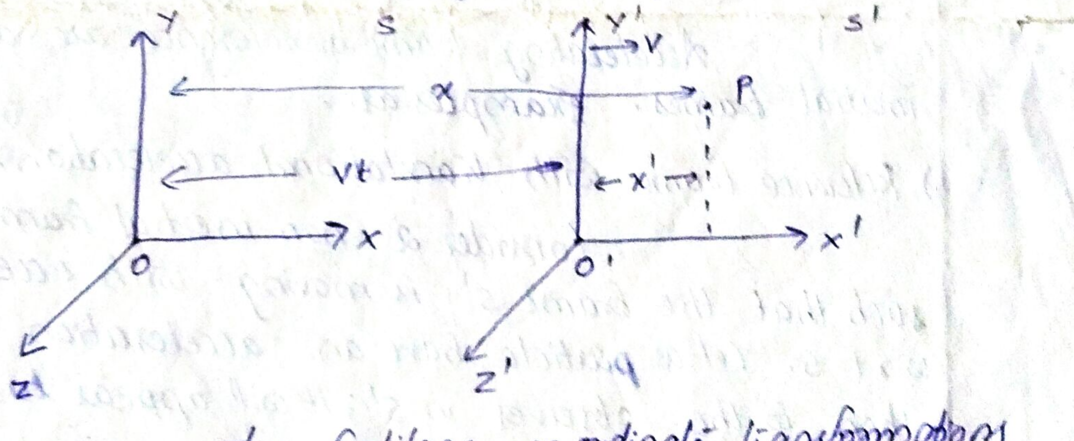
Special theory of relativity.

Inertial frame of reference:

→ is one in which Newton's 1st law of motion holds. In such a frame, an object at rest remains at rest and an object in motion continues to move at constant velocity if no force acts on it. Any frame of reference that moves at const velocity relative to an inertial frame is itself an inertial frame. All inertial frames are equally valid.

Galilean transformations

Let S & S' be two inertial frames. Let S be at rest and S' move with uniform velocity v along the +ve x direction. Let the origins of the 2 frames coincide at $t=0$. Suppose some event occurs at the point P. The observer O in the frame S specifies the co-ordinates (x,y,z,t) and the co-ordinates (x',y',z',t') by the observer in S'.



The Galilean co-ordinate transformations which relate the measurements are

$$x' = x - vt \qquad y' = y$$
$$z' = z \qquad t' = t$$

Inverse Galilean transformations can be written by changing primed into unprimed quantities & replacing v by $-v$

$$x = x' + vt' \qquad y = y'$$
$$z = z' \qquad t = t'$$

The transformation of velocities from one system to the other is obtained by taking time derivatives

$$\frac{dx'}{dt} = \frac{dx}{dt} - V \Rightarrow u' = u - V$$

To obtain acceleration, diff. again w.r.t

$$\frac{du_x'}{dt} = \frac{du_x}{dt} \Rightarrow a_x' = a_x \;,\; a_y' = a_y \;,\; a_z' = a_z$$

This implies acceleration remains invariant when passing from one inertial frame to another that is in uniform relative translational motion.

Since $F = ma \Rightarrow F = F'$

Q) Consider a ship moving with a uniform velocity of 18 m/s relative to the earth. Let a ball be rolled at a speed of 2 m/s relative to the ship, in the direction of motion of the ship. Find the speed of the ball relative to the earth.

$V = 18\ m/s$

$u' = 2\ m/s$

$u = u' + V = \underline{20\ m/s}$.

Non inertial frame and fictitious forces

Accelerating frame of reference are called non-inertial frames. Examples are.

1) Reference frame with translational acceleration.

Consider 2 non-inertial frames S & S' such that the frame S' is moving with acceleration a_0 w.r.t S. Let a particle have an acceleration a w.r.t S. Then to the observer in S', it will appear to have acceleration a' given by,

$$a' = a - a_0$$

If m is the mass of the particle, then force on the particle in S' is

$$F' = ma' = m(a - a_0)$$

$$F' = F - F_0$$

$F \rightarrow$ force seen by an observer in S

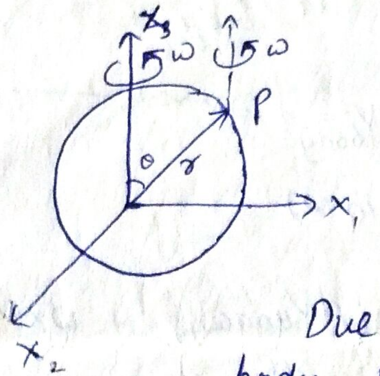
$F_0 \rightarrow$ force due to relative acceleration a_0 b/w the 2 frames

When $F = 0$, $F' = -F_0$

Thus particle seems to experience a force $-F_o$ when viewed from s' even when there is no force of it in s. Thus F_o is called fictitious or pseudoforce. These arise from the acceleration of the reference frame and go on increasing with enhanced acceleration.

2) Uniformly rotating frame : Coriolis & centrifugal force.

Let $x_1 x_2 x_3$ be an inertial frame s fixed in space and $x_1' x_2' x_3'$ be a reference frame s' that is fixed in a rigid body & is uniformly rotating in space w.r.t s with angular velocity ω. The unit vectors i_1, i_2, i_3 refer to the reference frame s and i_1', i_2', i_3' to the frame s'.



The position vector r of the point P is given by

$$r = x_1 i_1 + x_2 i_2 + x_3 i_3$$

$$r = x_1' i_1' + x_2' i_2' + x_3' i_3'$$

Due to rotational motion of the rigid body, the unit base vectors i_1', i_2', i_3' are continuously changing and in taking time derivatives the unit vectors are treated as variables

$$\therefore \frac{dr}{dt} = \frac{d}{dt}\left(x_1' i_1' + x_2' i_2' + x_3' i_3'\right)$$

$$= \frac{dx_1'}{dt} i_1' + \frac{dx_2'}{dt} i_2' + \frac{dx_3'}{dt} i_3' +$$

$$x_1' \frac{di_1'}{dt} + x_2' \frac{di_2'}{dt} + x_3' \frac{di_3'}{dt}$$

$$\dot{r} = \dot{x}_1' i_1' + \dot{x}_2' i_2' + \dot{x}_3' i_3' + x_1' \frac{di_1'}{dt} + x_2' \frac{di_2'}{dt} + x_3' \frac{di_3'}{dt}$$

The linear velocity v of a particle is expressed as

$$\frac{d\bar{r}}{dt} = \bar{\omega} \times \bar{r}$$

$$\frac{di_1'}{dt} = \omega \times i_1' \qquad \frac{di_2'}{dt} = \omega \times i_2'$$

$$\frac{di_3'}{dt} = \omega \times i_3'$$

⑥

④

$$\dot{r} = \dot{x}_1' i_1' + \dot{x}_2' i_2' + \dot{x}_3' i_3' + x_1'(\omega \times i_1') + x_2'(\omega \times i_2') + x_3' \ldots$$

This can be written as,

$$\left(\frac{dr}{dt}\right)_{space} = \left(\frac{dr}{dt}\right)_{body} + \omega \times r \quad \text{——} (1)$$

$\left(\frac{dr}{dt}\right)_{space} \rightarrow$ velocity of rigid body w.r.t S

$\left(\frac{dr}{dt}\right)_{body} \rightarrow$ velocity w.r.t S'

It can expres as operator form,

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} = (\omega \times) \quad \text{——} (2)$$

$$(1) \Rightarrow \quad V_{space} = V_{body} + \omega \times r \quad \text{——} (3)$$

Applying (3) in (2)

$$\left(\frac{dV_{space}}{dt}\right)_{space} = \left(\frac{dV_{space}}{dt}\right)_{body} + \omega \times V_{space}$$

$$a_{space} = \frac{d}{dt}\left(V_{body} + (\omega \times r)\right)_{body} + \omega \times \left(V_{body} + \omega \times r\right)$$

$$a_{space} = a_{body} + \frac{d}{dt}(\omega \times r)_{body} + \omega \times V_{body} + \omega \times (\omega \times r)$$

$$a_{space} = a_{body} + \omega \times V_{body} + \frac{d\omega}{dt} \times r + \omega \times V_{body} + \omega \times (\omega \times r)$$

$$a_{space} = a_{body} + 2(\omega \times V_{body}) + \omega \times (\omega \times r) + \frac{d\omega}{dt} \times r$$

Eqn of motion in fixed space axes,

$$F_{space} = m a_{space}$$

$$\therefore F_{body} = m a_{body}$$

$$F_{body} = m\left(a_{space} - 2(\omega \times V_{body}) - \omega \times (\omega \times r) - \frac{d\omega}{dt} \times r\right)$$

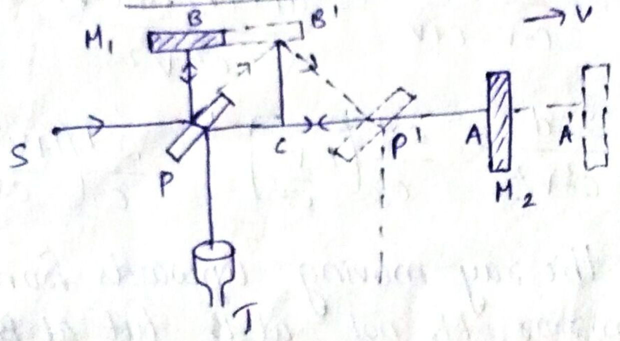
$$F_{body} = F_{space} - 2m(\omega \times V_{body}) - m\omega \times (\omega \times r) - m\frac{d\omega}{dt} \times r$$

This is the effective force in which the body appears to be moving to an observer in the rotating frame. $m\omega \times (\omega \times r)$ is the ordinary centrifugal force and is $\perp^r$ to ω. $-2m(\omega \times V_{body})$ is the coriolis force and is $\perp^r$ to both ω and V_{body}. Last term $(d\omega/dt) \times r$ is non zero only when $(d\omega/dt) \neq 0$ and will vanish when ω is const.

Thus the fictitious force is given by

$$F_0 = -2m(\omega \times V_{body}) - m\omega \times (\omega \times r)$$

Michelson-Morley experiment



A material medium is a necessity for the propagation of waves. It was considered that light propagates through ether as the sound waves propagate through air. Ether pervades all space. Then we can consider the relative velocity of earth w.r.t ether. If such a motion can be detected, we can choose a fixed frame of reference in a stationary ether. Michelson & Morley conducted an expt to find the existence of ether.

A beam of light from a monochromatic light source S falls on a half-silvered glass plate P, placed at an angle of 45° to the beam. The incident beam is split up into two parts by P. The reflected portion travels in a direction at right angles to the incident beam, falls normally at B on the plane mirror M_1 and is reflected back to P. It gets refracted through P enters the telescope T. The transmitted portion travels along the direction of the initial beam, falls normally on mirror M_2 at A and is reflected back at P.

⑥ After reflection from the back surface of P, it enters the telescope T. The two reflected beams interfere and the interference fringes are viewed with the help of T. One arm (PA) points in the direction of earth's motion round the sun and the other (PB) points $\perp^r$ to this motion.

Assume that the velocity of the ~~apparatus~~ Earth relative to fixed ether is v in the direction PA.

Let $PA = PB = d$.

$\therefore$ Time taken by light to travel from P to A $= \dfrac{d}{(c-v)}$

" " " A to P $= \dfrac{d}{c+v}$

$$\therefore \text{Total time}, t = \frac{d}{c-v} + \frac{d}{c+v} = \frac{d(c+v) + d(c-v)}{c^2 - v^2}$$

$$t = \frac{2cd}{c^2 - v^2} = \frac{2d}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1} = \frac{2d}{c}\left(1 + \frac{v^2}{c^2}\right) \quad \text{---(1)}$$

Now consider the ray moving upwards from P to B. It will strike the mirror M_1 not at B but at B' due to the motion of the earth. If t_1 is the time taken by the ray starting from P to reach M_1, then $PB' = ct_1$ & $BB' = vt_1$,

$$\therefore PB'^2 = BB'^2 + B'C^2$$

$$(ct_1)^2 = (vt_1)^2 + d^2$$

$$t_1^2(c^2 - v^2) = d^2$$

$$t_1 = \frac{d}{\sqrt{c^2 - v^2}}$$

$\therefore$ Total time taken by the ray to travel the whole path PB'P

$$t' = 2t_1 = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{1 - v^2/c^2}} = \frac{2d}{c}\left(1 + \frac{v^2}{2c^2}\right) \quad \text{---(2)}$$

$$\Delta t = t - t'$$

$$\Delta t = \frac{2d}{c}\left[1 + \frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2}\right] = \frac{2d}{c}\frac{v^2}{2c^2} = \frac{dv^2}{c^3} \quad \text{---(3)}$$

The distance travelled by light in time $\Delta t = c \times \Delta t = \dfrac{dv^2}{c^2}$ ---(4

This path difference may occur even when $PB \neq PA$. To elliminate such an error the apparatus is turned through 90° and the expt is repeated. Michelson and Morley expected a fringe shift of 0.4 in their apparatus, but they found nothing. The negative result shows that the ether hypothesis was wrong and thus no absolute space can be considered.

Postulates of the special theory of relativity

1) The principle of relativity.

The laws of Physics are the same in all inertial systems so that there is no preferred inertial frame and all the inertial frames are equivalent. Thus there is no such thing as absolute rest; there is no physical reasoning to prefer one inertial frame over the other.

2) The postulate of constancy of velocity of light.

The light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the source, intervening medium or observer.

Consequences of Einstein's postulates

The -ve result of the Michelson-Morley expt forced Einstein to conclude that the E.M laws hold in all inertial systems, with the value of the velocity of light, which is the same in all directions and is independent of the relative motion of the observer, medium and source. This invariance of the velocity of light c is embodied in the relationship as

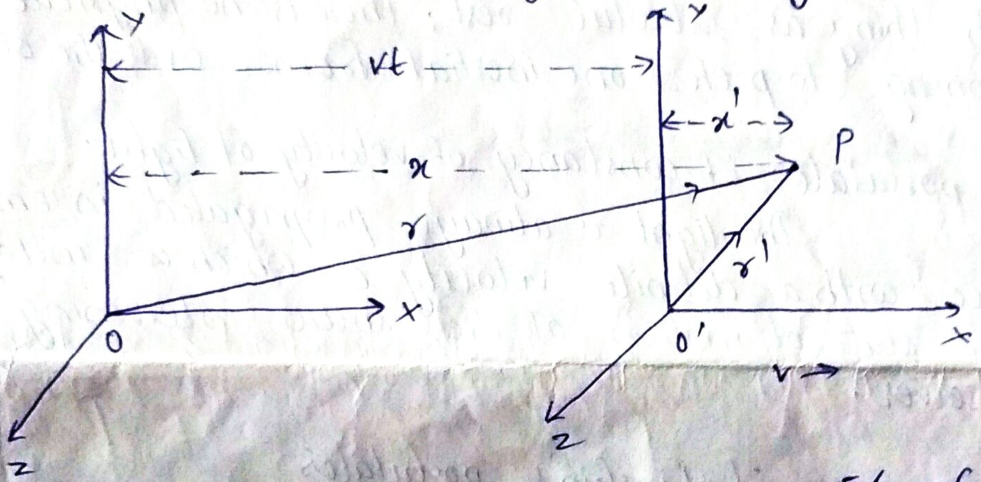
$$x^2+y^2+z^2-c^2t^2 = x'^2+y'^2+z'^2-c^2t'^2 = 0$$

where (x,y,z,t) refer to the termini of the light path in the unprimed system 's' and (x',y',z',t') to the termini in the primed system s' which is moving with velocity V relative to s. When x' is different from x, on account of relative motion in that direction, it will inevitably imply that t is different from t'. Thus new transformations, time will be no longer considered absolute for all observers in relative motion.

- Simultaneity of event
- Length contraction
- Time dilation.

Lorentz transformations

Consider two observers O and O' in two systems S and S'. System S' is moving with a constant velocity v relative to system S along the +ve x-axis.



Suppose we make measurements of time from the instant when the origins of S & S' just coincide ie, $t=0$ when O & O' coincide. Suppose a light pulse is emitted when O & O' coincide. The light pulse produced at $t=0$ will spread out as a growing sphere. The radius of the wavefront produced in this way will grow with speed c. After a time t, the observer O will note that the light has reached a point $P(x,y,z)$. For him, the distance of the point P is given by $r=ct$. From figure $r^2=x^2+y^2+z^2$

$$\therefore x^2+y^2+z^2 = c^2t^2 \quad \text{——— (1)}$$

|||ly the observer O' will note that the light has reached the same point P in a time t' with same velocity.

$\therefore x' = ct'$

$$\therefore x'^2 + y'^2 + z'^2 = c^2 t'^2 \quad \text{——— (2)}$$

(1) & (2) must be equal since both the observers are at the centre of the same expanding wavefront

$$x^2+y^2+z^2-c^2t^2 = x'^2+y'^2+z'^2-c^2t'^2 \quad \text{——— (3)}$$

Since there is no motion in Y & Z directions $y' = y$ and $z' = z$

$$\therefore (3) \Rightarrow \quad x^2 - c^2t^2 = x'^2 - c^2 t'^2 \quad \text{——— (4)}$$

The transformation eqn relating to x & x' can be written as

$$x' = k(x - vt) \quad \text{——— (5)}$$

The reason to take x' in this form is that the transformation must reduce to Galilean transformation when $v << c$.

$|||^{ly}$ $\quad t' = a(t - bx) \quad$ ——— (6) $\quad k, a, b \rightarrow$ consts

Sub. these values in (4)

$$x^2 - c^2t^2 = k^2(x-vt)^2 - c^2a^2(t-bx)^2$$

$$x^2 - c^2t^2 = k^2(x^2 - 2xvt + v^2t^2) - a^2c^2(t^2 - 2tbx + b^2x^2)$$

$$x^2 - c^2t^2 = x^2(k^2 - a^2b^2c^2) - 2(k^2v - a^2bc^2)xt - \left(a^2 - \frac{k^2v^2}{c^2}\right)c^2t^2$$

Equating coefficients of corresponding terms,

$$1 = k^2 - a^2b^2c^2$$

$$1 = a^2 - \frac{k^2v^2}{c^2}$$

$$0 = k^2v - a^2bc^2$$

Solving above eqns, for k, a & b,

$$k = a = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \& \quad b = v/c^2$$

$$\boxed{\therefore x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad y' = y \qquad z' = z}$$

$\rightarrow$ Lorentz transformation eqns.

The inverse eqns are

$$x = \frac{x'+vt'}{\sqrt{1-v^2/c^2}} \quad y = y' \quad z = z' \quad t = \frac{t'+vx'/c^2}{\sqrt{1-v^2/c^2}}$$

a) S.T. for values of $v<<c$, Lorentz transformation reduces to the Galilean transformation.

When $v<<c$ $\quad \frac{v}{c} \simeq 0$

$$\therefore \frac{1}{\sqrt{1-v^2/c^2}} \approx 1$$

$$\therefore x' = x - vt \quad y' = y \quad z' = z \quad t' = t'$$

Kinematic consequences of Lorentz transformations

1) Lorentz - Fitzgerald contraction / length contraction

Suppose there is a rod at rest in the system S and let the co-ordinates of its ends be x_1 and x_2 so that its length as measured by an observer in S is given by $l = x_2 - x_1$. The same rod is measured by an observer in S' at time t', to whom it appears to have length l'.

$$\therefore x_1 = \frac{x_1'+vt'}{\sqrt{1-v^2/c^2}} \qquad x_2 = \frac{x_2'+vt'}{\sqrt{1-v^2/c^2}}$$

$$\therefore x_2 - x_1 = \frac{x_2'-x_1'}{\sqrt{1-v^2/c^2}} \qquad \therefore l = \frac{l'}{\sqrt{1-v^2/c^2}}$$

$$\boxed{\therefore l' = l\sqrt{1-v^2/c^2}} \quad \text{or} \quad l = l_0\sqrt{1-\frac{v^2}{c^2}}$$

The length of an object in its rest frame is called its proper length so that the proper length is always the greatest and to any other observer who is moving with velocity v, the rod appears to be contracted in the ratio $\sqrt{1-v^2/c^2}$.

Q) A rod ~~&~~ 1m long is moving along its length with a velocity 0.6c. Calculate its length as it appears to an observer (a) On the earth (b) moving with the rod itself.

$l_0 = 1m$

$v = 0.6c$.

a) Let l be the length of the rod as it appears to an observer in the stationary reference frame of the earth

$$l = l_0 \sqrt{1 - v^2/c^2}$$

$$l = 1\sqrt{1 - 0.6^2} = \sqrt{1 - 0.36} = \underline{0.8m}$$

(b) $l_0 = 1m$.

Q) How fast would a rocket have to go relative to an observer for its length to be contracted to 99% of its length at rest?

$l = 0.99\, l_0 \qquad v = ?$

$$l = l_0 \sqrt{1 - v^2/c^2}$$

$$0.99 = \sqrt{1 - v^2/c^2}$$

$$1 - \frac{v^2}{c^2} = 0.9801$$

$$v^2 = 0.0199\, c^2$$

$$v = 0.141\, c = \underline{4.23 \times 10^7 \, m/s}$$

2) Time dilation

Let an observer in the system s' send light signals from the point $(x', 0, 0)$ at t_1' at a subsequent time t_2'. The interval $(t_2' - t_1')$ as observed by this observer will appear like an interval $(t_2 - t_1)$ to an observer in S.

$$\therefore t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \qquad t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$\therefore t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

$\tau_0 \rightarrow$ proper time.

$$\boxed{t = \frac{\tau'}{\sqrt{1 - v^2/c^2}}} \quad \text{or} \quad \tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}}$$

(12)

The time interval measured by a clock at rest relative to the observer is called proper time interval.

The proper time of a moving object is always less than the corresponding interval measured in a system at rest. It implies that a moving clock runs slower and this kinematical effect of relativity is called time dilation.

Q) A clock in a space ship emits signals at intervals of 1s as observed by an astronaut in the space ship. If the space ship travels with a speed of 3×10^7 m/s, what is the interval b/w successive signals as seen by an observer at the control centre on the ground?

$t_0 = 1s \qquad V = 3\times10^7 m/s$

$$t = \frac{t_0}{\sqrt{1-V^2/c^2}} = \frac{1}{\sqrt{1-\left(\frac{3\times10^7}{3\times10^8}\right)^2}}$$

$$= \frac{1}{\sqrt{1-\frac{1}{100}}} = \underline{\underline{1.005\ s}}$$

Twin paradox

If one clock remains at rest in an inertial frame, and another (which has been synchronised with the 1st one) is taken off to a distant planet on any sort of path and finally brought back to the starting point, the time elapsed by the moving clock will be less than time shown by the stationary clock. Further more there is no difference b/w the physical and biological clocks & one can take the heartbeats to be a clock. Accordingly an astronaut on return back to the earth from a long and fast journey will appear younger to himself having aged less than his twin brother who remained at home. However, the effects of time dilation being reciprocal in character should imply that the astronaut will appear older when viewed by the stay-at home twin. This is the twin paradox.

The paradox is resolved in the following way. The astronaut in a space ship experiences g loads in launching, turning and landing and weightlessness in free flight. The astronaut doesnot remain in the same inertial frame throughout the journey and as such we are not justified in using the formulae for time dilation from the special theory. However, by taking into account the noninertial nature of portions of the flight, it can be shown that time also slows down during the launching turning and landing part of the journey. Consequently both the frames of reference being not equivalent, the results of ageing cannot be reciprocal. Thus the stay-at home twin will age more than his journeying counterpart who makes a round trip and there is no paradox in the conclusion arrived at. $\Delta t_{space} = \Delta t_{earth}\sqrt{1-v^2/c^2}$

Q) A young man goes to the pole star and comes back to the earth on a rocket. Calculate the age difference b/w him and his twin brother who prefered to stay on the earth. The rocket velocity $v = \frac{4}{5}c$. and the distance b/w the earth & the pole star is 40 light yrs.

$$1\ ly = (c \times year)\ m.$$

$\therefore$ For the man who stays back on the earth the journey takes

$$= \frac{2 \times 40 \times c \times year}{\frac{4}{5}c}$$

$$= \underline{100\ years}.$$

To the man who makes the to & fro journey, the time interval appears to be as follows.

$$\tau = \tau_0 \sqrt{\cancel{1 - \frac{v^2}{c^2}}}$$

The distance measured by the astronaut.

$$l = l_0 \sqrt{1 - v^2/c^2}$$

$$= 40\ ly \sqrt{1 - \frac{16}{25}} = \underline{24\ lyrs}$$

$\therefore$ Time interval $= \frac{2 \times 24}{4/5} = \underline{60\ yrs}$.

9) On her 16^{th} birthday, a young lady decides that she will like to remain 16 for at least 10 years. She decides to go on a journey into outer space with uniform velocity. What is the minimum speed she must move relative to the laboratory so that when she returns after 10 years (relative to lab) she can still say, quite truthfully, that she is only 16.

The period of 10 yrs should appear like a day, which is nearly 1/360 years to the lady

$$\therefore \frac{1}{360} = 10 \sqrt{1 - \frac{v^2}{c^2}}$$

$$\left(\Delta t_{space} = \Delta t_{earth} \sqrt{1 - \frac{v^2}{c^2}} \right)$$

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{3600}$$

$$\cancel{1 - \frac{v^2}{c^2}} = \frac{1}{60} \Rightarrow \underline{v = 0.999c}$$

Transformation of velocities

The velocity of a material particle in frames can be related to its velocity in s'. Let its velocity in s & s' be $u(u_x, u_y, u_z)$ and $u'(u_x', y', u_z')$ resp.

Then $u_x = \frac{dx}{dt}$ $\quad u_x' = \frac{dx'}{dt'}$

from Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore dx' = \frac{dx - vdt}{\sqrt{1-v^2/c^2}}$$

$$dy' = dy \quad ; \quad dz' = dz$$

$$dt' = \frac{dt - vdx/c^2}{\sqrt{1-v^2/c^2}}$$

$$\therefore \frac{dx'}{dt'} = \frac{dx - vdt}{dt - vdx/c^2} = \frac{dx/dt - v}{1 - \frac{v}{c^2}\, dx/dt}$$

$$\therefore \boxed{u_x' = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}}$$

$$\frac{dy'}{dt'} = \frac{dy\sqrt{1-v^2/c^2}}{dt - vdx/c^2} = \frac{dy/dt\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{v}{c^2}\, dx/dt}$$

$$\boxed{u_y' = \frac{u_y\sqrt{1-v^2/c^2}}{1 - \frac{v}{c^2}u_x}}$$

$$\frac{dz'}{dt'} = \frac{dz\sqrt{1-v^2/c^2}}{dt - \frac{vdx}{c^2}} = \frac{dz/dt\sqrt{1-v^2/c^2}}{1 - \frac{v}{c^2}\, dx/dt}$$

$$\boxed{u_z' = \frac{u_z\sqrt{1-v^2/c^2}}{1 - \frac{v}{c^2}u_x}}$$

Inverse transformations are

$$u_x = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}} \qquad u_y = \frac{u_y'\sqrt{1-v^2/c^2}}{1 + \frac{v}{c^2}u_x'} \qquad u_z = \frac{u_z'\sqrt{1-v^2/c^2}}{1 + \frac{v}{c^2}u_x'}$$

In the non relativistic case $v/c \ll 1$

$$u_x = u_x' + v \qquad u_y = u_y' \qquad u_z = u_z'$$

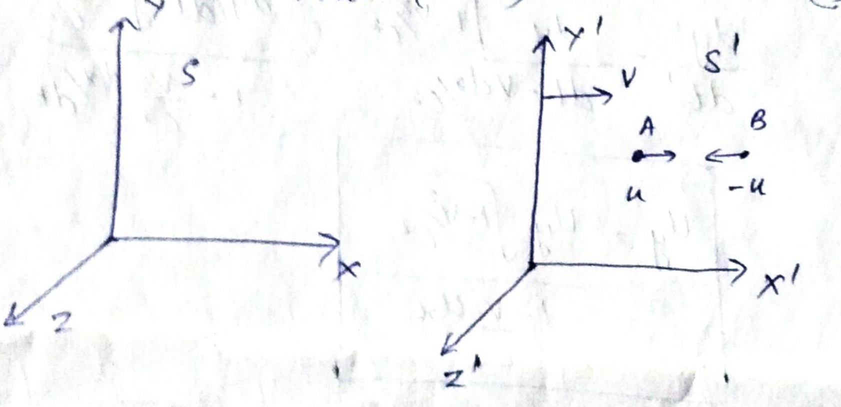
(17)

Variation of mass with velocity

Consider 2 systems S & S'. S' is moving with a constant velocity v relative to the system S, in the +ve x-direction. Suppose that in the system S', two exactly similar elastic balls A & B approach each other at equal speeds (u & $-u$). Let the mass of each ball be m in S'. They collide with each other and after collision coalesce into one body. According to the law of conservation of momentum,

Momentum of ball A + momentum of ball B
= Momentum of coalesced mass

$$mu + (-mu) = 0 \quad \text{---(1)}$$



Let us now consider the collision with reference to the system S. Let u_1 & u_2 be the velocities of the balls relative to S. Then

$$u_1 = \frac{u+v}{1+uv/c^2} \qquad u_2 = \frac{-u+v}{1-uv/c^2}$$

After collision, velocity of the coalesced mass is v relative to the system S.

Let mass of the ball travelling with velocity u_1 be m_1 and that of B with velocity u_2 be m_2 in the system S. Total momentum of the balls is conserved.

$$\therefore \quad m_1u_1 + m_2u_2 = (m_1+m_2)v$$

$$m_1\left(\frac{u+v}{1+\frac{uv}{c^2}}\right) + m_2\left(\frac{-u+v}{1-\frac{uv}{c^2}}\right) = (m_1+m_2)v$$

Let us consider the case of the motion of a particle along the x-axis $u_x = U \quad u_y = u_z = 0$

$$\therefore U = \frac{u' + V}{1 + \frac{V u'}{c^2}} \quad \text{or} \quad U' = \frac{U - V}{1 - \frac{UV}{c^2}}$$

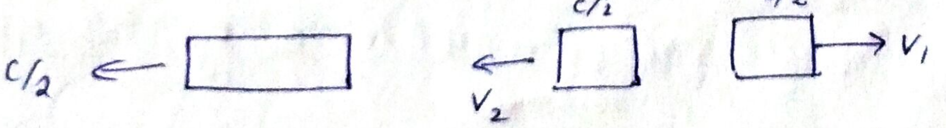
When $U = c$; $U' = \dfrac{c - V}{1 - \frac{V}{c}} = c$

Thus when a particle is moving with the velocity c w.r.t S, its velocity as observed from S' is still c which is consistent with the principle of constancy of light.

Q) A radioactive atom moves with a velocity $v = 0.1c$ along the x-axis of the system S. It emits a β-particle of velocity 0.95c relative to the system S' in which the radioactive atom is at rest. Find its speed relative to S.

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{0.95c + 0.1c}{1 + 0.95 \times 0.1} = \underline{\underline{0.963c}}$$

Q) A huge missile explodes and divides into two equal parts which take off with velocities c/2, one to the right & the other to the left. The piece that moves to the right again gets divided into two, by an explosion such that w.r.t its own rest frame the resulting 2 pieces take off with velocities c/2 to the right & left. Calculate the velocities of the last two pieces w.r.t the earth.



Let V_1 & V_2 denote the velocities of the last two pieces w.r.t the earth. Applying the formula for the combination of velocities

$v = c/2$
$v_x' = c/2$

$$V_1 = \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{c^2}{4c^2}} = \underline{\underline{\frac{4}{5}c}}$$

$v = c/2$
$v_x' = -c/2$

$$V_2 = \frac{\frac{c}{2} - \frac{c}{2}}{1 - \frac{c^2}{4c^2}} = \underline{\underline{0}}$$

$$m_1\left[\frac{u+v}{1+\frac{uv}{c^2}} - v\right] = m_2\left[v - \frac{-u+v}{1-\frac{uv}{c^2}}\right]$$

$$m_1\left[\frac{u+v-v-uv^2/c^2}{1+\frac{uv}{c^2}}\right] = m_2\left[\frac{v-\frac{uv^2}{c^2}+u-v}{1-uv/c^2}\right]$$

$$m_1\left[\frac{u\left(1-v^2/c^2\right)}{1+uv/c^2}\right] = m_2\left[\frac{u\left(1-v^2/c^2\right)}{1-uv/c^2}\right]$$

$$\frac{m_1}{m_2} = \frac{1+uv/c^2}{1-uv/c^2} \quad \text{--- (2)}$$

Also $$1-\frac{u_1^2}{c^2} = 1-\left(\frac{u+v}{1+uv/c^2}\right)^2\cdot\frac{1}{c^2}$$

$$= \frac{c^2\left[1+\frac{uv}{c^2}\right]^2-(u+v)^2}{c^2\left(1+uv/c^2\right)^2}$$

$$= \frac{c^2\left[1+\frac{2uv}{c^2}+\frac{u^2v^2}{c^4}\right]-\left(u^2+2uv+v^2\right)}{c^2\left(1+uv/c^2\right)^2}$$

$$= \frac{1+\frac{2uv}{c^2}+\frac{u^2v^2}{c^4}-\frac{u^2}{c^2}-\frac{2uv}{c^2}-\frac{v^2}{c^2}}{\left(1+uv/c^2\right)^2}$$

$$= \frac{1+\frac{u^2v^2}{c^4}-\frac{u^2}{c^2}-\frac{v^2}{c^2}}{\left(1+uv/c^2\right)^2} = \frac{\left(1-\frac{u^2}{c^2}\right)-\frac{v^2}{c^2}\left(1-\frac{u^2}{c^2}\right)}{\left(1+uv/c^2\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1-u^2/c^2\right)\left(1-v^2/c^2\right)}{\left(1+uv/c^2\right)^2} \quad \text{--- (3)}$$

lly $$1-\frac{u_2^2}{c^2} = \frac{\left(1-u^2/c^2\right)\left(1-v^2/c^2\right)}{\left(1-uv/c^2\right)^2} \quad \text{--- (4)}$$

(19)

$$\frac{(3)}{(4)} \cdot \frac{(4)}{(3)} \Rightarrow \frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}} = \frac{\left(1+\frac{uv}{c^2}\right)^2}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$\frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}} = \frac{1+uv/c^2}{1-uv/c^2} \quad \text{—(5)}$$

Sub (2) ; $\frac{m_1}{m_2} = \sqrt{\frac{1-u_2^2/c^2}{1-u_1^2/c^2}}$

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} \quad \text{—(6)}$$

Since LHS & RHS are independent of one another, the above eqn can be true only if each is a const

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_0$$

$m_0 \rightarrow$ rest mass ; corresponds to zero velocity

In general, $\boxed{m = \frac{m_0}{\sqrt{1-u^2/c^2}}}$

$m \rightarrow$ relativistic mass.

$u \rightarrow$ velocity of particle.

When $u \Rightarrow c$; $m \rightarrow \infty \Rightarrow$ object travelling at the speed of light would have infinite mass.

Q) At what speed is a particle moving if the mass is equal to 3 times its rest mass.

$$m = \frac{m_0}{\sqrt{1-u^2/c^2}} \qquad m = 3m_0$$

$$3 = \frac{1}{\sqrt{1-u^2/c^2}} \qquad 1-\frac{u^2}{c^2} = \frac{1}{9}$$

$$\frac{u^2}{c^2} = \frac{8}{9} \Rightarrow u = \sqrt{\frac{8}{9}}\,c = \underline{\underline{0.94\,c}}$$

Mass-energy equivalence

Let a force F be acting on a particle which gets displaced by a distance dl in the direction of the force, then the work done dW is given by the scalar product of F and dl ie,

$$dW = F \cdot dl$$

Assuming that the work done goes into increasing the K.E of the particle,

$$dT = F \cdot dl$$

$$\therefore \frac{dT}{dt} = F \cdot \frac{dl}{dt} = F \cdot u \qquad u \rightarrow \text{velocity of particle}$$

Sub $F = \frac{d(mu)}{dt}$

$$\therefore \frac{dT}{dt} = \frac{d(mu)}{dt} \cdot u$$

$$\frac{dT}{dt} = m\frac{du}{dt} \cdot u + u\frac{dm}{dt} \cdot u \qquad \text{—— (1)}$$

But $u^2 = u_x^2 + u_y^2 + u_z^2$

$$2u\frac{du}{dt} = 2u_x\frac{du_x}{dt} + 2u_y\frac{du_y}{dt} + 2u_z\frac{du_z}{dt}$$

$$\therefore u \cdot \frac{du}{dt} = u\frac{du}{dt} \qquad \text{—— (2)}$$

Also $\frac{dm}{dt} = \frac{dm}{du} \times \frac{du}{dt} = \frac{du}{dt} \frac{d}{du} \frac{m_0}{\sqrt{1-u^2/c^2}}$

$$\frac{dm}{dt} = \frac{du}{dt} \cdot \frac{m_0\, u/c^2}{\left(1-u^2/c^2\right)^{3/2}} \qquad \text{—— (3)}$$

Sub these in (1).

$$\frac{dT}{dt} = \frac{m_0}{\sqrt{1-u^2/c^2}}\, u\frac{du}{dt} + m_0\frac{du}{dt}\frac{u^3/c^2}{\left(1-u^2/c^2\right)^{3/2}}$$

$$= \frac{m_0\left(1-u^2/c^2\right) u\frac{du}{dt} + m_0\frac{du}{dt}\frac{u^3}{c^2}}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}$$

$$= \frac{m_0 v \frac{dv}{dt}\left(1-\frac{v^2}{c^2}+\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$= \frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \frac{dv}{dt}$$

$$\frac{dT}{dt} = \frac{d}{dt}\left[\frac{m_0 c^2}{\sqrt{1-v^2/c^2}}\right]$$

Integrating w.r.t t,

$$T = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} + C_1$$

When $T=0,\ v=0 \Rightarrow C_1 = -m_0 c^2$

$$\therefore T = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2$$

$$= m_0 c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

$$T = mc^2 - m_0 c^2$$

$$mc^2 = T + m_0 c^2$$

$\therefore$ Total energy, $\underline{E = mc^2}$

This eqn is the law of equivalence of mass and energy or the law of inertia of energy.

~~Transformation of relativistic momentum and energy~~

Q) Calculate the K.E of an $\bar{e}$ moving with a velocity of 0.98 times the velocity of light in the laboratory system.

$$T = mc^2 - m_0 c^2$$

m_0 = rest mass of $\bar{e}$ = 9.1×10^{-31} kg

$c = 3\times10^8$ m/s $\quad V = 0.98c$

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} = \frac{m_0}{\sqrt{1-\left(\frac{0.98c}{c}\right)^2}} = \frac{m_0}{\sqrt{1-0.9604}} = 5.02\,m_0$$

$$\therefore T = (5.02m_0 - m_0)c^2 = 4.02\,m_0c^2$$
$$= 3.296\times10^{-13}\ \text{J}$$

Transformation of relativistic momentum and energy.

In the inertial frame S, the particle momentum $\bar{P}$ is given by

$$P_x = mu_x = \frac{m_0u_x}{\sqrt{1-u^2/c^2}} \quad \text{——— (1)}$$

$$P_y = mu_y = \frac{m_0u_y}{\sqrt{1-U^2/c^2}}$$

$$P_z = mu_z = \frac{m_0u_z}{\sqrt{1-U^2/c^2}}$$

Total energy is, $E = mc^2 = \dfrac{m_0c^2}{\sqrt{1-U^2/c^2}}$

Identically the corresponding quantities in s' are,

$$P_x' = \frac{m_0u_x'}{\sqrt{1-U'^2/c^2}} \qquad P_y' = \frac{m_0U_y'}{\sqrt{1-U'^2/c^2}} \qquad P_z' = \frac{m_0U_z'}{\sqrt{1-U'^2/c^2}}$$

$$E' = \frac{m_0c^2}{\sqrt{1-U'^2/c^2}}$$

Consider eqn for P_x'

$$P_x' = \frac{m_0u_x'}{\sqrt{1-U'^2/c^2}} \quad \text{——— (1)}$$

We have $\dfrac{1}{\sqrt{1-U'^2/c^2}} = \dfrac{1-Vu_x/c^2}{\sqrt{1-V^2/c^2}\sqrt{1-u^2/c^2}}$

also $u_x' = \dfrac{u_x - V}{1-u_xV/c^2}$

sub. these two in (1),

$$P_x' = \frac{m_0(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)} \times \frac{(1 - vu_x/c^2)}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}$$

$$= \frac{m_0}{\sqrt{1 - v^2/c^2}} \cdot \frac{(u_x - v)}{\sqrt{1 - u^2/c^2}}$$

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \; ; \; \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

$$= \gamma m (u_x - v)$$

$$P_x' = \gamma m u_x - \gamma m v$$

$$\boxed{P_x' = \gamma \left[P_x - \frac{vE}{c^2}\right]}$$

III^{ly} substituting for $\frac{1}{\sqrt{1 - u'^2/c^2}}$ & $u_y' = u_y \frac{\sqrt{1 - v^2/c^2}}{1 - vu_x/c}$

into expression for P_y',

$$P_y' = \frac{m_0 \, u_y'}{\sqrt{1 - u'^2/c^2}}$$

$$= \frac{m_0}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}} (1 - vu_x/c^2) \times \frac{u_y\sqrt{1 - v^2/c^2}}{1 - vu_x/c^2}$$

$$P_y' = \frac{m_0 u_y}{\sqrt{1 - u^2/c^2}} = P_y$$

III^{ly} $\underline{P_z' = P_z}$

Lastly $E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1 - u'^2/c^2}} = \frac{m_0(1 - vu_x/c^2)\,c^2}{\sqrt{1 - v^2/c^2}\sqrt{1 - u^2/c^2}}$

$$= \gamma m c^2 (1 - vu_x/c^2)$$

$$\boxed{E' = \gamma (E - vP_x)}$$

∴ Inverse transformations are

$$P_x = \gamma(P_x' + VE'/c^2)$$

$$P_y = P_y' \qquad P_z = P_z'$$

$$E = \gamma[E' + VP_x']$$

Also,

$$P = \frac{m_0 u}{\sqrt{1-u^2/c^2}}$$

$$\therefore P^2c^2 + m_0^2c^4 = \frac{m_0^2 u^2 c^2}{(1-u^2/c^2)} + m_0^2c^4$$

$$= \frac{m_0^2u^2c^2 + m_0^2c^4 - m_0^2u^2c^2}{(1-u^2/c^2)}$$

$$= \frac{m_0^2c^4}{1-u^2/c^2} = E^2$$

$$\boxed{\begin{aligned} \therefore E^2 &= p^2c^2 + m_0^2c^4 \\ E &= \sqrt{p^2c^2 + m_0^2c^4} \end{aligned}}$$

Thus the sign of the energy may be positive or negative is a consequence of relativity.

Tachyons

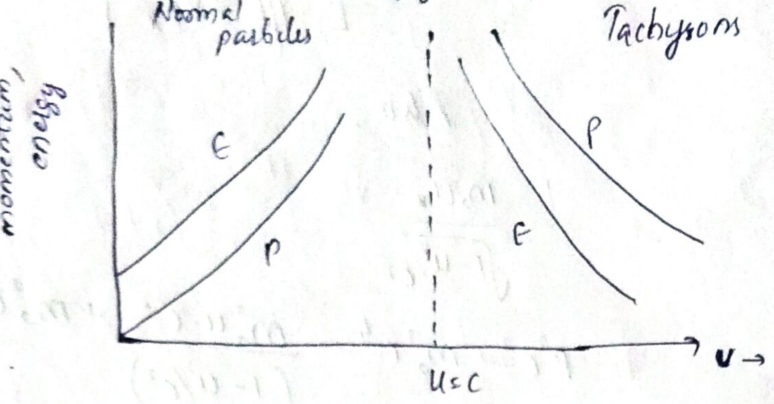
According to special theory of relativity, the relativistic mass, momentum and total energy of a particle are given by, $m = \gamma m_0$; $P = \gamma m_0 u$; $E = \gamma m_0 c^2$

where $m_0 \rightarrow$ mass measured in an inertial frame w.r.t which the particle is at rest.

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

Thus an infinite amount of energy is necessary for accelerating a particle up to the velocity of light since γ then become infinite. But some physicists had suggested that particles can indeed travel faster than light. They assume that their rest mass is imaginary.

since it is not observable and their energy & momentum are real. Such particles are called tachyons. Also on losing energy it gets speeded up until it travelling infinitely fast and then it has no energy at all.



According to special theory, as u increases, momentum, energy increase but never reach the asymptote $u=c$. The normal particles lie to the left of the line $u=c$ and the tachyons to the right.

Space-like and time like intervals

Suppose an event occurs at the location (x,y,z) and time t in a frame s and another event occurs at $(x+\delta x, y+\delta y, z+\delta z)$ and time $(t+\delta t)$. Let the co-ordinates of these events in s' be (x',y',z') and $(x'+\delta x', y+\delta y', z'+\delta z')$ at times t' & $\delta t'$ resp. Then according to Lorentz transformation

$\delta x = \gamma(\delta x'+v\delta t')$; $\delta y = \delta y'$; $\delta z = \delta z'$; $\delta t = \gamma(\delta t'+\delta x' v/c^2)$

but $\delta x^2+\delta y^2+\delta z^2 \neq \delta x'^2+\delta y'^2+\delta z'^2$ means that the spatial interval is not invariant under Lorentz transformation. However, the expression

$$\delta x^2+\delta y^2+\delta z^2-c^2\delta t^2$$
$$= \gamma^2\left[\delta x'+v\delta t'\right]^2+\delta y'^2+\delta z'^2-c^2\gamma^2\left[\delta t'+\frac{v\delta x'}{c^2}\right]^2$$
$$= \delta x'^2+\delta y'^2+\delta z'^2-c^2\delta t'^2$$

is invariant under Lorentz transformation. ∴ The interval b/w two events δs is defined as $\delta s^2 = \delta x^2+\delta y^2+\delta z^2-c^2\delta t^2$

The invariance of the interval means that it two events occur, 2 observers in s & s' will measure diff. distance & time seperations b/w the events but will measure same interval δs.

If δs^2 is positive $\Rightarrow \delta x^2+\delta y^2+\delta z^2 > c^2\delta t^2$, then the interval δs b/w the 2 events is called <u>space like</u> if δs^2 is negative, then the interval δs is called <u>time like</u>.