = r$ and $r > 1$.

Problem 5.10.15 (Sp86) *Let C be a simple closed contour enclosing the points $0, 1, 2, \ldots, k$ in the complex plane, with positive orientation. Evaluate the integrals*

$$I_k = \int_C \frac{dz}{z(z-1)\cdots(z-k)}, \qquad k = 0, 1, \ldots,$$

$$J_k = \int_C \frac{(z-1)\cdots(z-k)}{z}\,dz, \qquad k = 0, 1, \ldots.$$

Problem 5.10.16 (Sp86) *Evaluate*

$$\int_{|z|=1}(e^{2\pi z}+1)^{-2}\,dz$$

where the integral is taken in counterclockwise direction.

Problem 5.10.17 (Fa86) *Evaluate*

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{z^{11}}{12z^{12}-4z^9+2z^6-4z^3+1}\,dz$$

where the direction of integration is counterclockwise.

Problem 5.10.18 (Sp89) *Evaluate*

$$\int_C (2z-1)e^{z/(z-1)}\,dz$$

where C is the circle $|z| = 2$ with counterclockwise orientation.

Problem 5.10.19 (Fa90) *Evaluate the integral*

$$I = \frac{1}{2\pi i}\int_C \frac{dz}{(z-2)(1+2z)^2(1-3z)^3}$$

where C is the circle $|z| = 1$ with counterclockwise orientation.

Problem 5.10.20 (Fa91) *Evaluate the integral*

$$I = \frac{1}{2\pi i}\int_C \frac{z^{n-1}}{3z^n-1}\,dz,$$

where n is a positive integer, and C is the circle $|z| = 1$, with counterclockwise orientation.

Problem 5.10.21 (Fa92) *Evaluate*

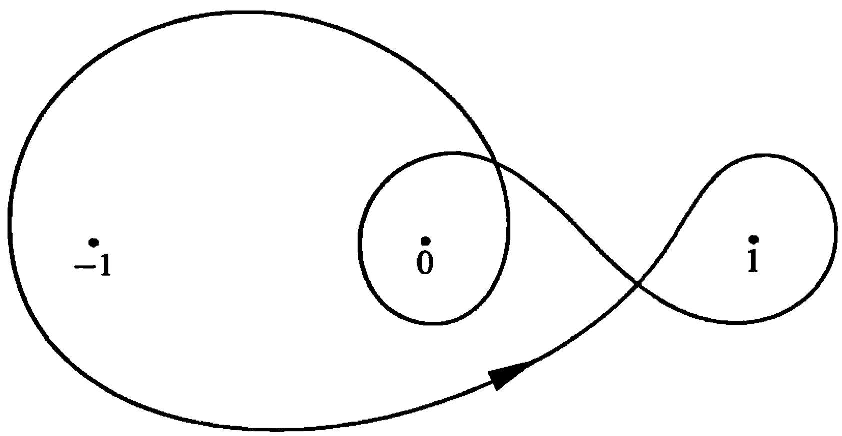
$$\int_C \frac{e^z}{z(2z+1)^2}\,dz,$$

where C is the unit circle with counterclockwise orientation.

Problem 5.10.22 (Fa01) *Let* $a = \dfrac{1+i}{2}$, $b = \dfrac{-1+i}{2}$. *Let Γ be the polygonal path in the complex plane with successive vertices* -1, $-1+i$, 1, $1+i$, -1. *Evaluate the integral*

$$I = \int_\Gamma \frac{1}{(z-a)^2(z-b)^3}\,dz\,.$$

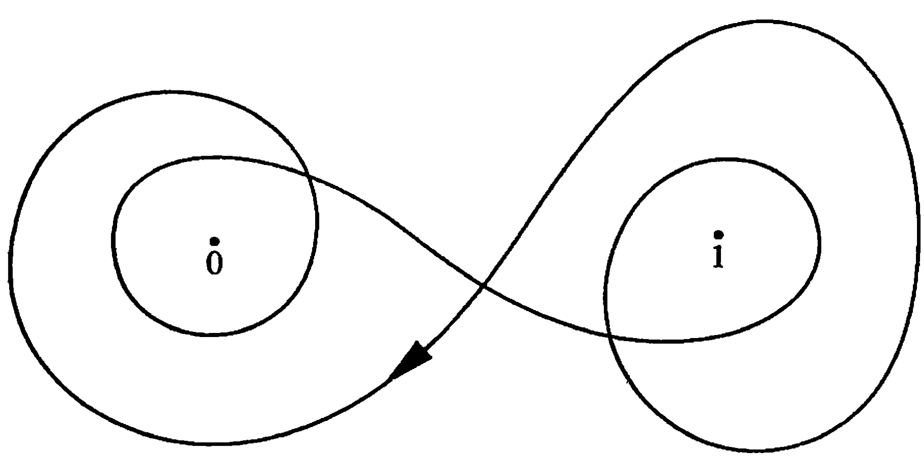
Problem 5.10.23 (Fa93) *Evaluate the integral* $\frac{1}{2\pi i}\int_\gamma f(z)\,dz$ *for the function* $f(z) = z^{-2}(1-z^2)^{-1}e^z$ *and the curve γ depicted by*



Problem 5.10.24 (Sp81) *Evaluate*

$$\int_C \frac{e^z-1}{z^2(z-1)}\,dz$$

where C is the closed curve shown below:



Problem 5.10.25 (Sp95) *Let n be a positive integer and* $0 < \theta < \pi$. *Prove that*

$$\frac{1}{2\pi i}\int_{|z|=2}\frac{z^n}{1-2z\cos\theta+z^2}dz=\frac{\sin n\theta}{\sin\theta}$$

where the circle $|z|=2$ *is oriented counterclockwise.*

Problem 5.10.26 (Su77, Fa84, Sp94, Sp96) *Use the Residue Theorem to evaluate the integral*

$$I(a)=\int_0^{2\pi}\frac{d\theta}{a+\cos\theta}$$

where a is real and $a > 1$. *Why the formula obtained for* $I(a)$ *is also valid for certain complex (nonreal) values of a ?*

Problem 5.10.27 (Fa78) *Evaluate*

$$\int_0^{2\pi}\frac{d\theta}{1-2r\cos\theta+r^2}$$

where $r^2 \neq 1$.

Problem 5.10.28 (Sp87) *Evaluate*

$$I=\int_0^{\pi}\frac{\cos 4\theta}{1+\cos^2\theta}\,d\theta\,.$$

Problem 5.10.29 (Fa87) *Evaluate the integral*

$$I=\int_0^{2\pi}\frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta\,.$$

Problem 5.10.30 (Sp00) *Evaluate*

$$I=\int_{|z|=1}\frac{\cos^3 z}{z^3}\,dz\,,$$

where the direction of integration is counterclockwise.

Problem 5.10.31 (Sp95) *Let n be a positive integer. Compute*

$$\int_0^{2\pi}\frac{1-\cos n\theta}{1-\cos\theta}d\theta\,.$$

Problem 5.10.32 (Fa94) *Evaluate the integrals*

$$\int_{-\pi}^{\pi}\frac{\sin n\theta}{\sin\theta}d\theta,\quad n=1,2,\ldots.$$

Problem 5.10.33 (Sp88) *For* $a > 1$ *and* $n = 0, 1, 2, \ldots$, *evaluate the integrals*

$$C_n(a)=\int_{-\pi}^{\pi}\frac{\cos n\theta}{a-\cos\theta}\,d\theta\,,\qquad S_n(a)=\int_{-\pi}^{\pi}\frac{\sin n\theta}{a-\cos\theta}\,d\theta\,.$$

5.11 Integrals Along the Real Axis

Problem 5.11.1 (Sp86) *Let the complex valued functions f_n, $n \in \mathbb{Z}$, be defined on $\mathbb{R}$ by*

$$f_n(x) = \frac{(x-i)^n}{\sqrt{\pi}(x+i)^{n+1}} \, .$$

Prove that these functions are orthonormal; *that is,*

$$\int_{-\infty}^{\infty} f_m(x)\overline{f_n(x)}\, dx = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

Problem 5.11.2 (Fa85) *Evaluate the integral*

$$\int_0^{\infty} \frac{1-\cos ax}{x^2}\, dx$$

for $a \in \mathbb{R}$.

Problem 5.11.3 (Sp00) *Evaluate the integrals*

$$I(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{(x+i)^2}\, dx \, , \qquad -\infty < t < \infty \, .$$

Problem 5.11.4 (Fa01) *Evaluate the integrals* $F(t) = \displaystyle\int_{-\infty}^{\infty} \frac{e^{-itx}}{(x+i)^3}\, dx$, $-\infty < t < \infty$.

Problem 5.11.5 (Sp78, Sp83, Sp97) *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx \, .$$

Problem 5.11.6 (Fa82, Sp92) *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\, dx \, .$$

Problem 5.11.7 (Sp93) *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1+x^2)^2}\, dx \, .$$

Problem 5.11.8 (Sp81) *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(1+x^2)^2}\, dx \, .$$

Problem 5.11.9 (Sp90, Fa92) *Let a be a positive real number. Evaluate the improper integral*

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx\,.$$

Problem 5.11.10 (Sp91) *Prove that*

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin x}{x-3i}\,dx$$

exists and find its value.

Problem 5.11.11 (Sp83) *Evaluate*

$$\int_{-\infty}^{\infty}\frac{\sin x}{x(x-\pi)}\,dx\,.$$

Problem 5.11.12 (Fa02) *Evaluate*

$$\int_{-\infty}^{\infty}\frac{\cos x}{(1+x^2)^3}\,dx\,.$$

Problem 5.11.13 (Fa97) *Evaluate the integral*

$$\int_{-\infty}^{\infty}\frac{\cos kx}{1+x+x^2}\,dx$$

where $k \geqslant 0$.

Problem 5.11.14 (Fa82) *Evaluate*

$$\int_{-\infty}^{\infty}\frac{\cos \pi x}{4x^2-1}\,dx\,.$$

Problem 5.11.15 (Sp77, Fa81, Sp82) *Evaluate*

$$\int_{-\infty}^{\infty}\frac{\cos nx}{x^4+1}\,dx\,.$$

Problem 5.11.16 (Sp79) *Evaluate*

$$\int_0^\infty \frac{x^2+1}{x^4+1}\,dx\,.$$

Problem 5.11.17 (Sp02) *Evaluate the integrals* $I(a)=\displaystyle\int_{-\infty}^{\infty}\frac{\cos^3 x}{a^2+x^2}\,dx$ *for $a>0$.*

Problem 5.11.18 (Su84) *Evaluate*

$$\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+4x+20}\,dx\,.$$

Problem 5.11.19 (Fa84) *Evaluate*

$$\int_0^\infty \frac{x - \sin x}{x^3}\, dx\,.$$

Problem 5.11.20 (Fa84) *Evaluate*

$$\int_{-\infty}^\infty \frac{dx}{(1 + x + x^2)^2}\,.$$

Problem 5.11.21 (Fa79, Fa80, Sp85, Su85, Fa98, Sp99) *Prove that*

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx = \frac{\pi}{\sin \pi\alpha}.$$

What restrictions must be placed on α*?*

Problem 5.11.22 (Fa96) *Evaluate the integral*

$$I = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx\,.$$

Problem 5.11.23 (Sp01) *Evaluate*

$$\int_0^\infty \frac{1}{1+x^5} dx\,.$$

Problem 5.11.24 (Fa77, Su82, Fa97) *Evaluate*

$$\int_{-\infty}^\infty \frac{dx}{1+x^{2n}}$$

where n *is a positive integer.*

Problem 5.11.25 (Fa03) *Evaluate* $\displaystyle\int_{-\infty}^\infty \frac{x^2}{x^n+1}\, dx$*, where* $n \geq 4$ *is an even integer.*

Problem 5.11.26 (Fa88) *Prove that*

$$\int_0^\infty \frac{x}{e^x - e^{-x}}\, dx = \frac{\pi^2}{8}.$$

Problem 5.11.27 (Fa93) *Evaluate*

$$\int_{-\infty}^\infty \frac{e^{-ix}}{x^2 - 2x + 4}\, dx\,.$$

Problem 5.11.28 (Fa86) *Evaluate*

$$\int_0^\infty \frac{\log x}{(x^2+1)(x^2+4)}\, dx\,.$$

Problem 5.11.29 (Fa94) *Evaluate*

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx\,.$$

Problem 5.11.30 (Fa83) *Evaluate*

$$\int_0^\infty (\operatorname{sech} x)^2 \cos \lambda x\,dx$$

where λ is a real constant and

$$\operatorname{sech} x = \frac{2}{e^x+e^{-x}}.$$

Problem 5.11.31 (Sp85) *Prove that*

$$\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{1}{2}\sqrt{\pi}e^{-b^2}.$$

What restrictions, if any, need be placed on b?

Problem 5.11.32 (Sp03) *Evaluate*

$$\int_0^\infty e^{-x^2}\cos x^2\,dx.$$

Problem 5.11.33 (Sp97) *Prove that*

$$\int_{-\infty}^\infty \frac{e^{-(t-i\gamma)^2/2}}{\sqrt{2\pi}}\,dt$$

is independent of the real parameter γ.

6
Algebra

6.1 Examples of Groups and General Theory

Problem 6.1.1 (Sp77) *Let G be the collection of 2×2 real matrices with nonzero determinant. Define the product of two elements in G as the usual matrix product.*

1. *Show that G is a group.*
2. *Find the* center *Z of G; that is, the set of all elements z of G such that $az = za$ for all $a \in G$.*
3. *Show that the set O of real orthogonal matrices is a subgroup of G (a matrix is* orthogonal *if $AA^t = I$, where A^t denotes the transpose of A). Show by example that O is not a normal subgroup.*
4. *Find a nontrivial homomorphism from G onto an abelian group.*

Problem 6.1.2 (Fa77) *Let G be the set of 3×3 real matrices with zeros below the diagonal and ones on the diagonal.*

1. *Prove G is a group under matrix multiplication.*
2. *Determine the center of G.*

Problem 6.1.3 (Su78) *For each of the following either give an example or else prove that no such example is possible.*

1. *A nonabelian group.*

2. *A finite abelian group that is not cyclic.*

3. *An infinite group with a subgroup of index* 5.

4. *Two finite groups that have the same order but are not isomorphic.*

5. *A group* G *with a subgroup* H *that is* not *normal.*

6. *A nonabelian group with no normal subgroups except the whole group and the unit element.*

7. *A group* G *with a normal subgroup* H *such that the factor group* G/H *is* not *isomorphic to any subgroup of* G.

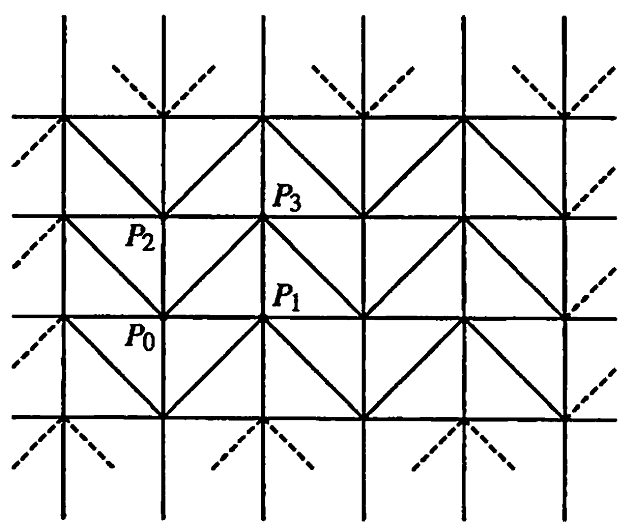
8. *A group* G *with a subgroup* H *which has index* 2 *but is* not *normal.*

Problem 6.1.4 (Fa80) *Let* R *be a ring with multiplicative identity* 1. *Call* $x \in R$ *a* unit *if* $xy = yx = 1$ *for some* $y \in R$. *Let* $G(R)$ *denote the set of units.*

1. *Prove* $G(R)$ *is a multiplicative group.*

2. *Let* R *be the ring of complex numbers* $a + bi$, *where* a *and* b *are integers. Prove* $G(R)$ *is isomorphic to* $\mathbb{Z}_4$ *(the additive group of integers modulo* 4*).*

Problem 6.1.5 (Sp83) *In the triangular network in* $\mathbb{R}^2$ *depicted below, the points* P_0, P_1, P_2, *and* P_3 *are respectively* $(0,0)$, $(1,0)$, $(0,1)$, *and* $(1,1)$. *Describe the structure of the group of all Euclidean transformations of* $\mathbb{R}^2$ *which leave this network invariant.*



Problem 6.1.6 (Fa90) *Does the set* $G = \{a \in \mathbb{R} \mid a > 0, a \neq 1\}$ *form a group with the operation* $a * b = a^{\log b}$?

Problem 6.1.7 (Sp81) *Let G be a finite group. A* conjugacy class *is a set of the form*

$$C(a) = \{bab^{-1} \mid b \in G\}$$

for some $a \in G$.

1. *Prove that the number of elements in a conjugacy class divides the order of G.*

2. *Do all conjugacy classes have the same number of elements?*

3. *If G has only two conjugacy classes, prove G has order 2.*

Problem 6.1.8 (Sp91) *Let G be a finite nontrivial group with the property that for any two elements a and b in G different from the identity, there is an element c in G such that $b = c^{-1}ac$. Prove that G has order 2.*

Problem 6.1.9 (Sp99) *Let G be a finite group, with identity e. Suppose that for every $a, b \in G$ distinct from e, there is an automorphism σ of G such that $\sigma(a) = b$. Prove that G is abelian.*

Problem 6.1.10 (Sp84) *For a p-group of order p^4, assume the center of G has order p^2. Determine the number of conjugacy classes of G.*

Problem 6.1.11 (Sp85) *In a commutative group G, let the element a have order r, let b have order s $(r, s < \infty)$, and assume that the greatest common divisor of r and s is 1. Show that ab has order rs.*

Problem 6.1.12 (Fa85) *Let G be a group. For any subset X of G, define its* centralizer *$C(X)$ to be $\{y \in G \mid xy = yx, \text{ for all } x \in X\}$. Prove the following:*

1. *If $X \subset Y$, then $C(Y) \subset C(X)$.*

2. $X \subset C(C(X))$.

3. $C(X) = C(C(C(X)))$.

Problem 6.1.13 (Sp88) *Let D be a group of order $2n$, where n is odd, with a subgroup H of order n satisfying $xhx^{-1} = h^{-1}$ for all h in H and all x in $D \setminus H$. Prove that H is commutative and that every element of $D \setminus H$ is of order 2.*

6.2 Homomorphisms and Subgroups

Problem 6.2.1 (Fa78) *How many homomorphisms are there from the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the symmetric group on three objects?*

Problem 6.2.2 (Fa03) *1. Let G be a finite group and let X be the set of pairs of commuting elements of G:*

$$X = \{(g, h) \in G \times G : gh = hg\}.$$

Prove that $|X| = c|G|$ where c is the number of conjugacy classes in G.

2. *Compute the number of pairs of commuting permutations on five letters.*

Problem 6.2.3 (Fa03) *The set of 5×5 complex matrices A satisfying $A^3 = A^2$ is a union of conjugacy classes. How many conjugacy classes?*

Problem 6.2.4 (Sp90) *Let $\mathbb{C}^*$ be the multiplicative group of nonzero complex numbers. Suppose that H is a subgroup of finite index of $\mathbb{C}^*$. Prove that $H = \mathbb{C}^*$.*

Problem 6.2.5 (Su80) *Let G be a finite group and $H \subset G$ a subgroup.*

1. *Show that the number of subgroups of G of the form xHx^{-1} for some $x \in G$ is $\leqslant$ the index of H in G.*

2. *Prove that some element of G is not in any subgroup of the form xHx^{-1}, $x \in G$.*

Problem 6.2.6 (Su79) *Prove that the group of automorphisms of a cyclic group of prime order p is cyclic and find its order.*

Problem 6.2.7 (Su81) *Let G be a finite group, and let φ be an automorphism of G which leaves fixed only the identity element of G.*

1. *Show that every element of G may be written in the form $g^{-1}\varphi(g)$.*

2. *If φ has order 2 (i.e., $\varphi \cdot \varphi = \mathrm{id}$) show that φ is given by the formula $g \mapsto g^{-1}$ and that G is an abelian group whose order is odd.*

Problem 6.2.8 (Fa79, Sp88, Fa91) *Prove that every finite group of order > 2 has a nontrivial automorphism.*

Problem 6.2.9 (Fa90) *Let A be an additively written abelian group, and $u, v : A \to A$ homomorphisms. Define the group homomorphisms $f, g : A \to A$ by*

$$f(a) = a - v(u(a)), \qquad g(a) = a - u(v(a)) \qquad (a \in A).$$

Prove that the kernel of f is isomorphic to the kernel of g.

Problem 6.2.10 (Su81) *Let G be an additive group, and $u, v : G \to G$ homomorphisms. Show that the map $f : G \to G$, $f(x) = x - v(u(x))$ is surjective if the map $h : G \to G$, $h(x) = x - u(v(x))$ is surjective.*

Problem 6.2.11 (Sp83) *Let H be the group of integers* mod p, *under addition, where p is a prime number. Suppose that n is an integer satisfying $1 \leqslant n \leqslant p$, and let G be the group $H \times H \times \cdots \times H$ (n factors). Show that G has no automorphism of order p^2.*

Problem 6.2.12 (Sp03) 1. *Suppose that H_1 and H_2 are subgroups of a group G such that $H_1 \cup H_2$ is a subgroup of G. Prove that either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.*

2. *Show that for each integer $n \geqslant 3$, there exists a group G with subgroups $H_1, H_2, \ldots, H_n$, such that no H_i is contained in any other, and such that $H_1 \cup H_2 \cup \ldots \cup H_n$ is a subgroup of G.*

Problem 6.2.13 (Fa84) *Let G be a group and H a subgroup of index $n < \infty$. Prove or disprove the following statements:*

1. *If $a \in G$, then $a^n \in H$.*

2. *If $a \in G$, then for some k, $1 \leqslant k \leqslant n$, we have $a^k \in H$.*

Problem 6.2.14 (Fa78) *Find all automorphisms of the additive group of rational numbers.*

Problem 6.2.15 (Sp01) *Let α and β be real numbers such that the subgroup Γ of $\mathbb{R}$ generated by α and β is closed. Prove that α and β are linearly dependent over $\mathbb{Q}$.*

Problem 6.2.16 (Sp00) *Prove that the group $G = \mathbb{Q}/\mathbb{Z}$ has no proper subgroup of finite index.*

Problem 6.2.17 (Fa87, Fa93) *Let A be the group of rational numbers under addition, and let M be the group of positive rational numbers under multiplication. Determine all homomorphisms $\varphi : A \to M$.*

Problem 6.2.18 (Sp00) *Suppose that H_1 and H_2 are distinct subgroups of a group G such that $[G : H_1] = [G : H_2] = 3$. What are the possible values of $[G : H_1 \cap H_2]$?*

Problem 6.2.19 (Fa92) *Let G be a group and H and K subgroups such that H has a finite index in G. Prove that $K \cap H$ has a finite index in K.*

Problem 6.2.20 (Fa94) *Suppose the group G has a nontrivial subgroup H which is contained in every nontrivial subgroup of G. Prove that H is contained in the center of G.*

Problem 6.2.21 (Fa00) *Show that for each positive integer k there exists a positive integer N such that there are at least k nonisomorphic groups of order N.*

Problem 6.2.22 (Fa95) *Let G be a group generated by n elements. Find an upper bound $N(n, k)$ for the number of subgroups H of G with the index $[G : H] = k$.*

6.3 Cyclic Groups

Problem 6.3.1 (Su77, Sp92) *1. Prove that every finitely generated subgroup of $\mathbb{Q}$, the additive group of rational numbers, is cyclic.*

2. *Does the same conclusion hold for finitely generated subgroups of $\mathbb{Q}/\mathbb{Z}$, where $\mathbb{Z}$ is the group of integers?*

Note: See also Problems 6.6.3 and 6.7.2.

Problem 6.3.2 (Sp98) *Let G be the group $\mathbb{Q}/\mathbb{Z}$. Show that for every positive integer t, G has a unique cyclic subgroup of order t.*

Problem 6.3.3 (Fa99) *Show that a group G is isomorphic to a subgroup of the additive group of the rationals if and only if G is countable and every finite subset of G is contained in an infinite cyclic subgroup of G.*

Problem 6.3.4 (Su85) *1. Let G be a cyclic group, and let $a, b \in G$ be elements which are not squares. Prove that ab is a square.*

2. *Give an example to show that this result is false if the group is not cyclic.*

Problem 6.3.5 (Sp82) *Prove that any group of order 77 is cyclic.*

Problem 6.3.6 (Fa91) *Let G be a group of order $2p$, where p is an odd prime. Assume that G has a normal subgroup of order 2. Prove that G is cyclic.*

Problem 6.3.7 (Fa97) *A finite abelian group G has the property that for each positive integer n the set $\{x \in G \mid x^n = 1\}$ has at most n elements. Prove that G is cyclic, and deduce that every finite field has cyclic multiplicative group.*

Problem 6.3.8 (Fa00) *Let G be a finite group of order n with the property that for each divisor d of n there is at most one subgroup in G of order d. Show G is cyclic.*

6.4 Normality, Quotients, and Homomorphisms

Problem 6.4.1 (Fa78) *Let H be a subgroup of a finite group G.*

1. *Show that H has the same number of left cosets as right cosets.*
2. *Let G be the group of symmetries of the square. Find a subgroup H such that $xH \neq Hx$ for some x.*

Problem 6.4.2 (Fa80) *Let G be the group of orthogonal transformations of $\mathbb{R}^3$ to $\mathbb{R}^3$ with determinant 1. Let $v \in \mathbb{R}^3$, $|v| = 1$, and let $H_v = \{T \in G \mid Tv = v\}$.*

1. *Show that H_v is a subgroup of G.*

2. *Let $S_v = \{T \in G \mid T$ is a rotation of $180°$ about a line orthogonal to $v\}$. Show that S_v is a coset of H_v in G.*

Problem 6.4.3 (Su84) *Show that if a subgroup H of a group G has just one left coset different from itself, then it is a normal subgroup of G.*

Problem 6.4.4 (Su85) *Let G be a group of order 120, let H be a subgroup of order 24, and assume that there is at least one left coset of H (other than H itself) which is equal to some right coset of H. Prove that H is a normal subgroup of G.*

Problem 6.4.5 (Fa02) *Let G be a group of order 112. Prove that G has a non-trivial normal subgroup.*

Problem 6.4.6 (Fa02) *List all groups of order $\leqslant 6$ up to isomorphism. Find a group of order 120 that contains as a subgroup an isomorphic copy of each of them. Prove that no group of order < 120 has the preceding property.*

Problem 6.4.7 (Sp89) *For G a group and H a subgroup, let $C(G, H)$ denote the collection of left cosets of H in G. Prove that if H and K are two subgroups of G of infinite index, then G is not a finite union of cosets from $C(G, H) \cup C(G, K)$.*

Problem 6.4.8 (Fa82, Fa92) *Let*

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R},\ a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}.$$

1. *Show that N is a normal subgroup of G and prove that G/N is isomorphic to $\mathbb{R}$.*

2. *Find a normal subgroup N' of G satisfying $N \subset N' \subset G$ (where the inclusions are proper), or prove that there is no such subgroup.*

Problem 6.4.9 (Sp86) *Let $\mathbb{Z}^2$ be the group of lattice points in the plane (ordered pairs of integers, with coordinatewise addition as the group operation). Let H_1 be the subgroup generated by the two elements $(1, 2)$ and $(4, 1)$, and H_2 the subgroup generated by the two elements $(3, 2)$ and $(1, 3)$. Are the quotient groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ isomorphic?*

Problem 6.4.10 (Sp78, Fa81) *Let G be a group of order 10 which has a normal subgroup of order 2. Prove that G is abelian.*

Problem 6.4.11 (Sp79, Fa81) *Let G be a group with three normal subgroups N_1, N_2, and N_3. Suppose $N_i \cap N_j = \{e\}$ and $N_i N_j = G$ for all i, j with $i \neq j$. Show that G is abelian and N_i is isomorphic to N_j for all i, j.*

Problem 6.4.12 (Fa97) *Suppose H_i is a normal subgroup of a group G for $1 \leqslant i \leqslant k$, such that $H_i \cap H_j = \{1\}$ for $i \neq j$. Prove that G contains a subgroup isomorphic to $H_1 \times H_2 \times \cdots \times H_k$ if $k = 2$, but not necessarily if $k \geqslant 3$.*

Problem 6.4.13 (Sp80) *G is a group of order n, H a proper subgroup of order m, and $(n/m)! < 2n$. Prove G has a proper normal subgroup different from the identity.*

Problem 6.4.14 (Sp82, Sp93) *Prove that if G is a group containing no subgroup of index 2, then any subgroup of index 3 is normal.*

Problem 6.4.15 (Sp03) *Suppose G is a nonabelian simple group, and A is its automorphism group. Show that A contains a normal subgroup isomorphic to G.*

Problem 6.4.16 (Sp89) *Let G be a group whose order is twice an odd number. For g in G, let λ_g denote the permutation of G given by $\lambda_g(x) = gx$ for $x \in G$.*

1. *Let g be in G. Prove that the permutation λ_g is even if and only if the order of g is odd.*

2. *Let $N = \{g \in G \mid \text{order}(g) \text{ is odd}\}$. Prove that N is a normal subgroup of G of index 2.*

Problem 6.4.17 (Fa89) *Let G be a group, G' its commutator subgroup, and N a normal subgroup of G. Suppose that N is cyclic. Prove that $gn = ng$ for all $g \in G'$ and all $n \in N$.*

Problem 6.4.18 (Fa90) *Let G be a group and N be a normal subgroup of G with $N \neq G$. Suppose that there does not exist a subgroup H of G satisfying $N \subset H \subset G$ and $N \neq H \neq G$. Prove that the index of N in G is finite and equal to a prime number.*

Problem 6.4.19 (Sp94) *Let G be a group having a subgroup A of finite index. Prove that there is a normal subgroup N of G contained in A such that N is of finite index in G.*

Problem 6.4.20 (Sp97) *Let H be the quotient of an abelian group G by a subgroup K. Prove or disprove each of the following statements:*

1. *If H is finite cyclic then G is isomorphic to the direct product of H and K.*

2. *If H is a direct product of infinite cyclic groups then G is isomorphic to the direct product of H and K.*

6.5 S_n, A_n, D_n, ...

Problem 6.5.1 (Fa80) *Let $\mathbf{F}_2 = \{0, 1\}$ be the field with two elements. Let G be the group of invertible 2×2 matrices with entries in $\mathbf{F}_2$. Show that G is isomorphic to S_3, the group of permutations of three objects.*

Problem 6.5.2 (Su84) *Let S_n denote the group of permutations of n objects. Find four different subgroups of S_4 isomorphic to S_3 and nine isomorphic to S_2.*

Problem 6.5.3 (Fa86) *Let G be a subgroup of S_5, the group of all permutations of five objects Prove that if G contains a 5-cycle and a 2-cycle, then $G = S_5$.*

Problem 6.5.4 (Fa85) *Let G be a subgroup of the symmetric group on six objects, S_6. Assume that G has an element of order 6. Prove that G has a normal subgroup H of index 2.*

Problem 6.5.5 (Sp79) *Let S_7 be the group of permutations of a set of seven objects. Find all n such that some element of S_7 has order n.*

Problem 6.5.6 (Sp80) *S_9 is the group of permutations of 9 objects.*

1. *Exhibit an element of S_9 of order 20.*

2. *Prove that no element of S_9 has order 18.*

Problem 6.5.7 (Sp88, Sp01) *Let S_9 denote the group of permutations of nine objects and let A_9 be the subgroup consisting of all even permutations. Denote by $1 \in S_9$ the identity permutation. Determine the minimum of all positive integers m such that every $\sigma \in S_9$ satisfies $\sigma^m = 1$. Determine also the minimum of all positive integers m such that every $\sigma \in A_9$ satisfies $\sigma^m = 1$.*

Problem 6.5.8 (Sp92) *Let S_{999} denote the group of permutations of 999 objects, and let $G \subset S_{999}$ be an abelian subgroup of order 1111. Prove that there exists $i \in \{1, \ldots, 999\}$ such that for all $\sigma \in G$, one has $\sigma(i) = i$.*

Problem 6.5.9 (Sp02) *Prove that S_n, the group of permutations of $\{1, 2, \ldots, n\}$, is isomorphic to a subgroup of A_{n+2}, the alternating subgroup of S_{n+2}.*

Problem 6.5.10 (Fa81, Sp95) *Let S_n be the group of all permutations of n objects and let G be a subgroup of S_n of order p^k, where p is a prime not dividing n. Show that G has a* fixed point; *that is, one of the objects is left fixed by every element of G.*

Problem 6.5.11 (Sp80) *Let G be a subgroup of S_n the group of permutations of n objects. Assume G is* transitive; *that is, for any x and y in S, there is some $\sigma \in G$ with $\sigma(x) = y$.*

1. *Prove that n divides the order of G.*

2. *Suppose $n = 4$. For which integers $k \geqslant 1$ can such a G have order $4k$?*

Problem 6.5.12 (Su83) *Let G be a transitive subgroup of the group S_n of permutations of n objects $\{1, \ldots, n\}$. Suppose that G is a simple group and that $\sim$ is an equivalence relation on $\{1, \ldots, n\}$ such that $i \sim j$ implies that $\sigma(i) \sim \sigma(j)$ for all $\sigma \in G$. What can one conclude about the relation $\sim$?*

Problem 6.5.13 (Sp89) *Let D_n be the dihedral group, the group of rigid motions of a regular n-gon ($n \geqslant 3$). (It is a noncommutative group of order $2n$.) Determine its center $Z = \{c \in D_n \mid cx = xc \ for \ all \ x \in D_n\}$.*

Problem 6.5.14 (Fa92) *How many Sylow 2-subgroups does the dihedral group D_n of order $2n$ have, when n is odd?*

6.6 Direct Products

Problem 6.6.1 (Fa83) *Let G be a finite group and suppose that $G \times G$ has exactly four normal subgroups. Show that G is simple and nonabelian.*

Problem 6.6.2 (Sp99) *Let G be a finite simple group of order n. Determine the number of normal subgroups in the direct product $G \times G$.*

Problem 6.6.3 (Sp91) *Prove that $\mathbb{Q}$, the additive group of rational numbers, cannot be written as the direct sum of two nontrivial subgroups.*
Note: See also Problems 6.3.1 and 6.7.2.

Problem 6.6.4 (Su79, Fa93) *Let A, B, and C be finite abelian groups such that $A \times B$ and $A \times C$ are isomorphic. Prove that B and C are isomorphic.*

Problem 6.6.5 (Su83) *Let G_1, G_2, and G_3 be finite groups, each of which is generated by its commutators (elements of the form $xyx^{-1}y^{-1}$). Let A be a subgroup of $G_1 \times G_2 \times G_3$, which maps surjectively, by the natural projection map, to the partial products $G_1 \times G_2$, $G_1 \times G_3$ and $G_2 \times G_3$. Show that A is equal to $G_1 \times G_2 \times G_3$.*

Problem 6.6.6 (Fa82) *Let A be a subgroup of an abelian group B. Assume that A is a direct summand of B, i.e., there exists a subgroup X of B such that $A \cap X = 0$ and such that $B = X + A$. Suppose that C is a subgroup of B and satisfying $A \subset C \subset B$. Is A necessarily a direct summand of C?*

Problem 6.6.7 (Fa87, Sp96) *Let G and H be finite groups of relatively prime order. Show that $\mathrm{Aut}(G \times H)$, the group of automorphisms of $G \times H$, is isomorphic to the direct product of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$.*

Problem 6.6.8 (Fa03) *Let p be a prime, and let G be the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. How many automorphisms does G have?*

6.7 Free Groups, Generators, and Relations

Problem 6.7.1 (Sp77) *Let $\mathbb{Q}_+$ be the multiplicative group of positive rational numbers.*

1. *Is $\mathbb{Q}_+$ torsion free?*

2. *Is $\mathbb{Q}_+$ free?*

Problem 6.7.2 (Sp86) *Prove that the additive group of $\mathbb{Q}$, the rational number field, is not finitely generated.*
Note: See also Problems 6.3.1 and 6.6.3.

Problem 6.7.3 (Fa79, Fa82) *Let G be the abelian group defined by generators x, y, and z, and relations*

$$\begin{aligned} 15x + 3y &= 0 \\ 3x + 7y + 4z &= 0 \\ 18x + 14y + 8z &= 0. \end{aligned}$$

1. *Express G as a direct product of two cyclic groups.*
2. *Express G as a direct product of cyclic groups of prime power order.*
3. *How many elements of G have order 2?*

Problem 6.7.4 (Sp82, Sp93) *Suppose that the group G is generated by elements x and y that satisfy $x^5y^3 = x^8y^5 = 1$. Is G the trivial group?*

Problem 6.7.5 (Su82) *Let G be a group with generators a and b satisfying*

$$a^{-1}b^2a = b^3, \qquad b^{-1}a^2b = a^3.$$

Is G trivial?

Problem 6.7.6 (Fa88, Fa97) *Let the group G be generated by two elements, a and b, both of order 2. Prove that G has a subgroup of index 2.*

Problem 6.7.7 (Fa89) *Let G_n be the free group on n generators. Show that G_2 and G_3 are not isomorphic.*

Problem 6.7.8 (Sp83) *Let G be an abelian group which is generated by, at most, n elements. Show that each subgroup of G is again generated by, at most, n elements.*

Problem 6.7.9 (Sp84) *Determine all finitely generated abelian groups G which have only finitely many automorphisms.*

Problem 6.7.10 (Fa89) *Let A be a finite abelian group, and m the maximum of the orders of the elements of A. Put $S = \{a \in A \mid |a| = m\}$. Prove that A is generated by S.*

6.8 Finite Groups

Problem 6.8.1 (Sp91) *List, to within isomorphism, all the finite groups whose orders do not exceed 5.*

Problem 6.8.2 (Fa84) *Show that all groups of order* $\leqslant 5$ *are commutative. Give an example of a noncommutative group of order* 6.

Problem 6.8.3 (Fa80) *Prove that any group of order* 6 *is isomorphic to either* $\mathbb{Z}_6$ *or* S_3 *(the group of permutations of three objects).*

Problem 6.8.4 (Sp87)

1. *Show that, to within isomorphism, there is just one noncyclic group* G *of order* 4.
2. *Show that the group of automorphisms of* G *is isomorphic to the permutation group* S_3.

Problem 6.8.5 (Fa88) *Find all abelian groups of order* 8, *up to isomorphism. Then identify which type occurs in each of*

1. $(\mathbb{Z}_{15})^*$,
2. $(\mathbb{Z}_{17})^* / (\pm 1)$,
3. *the roots of* $z^8 - 1$ *in* $\mathbb{C}$,
4. $\mathbf{F}_8^+$,
5. $(\mathbb{Z}_{16})^*$.

$\mathbf{F}_8$ *is the field of eight elements, and* $\mathbf{F}_8^+$ *is its underlying additive group;* R^* *is the group of invertible elements in the ring* R, *under multiplication.*

Problem 6.8.6 (Sp90, Fa93, Sp94) *Show that there are at least two nonisomorphic nonabelian groups of order* 24, *of order* 30 *and order* 40.

Problem 6.8.7 (Fa03) *List eight groups of order* 36 *and prove that they are not isomorphic.*

Problem 6.8.8 (Fa97) *Prove that if* p *is prime then every group of order* p^2 *is abelian.*

Problem 6.8.9 (Sp93) *Classify up to isomorphism all groups of order* 45.

Problem 6.8.10 (Sp02) *Let* G *be a group of order 56 having at least 7 elements of order 7. Prove that* G *has only one Sylow 2-subgroup* P, *and that all nonidentity elements of* P *have order 2.*

Problem 6.8.11 (Sp79, Sp97) *Classify all abelian groups of order* 80 *up to isomorphism.*

Problem 6.8.12 (Fa88) *Find (up to isomorphism) all groups of order* $2p$, *where* p *is a prime* ($p \geqslant 2$).

Problem 6.8.13 (Sp87) *Prove that any finite group of order* n *is isomorphic to a subgroup of* $\mathbb{O}(n)$, *the group of* $n \times n$ *orthogonal real matrices.*

Problem 6.8.14 (Fa98) *Suppose that G is a finite group such that every Sylow subgroup is normal and abelian. Show that G is abelian.*

Problem 6.8.15 (Sp01) *If G is a finite group, must $S = \{g^2 \mid g \in G\}$ be a subgroup? Provide a proof or a counterexample.*

Problem 6.8.16 (Fa99) *Let G be a finite group acting transitively on a set X of size at least 2. Prove that some element g of G acts without fixed points.*

Problem 6.8.17 (Fa02) *Let G be a finite non-Abelian group of order n. Show that there exists an integer d satisfying $2 \leqslant d \leqslant n/2$ and a set P of cardinality d, such that G acts transitively on P.*

Problem 6.8.18 (Su80, Fa96) *Prove that every finite group is isomorphic to*

1. *A group of permutations;*
2. *A group of even permutations.*

Problem 6.8.19 (Sp00) *Let G be a finite group and p a prime number. Suppose a and b are elements of G of order p such that b is not in the subgroup generated by a. Prove that G contains at least $p^2 - 1$ elements of order p.*

Problem 6.8.20 (Fa01) *Find all finite abelian groups G (up to isomorphism) such that the group of automorphisms of G has odd order.*

Problem 6.8.21 (Fa03) *Let n be a positive integer. Let $\phi(n)$ be the Euler phi function, so $\phi(n) = \#(\mathbb{Z}_n)^*$. Prove that if $\gcd(n, \phi(n)) > 1$, then there exists a noncyclic group of order n.*

6.9 Rings and Their Homomorphisms

Problem 6.9.1 (Fa80) *Let $M_{2\times2}$ be the ring of real 2×2 matrices and $S \subset M_{2\times2}$ the subring of matrices of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

1. *Exhibit an isomorphism between S and $\mathbb{C}$.*
2. *Prove that*
$$A = \begin{pmatrix} 0 & 3 \\ -4 & 1 \end{pmatrix}$$
lies in a subring isomorphic to S.
3. *Prove that there is an $X \in M_{2\times2}$ such that $X^4 + 13X = A$.*

Problem 6.9.2 (Sp03) *Let $M_{2\times2}(\mathbb{Q})$ denote the ring of 2×2 matrices with entries in $\mathbb{Q}$. Let R be the set of matrices in $M_{2\times2}(\mathbb{Q})$ that commute with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

1. *Prove that R is a subring of $M_{2\times 2}(\mathbb{Q})$.*

2. *Prove that R is isomorphic to the ring $\mathbb{Q}[x]/\langle x^2\rangle$.*

Problem 6.9.3 (Sp86) *Prove that there exists only one automorphism of the field of real numbers; namely the identity automorphism.*

Problem 6.9.4 (Sp86) *Suppose addition and multiplication are defined on $\mathbb{C}^n$, complex n-space, coordinatewise, making $\mathbb{C}^n$ into a ring. Find all ring homomorphisms of $\mathbb{C}^n$ onto $\mathbb{C}$.*

Problem 6.9.5 (Fa88) *Let R be a finite ring. Prove that there are positive integers m and n with $m > n$ such that $x^m = x^n$ for every x in R.*

Problem 6.9.6 (Sp89) *Let R be a ring with at least two elements. Suppose that for each nonzero a in R there is a unique b in R (depending on a) with $aba = a$. Show that R is a division ring.*

Problem 6.9.7 (Sp91) *Let p be a prime number and R a ring with identity containing p^2 elements. Prove that R is commutative.*

Problem 6.9.8 (Fa00) *Let R be a ring with identity, having fewer than eight elements. Prove that R is commutative.*

Problem 6.9.9 (Sp03) *Let R be the set of complex numbers of the form*

$$a + 3bi, \quad a, b \in \mathbb{Z}.$$

Prove that R is a subring of $\mathbb{C}$, and that R is an integral domain but not a unique factorization domain.

Problem 6.9.10 (Fa93) *Let R be a commutative ring with identity. Let G be a finite subgroup of R^*, the group of units of R. Prove that if R is an integral domain, then G is cyclic.*

Problem 6.9.11 (Fa98) *Let R be a finite ring with identity. Let a be an element of R which is not a zero divisor. Show that a is invertible.*

Problem 6.9.12 (Fa94) *Let R be a ring with identity, and let u be an element of R with a right inverse. Prove that the following conditions on u are equivalent:*

1. *u has more than one right inverse;*

2. *u is a zero divisor;*

3. *u is not a unit.*

Problem 6.9.13 (Su81, Sp93) *Show that no commutative ring with identity has additive group isomorphic to $\mathbb{Q}/\mathbb{Z}$.*

Problem 6.9.14 (Sp81) *Let D be an ordered integral domain and $a \in D$. Prove that*

$$a^2 - a + 1 > 0.$$

Problem 6.9.15 (Fa95) *Prove that $\mathbb{Q}[x, y]/\langle x^2 + y^2 - 1\rangle$ is an integral domain and that its field of fractions is isomorphic to the field of rational functions $\mathbb{Q}(t)$.*

Problem 6.9.16 (Sp00) *Find the cardinality of the set of all subrings of $\mathbb{Q}$, the field of rational numbers.*

6.10 Ideals

Problem 6.10.1 (Sp98) *Let A be the ring of real 2×2 matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. What are the 2-sided ideals in A?*

Problem 6.10.2 (Fa79, Fa87) *Let $M_{n\times n}(\mathbf{F})$ be the ring of $n \times n$ matrices over a field $\mathbf{F}$. Prove that it has no 2-sided ideals except $M_{n\times n}(\mathbf{F})$ and $\{0\}$.*

Problem 6.10.3 (Fa83, Su85) *Let $M_{n\times n}(\mathbf{F})$ be the ring of $n\times n$ matrices over a field $\mathbf{F}$. For $n \geqslant 1$ does there exist a ring homomorphism from $M_{(n+1)\times(n+1)}(\mathbf{F})$ onto $M_{n\times n}(\mathbf{F})$?*

Problem 6.10.4 (Sp97) *Let R be the ring of $n \times n$ matrices over a field. Suppose S is a ring and $h : R \to S$ is a homomorphism. Show that h is either injective or zero.*

Problem 6.10.5 (Sp84) *Let $\mathbf{F}$ be a field and let X be a finite set. Let $R(X, \mathbf{F})$ be the ring of all functions from X to $\mathbf{F}$, endowed with the pointwise operations. What are the maximal ideals of $R(X, \mathbf{F})$?*

Problem 6.10.6 (Fa00) *Suppose V is a vector space over a field $\mathbb{K}$. If U and W are subspaces, let $E(U, W)$ be the set of linear endomorphisms F of V over $\mathbb{K}$ with the property that the image of FU in V/W is finite dimensional. Show that $E(U, U)$ is a subring of the ring of endomorphisms of V with two-sided ideals $E(V, U)$ and $E(U, 0)$.*

Problem 6.10.7 (Sp88) *Let R be a commutative ring with identity element and $a \in R$. Let n and m be positive integers, and write $d = \gcd\{n, m\}$. Prove that the ideal of R generated by $a^n - 1$ and $a^m - 1$ is the same as the ideal generated by $a^d - 1$.*

Problem 6.10.8 (Sp89)

1. *Let R be a commutative ring with identity containing an element a with $a^3 = a + 1$. Further, let $\mathfrak{I}$ be an ideal of R of index < 5 in R. Prove that $\mathfrak{I} = R$.*
2. *Show that there exists a commutative ring with identity that has an element a with $a^3 = a + 1$ and that contains an ideal of index 5.*

Note: The term index *is used here exactly as in group theory; namely the index of $\mathfrak{I}$ in R means the order of $R/\mathfrak{I}$.*

Problem 6.10.9 (Sp90) *Let R be a commutative ring with 1, and R^* be its group of units. Suppose that the additive group of R is generated by $\{u^2 \mid u \in R^*\}$. Prove that R has, at most, one ideal $\mathfrak{I}$ for which $R/\mathfrak{I}$ has cardinality 3.*

Problem 6.10.10 (Sp01) *Find all commutative rings R with identity such that R has a unique maximal ideal and such that the group of units of R is trivial.*

Problem 6.10.11 (Fa90) *Let R be a ring with identity, and let $\mathfrak{I}$ be the left ideal of R generated by $\{ab - ba \mid a, b \in R\}$. Prove that $\mathfrak{I}$ is a two-sided ideal.*

Problem 6.10.12 (Fa99) *Let R be a ring with identity element. Suppose that $\mathfrak{I}_1$, $\mathfrak{I}_2, \ldots, \mathfrak{I}_n$ are left ideals in R such that $R = \mathfrak{I}_1 \oplus \mathfrak{I}_2 \oplus \cdots \oplus \mathfrak{I}_n$ (as additive groups). Prove that there are elements $u_i \in \mathfrak{I}_i$ such that for any elements $a_i \in \mathfrak{I}_i$, $a_i u_i = a_i$ and $a_i u_j = 0$ if $j \neq i$.*

Problem 6.10.13 (Sp95) *Suppose that R is a subring of a commutative ring S and that R is of finite index n in S. Let m be an integer that is relatively prime to n. Prove that the natural map $R/mR \to S/mS$ is a ring isomorphism.*

Problem 6.10.14 (Sp81) *Let $\mathbf{M}$ be one of the following fields: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbf{F}_9$ (the field with nine elements). Let $\mathfrak{I} \subset \mathbf{M}[x]$ be the ideal generated by $x^4 + 2x - 2$. For which choices of $\mathbf{M}$ is the ring $\mathbf{M}[x]/\mathfrak{I}$ a field?*

Problem 6.10.15 (Sp84) *Let R be a principal ideal domain and let $\mathfrak{I}$ and $\mathfrak{J}$ be nonzero ideals in R. Show that $\mathfrak{I}\mathfrak{J} = \mathfrak{I} \cap \mathfrak{J}$ if and only if $\mathfrak{I} + \mathfrak{J} = R$.*

6.11 Polynomials

Problem 6.11.1 (Fa77) *Suppose the nonzero complex number α is a root of a polynomial of degree n with rational coefficients. Prove that $1/\alpha$ is also a root of a polynomial of degree n with rational coefficients.*

Problem 6.11.2 (Su85) *By the Fundamental Theorem of Algebra, the polynomial $x^3 + 2x^2 + 7x + 1$ has three complex roots, α_1, α_2, and α_3. Compute $\alpha_1^3 + \alpha_2^3 + \alpha_3^3$.*

Problem 6.11.3 (Sp85) *Let $\zeta = e^{\frac{2\pi i}{7}}$ be a primitive 7^{th} root of unity. Find a cubic polynomial with integer coefficients having $\alpha = \zeta + \zeta^{-1}$ as a root.*

Problem 6.11.4 (Sp92, Su77, Fa81)

1. *Prove that $\alpha = \sqrt{5} + \sqrt{7}$ is algebraic over $\mathbb{Q}$, by explicitly finding a polynomial $f(x)$ in $\mathbb{Q}[x]$ of degree 4 having α as a root.*

2. *Prove that $f(x)$ is irreducible over $\mathbb{Q}$.*

Problem 6.11.5 (Fa90) *Prove that $\sqrt{2}+\sqrt[3]{3}$ is irrational.*

Problem 6.11.6 (Su85) *Let $P(z)$ be a polynomial of degree $< k$ with complex coefficients. Let $\omega_1, \ldots, \omega_k$ be the k^{th} roots of unity in $\mathbb{C}$. Prove that*

$$\frac{1}{k}\sum_{i=1}^{k} P(\omega_i) = P(0).$$

Problem 6.11.7 (Fa95) *Let $f(x) \in \mathbb{Q}[x]$ be a polynomial with rational coefficients. Show that there is a $g(x) \in \mathbb{Q}[x]$, $g \neq 0$, such that $f(x)g(x) = a_2x^2 + a_3x^3 + a_5x^5 + \cdots + a_px^p$ is a polynomial in which only prime exponents appear.*

Problem 6.11.8 (Fa91) *Let $\mathfrak{I}$ be the ideal in the ring $\mathbb{Z}[x]$ generated by $x-7$ and 15. Prove that the quotient ring $\mathbb{Z}[x]/\mathfrak{I}$ is isomorphic to $\mathbb{Z}_{15}$.*

Problem 6.11.9 (Fa92) *Let $\mathfrak{I}$ denote the ideal in $\mathbb{Z}[x]$, the ring of polynomials with coefficients in $\mathbb{Z}$, generated by x^3+x+1 and 5. Is $\mathfrak{I}$ a prime ideal?*

Problem 6.11.10 (Su77) *In the ring $\mathbb{Z}[x]$ of polynomials in one variable over the integers, show that the ideal $\mathfrak{I}$ generated by 5 and x^2+2 is a maximal ideal.*

Problem 6.11.11 (Sp78) *let $\mathbb{Z}_n$ denote the ring of integers modulo n. Let $\mathbb{Z}_n[x]$ be the ring of polynomials with coefficients in $\mathbb{Z}_n$. Let $\mathfrak{I}$ denote the ideal in $\mathbb{Z}_n[x]$ generated by x^2+x+1.*

1. *For which values of n, $1 \leqslant n \leqslant 10$, is the quotient ring $\mathbb{Z}_n[x]/\mathfrak{I}$ a field?*
2. *Give the multiplication table for $\mathbb{Z}_2/\mathfrak{I}$.*

Problem 6.11.12 (Sp86) *Let $\mathbb{Z}$ be the ring of integers, p a prime, and $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field of p elements. Let x be an indeterminate, and set $R_1 = \mathbf{F}_p[x]/\langle x^2-2\rangle$, $R_2 = \mathbf{F}_p[x]/\langle x^2-3\rangle$. Determine whether the rings R_1 and R_2 are isomorphic in each of the cases $p = 2, 5, 11$.*

Problem 6.11.13 (Fa79, Su80, Fa82) *Consider the polynomial ring $\mathbb{Z}[x]$ and the ideal $\mathfrak{I}$ generated by 7 and $x-3$.*

1. *Show that for each $r \in \mathbb{Z}[x]$, there is an integer α satisfying $0 \leqslant \alpha \leqslant 6$ such that $r-\alpha \in \mathfrak{I}$.*
2. *Find α in the special case $r = x^{250} + 15x^{14} + x^2 + 5$.*

Problem 6.11.14 (Fa96) *Let $\mathbb{Z}[x]$ be the ring of polynomials in the indeterminate x with coefficients in the ring $\mathbb{Z}$ of integers. Let $\mathfrak{I} \subset \mathbb{Z}[x]$ be the ideal generated by 13 and $x-4$. Find an integer m such that $0 \leqslant m \leqslant 12$ and*

$$(x^{26}+x+1)^{73} - m \in \mathfrak{I}.$$

Problem 6.11.15 (Fa03) *Give an example, with proof, of a nonconstant irreducible polynomial $f(x)$ over $\mathbb{Q}$ with the property that $f(x)$ does not factor into linear factors over the field $K = \mathbb{Q}[x]/(f(x))$.*

Problem 6.11.16 (Sp77)

1. *In $\mathbb{R}[x]$, consider the set of polynomials $f(x)$ for which $f(2) = f'(2) = f''(2) = 0$. Prove that this set forms an ideal and find its monic generator.*
2. *Do the polynomials such that $f(2) = 0$ and $f'(3) = 0$ form an ideal?*

Problem 6.11.17 (Sp94) *Find all automorphisms of $\mathbb{Z}[x]$, the ring of polynomials over $\mathbb{Z}$.*

Problem 6.11.18 (Su78) *Let R denote the ring of polynomials over a field $\mathbf{F}$. Let $p_1, \dots, p_n$ be elements of R. Prove that the greatest common divisor of $p_1, \dots, p_n$ is 1 if and only if there is an $n \times n$ matrix over R of determinant 1 whose first row is $(p_1, \dots, p_n)$.*

Problem 6.11.19 (Sp79) *Let $f(x)$ be a polynomial over $\mathbb{Z}_p$, the field of integers* $\bmod p$. *Let $g(x) = x^p - x$. Show that the greatest common divisor of $f(x)$ and $g(x)$ is the product of the distinct linear factors of $f(x)$.*

Problem 6.11.20 (Su79) *Let $\mathbf{F}$ be a subfield of a field $\mathbf{K}$. Let p and q be polynomials over $\mathbf{F}$. Prove that their greatest common divisor in the ring of polynomials over $\mathbf{F}$ is the same as their* gcd *in the ring of polynomials over $\mathbf{K}$.*

Problem 6.11.21 (Su81, Su82) *Show that $x^{10}+x^9+x^8+\cdots+x+1$ is irreducible over $\mathbb{Q}$. How about $x^{11}+x^{10}+\cdots+x+1$?*

Problem 6.11.22 (Su84) *Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}[x]$ the polynomial ring over $\mathbb{Z}$. Show that*

$$x^6 + 539x^5 - 511x + 847$$

is irreducible in $\mathbb{Z}[x]$.

Problem 6.11.23 (Sp82) *Prove that the polynomial x^4+x+1 is irreducible over $\mathbb{Q}$.*

Problem 6.11.24 (Fa83, Fa86) *Prove that if p is a prime number, then the polynomial*

$$f(x) = x^{p-1} + x^{p-2} + \cdots + 1$$

is irreducible in $\mathbb{Q}[x]$.

Problem 6.11.25 (Sp96) *Prove that $f(x) = x^4+x^3+x^2+6x+1$ is irreducible over $\mathbb{Q}$.*

Problem 6.11.26 (Sp01) *Prove that the polynomial $f(x) = 16x^5 - 125x^4 + 50x^3 - 100x^2 + 75x + 25$ is irreducible over the rationals.*

Problem 6.11.27 (Su84) *Let $\mathbb{Z}_3$ be the field of integers* mod 3 *and $\mathbb{Z}_3[x]$ the corresponding polynomial ring. Decompose $x^3 + x + 2$ into irreducible factors in $\mathbb{Z}_3[x]$.*

Problem 6.11.28 (Sp85) *Factor $x^4 + x^3 + x + 3$ completely in $\mathbb{Z}_5[x]$.*

Problem 6.11.29 (Fa85)
1. *How many different monic irreducible polynomials of degree 2 are there over the field $\mathbb{Z}_5$?*
2. *How many different monic irreducible polynomials of degree 3 are there over the field $\mathbb{Z}_5$?*

Problem 6.11.30 (Sp78) *Is x^4+1 irreducible over the field of real numbers? The field of rational numbers? A field with 16 elements?*

Problem 6.11.31 (Sp81) *Decompose $x^4 - 4$ and $x^3 - 2$ into irreducibles over $\mathbb{R}$, over $\mathbb{Z}$, and over $\mathbb{Z}_3$ (the integers modulo 3).*

Problem 6.11.32 (Fa84) *Let a be an element in a field $\mathbf{F}$ and let p be a prime. Assume a is not a p^{th} power. Show that the polynomial $x^p - a$ is irreducible in $\mathbf{F}[x]$.*

Problem 6.11.33 (Sp92) *Let p be a prime integer, $p \equiv 3 \pmod 4$, and let $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$. If $x^4 + 1$ factors into a product $g(x)h(x)$ of two quadratic polynomials in $\mathbf{F}_p[x]$, prove that $g(x)$ and $h(x)$ are both irreducible over $\mathbf{F}_p$.*

Problem 6.11.34 (Fa88) *Let n be a positive integer and let f be a polynomial in $\mathbb{R}[x]$ of degree n. Prove that there are real numbers $a_0, a_1, \ldots, a_n$, not all equal to zero, such that the polynomial*

$$\sum_{i=0}^{n} a_i x^{2^i}$$

is divisible by f.

Problem 6.11.35 (Fa89) *Let $\mathbf{F}$ be a field, $\mathbf{F}[x]$ the polynomial ring in one variable over $\mathbf{F}$, and R a subring of $\mathbf{F}[x]$ with $\mathbf{F} \subset R$. Prove that there exists a finite set $\{f_1, f_2, \ldots, f_n\}$ of elements of $\mathbf{F}[x]$ such that $R = \mathbf{F}[f_1, f_2, \ldots, f_n]$.*

Problem 6.11.36 (Sp87) *Let $\mathbf{F}$ be a finite field with q elements and let x be an indeterminate. For f a polynomial in $\mathbf{F}[x]$, let φ_f denote the corresponding function of $\mathbf{F}$ into $\mathbf{F}$, defined by $\varphi_f(a) = f(a)$, $(a \in \mathbf{F})$. Prove that if φ is any function of $\mathbf{F}$ into $\mathbf{F}$, then there is an f in $\mathbf{F}[x]$ such that $\varphi = \varphi_f$. Prove that f is uniquely determined by φ to within addition of a multiple of $x^q - x$.*

6.12 Fields and Their Extensions

Problem 6.12.1 (Su78, Fa87, Sp93) *Let R be the set of 2×2 matrices of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a, b are elements of a given field $\mathbf{F}$. Show that with the usual matrix operations, R is a commutative ring with identity. For which of the following fields $\mathbf{F}$ is R a field: $\mathbf{F} = \mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_5$, $\mathbb{Z}_7$?

Problem 6.12.2 (Fa83) *Prove that every finite integral domain is a field.*

Problem 6.12.3 (Sp77, Sp78) *Let $\mathbf{F} \subset \mathbf{K}$ be fields, and a and b elements of $\mathbf{K}$ which are algebraic over $\mathbf{F}$. Show that $a + b$ is algebraic over $\mathbf{F}$.*

Problem 6.12.4 (Fa78, Fa85) *Prove that every finite multiplicative group of complex numbers is cyclic.*

Problem 6.12.5 (Sp87, Fa95) *Let $\mathbf{F}$ be a field. Prove that every finite subgroup of the multiplicative group of nonzero elements of $\mathbf{F}$ is cyclic.*

Problem 6.12.6 (Fa02) *Let $\mathbf{K}$ be a field such that the additive group of $\mathbf{K}$ is finitely generated as a group. Prove that $\mathbf{K}$ is finite.*

Problem 6.12.7 (Sp85) *Let $\mathbf{F} = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$. Prove that $\mathbf{F}$ is a field and each element in $\mathbf{F}$ has a unique representation as $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ with $a, b, c \in \mathbb{Q}$. Find $(1 - \sqrt[3]{2})^{-1}$ in $\mathbf{F}$.*

Problem 6.12.8 (Sp85) *Let $\mathbf{F}$ be a finite field. Give a complete proof of the fact that the number of elements of $\mathbf{F}$ is of the form p^r, where $p \geqslant 2$ is a prime number and r is an integer $\geqslant 1$.*

Problem 6.12.9 (Fa02) *Let $\mathbf{K}$ be a field and $\mathbf{L} \subset \mathbf{K}$ a subfield containing $\{a^2 \mid a \in \mathbf{K}\}$. Prove that if the characteristic of $\mathbf{K}$ is not 2 then $\mathbf{L} = \mathbf{K}$ and that the same conclusion holds for finite fields $\mathbf{K}$ of characteristic 2, but not in general for fields of characteristic 2.*

Problem 6.12.10 (Su85) *Let $\mathbf{F}$ be a field of characteristic $p > 0$, $p \neq 3$. If α is a zero of the polynomial $f(x) = x^p - x + 3$ in an extension field of $\mathbf{F}$, show that $f(x)$ has p distinct zeros in the field $\mathbf{F}(\alpha)$.*

Problem 6.12.11 (Fa99) *Let $\mathbf{K}$ be the field $\mathbb{Q}(\sqrt[10]{2})$. Prove that $\mathbf{K}$ has degree 10 over $\mathbb{Q}$, and that the group of automorphisms of $\mathbf{K}$ has order 2.*

Problem 6.12.12 (Fa85) *Let $f(x) = x^5 - 8x^3 + 9x - 3$ and $g(x) = x^4 - 5x^2 - 6x + 3$. Prove that there is an integer d such that the polynomials $f(x)$ and $g(x)$ have a common root in the field $\mathbb{Q}(\sqrt{d})$. What is d?*

Problem 6.12.13 (Fa86) *Let* $\mathbf{F}$ *be a field containing* $\mathbb{Q}$ *such that* $[\mathbf{F} : \mathbb{Q}] = 2$. *Prove that there exists a unique integer* m *such that* m *has no multiple prime factors and* $\mathbf{F}$ *is isomorphic to* $\mathbb{Q}(\sqrt{m})$.

Problem 6.12.14 (Sp96) *Exhibit infinitely many pairwise nonisomorphic quadratic extensions of* $\mathbb{Q}$ *and show they are pairwise nonisomorphic.*

Problem 6.12.15 (Fa94) *Let* $\mathbb{Q}$ *be the field of rational numbers. For* θ *a real number, let* $\mathbf{F}_\theta = \mathbb{Q}(\sin\theta)$ *and* $\mathbf{E}_\theta = \mathbb{Q}\left(\sin\frac{\theta}{3}\right)$. *Show that* $\mathbf{E}_\theta$ *is an extension field of* $\mathbf{F}_\theta$, *and determine all possibilities for* $\dim_{\mathbf{F}_\theta}\mathbf{E}_\theta$.

Problem 6.12.16 (Fa98) *Show that the field* $\mathbb{Q}(t_1, \ldots, t_n)$ *of rational functions in* n *variables over the rational numbers is isomorphic to a subfield of* $\mathbb{R}$.

Problem 6.12.17 (Sp99) *Let* $f(x) \in \mathbb{Q}[x]$ *be an irreducible polynomial of degree* $n \geqslant 3$. *Let* L *be the splitting field of* f, *and let* $\alpha \in L$ *be a zero of* f. *Given that* $[L : \mathbb{Q}] = n!$, *prove that* $\mathbb{Q}(\alpha^4) = \mathbb{Q}(\alpha)$.

Problem 6.12.18 (Fa01) *Let* $\mathbf{K}$ *be a field. For what pairs of positive integers* (a, b) *is the subring* $\mathbf{K}[t^a, t^b]$ *of* $\mathbf{K}[t]$ *a unique factorization domain?*

Problem 6.12.19 (Sp95) *Let* $\mathbf{F}$ *be a finite field of cardinality* p^n, *with* p *prime and* $n > 0$, *and let* G *be the group of invertible* 2×2 *matrices with coefficients in* $\mathbf{F}$.

1. *Prove that* G *has order* $(p^{2n} - 1)(p^{2n} - p^n)$.
2. *Show that any* p*-Sylow subgroup of* G *is isomorphic to the additive group of* $\mathbf{F}$.

Problem 6.12.20 (Sp02) *Let* p *be a prime and* k, n *positive integers. Prove that the group* $GL_n(\mathbf{F}_p)$ *of invertible* $n \times n$ *matrices over* $\mathbf{F}_p$ *(the field of* p *elements) contains an element of order* p^k *if and only if* $n > p^{k-1}$.

Problem 6.12.21 (Fa01) *Let* $\mathbf{F}$ *be a field, and let* G *be the group of* 2×2 *upper-triangular matrices over* $\mathbf{F}$ *of determinant* 1.

1. *Determine the commutator subgroup of* G.
2. *Suppose* $\mathbf{F}$ *is the finite field of order* p^k. *Determine for this case the minimum number of generators for the subgroup found in (i).*

Problem 6.12.22 (Fa94) *Let* p *be an odd prime and* $\mathbf{F}_p$ *the field of* p *elements. How many elements of* $\mathbf{F}_p$ *have square roots in* $\mathbf{F}_p$? *How many have cube roots in* $\mathbf{F}_p$?

Problem 6.12.23 (Sp94) *Let* $\mathbf{F}$ *be a finite field with* q *elements. Say that a function* $f : \mathbf{F} \to \mathbf{F}$ *is a* polynomial function *if there are elements* $a_0, a_1, \ldots, a_n$ *of* $\mathbf{F}$ *such that* $f(x) = a_0 + a_1x + \cdots + a_nx^n$ *for all* $x \in \mathbf{F}$. *How many polynomial functions are there?*

Problem 6.12.24 (Sp95) *Let* $\mathbf{F}$ *be a finite field, and suppose that the subfield of* $\mathbf{F}$ *generated by* $\{x^3 \mid x \in \mathbf{F}\}$ *is different from* $\mathbf{F}$. *Show that* $\mathbf{F}$ *has cardinality* 4.

Problem 6.12.25 (Sp97) *Suppose that* A *is a commutative algebra with identity over* $\mathbb{C}$ *(i.e.,* A *is a commutative ring containing* $\mathbb{C}$ *as a subring with identity). Suppose further that* $a^2 \neq 0$ *for all nonzero elements* $a \in A$. *Show that if the dimension of* A *as a vector space over* $\mathbb{C}$ *is finite and at least two, then the equations* $a^2 = a$ *is satisfied by at least three distinct elements* $a \in A$.

Problem 6.12.26 (Sp02) *Let* A *be a commutative ring with* 1. *Prove that the group* A^* *of units of* A *does not have exactly 5 elements.*

Problem 6.12.27 (Fa00) *Suppose* $\mathbf{K}$ *is a field and* R *is a nonzero* $\mathbf{K}$*-algebra generated by two elements* a *and* b *which satisfy* $a^2 = b^2 = 0$ *and* $(a+b)^2 = 1$. *Show* R *is isomorphic to* $M_2(\mathbf{K})$ *(the algebra of* 2×2 *matrices over* K*).*

6.13 Elementary Number Theory

Problem 6.13.1 (Fa86) *Prove that if six people are riding together in an Evans Hall elevator, there is either a three-person subset of mutual friends (each knows the other two) or a three-person subset of mutual strangers (each knows neither of the other two).*

Problem 6.13.2 (Fa03) *Let* $u_{m,n}$ *be an array of numbers for* $1 \leq m \leq N$ *and* $1 \leq n \leq N$. *Suppose that* $u_{m,n} = 0$ *when* m *is* 1 *or* N, *or when* n *is* 1 *or* N. *Suppose also that*

$$u_{m,n} = \frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1}\right)$$

whenever $1 < m < N$ *and* $1 < n < N$. *Show that all the* $u_{m,n}$ *are zero.*

Problem 6.13.3 (Sp02) *How many functions* $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ *have a range of size exactly 3 ?*

Problem 6.13.4 (Sp98) *Let* $m \geqslant 0$ *be an integer. Let* $a_1, a_2, \ldots, a_m$ *be integers and let*

$$f(x) = \sum_{i=1}^{m} \frac{a_i x^i}{i!}$$

Show that if $d \geqslant 0$ *is an integer then* $f(x)^d/d!$ *can be expressed in the form*

$$\sum_{i=0}^{md} \frac{b_i x^i}{i!}.$$

where the b_i *are integers.*

Problem 6.13.5 (Fa03) *Let $f : [0,1] \to [0,1]$ be an increasing (not strictly increasing) function such that*

$$f\left(\sum_{j=1}^{\infty} a_j 3^{-j}\right) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$$

whenever the a_j are 0 or 2. Prove that there is a constant C_0 such that

$$|f(x) - f(y)| \le C_0 |x - y|^{\log 2/\log 3}$$

for all $x, y \in [0, 1]$.

Problem 6.13.6 (Sp77) *Let p be an odd prime. Let $Q(p)$ be the set of integers a, $0 \leqslant a \leqslant p-1$, for which the congruence*

$$x^2 \equiv a \pmod{p}$$

has a solution. Show that $Q(p)$ has cardinality $(p+1)/2$.

Problem 6.13.7 (Su77) *Let p be an odd prime. If the congruence $x^2 \equiv -1 \pmod{p}$ has a solution, show that $p \equiv 1 \pmod{4}$.*

Problem 6.13.8 (Sp80) *Let $n \geqslant 2$ be an integer such that $2^n + n^2$ is prime. Prove that*

$$n \equiv 3 \pmod{6}.$$

Problem 6.13.9 (Fa77)

1. *Show that the set of all units in a ring with unity form a group under multiplication. (A* unit *is an element having a two-sided multiplicative inverse.)*
2. *In the ring $\mathbb{Z}_n$ of integers* mod$n$*, show that k is a unit if and only if k and n are relatively prime.*
3. *Suppose $n = pq$, where p and q are primes. Prove that the number of units in $\mathbb{Z}_n$ is $(p-1)(q-1)$.*

Problem 6.13.10 (Su79) *Which rational numbers t are such that*

$$3t^3 + 10t^2 - 3t$$

is an integer?

Problem 6.13.11 (Fa96) *Show the denominator of $\binom{1/2}{n}$ is a power of 2 for all integers n.*

Problem 6.13.12 (Su82) *Let n be a positive integer.*

1. *Show that the binomial coefficient*

$$c_n = \binom{2n}{n}$$

is even.

2. *Prove that c_n is divisible by 4 if and only if n is not a power of 2.*

Problem 6.13.13 (Sp83) *Suppose that $n > 1$ is an integer. Prove that the sum*

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer.

Problem 6.13.14 (Fa84, Fa96) *Let* gcd *abbreviate greatest common divisor and* lcm *abbreviate least common multiple. For three nonzero integers a, b, c, show that*

$$\gcd\{a, \operatorname{lcm}\{b, c\}\} = \operatorname{lcm}\{\gcd\{a, b\}, \gcd\{a, c\}\}.$$

Problem 6.13.15 (Sp92) *Let $a_1, a_2, \ldots, a_{10}$ be integers with $1 \leqslant a_i \leqslant 25$, for $1 \leqslant i \leqslant 10$. Prove that there exist integers $n_1, n_2, \ldots, n_{10}$, not all zero, such that*

$$\prod_{i=1}^{10} a_i^{n_i} = 1.$$

Problem 6.13.16 (Su83) *The number* 21982145917308330487013369 *is the thirteenth power of a positive integer. Which positive integer?*

Problem 6.13.17 (Sp96) *Determine the rightmost decimal digit of*

$$A = 17^{17^{17}}.$$

Problem 6.13.18 (Sp88) *Determine the last digit of*

$$23^{23^{23^{23}}}$$

in the decimal system.

Problem 6.13.19 (Sp88) *Show that one can represent the set of nonnegative integers, $\mathbb{Z}_+$, as the union of two disjoint subsets N_1 and N_2 ($N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2 = \mathbb{Z}_+$) such that neither N_1 nor N_2 contains an infinite arithmetic progression.*

Problem 6.13.20 (Fa89) *Let φ be Euler's totient function; so if n is a positive integer, then $\varphi(n)$ is the number of integers m for which $1 \leqslant m \leqslant n$ and* $\gcd\{n, m\} = 1$. *Let a and k be two integers, with $a > 1$, $k > 0$. Prove that k divides $\varphi(a^k - 1)$.*

Problem 6.13.21 (Sp90) *Determine the greatest common divisor of the elements of the set* $\{n^{13} - n \mid n \in \mathbb{Z}\}$.

Problem 6.13.22 (Sp91) *For* n *a positive integer, let* $d(n)$ *denote the number of positive integers that divide* n. *Prove that* $d(n)$ *is odd if and only if* n *is a perfect square.*

Problem 6.13.23 (Fa01) *Which of the numbers 0,1,2,3,4,5,6,7,8,9 occur as the last digit of* n^n *for infinitely many positive integers* n*?*

Problem 6.13.24 (Sp03) *Let* $N = 30030$, *which is the product of the first six primes. How many nonnegative integers* x *less than* N *have the property that* N *divides* $x^3 - 1$*?*

7
Linear Algebra

7.1 Vector Spaces

Problem 7.1.1 (Sp99) *Let p, q, r and s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly dependent?*

1. *At 1 each of the polynomials has the value 0.*
2. *At 0 each of the polynomials has the value 1.*

Problem 7.1.2 (Sp03) *For an analytic function h on $\mathbb{C}$, let $h^{(i)}$ denote its i-th derivative, with $h^{(0)} = h$. Suppose that f and g are analytic functions on $\mathbb{C}$ satisfying*

$$\begin{aligned} f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0 f^{(0)} &= 0 \\ g^{(m)} + b_{m-1}g^{(m-1)} + \cdots + b_0 g &= 0 \end{aligned}$$

for some constants $a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1} \in \mathbb{C}$. Show that the product function $F = fg$ satisfies

$$c_{mn}F^{(mn)} + c_{mn-1}F^{(mn-1)} + \cdots + c_0 F = 0$$

for some constants $c_0, \dots, c_{mn} \in \mathbb{C}$ not all zero.

Problem 7.1.3 (Su79, Sp82, Sp83, Su84, Fa91, Fa98) *Let $\mathbf{F}$ be a finite field with q elements and let V be an n-dimensional vector space over $\mathbf{F}$.*

1. *Determine the number of elements in V.*

2. *Let $GL_n(\mathbf{F})$ denote the group of all $n \times n$ nonsingular matrices over $\mathbf{F}$. Determine the order of $GL_n(\mathbf{F})$.*

3. *Let $SL_n(\mathbf{F})$ denote the subgroup of $GL_n(\mathbf{F})$ consisting of matrices with determinant 1. Find the order of $SL_n(\mathbf{F})$.*

Problem 7.1.4 (Sp01) *Let $\mathbf{F}$ be a finite field of order q, and let V be a two dimensional vector space over $\mathbf{F}$. Find the number of endomorphisms of V that fix at least one nonzero vector.*

Problem 7.1.5 (Sp97) *Let $GL_2(\mathbb{Z}_m)$ denote the multiplicative group of invertible 2×2 matrices over the ring of integers modulo m. Find the order of $GL_2(\mathbb{Z}_{p^n})$ for each prime p and positive integer n.*

Problem 7.1.6 (Fa00) *Let $\mathbf{F}_p$ denote the field of p elements (p prime). Let n be a positive integer. Prove that there is a transformation $A \in GL_n(\mathbf{F}_p)$ (the group of invertible linear transformations from $(\mathbf{F}_p)^n$ into itself) which, as a permutation of the nonzero vectors of $(\mathbf{F}_p)^n$, acts as a single cycle of length $p^n - 1$.*

Problem 7.1.7 (Sp96) *Let G be the group of 2×2 matrices with determinant 1 over the four-element field $\mathbf{F}$. Let S be the set of lines through the origin in $\mathbf{F}^2$. Show that G acts* faithfully *on S. (The action is faithful if the only element of G which fixes every element of S is the identity.)*

Problem 7.1.8 (Su77) *Prove the following statements about the polynomial ring $\mathbf{F}[x]$, where $\mathbf{F}$ is any field.*

1. *$\mathbf{F}[x]$ is a vector space over $\mathbf{F}$.*

2. *The subset $\mathbf{F}_n[x]$ of polynomials of degree $\leqslant n$ is a subspace of dimension $n+1$ in $\mathbf{F}[x]$.*

3. *The polynomials $1, x-a, \ldots, (x-a)^n$ form a basis of $\mathbf{F}_n[x]$ for any $a \in \mathbf{F}$.*

Problem 7.1.9 (Sp02) *Let U, V, W be finite-dimensional subspaces of a vector space. Prove that* $\dim(U)+\dim(V)+\dim(W)-\dim(U+V+W) \geq \max\{\dim(U\cap V), \dim(U \cap W), \dim(V \cap W)\}$.

Problem 7.1.10 (Su84) *Suppose V is an n-dimensional vector space over the field $\mathbf{F}$. Let $W \subset V$ be a subspace of dimension $r < n$. Show that*

$$W = \bigcap \{U \mid U \text{ is an } (n-1)-\text{dimensional subspace of } V \text{ and } W \subset U\}.$$

Problem 7.1.11 (Sp80, Fa89) *Show that a vector space over an infinite field cannot be the union of a finite number of proper subspaces.*

Problem 7.1.12 (Fa88) *Let A be a complex $n \times n$ matrix, and let $C(A)$ be the* commutant *of A; that is, the set of complex $n \times n$ matrices B such that $AB = BA$. (It is obviously a subspace of $M_{n\times n}$, the vector space of all complex $n \times n$ matrices.) Prove that* $\dim C(A) \geqslant n$.

Problem 7.1.13 (Sp89, Fa97) *Let S be the subspace of $M_{n\times n}$ (the vector space of all real $n \times n$ matrices) generated by all matrices of the form $AB - BA$ with A and B in $M_{n\times n}$. Prove that* $\dim(S) = n^2 - 1$.

Problem 7.1.14 (Sp90) *Let A and B be subspaces of a finite-dimensional vector space V such that $A + B = V$. Write $n = \dim V$, $a = \dim A$, and $b = \dim B$. Let S be the set of those endomorphisms f of V for which $f(A) \subset A$ and $f(B) \subset B$. Prove that S is a subspace of the set of all endomorphisms of V, and express the dimension of S in terms of n, a, and b.*

Problem 7.1.15 (Sp81) *Let T be a linear transformation of a vector space V into itself. Suppose $x \in V$ is such that $T^m x = 0$, $T^{m-1}x \neq 0$ for some positive integer m. Show that $x, Tx, \ldots, T^{m-1}x$ are linearly independent.*

Problem 7.1.16 (Fa97) *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct real numbers. Show that the n exponential functions $e^{\alpha_1 t}, e^{\alpha_2 t}, \ldots, e^{\alpha_n t}$ are linearly independent over the real numbers.*

Problem 7.1.17 (Su83) *Let V be a real vector space of dimension n with a positive definite inner product. We say that two bases (a_i) and (b_i) have the same orientation if the matrix of the change of basis from (a_i) to (b_i) has a positive determinant. Suppose now that (a_i) and (b_i) are orthonormal bases with the same orientation. Show that $(a_i + 2b_i)$ is again a basis of V with the same orientation as (a_i).*

7.2 Rank and Determinants

Problem 7.2.1 (Sp78, Fa82, Fa86) *Let M be a matrix with entries in a field* $\mathbf{F}$. *The* row rank *of M over $\mathbf{F}$ is the maximal number of rows which are linearly independent (as vectors) over $\mathbf{F}$. The* column rank *is similarly defined using columns instead of rows.*

1. *Prove row rank = column rank.*
2. *Find a maximal linearly independent set of columns of*

$$\begin{pmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & -4 & 4 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

taking $\mathbf{F} = \mathbb{R}$.

3. *If* $\mathbb{F}$ *is a subfield of* $\mathbb{K}$*, and* M *has entries in* $\mathbb{F}$*, how is the row rank of* M *over* $\mathbb{F}$ *related to the row rank of* M *over* $\mathbb{K}$*?*

Problem 7.2.2 (Fa02) *Let the* $n \times n$ *real matrix* A *be diagonalizable and have a one-dimensional null space. Prove that a nonzero left null vector of* A *cannot be orthogonal to a nonzero right null vector of* A*.*

Problem 7.2.3 (Su85, Fa89) *Let* A *be an* $n \times n$ *real matrix and* A^t *its transpose. Show that* $A^t A$ *and* A^t *have the same range.*

Problem 7.2.4 (Sp97) *Suppose that* P *and* Q *are* $n \times n$ *matrices such that* $P^2 = P$*,* $Q^2 = Q$*, and* $1 - (P + Q)$ *is invertible. Show that* P *and* Q *have the same rank.*

Problem 7.2.5 (Fa03) *Let* $A(m, n)$ *be the* $m \times n$ *matrix with entries*

$$a_{ij} = j^i \quad (0 \le i \le m-1,\ 0 \le j \le n-1),$$

where $0^0 = 1$ *by definition. Regarding the entries of* $A(m, n)$ *as representing congruence classes (mod* p*), determine the rank of* $A(m, n)$ *over the finite field* $\mathbb{F}_p = \mathbb{Z}_p$ *for all* $m, n \geq 1$ *and all primes* p*.*

Problem 7.2.6 (Sp01) *Let* A *be an* $n \times n$ *matrix over a field* K*. Prove that*

$$\operatorname{rank} A^2 - \operatorname{rank} A^3 \leqslant \operatorname{rank} A - \operatorname{rank} A^2$$

Problem 7.2.7 (Sp91) *Let* T *be a real, symmetric,* $n \times n$*,* tridiagonal *matrix:*

$$T = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}$$

(All entries not on the main diagonal or the diagonals just above and below the main one are zero.) Assume $b_j \neq 0$ *for all* j*.*

Prove:

1. $\operatorname{rank} T \geqslant n - 1$.
2. T *has* n *distinct eigenvalues.*

Problem 7.2.8 (Sp83) *Let* $A = (a_{ij})$ *be an* $n \times n$ *real matrix satisfying the conditions:*

$$a_{ii} > 0 \quad (1 \leqslant i \leqslant n),$$

$$a_{ij} \leqslant 0 \quad (i \neq j,\ 1 \leqslant i, j \leqslant n),$$

$$\sum_{i=1}^{n} a_{ij} > 0 \quad (1 \leqslant j \leqslant n).$$

Show that $\det(A) > 0$.

Problem 7.2.9 (Sp91) *Let $A = (a_{ij})_{i,j=1}^r$ be a square matrix with integer entries.*

1. *Prove that if an integer n is an eigenvalue of A, then n is a divisor of* $\det A$, *the determinant of A.*

2. *Suppose that n is an integer and that each row of A has sum n:*

$$\sum_{j=1}^{r} a_{ij} = n, \qquad 1 \leqslant i \leqslant r.$$

Prove that n is a divisor of $\det A$.

Problem 7.2.10 (Fa01) *Let A be a symmetric $n \times n$ matrix over $\mathbb{R}$ of rank $n-1$. Prove there is a k in $\{1, 2, \ldots, n\}$ such that the matrix resulting from deletion of the k^{th} row and k^{th} column from A has rank $n-1$.*

Problem 7.2.11 (Fa84) *Let $\mathbb{R}[x_1, \ldots, x_n]$ be the polynomial ring over the real field $\mathbb{R}$ in the n variables $x_1, \ldots, x_n$. Let the matrix A be the $n \times n$ matrix whose i^{th} row is $(1, x_i, x_i^2, \ldots, x_i^{n-1})$, $i = 1, \ldots, n$. Show that*

$$\det A = \prod_{i>j} (x_i - x_j).$$

Problem 7.2.12 (Sp77) *A matrix of the form*

$$\begin{pmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{pmatrix}$$

where the a_i are complex numbers, is called a Vandermonde matrix.

1. *Prove that the Vandermonde matrix is invertible if $a_0, a_1, \ldots, a_n$ are all different.*

2. *If $a_0, a_1, \ldots, a_n$ are all different, and $b_0, b_1, \ldots, b_n$ are complex numbers, prove that there is a unique polynomial f of degree n with complex coefficients such that $f(a_0) = b_0$, $f(a_1) = b_1, \ldots, f(a_n) = b_n$.*

Problem 7.2.13 (Fa03) *Let A and B be $n \times n$ complex unitary matrices. Prove that $|\det(A + B)| \leq 2^n$.*

Problem 7.2.14 (Sp90) *Give an example of a continuous function $v : \mathbb{R} \to \mathbb{R}^3$ with the property that $v(t_1)$, $v(t_2)$, and $v(t_3)$ form a basis for $\mathbb{R}^3$ whenever t_1, t_2, and t_3 are distinct points of $\mathbb{R}$.*

Problem 7.2.15 (Fa95) *Let $f_1, f_2, \dots, f_n$ be continuous real valued functions on $[a, b]$. Show that the set $\{f_1, \dots, f_n\}$ is linearly dependent on $[a, b]$ if and only if*

$$\det\left(\int_a^b f_i(x)f_j(x)dx\right) = 0 .$$

Problem 7.2.16 (Fa81) *Let $M_{2\times2}$ be the vector space of all real 2×2 matrices. Let*

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$$

and define a linear transformation $L : M_{2\times2} \to M_{2\times2}$ by $L(X) = AXB$. Compute the trace and the determinant of L.

Problem 7.2.17 (Su82) *Let V be the vector space of all real 3×3 matrices and let A be the diagonal matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate the determinant of the linear transformation T on V defined by $T(X) = \frac{1}{2}(AX + XA)$.

Problem 7.2.18 (Sp80) *Let $M_{3\times3}$ denote the vector space of real 3×3 matrices. For any matrix $A \in M_{3\times3}$, define the linear operator $L_A : M_{3\times3} \to M_{3\times3}$, $L_A(B) = AB$. Suppose that the determinant of A is 32 and the minimal polynomial is $(t-4)(t-2)$. What is the trace of L_A?*

Problem 7.2.19 (Su81) *Let S denote the vector space of real $n\times n$ skew-symmetric matrices. For a nonsingular matrix A, compute the determinant of the linear map $T_A : S \to S$, $T_A(X) = AXA^t$.*

Problem 7.2.20 (Fa94) *Let $M_{7\times7}$ denote the vector space of real 7×7 matrices. Let A be a diagonal matrix in $M_{7\times7}$ that has $+1$ in four diagonal positions and -1 in three diagonal positions. Define the linear transformation T on $M_{7\times7}$ by $T(X) = AX - XA$. What is the dimension of the range of T?*

Problem 7.2.21 (Fa93) *Let $\mathbf{F}$ be a field. For m and n positive integers, let $M_{m\times n}$ be the vector space of $m \times n$ matrices over $\mathbf{F}$. Fix m and n, and fix matrices A and B in $M_{m\times n}$. Define the linear transformation T from $M_{n\times m}$ to $M_{m\times n}$ by*

$$T(X) = AXB.$$

Prove that if $m \neq n$, then T is not invertible.

7.3 Systems of Equations

Problem 7.3.1 (Su77) *Determine all solutions to the following infinite system of linear equations in the infinitely many unknowns $x_1, x_2, \ldots$:*

$$\begin{array}{ccccccc} x_1 & + & x_3 & + & x_5 & = & 0 \\ x_2 & + & x_4 & + & x_6 & = & 0 \\ x_3 & + & x_5 & + & x_7 & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

How many free parameters are required?

Problem 7.3.2 (Fa77, Su78)

1. *Using only the axioms for a field $\mathbf{F}$, prove that a system of m homogeneous linear equations in n unknowns with $m < n$ and coefficients in $\mathbf{F}$ has a nonzero solution.*
2. *Use Part 1 to show that if V is a vector space over $\mathbf{F}$ which is spanned by a finite number of elements, then every maximal linearly independent subset of V has the same number of elements.*

Problem 7.3.3 (Sp88, Sp96) *If a finite homogeneous system of linear equations with rational coefficients has a nontrivial complex solution, need it have a nontrivial rational solution? Give a proof or a counterexample.*

Problem 7.3.4 (Sp84, Sp87) *Let A be a real $m \times n$ matrix with rational entries and let b be an m-tuple of rational numbers. Assume that the system of equations $Ax = b$ has a solution x in complex n-space $\mathbb{C}^n$. Show that the equation has a solution vector with rational components, or give a counterexample.*

7.4 Linear Transformations

Problem 7.4.1 (Fa77) *Let E and F be vector spaces (not assumed to be finite-dimensional). Let $S : E \to F$ be a linear transformation.*

1. *Prove $S(E)$ is a vector space.*
2. *Show S has a kernel $\{0\}$ if and only if S is injective (i.e., one-to-one).*
3. *Assume S is injective; prove $S^{-1} : S(E) \to E$ is linear.*

Problem 7.4.2 (Sp82) *Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Prove that*

$$\dim(\ker T) + \dim(\operatorname{range} T) = \dim V .$$

Problem 7.4.3 (Fa99) *Let V and W be finite dimensional vector spaces, let X be a subspace of W, and let $T : V \to W$ be a linear map. Prove that the dimension of $T^{-1}(X)$ is at least* $\dim V - \dim W + \dim X$.

Problem 7.4.4 (Fa98) *Let A and B be linear transformations on a finite dimensional vector space V. Prove that* $\dim\ker(AB) \leqslant \dim\ker A + \dim\ker B$.

Problem 7.4.5 (Fa02) *Let V and W be vector spaces over a field K. Assume $A : V \to W$ and $B : V \to W$ are linear transformations such that A has rank at least 2, and for every vector v in V the vectors Av and Bv are linearly dependent. Prove that the linear transformations A and B are linearly dependent.*

Problem 7.4.6 (Sp95) *Suppose that $W \subset V$ are finite-dimensional vector spaces over a field, and let $L\colon V \to V$ be a linear transformation with $L(V) \subset W$. Denote the restriction of L to W by L_W. Prove that* $\det(1 - tL) = \det(1 - tL_W)$.

Problem 7.4.7 (Fa00) *Let V be a finite-dimensional vector space, and let $f : V \to V$ be a linear transformation. Let W denote the image of f. Prove that the restriction of f to W, considered as an endomorphism of W, has the same trace as $f : V \to V$.*

Problem 7.4.8 (Fa99) *Let $T : V \to V$ be a linear operator on an n dimensional vector space V over a field $\mathbf{F}$. Prove that T has an invariant subspace W other than $\{0\}$ and V if and only if the characteristic polynomial of T has a factor $f \in \mathbf{F}[t]$ with $0 < deg\ f < n$.*

Problem 7.4.9 (Fa00) *Let $T\colon\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, where $n>1$. Prove that there is a 2-dimensional subspace $M \subseteq \mathbb{R}^n$ such that $T(M) \subseteq M$.*

Problem 7.4.10 (Sp95) *Let V be a finite-dimensional vector space over a field $\mathbf{F}$, and let $L : V \to V$ be a linear transformation. Suppose that the characteristic polynomial χ of L is written as $\chi = \chi_1\chi_2$, where χ_1 and χ_2 are two relatively prime polynomials with coefficients in $\mathbf{F}$. Show that V can be written as the direct sum of two subspaces V_1 and V_2 with the property that $\chi_i(L)V_i = 0$ for $i = 1, 2$.*

Problem 7.4.11 (Su79) *Let E be a three-dimensional vector space over $\mathbb{Q}$. Suppose $T : E \to E$ is a linear transformation and $Tx = y$, $Ty = z$, $Tz = x + y$, for certain $x, y, z \in E$, $x \neq 0$. Prove that x, y, and z are linearly independent.*

Problem 7.4.12 (Su80) *Let $T : V \to V$ be an invertible linear transformation of a vector space V. Denote by G the group of all maps $f_{k,a} : V \to V$ where $k \in \mathbb{Z}$, $a \in V$, and for $x \in V$,*

$$f_{k,a}(x) = T^k x + a \qquad (x \in V).$$

Prove that the commutator subgroup G' of G is isomorphic to the additive group of the vector space $(T - I)V$, the image of $T - I$. (G' is generated by all $ghg^{-1}h^{-1}$, g and h in G.)

Problem 7.4.13 (Sp86) *Let V be a finite-dimensional vector space and A and B two linear transformations of V into itself such that $A^2 = B^2 = 0$ and $AB + BA = I$.*

1. *Prove that if N_A and N_B are the respective null spaces of A and B, then $N_A = AN_B$, $N_B = BN_A$, and $V = N_A \oplus N_B$.*
2. *Prove that the dimension of V is even.*
3. *Prove that if the dimension of V is 2, then V has a basis with respect to which A and B are represented by the matrices*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Problem 7.4.14 (Su84) *Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $n \geqslant 2$, be a linear transformation of rank $n-1$. Let $f(v) = (f_1(v), f_2(v), \ldots, f_n(v))$ for $v \in \mathbb{R}^m$. Show that a necessary and sufficient condition for the system of inequalities $f_i(v) > 0$, $i = 1, \ldots, n$, to have no solution is that there exist real numbers $\lambda_i \geqslant 0$, not all zero, such that*

$$\sum_{i=1}^{n} \lambda_i f_i = 0.$$

Problem 7.4.15 (Sp95) *Let n be a positive integer, and let $S \subset \mathbb{R}^n$ a finite subset with $0 \in S$. Suppose that $\varphi : S \to S$ is a map satisfying*

$$\varphi(0) = 0,$$
$$d(\varphi(s), \varphi(t)) = d(s,t) \qquad \textit{for all} \quad s, t \in S,$$

where $d(\ ,\ )$ denotes Euclidean metric. Prove that there is a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ whose restriction to S is φ.

Problem 7.4.16 (Sp86) *Consider $\mathbb{R}^2$ be equipped with the Euclidean metric $d(x,y) = \|x - y\|$. Let T be an isometry of $\mathbb{R}^2$ into itself. Prove that T can be represented as $T(x) = a + U(x)$, where a is a vector in $\mathbb{R}^2$ and U is an orthogonal linear transformation.*

Problem 7.4.17 (Sp88) *Let X be a set and V a real vector space of real valued functions on X of dimension n, $0 < n < \infty$. Prove that there are n points $x_1, x_2, \ldots, x_n$ in X such that the map $f \mapsto (f(x_1), \ldots, f(x_n))$ of V to $\mathbb{R}^n$ is an isomorphism.*

Problem 7.4.18 (Sp97) *Suppose that X is a topological space and V is a finite-dimensional subspace of the vector space of continuous real valued functions on X. Prove that there exist a basis $\{f_1, \ldots, f_n\}$ for V and points $x_1, \ldots, x_n$ in X such that $f_i(x_j) = \delta_{ij}$.*

Problem 7.4.19 (Fa90) *Let n be a positive integer and let P_{2n+1} be the vector space of real polynomials whose degrees are, at most, $2n+1$. Prove that there exist unique real numbers $c_1, \ldots, c_n$ such that, for all $p \in P_{2n+1}$.*

$$\int_{-1}^{1} p(x)\,dx = 2p(0) + \sum_{k=1}^{n} c_k (p(k) + p(-k) - 2p(0))$$

Problem 7.4.20 (Sp94) *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a diagonalizable linear transformation. Prove that there is an orthonormal basis for $\mathbb{R}^n$ with respect to which T has an upper-triangular matrix.*

Problem 7.4.21 (Fa77) *Let P be a linear operator on a finite-dimensional vector space over a finite field. Show that if P is invertible, then $P^n = I$ for some positive integer n.*

Problem 7.4.22 (Fa82) *Let A be an $n \times n$ complex matrix, and let B be the Hermitian transpose of A (i.e., $b_{ij} = \overline{a}_{ji}$). Suppose that A and B commute with each other. Consider the linear transformations α and β on $\mathbb{C}^n$ defined by A and B. Prove that α and β have the same image and the same kernel.*

Problem 7.4.23 (Su79, Fa96) *Prove that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has*

1. *a one-dimensional invariant subspace, and*
2. *a two-dimensional invariant subspace.*

Problem 7.4.24 (Fa83) *Let A be a linear transformation on $\mathbb{R}^3$ whose matrix (relative to the usual basis for $\mathbb{R}^3$) is both symmetric and orthogonal. Prove that A is either plus or minus the identity, or a rotation by $180°$ about some axis in $\mathbb{R}^3$, or a reflection about some two-dimensional subspace of $\mathbb{R}^3$.*

Problem 7.4.25 (Fa84) *Let θ and φ be fixed, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \varphi \leqslant 2\pi$ and let R be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose matrix in the standard basis $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Let S be the linear transformation of $\mathbb{R}^3$ to $\mathbb{R}^3$ whose matrix with respect to the basis

$$\left\{ \frac{1}{\sqrt{2}}(\vec{\imath} + \vec{k}), \vec{\jmath}, \frac{1}{\sqrt{2}}(\vec{\imath} - \vec{k}) \right\}$$

is

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that $T = R \circ S$ leaves a line invariant.

Problem 7.4.26 (Sp86) *Let $e = (a, b, c)$ be a unit vector in $\mathbb{R}^3$ and let T be the linear transformation on $\mathbb{R}^3$ of rotation by $180°$ about e. Find the matrix for T with respect to the standard basis.*

Problem 7.4.27 (Su80) *Exhibit a real 3×3 matrix having minimal polynomial* $(t^2+1)(t-10)$, *which, as a linear transformation of* $\mathbb{R}^3$, *leaves invariant the line* L *through* $(0,0,0)$ *and* $(1,1,1)$ *and the plane through* $(0,0,0)$ *perpendicular to* L.

Problem 7.4.28 (Su77) *Show that every rotation of* $\mathbb{R}^3$ *has an* axis; *that is, given a 3×3 real matrix* A *such that* $A^t = A^{-1}$ *and* $\det A > 0$, *prove that there is a nonzero vector* v *such that* $Av = v$.

Problem 7.4.29 (Sp93) *Let* P *be the vector space of polynomials over* $\mathbb{R}$. *Let the linear transformation* $E : P \to P$ *be defined by* $Ef = f + f'$, *where* f' *is the derivative of* f. *Prove that* E *is invertible.*

Problem 7.4.30 (Fa84) *Let* P_n *be the vector space of all real polynomials with degrees at most* n. *Let* $D : P_n \to P_n$ *be given by differentiation:* $D(p) = p'$. *Let* π *be a real polynomial. What is the minimal polynomial of the transformation* $\pi(D)$?

Problem 7.4.31 (Su77) *Let* V *be the vector space of all polynomials of degree* $\leqslant 10$, *and let* D *be the differentiation operator on* V *(i.e.,* $Dp(x) = p'(x)$*).*

1. *Show that* $\operatorname{tr} D = 0$.

2. *Find all eigenvectors of* D *and* e^D.

Problem 7.4.32 (Sp02) *Let* $x_0, x_1, \ldots, x_n$ *be distinct points of* $\mathbb{R}$. *Prove that there are unique real numbers* $a_0, a_1, \ldots, a_n$ *such that*

$$\int_0^1 p(t)dt = \sum_{j=0}^{n} a_j\, p(x_j)$$

for all polynomials of degree n *or less.*

Problem 7.4.33 (Sp00) *Let* $I_1, \ldots, I_n$ *be disjoint closed nonempty subintervals of* $\mathbb{R}$.

1. *Prove that if* p *is a real polynomial of degree less than* n *such that*

$$\int_{I_j} p(x)dx = 0\,, \qquad \text{for } j = 1, \ldots, n$$

then $p = 0$.

2. *Prove that there is a nonzero real polynomial* p *of degree* n *that satisfies all the above equations.*

7.5 Eigenvalues and Eigenvectors

Problem 7.5.1 (Fa77) *Let M be a real 3×3 matrix such that $M^3 = I$, $M \neq I$.*

1. *What are the eigenvalues of M?*
2. *Give an example of such a matrix.*

Problem 7.5.2 (Fa79) *Let N be a linear operator on an n-dimensional vector space, $n > 1$, such that $N^n = 0$, $N^{n-1} \neq 0$. Prove there is no operator X with $X^2 = N$.*

Problem 7.5.3 (Sp89) *Let $\mathbf{F}$ be a field, n and m positive integers, and A an $n \times n$ matrix with entries in $\mathbf{F}$ such that $A^m = 0$. Prove that $A^n = 0$.*

Problem 7.5.4 (Fa98) *Let B be a 3×3 matrix whose null space is 2-dimensional, and let $\chi(\lambda)$ be the characteristic polynomial of B. For each assertion below, provide either a proof or a counterexample.*

1. *λ^2 is a factor of $\chi(\lambda)$.*
2. *The trace of B is an eigenvalue of B.*
3. *B is diagonalizable.*

Problem 7.5.5 (Sp99) *Suppose that the minimal polynomial of a linear operator T on a seven-dimensional vector space is x^2. What are the possible values of the dimension of the kernel of T?*

Problem 7.5.6 (Sp03) *Let L be a real symmetric $n \times n$ matrix with 0 as a simple eigenvalue, and let $v \in \mathbb{R}^n$.*

1. *Show that for sufficiently small positive real ε, the equation $Lx + \varepsilon x = v$ has a unique solution $x = x(\varepsilon) \in \mathbb{R}^n$.*
2. *Evaluate $\lim_{\varepsilon\to 0^+} \varepsilon x(\varepsilon)$ in terms of v, the eigenvectors of L, and the inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$.*

Problem 7.5.7 (Su81, Su82) *Let V be a finite-dimensional vector space over the rationals $\mathbb{Q}$ and let M be an automorphism of V such that M fixes no nonzero vector in V. Suppose that M^p is the identity map on V, where p is a prime number. Show that the dimension of V is divisible by $p - 1$.*

Problem 7.5.8 (Fa92) *Let $\mathbf{F}$ be a field, V a finite-dimensional vector space over $\mathbf{F}$, and T a linear transformation of V into V whose minimum polynomial, μ, is irreducible over $\mathbf{F}$.*

1. *Let v be a nonzero vector in V and let V_1 be the subspace spanned by v and its images under the positive powers of T. Prove that $\dim V_1 = \deg \mu$.*
2. *Prove that $\deg \mu$ divides $\dim V$.*

Problem 7.5.9 (Fa02) *Suppose A and M are $n \times n$ matrices over $\mathbb{C}$, A is invertible and $AMA^{-1} = M^2$. Prove the nonzero eigenvalues of M are roots of unity.*

Problem 7.5.10 (Su79, Fa93) *Prove that the matrix*

$$\begin{pmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{pmatrix}$$

has two positive and two negative eigenvalues (counting multiplicities).

Problem 7.5.11 (Fa94) *Prove that the matrix*

$$\begin{pmatrix} 1 & 1.00001 & 1 \\ 1.00001 & 1 & 1.00001 \\ 1 & 1.00001 & 1 \end{pmatrix}$$

has one positive eigenvalue and one negative eigenvalue.

Problem 7.5.12 (Sp85) *For arbitrary elements a, b, and c in a field $\mathbf{F}$, compute the minimal polynomial of the matrix*

$$\begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

Problem 7.5.13 (Fa01) *Let A be an $n \times n$ complex matrix such that the three matrices $A + I$, $A^2 + I$, $A^3 + I$ are all unitary. Prove that A is the zero matrix.*

Problem 7.5.14 (Sp03) *Let k be a field, and let $n \geqslant 1$. Prove that the following properties of an $n \times n$ matrix A with entries in k are equivalent:*

- *A is a scalar multiple of the identity matrix.*
- *Every nonzero vector $v \in k^n$ is an eigenvector of A.*

Problem 7.5.15 (Fa85, Sp97, Fa98) *Suppose that A and B are endomorphisms of a finite-dimensional vector space V over a field $\mathbf{F}$. Prove or disprove the following statements:*

1. *Every eigenvector of AB is also an eigenvector of BA.*
2. *Every eigenvalue of AB is also an eigenvalue of BA.*

Problem 7.5.16 (Sp78, Sp98) *Let A and B denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that A and B have a common eigenvector in $\mathbb{R}^n$.*

Problem 7.5.17 (Sp86) *Let S be a nonempty commuting set of $n \times n$ complex matrices ($n \geqslant 1$). Prove that the members of S have a common eigenvector.*

Problem 7.5.18 (Sp84) *Let A and B be complex $n \times n$ matrices such that $AB = BA^2$, and assume A has no eigenvalues of absolute value 1. Prove that A and B have a common (nonzero) eigenvector.*

Problem 7.5.19 (Su78) *Let V be a finite-dimensional vector space over an algebraically closed field. A linear operator $T : V \to V$ is called completely reducible if whenever a linear subspace $E \subset V$ is invariant under T, that is $T(E) \subset E$, there is a linear subspace $F \subset V$ which is invariant under T and such that $V = E \oplus F$. Prove that T is completely reducible if and only if V has a basis of eigenvectors.*

Problem 7.5.20 (Fa79, Su81) *Let V be the vector space of sequences (a_n) of complex numbers. The shift operator $S : V \to V$ is defined by*

$$S((a_1, a_2, a_3, \ldots)) = (a_2, a_3, a_4, \ldots).$$

1. *Find the eigenvectors of S.*
2. *Show that the subspace W consisting of the sequences (x_n) with $x_{n+2} = x_{n+1} + x_n$ is a two-dimensional, S-invariant subspace of V and exhibit an explicit basis for W.*
3. *Find an explicit formula for the n^{th} Fibonacci number f_n, where $f_2 = f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geqslant 1$.*

Note: See also Problem 1.3.12.

Problem 7.5.21 (Fa82) *Let T be a linear transformation on a finite-dimensional $\mathbb{C}$-vector space V, and let f be a polynomial with coefficients in $\mathbb{C}$. If λ is an eigenvalue of T, show that $f(\lambda)$ is an eigenvalue of $f(T)$. Is every eigenvalue of $f(T)$ necessarily obtained in this way?*

Problem 7.5.22 (Fa83, Sp96) *Let A be the $n \times n$ matrix which has zeros on the main diagonal and ones everywhere else. Find the eigenvalues and eigenspaces of A and compute* $\det(A)$.

Problem 7.5.23 (Sp85) *Let A and B be two $n \times n$ self-adjoint (i.e., Hermitian) matrices over $\mathbb{C}$ and assume A is positive definite. Prove that all eigenvalues of AB are real.*

Problem 7.5.24 (Fa84) *Let a, b, c, and d be real numbers, not all zero. Find the eigenvalues of the following 4×4 matrix and describe the eigenspace decomposition of $\mathbb{R}^4$:*

$$\begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}.$$

Problem 7.5.25 (Sp81) *Show that the following three conditions are all equivalent for a real 3×3 symmetric matrix A, whose eigenvalues are λ_1, λ_2, and λ_3:*

1. $\operatorname{tr} A$ *is not an eigenvalue of A.*
2. $(a+b)(b+c)(a+c) \neq 0.$
3. *The map $L : S \to S$ is an isomorphism, where S is the space of 3×3 real skew-symmetric matrices and $L(W) = AW + WA$.*

Problem 7.5.26 (Su84) *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a real matrix with $a, b, c, d > 0$. Show that A has an eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

with $x, y > 0$.

Problem 7.5.27 (Sp90) *Let n be a positive integer, and let $A = (a_{ij})_{i,j=1}^n$ be the $n\times n$ matrix with $a_{ii} = 2$, $a_{i\,i\pm1} = -1$, and $a_{ij} = 0$ otherwise; that is,*

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Prove that every eigenvalue of A is a positive real number.

Problem 7.5.28 (Sp92) *Let A be a real symmetric $n \times n$ matrix with nonnegative entries. Prove that A has an eigenvector with nonnegative entries.*

Problem 7.5.29 (Sp00) *Let A_n be the $n \times n$ matrix whose entries a_{jk} are given by*

$$a_{jk} = \begin{cases} 1 & \text{if } \ |j-k| = 1 \\ 0 & \quad\text{otherwise}. \end{cases}$$

Prove that the eigenvalues of A are symmetric with respect to the origin.

Problem 7.5.30 (Fa91) *Let $A = (a_{ij})_{i,j=1}^n$ be a real $n \times n$ matrix with nonnegative entries such that*

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad (1 \leqslant i \leqslant n).$$

Prove that no eigenvalue of A has absolute value greater than 1.

Problem 7.5.31 (Sp01) *Let S be a* special orthogonal $n \times n$ *matrix, a real $n \times n$ matrix satisfying $S^t S = I$ and $\det(S) = 1$.*

1. *Prove that if n is odd then 1 is an eigenvalue of S.*

2. *Prove that if n is even then 1 need not be an eigenvalue of S.*

Problem 7.5.32 (Sp85, Fa88) *Let A and B be two $n \times n$ self-adjoint (i.e., Hermitian) matrices over $\mathbb{C}$ such that all eigenvalues of A lie in $[a, a']$ and all eigenvalues of B lie in $[b, b']$. Show that all eigenvalues of $A + B$ lie in $[a + b, a' + b']$.*

Problem 7.5.33 (Fa85) *Let k be real, n an integer $\geqslant 2$, and let $A = (a_{ij})$ be the $n \times n$ matrix such that all diagonal entries $a_{ii} = k$, all entries $a_{i\, i\pm 1}$ immediately above or below the diagonal equal 1, and all other entries equal 0. For example, if $n = 5$,*

$$A = \begin{pmatrix} k & 1 & 0 & 0 & 0 \\ 1 & k & 1 & 0 & 0 \\ 0 & 1 & k & 1 & 0 \\ 0 & 0 & 1 & k & 1 \\ 0 & 0 & 0 & 1 & k \end{pmatrix}.$$

Let λ_{min} and λ_{max} denote the smallest and largest eigenvalues of A, respectively. Show that $\lambda_{min} \leqslant k - 1$ and $\lambda_{max} \geqslant k + 1$.

Problem 7.5.34 (Fa87) *Let A and B be real $n \times n$ symmetric matrices with B positive definite. Consider the function defined for $x \neq 0$ by*

$$G(x) = \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle}.$$

1. *Show that G attains its maximum value.*

2. *Show that any maximum point U for G is an eigenvector for a certain matrix related to A and B and show which matrix.*

Problem 7.5.35 (Fa90) *Let A be a real symmetric $n \times n$ matrix that is positive definite. Let $y \in \mathbb{R}^n$, $y \neq 0$. Prove that the limit*

$$\lim_{m\to\infty} \frac{y^t A^{m+1} y}{y^t A^m y}$$

exists and is an eigenvalue of A.

7.6 Canonical Forms

Problem 7.6.1 (Sp90, Fa93) *Let A be a complex $n \times n$ matrix that has* finite order; *that is, $A^k = I$ for some positive integer k. Prove that A is diagonalizable.*

Problem 7.6.2 (Sp84) *Prove, or supply a counterexample: If A is an invertible $n \times n$ complex matrix and some power of A is diagonal, then A can be diagonalized.*

Problem 7.6.3 (Fa78) *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Express A^{-1} as a polynomial in A with real coefficients.

Problem 7.6.4 (Sp81) *For $x \in \mathbb{R}$, let*

$$A_x = \begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}.$$

1. *Prove that* $\det(A_x) = (x-1)^3(x+3)$.
2. *Prove that if $x \neq 1, -3$, then $A_x^{-1} = -(x-1)^{-1}(x+3)^{-1}A_{-x-2}$.*

Problem 7.6.5 (Sp88) *Compute A^{10} for the matrix*

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Problem 7.6.6 (Fa87) *Calculate A^{100} and A^{-7}, where*

$$A = \begin{pmatrix} 3/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Problem 7.6.7 (Sp96) *Prove or disprove: For any 2×2 matrix A over $\mathbb{C}$, there is a 2×2 matrix B such that $A = B^2$.*

Problem 7.6.8 (Su85) 1. *Show that a real 2×2 matrix A satisfies $A^2 = -I$ if and only if*

$$A = \begin{pmatrix} \pm\sqrt{pq-1} & -p \\ q & \mp\sqrt{pq-1} \end{pmatrix}$$

where p and q are real numbers such that $pq \geqslant 1$ and both upper or both lower signs should be chosen in the double signs.

2. *Show that there is no real 2×2 matrix A such that*

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\varepsilon \end{pmatrix}$$

with $\varepsilon > 0$.

Problem 7.6.9 (Fa96) *Is there a real* 2×2 *matrix* A *such that*

$$A^{20} = \begin{pmatrix} -1 & 0 \\ 0 & -1-\varepsilon \end{pmatrix} ?$$

Exhibit such an A *or prove there is none.*

Problem 7.6.10 (Sp88) *For which positive integers* n *is there a* 2×2 *matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries and order n*; that is,* $A^n = I$ *but* $A^k \neq I$ *for* $0 < k < n$*?*
Note: See also Problem 7.7.8.

Problem 7.6.11 (Sp92) *Find a square root of the matrix*

$$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}.$$

How many square roots does this matrix have?

Problem 7.6.12 (Sp92) *Let* A *denote the matrix*

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For which positive integers n *is there a complex* 4×4 *matrix* X *such that* $X^n = A$*?*

Problem 7.6.13 (Sp88) *Prove or disprove: There is a real* $n \times n$ *matrix* A *such that*

$$A^2 + 2A + 5I = 0$$

if and only if n *is even.*

Problem 7.6.14 (Su83) *Let* A *be an* $n \times n$ *Hermitian matrix satisfying the condition*

$$A^5 + A^3 + A = 3I .$$

Show that $A = I$.

Problem 7.6.15 (Su80) *Which of the following matrix equations have a real matrix solution* X*? (It is not necessary to exhibit solutions.)*

1.

$$X^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix},$$

2.

$$2X^5 + X = \begin{pmatrix} 3 & 5 & 0 \\ 5 & 1 & 9 \\ 0 & 9 & 0 \end{pmatrix},$$

3.

$$X^6 + 2X^4 + 10X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

4.

$$X^4 = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Problem 7.6.16 (Sp80) *Find a real matrix B such that*

$$B^4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Problem 7.6.17 (Fa87) *Let V be a finite-dimensional vector space and $T: V \to V$ a diagonalizable linear transformation. Let $W \subset V$ be a linear subspace which is mapped into itself by T. Show that the restriction of T to W is diagonalizable.*

Problem 7.6.18 (Fa89) *Let A and B be diagonalizable linear transformations of $\mathbb{R}^n$ into itself such that $AB = BA$. Let E be an eigenspace of A. Prove that the restriction of B to E is diagonalizable.*

Problem 7.6.19 (Fa83, Sp87, Fa99) *Let V be a finite-dimensional complex vector space and let A and B be linear operators on V such that $AB = BA$. Prove that if A and B can each be diagonalized, then there is a basis for V which simultaneously diagonalizes A and B.*

Problem 7.6.20 (Sp80) *Let A and B be $n \times n$ complex matrices. Prove or disprove each of the following statements:*

1. *If A and B are diagonalizable, so is $A + B$.*
2. *If A and B are diagonalizable, so is AB.*
3. *If $A^2 = A$, then A is diagonalizable.*
4. *If A is invertible and A^2 is diagonalizable, then A is diagonalizable.*

Problem 7.6.21 (Fa77) *Let*

$$A = \begin{pmatrix} 7 & 15 \\ -2 & -4 \end{pmatrix}.$$

Find a real matrix B such that $B^{-1}AB$ is diagonal.

Problem 7.6.22 (Su77) *Let $A : \mathbb{R}^6 \to \mathbb{R}^6$ be a linear transformation such that $A^{26} = I$. Show that $\mathbb{R}^6 = V_1 \oplus V_2 \oplus V_3$, where V_1, V_2, and V_3 are two-dimensional invariant subspaces for A.*

Problem 7.6.23 (Sp78, Sp82, Su82, Fa90) *Determine the Jordan Canonical Form of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Problem 7.6.24 (Su83) *Find the eigenvalues, eigenvectors, and the Jordan Canonical Form of*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

considered as a matrix with entries in $\mathbf{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

Problem 7.6.25 (Su83) *Let A be an $n \times n$ complex matrix, and let χ and μ be the characteristic and minimal polynomials of A. Suppose that*

$$\chi(x) = \mu(x)(x - i),$$

$$\mu(x)^2 = \chi(x)(x^2 + 1).$$

Determine the Jordan Canonical Form of A.

Problem 7.6.26 (Fa78, Fa84) *Let M be the $n \times n$ matrix over a field $\mathbf{F}$, all of whose entries are equal to* 1.

1. *Find the characteristic polynomial of M.*
2. *Is M diagonalizable?*
3. *Find the Jordan Canonical Form of M and discuss the extent to which the Jordan form depends on the characteristic of the field $\mathbf{F}$.*

Problem 7.6.27 (Fa86) *Let $M_{2\times 2}$ denote the vector space of complex 2×2 matrices. Let*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let the linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ be defined by $T(X) = XA - AX$. Find the Jordan Canonical Form for T.

Problem 7.6.28 (Sp01) *Let M_n be the vector space of $n \times n$ complex matrices. For A in M_n define the linear transformation of T_A on M_n by $T_A(X) = AX - XA$. Prove that the rank of T_A is at most $n^2 - n$.*

Problem 7.6.29 (Fa01) *Let A be an $n \times n$ matrix with real entries, let $\xi(t)$ denote its characteristic polynomial, and let $g(t) \in \mathbb{R}[t]$ be a polynomial of degree $n - 1$ dividing $\xi(t)$. What are the possibilities for the rank of $g(A)$?*

Problem 7.6.30 (Fa88) *Find the Jordan Canonical Form of the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Problem 7.6.31 (Sp02) *Let A and B be complex matrices of sizes 3×5 and 5×3, respectively, such that*

$$AB = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Find the Jordan canonical form of BA.

Problem 7.6.32 (Sp99) *Let $A = (a_{ij})$ be a $n \times n$ complex matrix such that $a_{ij} \neq 0$ if $i = j + 1$ but $a_{ij} = 0$ if $i \geqslant j + 2$. Prove that A cannot have more than one Jordan block for any eigenvalue.*

Problem 7.6.33 (Fa89) *Let A be a real, upper-triangular, $n \times n$ matrix that commutes with its transpose. Prove that A is diagonal.*

Problem 7.6.34 (Su78)

1. *Prove that a linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$ is diagonalizable if for all $\lambda \in \mathbb{C}$, $\ker(T - \lambda I)^n = \ker(T - \lambda I)$, where I is the $n \times n$ identity matrix.*
2. *Show that T is diagonalizable if T commutes with its conjugate transpose T^* (i.e., $(T^*)_{jk} = \overline{T_{kj}}$).*

Problem 7.6.35 (Fa79) *Let A be an $n\times n$ complex matrix. Prove there is a unitary matrix U such that $B = UAU^{-1}$ is upper-triangular: $B_{jk} = 0$ for $j > k$.*

Problem 7.6.36 (Sp81) *Let b be a real nonzero $n \times 1$ matrix (a column vector). Set $M = bb^t$ (an $n \times n$ matrix) where b^t denotes the transpose of b.*

1. *Prove that there is an orthogonal matrix Q such that $QMQ^{-1} = D$ is diagonal, and find D.*
2. *Describe geometrically the linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$.*

Problem 7.6.37 (Sp83) *Let M be an invertible real $n \times n$ matrix. Show that there is a decomposition $M = UT$ in which U is an $n \times n$ real orthogonal matrix and T is upper-triangular with positive diagonal entries. Is this decomposition unique?*

Problem 7.6.38 (Su85) *Let A be a nonsingular real $n\times n$ matrix. Prove that there exists a unique orthogonal matrix Q and a unique positive definite symmetric matrix B such that $A = QB$.*

Problem 7.6.39 (Fa03) *Let A be a 2×2 matrix with complex entries. Prove that the series $I + A + A^2 + \ldots$ converges if and only if every eigenvalue of A has absolute value less than 1.*

Problem 7.6.40 (Sp95) *Let A be the 3×3 matrix*

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Determine all real numbers a for which the limit $\lim_{n\to\infty} a^n A^n$ exists and is nonzero (as a matrix).

Problem 7.6.41 (Fa96) *Suppose p is a prime. Show that every element of $GL_2(\mathbf{F}_p)$ has order dividing either $p^2 - 1$ or $p(p-1)$.*

7.7 Similarity

Problem 7.7.1 (Fa80, Fa92) *Are the matrices give below similar ?*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 7.7.2 (Fa78, Sp79) *Which of the following matrices are similar as matrices over $\mathbb{R}$?*

$$(a) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (b) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(d) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad (e) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (f) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 7.7.3 (Sp00) *Are the 4×4 matrices*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

similar?

Problem 7.7.4 (Sp79) *Let M be an $n \times n$ complex matrix. Let G_M be the set of complex numbers λ such that the matrix λM is similar to M.*

1. *What is G_M if*

$$M = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}?$$

2. *Assume M is* not *nilpotent. Prove G_M is finite.*

Problem 7.7.5 (Su80, Fa96) *Let A and B be real 2×2 matrices such that $A^2 = B^2 = I$ and $AB + BA = 0$. Prove there exists a real nonsingular matrix T with*

$$TAT^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad TBT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 7.7.6 (Su79, Fa82) *Let A and B be $n \times n$ matrices over a field $\mathbf{F}$ such that $A^2 = A$ and $B^2 = B$. Suppose that A and B have the same rank. Prove that A and B are similar.*

Problem 7.7.7 (Fa97) *Prove that if A is a 2×2 matrix over the integers such that $A^n = I$ for some strictly positive integer n, then $A^{12} = I$.*

Problem 7.7.8 (Su78) *Let G be a finite multiplicative group of 2×2 integer matrices.*

1. *Let $A \in G$. What can one prove about*

   (i) det *A?*

   (ii) the (real or complex) eigenvalues of A?

   (iii) the Jordan or Rational Canonical Form of A?

   (iv) the order of A?

2. *Find all such groups up to isomorphism.*

Note: See also Problem 7.6.10.

Problem 7.7.9 (Fa80) *Exhibit a set of 2×2 real matrices with the following property: A matrix A is similar to exactly one matrix in S provided A is a 2×2 invertible matrix of integers with all the roots of its characteristic polynomial on the unit circle.*

Problem 7.7.10 (Fa81, Su81, Sp84, Fa87, Fa95) *Let A and B be two real $n \times n$ matrices. Suppose there is a complex invertible $n \times n$ matrix U such that $A = UBU^{-1}$. Show that there is a real invertible $n \times n$ matrix V such that $A = VBV^{-1}$. (In other words, if two real matrices are similar over $\mathbb{C}$, then they are similar over $\mathbb{R}$.)*

Problem 7.7.11 (Sp91) *Let A be a linear transformation on an n-dimensional vector space over $\mathbb{C}$ with characteristic polynomial $(x-1)^n$. Prove that A is similar to A^{-1}.*

Problem 7.7.12 (Sp94) *Prove or disprove: A square complex matrix, A, is similar to its transpose, A^t.*

Problem 7.7.13 (Sp79) *Let M be a real nonsingular 3×3 matrix. Prove there are real matrices S and U such that $M = SU = US$, all the eigenvalues of U equal 1, and S is diagonalizable over $\mathbb{C}$.*

Problem 7.7.14 (Sp77, Sp93, Fa94) *Find a list of real matrices, as long as possible, such that*

- *the characteristic polynomial of each matrix is $(x-1)^5(x+1)$,*
- *the minimal polynomial of each matrix is $(x-1)^2(x+1)$,*
- *no two matrices in the list are similar to each other.*

Problem 7.7.15 (Fa95) *Let A and B be nonsimilar $n \times n$ complex matrices with the same minimal and the same characteristic polynomial. Show that $n \geqslant 4$ and the minimal polynomial is not equal to the characteristic polynomial.*

Problem 7.7.16 (Sp98) *Let A be an $n \times n$ complex matrix with $\operatorname{tr} A = 0$. Show that A is similar to a matrix with all 0's along the main diagonal.*

Problem 7.7.17 (Sp99) *Let M be a 3×3 matrix with entries in the polynomial ring $\mathbb{R}[t]$ such that $M^3 = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$. Let N be the matrix with real entries obtained by substituting $t = 0$ in M. Prove that N is similar to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.*

Problem 7.7.18 (Sp99) *Let M be a square complex matrix, and let $S = \{XMX^{-1} \mid X \text{ is non-singular}\}$ be the set of all matrices similar to M. Show that M is a nonzero multiple of the identity matrix if and only if no matrix in S has a zero anywhere on its diagonal.*

Problem 7.7.19 (Sp00) *Let A be a complex $n \times n$ matrix such that the sequence $(A^n)_{n=1}^{\infty}$ converges to a matrix B. Prove that B is similar to a diagonal matrix with zeros and ones along the main diagonal.*

7.8 Bilinear, Quadratic Forms, and Inner Product Spaces

Problem 7.8.1 (Fa98) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that*

1. *the function g defined by $g(x, y) = f(x + y) - f(x) - f(y)$ is* bilinear,
2. *for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $f(tx) = t^2 f(x)$.*

Show that there is a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $f(x) = \langle x, Ax\rangle$ *where* $\langle\cdot,\cdot\rangle$ *is the usual inner product on* $\mathbb{R}^n$ *(in other words,* f *is a quadratic form).*

Problem 7.8.2 (Sp98) *Let* $A, B, \dots, F$ *be real coefficients. Show that the quadratic form*

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2$$

is positive definite if and only if

$$A > 0, \qquad \begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0, \qquad \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} > 0.$$

Problem 7.8.3 (Fa00) *Find all real numbers* t *for which the quadratic form* Q_t *on* $\mathbb{R}^3$*, defined by*

$$Q_t(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + 3x_3^2 + 2tx_1x_2 + 2x_1x_3\,,$$

is positive definite.

Problem 7.8.4 (Fa90) *Let* $\mathbb{R}^3$ *be 3-space with the usual inner product, and* $(a, b, c) \in \mathbb{R}^3$ *a vector of length* 1*. Let* W *be the plane defined by* $ax+by+cz = 0$*. Find, in the standard basis, the matrix representing the orthogonal projection of* $\mathbb{R}^3$ *onto* W*.*

Problem 7.8.5 (Fa93) *Let* w *be a positive continuous function on* $[0, 1]$*,* n *a positive integer, and* P_n *the vector space of real polynomials whose degrees are at most* n*, equipped with the inner product*

$$\langle p, q\rangle = \int_0^1 p(t)q(t)w(t)\,dt\,.$$

1. *Prove that* P_n *has an orthonormal basis* $p_0, p_1, \dots, p_n$ *(i.e.,* $\langle p_j, p_k\rangle = 1$ *for* $j = k$ *and* 0 *for* $j \neq k$*) such that* $\deg p_k = k$ *for each* k*.*

2. *Prove that* $\langle p_k, p_k'\rangle = 0$ *for each* k*.*

Problem 7.8.6 (Sp98) *For continuous real valued functions* f, g *on the interval* $[-1, 1]$ *define the inner product* $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$*. Find that polynomial of the form* $p(x) = a + bx^2 - x^4$ *which is orthogonal on* $[-1, 1]$ *to all lower order polynomials.*

Problem 7.8.7 (Su80, Fa92) *Let* E *be a finite-dimensional vector space over a field* $\mathbf{F}$*. Suppose* $B : E \times E \to \mathbf{F}$ *is a bilinear map (not necessarily symmetric). Define subspaces*

$$E_1 = \{x \in E \mid B(x, y) = 0 \ for\ \ all\ y \in E\},$$

$$E_2 = \{y \in E \mid B(x, y) = 0 \ for\ \ all\ x \in E\}$$

Prove that $\dim E_1 = \dim E_2$*.*

Problem 7.8.8 (Su82) *Let A be a real $n \times n$ matrix such that $\langle Ax, x\rangle \geqslant 0$ for every real n-vector x. Show that $Au = 0$ if and only if $A^t u = 0$.*

Problem 7.8.9 (Fa85) *An $n\times n$ real matrix T is* positive definite *if T is symmetric and $\langle Tx, x\rangle > 0$ for all nonzero vectors $x \in \mathbb{R}^n$, where $\langle u, v\rangle$ is the standard inner product. Suppose that A and B are two positive definite real matrices.*

1. *Show that there is a basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{R}^n$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that, for $1 \leqslant i, j \leqslant n$:*

$$\langle Av_i, v_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and

$$\langle Bv_i, v_j\rangle = \begin{cases} \lambda_i & i = j \\ 0 & i \neq j \end{cases}$$

2. *Deduce from Part 1 that there is an invertible real matrix U such that $U^t AU$ is the identity matrix and $U^t BU$ is diagonal.*

Problem 7.8.10 (Sp83) *Let V be a real vector space of dimension n, and let $S : V \times V \to \mathbb{R}$ be a nondegenerate bilinear form. Suppose that W is a linear subspace of V such that the restriction of S to $W \times W$ is identically 0. Show that* $\dim W \leqslant n/2$.

Problem 7.8.11 (Fa85) *Let A be the symmetric matrix*

$$\frac{1}{6}\begin{pmatrix} 13 & -5 & -2 \\ -5 & 13 & -2 \\ -2 & -2 & 10 \end{pmatrix}.$$

Denote by v the column vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

in $\mathbb{R}^3$, and by x^t its transpose (x, y, z). Let $\|v\|$ denote the length of the vector v. As v ranges over the set of vectors for which $v^t Av = 1$, show that $\|v\|$ is bounded, and determine its least upper bound.

Problem 7.8.12 (Fa97) *Define the* index *of a real symmetric matrix A to be the number of strictly positive eigenvalues of A minus the number of strictly negative eigenvalues. Suppose A, and B are real symmetric $n \times n$ matrices such that $x^t Ax \leqslant x^t Bx$ for all $n \times 1$ matrices x. Prove the the index of A is less than or equal to the index of B.*

Problem 7.8.13 (Fa78) *For $x, y \in \mathbb{C}^n$, let $\langle x, y\rangle$ be the Hermitian inner product $\sum_j x_j \overline{y}_j$. Let T be a linear operator on $\mathbb{C}^n$ such that $\langle Tx, Ty\rangle = 0$ if $\langle x, y\rangle = 0$. Prove that $T = kS$ for some scalar k and some operator S which is unitary: $\langle Sx, Sy\rangle = \langle x, y\rangle$ for all x and y.*

Problem 7.8.14 (Sp79) *Let E denote a finite-dimensional complex vector space with a Hermitian inner product $\langle x, y\rangle$.*

1. *Prove that E has an orthonormal basis.*
2. *Let $f : E \to \mathbb{C}$ be such that $f(x, y)$ is linear in x and conjugate linear in y. Show there is a linear map $A : E \to E$ such that $f(x, y) = \langle Ax, y\rangle$.*

Problem 7.8.15 (Fa86) *Let a and b be real numbers. Prove that there are two orthogonal unit vectors u and v in $\mathbb{R}^3$ such that $u = (u_1, u_2, a)$ and $v = (v_1, v_2, b)$ if and only if $a^2 + b^2 \leqslant 1$.*

7.9 General Theory of Matrices

Problem 7.9.1 (Fa01) *Prove that a commutative $\mathbb{C}$-algebra of 2×2 complex matrices has dimension at most 2 over $\mathbb{C}$.*

Problem 7.9.2 (Fa81) *Prove the following three statements about real $n \times n$ matrices.*

1. *If A is an orthogonal matrix whose eigenvalues are all different from -1, then $I + A$ is nonsingular and $S = (I - A)(I + A)^{-1}$ is skew-symmetric.*
2. *If S is a skew-symmetric matrix, then $A = (I-S)(I+S)^{-1}$ is an orthogonal matrix with no eigenvalue equal to -1.*
3. *The correspondence $A \leftrightarrow S$ from Parts 1 and 2 is one-to-one.*

Problem 7.9.3 (Fa79) *Let B denote the matrix*

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

where a, b, and c are real and $|a|$, $|b|$, and $|c|$ are distinct. Show that there are exactly four symmetric matrices of the form BQ, where Q is a real orthogonal matrix of determinant 1.

Problem 7.9.4 (Sp79) *Let P be a $n \times n$ real matrix such that $x^t Py = -y^t Px$ for all column vectors x, y in $\mathbb{R}^n$. Prove that P is skew-symmetric.*

Problem 7.9.5 (Fa79) *Let A be a real* skew-symmetric *matrix* $(A_{ij} = -A_{ji})$. *Prove that A has even rank.*

Problem 7.9.6 (Fa98) *A real symmetric $n \times n$ matrix A is called* positive semi-definite *if $x^t Ax \geqslant 0$ for all $x \in \mathbb{R}^n$. Prove that A is positive semi-definite if and only if $\operatorname{tr} AB \geqslant 0$ for every real symmetric positive semi-definite $n\times n$ matrix B.*

Problem 7.9.7 (Fa80, Sp96) *Suppose that A and B are real matrices such that $A^t = A$,*

$$v^t A v \geqslant 0$$

for all $v \in \mathbb{R}^n$ and

$$AB + BA = 0.$$

Show that $AB = BA = 0$ and give an example where neither A nor B is zero.

Problem 7.9.8 (Fa02) *Let A and B be $n \times n$ matrices over $\mathbb{R}$ such that $A + B$ is invertible. Prove that*

$$A(A+B)^{-1}B = B(A+B)^{-1}A\,.$$

Problem 7.9.9 (Sp78) *Suppose A is a real $n \times n$ matrix.*

1. *Is it true that A must commute with its transpose?*
2. *Suppose the columns of A (considered as vectors) form an orthonormal set; is it true that the rows of A must also form an orthonormal set?*

Problem 7.9.10 (Sp98) *Let $M_1 = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right)$, $M_2 = \left(\begin{smallmatrix} 5 & 7 \\ -3 & -4 \end{smallmatrix}\right)$, $M_3 = \left(\begin{smallmatrix} 5 & 6.9 \\ -3 & -4 \end{smallmatrix}\right)$. For which (if any) i, $1 \leqslant i \leqslant 3$, is the sequence (M_i^n) bounded away from ∞? For which i is the sequence bounded away from 0?*

Problem 7.9.11 (Su83) *Let A be an $n\times n$ complex matrix, all of whose eigenvalues are equal to 1. Suppose that the set $\{A^n \mid n = 1, 2, \ldots\}$ is bounded. Show that A is the identity matrix.*

Problem 7.9.12 (Fa81) *Consider the complex 3×3 matrix*

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix},$$

where $a_0, a_1, a_2 \in \mathbb{C}$.

1. *Show that $A = a_0 I_3 + a_1 E + a_2 E^2$, where*

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

2. *Use Part 1 to find the complex eigenvalues of A.*
3. *Generalize Parts 1 and 2 to $n\times n$ matrices.*

Problem 7.9.13 (Su78) *Let A be a $n \times n$ real matrix.*

1. *If the sum of each column element of A is 1 prove that there is a nonzero column vector x such that $Ax = x$.*

2. *Suppose that $n = 2$ and all entries in A are positive. Prove there is a nonzero column vector y and a number $\lambda > 0$ such that $Ay = \lambda y$.*

Problem 7.9.14 (Sp89) *Let the real $2n \times 2n$ matrix X have the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, and D are $n \times n$ matrices that commute with one another. Prove that X is invertible if and only if $AD - BC$ is invertible.

Problem 7.9.15 (Sp03) *Let $GL_2(\mathbb{C})$ denote the group of invertible 2×2 matrices with coefficients in the field of complex numbers. Let $PGL_2(\mathbb{C})$ denote the quotient of $GL_2(\mathbb{C})$ by the normal subgroup $\left\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \mid \lambda \in \mathbb{C}^*\right\}$. Let n be a positive integer, and suppose that a, b are elements of $PGL_2(\mathbb{C})$ of order exactly n. Prove that there exists $c \in PGL_2(\mathbb{C})$ such that cac^{-1} is a power of b.*

Problem 7.9.16 (Sp89) *Let $B = (b_{ij})_{i,j=1}^{20}$ be a real 20×20 matrix such that*

$$b_{ii} = 0 \quad for \quad 1 \leqslant i \leqslant 20,$$

$$b_{ij} \in \{1, -1\} \quad for \quad 1 \leqslant i, j \leqslant 20, \quad i \neq j.$$

Prove that B is nonsingular.

Problem 7.9.17 (Sp80) *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Show that every real matrix B such that $AB = BA$ has the form $sI + tA$, where $s, t \in \mathbb{R}$.

Problem 7.9.18 (Su84) *Let A be a 2×2 matrix over $\mathbb{C}$ which is not a scalar multiple of the identity matrix I. Show that any 2×2 matrix X over $\mathbb{C}$ commuting with A has the form $X = \alpha I + \beta A$, where $\alpha, \beta \in \mathbb{C}$.*

Problem 7.9.19 (Sp02) 1. *Determine the commutant of the $n \times n$ Jordan matrix*

$$AB = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

In particular, determine the dimension of the commutant as a complex vector space.

2. *What is the dimension of the commutant of the* $2n \times 2n$ *matrix*

$$J \oplus J = AB = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} ?$$

Problem 7.9.20 (Fa96) *Let*

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

Show that every real matrix B such that $AB = BA$ has the form

$$B = aI + bA + cA^2$$

for some real numbers a, b, and c.

Problem 7.9.21 (Sp77, Su82) *A square matrix A is nilpotent if $A^k = 0$ for some positive integer k.*

1. *If A and B are nilpotent, is $A + B$ nilpotent?*

2. *Prove: If A and B are nilpotent matrices and $AB = BA$, then $A + B$ is nilpotent.*

3. *Prove: If A is nilpotent then $I + A$ and $I - A$ are invertible.*

Problem 7.9.22 (Sp77) *Consider the family of square matrices $A(\theta)$ defined by the solution of the matrix differential equation*

$$\frac{dA(\theta)}{d\theta} = BA(\theta)$$

with the initial condition $A(0) = I$, where B is a constant square matrix.

1. *Find a property of B which is necessary and sufficient for $A(\theta)$ to be* orthogonal *for all θ; that is, $A(\theta)^t = A(\theta)^{-1}$, where $A(\theta)^t$ denotes the transpose of $A(\theta)$.*

2. *Find the matrices $A(\theta)$ corresponding to*

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and give a geometric interpretation.

Problem 7.9.23 (Su77) *Let A be an $r \times r$ matrix of real numbers. Prove that the infinite sum*

$$e^A = I + A + \frac{A^2}{2} + \cdots + \frac{A^n}{n!} + \cdots$$

of matrices converges (i.e., for each i, j, the sum of $(i, j)^{th}$ entries converges), and hence that e^A is a well-defined matrix.

Problem 7.9.24 (Sp97) *Show that*

$$\det(\exp(M)) = e^{\operatorname{tr}(M)}$$

for any complex $n \times n$ matrix M, where $\exp(M)$ *is defined as in Problem 7.9.23.*

Problem 7.9.25 (Fa77) *Let T be an $n \times n$ complex matrix. Show that*

$$\lim_{k \to \infty} T^k = 0$$

if and only if all the eigenvalues of T have absolute value less than 1.

Problem 7.9.26 (Sp82) *Let A and B be $n \times n$ complex matrices. Prove that*

$$|\operatorname{tr}(AB^*)|^2 \leqslant \operatorname{tr}(AA^*)\operatorname{tr}(BB^*).$$

Problem 7.9.27 (Fa84) *Let A and B be $n \times n$ real matrices, and k a positive integer. Find*

1.

$$\lim_{t \to 0} \frac{1}{t}\left((A + tB)^k - A^k\right).$$

2.

$$\left.\frac{d}{dt}\operatorname{tr}(A + tB)^k\right|_{t=0}.$$

Problem 7.9.28 (Fa91) *1. Prove that any real $n \times n$ matrix M can be written as $M = A + S + cI$, where A is antisymmetric, S is symmetric, c is a scalar, I is the identity matrix, and* $\operatorname{tr} S = 0$.

2. Prove that with the above notation,

$$\operatorname{tr}(M^2) = \operatorname{tr}(A^2) + \operatorname{tr}(S^2) + \frac{1}{n}(\operatorname{tr} M)^2.$$

Problem 7.9.29 (Fa99) *Let A be an $n \times n$ complex matrix such that* $\operatorname{tr} A^k = 0$ *for $k = 1, \ldots, n$. Prove that A is nilpotent.*

Problem 7.9.30 (Sp98) *Let N be a nilpotent complex matrix. Let r be a positive integer. Show that there is a $n \times n$ complex matrix A with*

$$A^r = I + N.$$

Problem 7.9.31 (Fa94) *Let $A = \left(a_{ij}\right)_{i,j=1}^n$ be a real $n \times n$ matrix such that $a_{ii} \geqslant 1$ for all i, and*

$$\sum_{i \neq j} a_{ij}^2 < 1.$$

Prove that A is invertible.

Problem 7.9.32 (Fa95) *Show that an $n \times n$ matrix of complex numbers A satisfying*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for $1 \leqslant i \leqslant n$ must be invertible.

Problem 7.9.33 (Sp93) *Let $A = (a_{ij})$ be an $n \times n$ matrix such that $\sum_{j=1}^{n} |a_{ij}| < 1$ for each i. Prove that $I - A$ is invertible.*

Problem 7.9.34 (Sp94) *Let A be a real $n \times n$ matrix. Let M denote the maximum of the absolute values of the eigenvalues of A.*

1. *Prove that if A is symmetric, then $\|Ax\| \leqslant M\|x\|$ for all x in $\mathbb{R}^n$. (Here, $\|\cdot\|$ denotes the Euclidean norm.)*

2. *Prove that the preceding inequality can fail if A is not symmetric.*

Problem 7.9.35 (Sp00) *Let A be an $n \times n$ matrix over $\mathbb{C}$ whose minimal polynomial μ has degree k.*

1. *Prove that, if the point λ of $\mathbb{C}$ is not an eigenvalue of A, then there is a polynomial p_λ of degree $k-1$ such that $p_\lambda(A) = (A - \lambda I)^{-1}$.*

2. *Let $\lambda_1, \ldots, \lambda_k$ be distinct points of $\mathbb{C}$ that are not eigenvalues of A. Prove that there are complex numbers $c_1, \ldots, c_k$ such that*

$$\sum_{j=1}^{k} c_j (A - \lambda_j I)^{-1} = I\,.$$

Problem 7.9.36 (Sp99) *Let $\|x\|$ denote the Euclidean norm of a vector x. Show that for any real $m \times n$ matrix M there is a unique non-negative scalar σ, and (possibly non-unique) unit vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that*

1. *$\|Mx\| \leqslant \sigma\|x\|$ for all $x \in \mathbb{R}^n$,*

2. *$Mu = \sigma v$,*

3. *$M^T v = \sigma u$ (where M^T is the transpose of M).*