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LAGRANGIAN DYNAMICS

Degrees of freedom

The minimum no: of independent variables or co-ordinates required to specify the position of a dynamical system, consisting of one or more particles is called the no: of degrees of freedom of the system. If a system consists of N particles, moving freely in space, we need 3N independent co-ordinates to describe its position.

The configuration of the system of N particles, moving freely in space, may be represented by the position of a single point in 3N dimensional space, which is called configuration space.

Constraints

The limitations on the motion of a system are called constraints and the motion is said to be constrained motion. Constraints limit the motion of a system and the no: of independent co-ordinates needed to describe the motion, is reduced.

Classification :-

I (a) Holonomic constraints :- If the constraints are expressed in the form of eqns of the form $f(r_1, r_2, r_3, \dots t) = 0$, then they are called holonomic constraints. If there are m constraints then the degrees of freedom reduced to $(3N-m)$.

(b) Nonholonomic constraints :- The constraints which are not expressible in the form of $f(r_1, r_2, \dots t) = 0$

are called nonholonomic. For eg. the motion of a particle, placed on the surface of a sphere of radius 'a' will be described by,

$|\bar{r}| \geq a$ or $r - a \geq 0$.

$\bar{r} \rightarrow$ position vector of the particle.

II (a) <u>Rheonomous</u> :- constraint relations depend explicitly on time.

b) <u>Scleronomic</u> :- constraint relations donot explicitly depend on time.

III (a) <u>Conservative</u> :- Total mechanical energy of the system is conserved while performing the constrainted motion. Constraint forces do not do any work

(b) <u>Dissipative</u> :- constraint forces do work and the total mechanical energy is not conserved.

<u>Forces of constraint</u>

Constraints are always related to forces which restrict the motion of the system. These forces are called forces of constraint. Usually the constraint forces act in a direction $\perp^r$ to the surface of constraints while the motion of the object is parallel to the surface. In such cases the work done by the forces of constraint is zero. These constraints are termed as <u>workless</u> or <u>ideal constraints</u>.

<u>Generalised co-ordinates</u>

The name generalised co-ordinates is given to a set of independent co-ordinates sufficient in no:- to describe completely the state of configuration of a dynamical system. These co-ordinates are ~~define~~ denoted as

$$q_1, q_2, q_3 \cdots q_n \quad n \rightarrow \text{total nos of generalised co-ordinates}$$

The generalised co-ordinates for a system of N particles, constrained by 'm' equations are $n = 3N - m$. It is not necessary that these co-ordinates should be rectangular, spherical or cylindrical. Infact, the quantities like length, $(\text{length})^2$, angle, energy or a dimensionless quantity may be used as generalised co-ordinates; but they should completely describe the state of the system.

For a system of N particles, if x_i, y_i, z_i are the cartesian co-ordinates of the ith particle then these co-ordinates in terms of the generalised co-ordinates q_k can be expressed as,

$$x_i = x_i(q_1, q_2, q_3 \cdots q_n, t)$$

$$y_i = y_i(q_1, q_2, q_3 \cdots q_n, t)$$

$$z_i = z_i(q_1, q_2, q_3 \cdots q_n, t)$$

In general, $\quad r_i = r_i(q_1, q_2, q_3 \cdots q_n, t)$

Degrees of freedom; examples :-

1) A particle moving on the circumference of a circle = 1
2) Five particles moving freely in a plane = 10
3) 2 particles connected by a rigid rod moving freely in a plane = $2 \times 2 - 1 = 3$.
4) Rigid body :- A rigid body is a system of particles in which the distance b/w any two particles remain

fixed throughout the motion. Let us consider 3 noi. collinear particles P_1, P_2, P_3 of a rigid body. As each particle has 3 degrees of freedom and there are 3 constraints of the form,

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 = r_{12}^2$$

$$(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2 = r_{23}^2$$

$$(x_1-x_3)^2+(y_1-y_3)^2+(z_1-z_3)^2 = r_{13}^2$$

Hence the degrees of freedom for these particles are $3\times3-3 = 6$. The consideration of any other particle in the body needs three co-ordinates say $P_i(x_i, y_i, z_i)$ and obviously there are three eqns of constraints because the distances of P_i from P_1, P_2, P_3 are fixed. Hence any other particle will not add any new degree of freedom to the 6 degrees of freedom of the 3-particle system of the body.

Principal of virtual work

In order to investigate the properties of a system, we can imagine arbitrary instantaneous change in the position vectors of the particles of the system eg: virtual displacement. An infinitesimal virtual displacement of i^{th} particle of a system of N particles is denoted by δr_i. This is the pt displacement of position co-ordinates only and doesnot involve variation of time ie,

$$\delta r_i = \delta r_i(q_1, q_2 \cdots q_n)$$

suppose the system is in equilibrium, then the total force on any particle is zero ie,

$$F_i = 0 \qquad i = 1,2,3,\dots N \quad \text{———} (1)$$

The virtual work of the force F_i in the virtual displacement δr_i will also be zero

$$\delta W_i = F_i \cdot \delta r_i = 0$$

||| the sum of virtual work for all the particles must vanish ie,

$$\delta W = \sum_{i=1}^{N} F_i \cdot \delta r_i = 0 \quad \text{———} (2)$$

This is the principal of virtual work which states that the work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium.

The total force F_i on the i^{th} particle can be expressed as,

$$F_i = F_i^a + f_i$$

$F_i^a \rightarrow$ applied force

$f_i \rightarrow$ force of constraint

$$\therefore (2) \rightarrow \sum_{i=1}^{N} F_i^a \cdot \delta r_i + \sum_{i=1}^{N} f_i \cdot \delta r_i = 0$$

We restrict ourselves to the systems where the virtual work of the forces of constraints is zero.

$$\text{Then} \quad \sum_{i=1}^{N} F_i^a \cdot \delta r_i = 0 \quad \text{———} (3)$$

ie, for equilibrium of a system, the virtual work of applied forces is zero. [For static case]

D'Alembert's principle.

According to Newton's 2nd law of motion, the force acting on the ith particle is

$$F_i = \frac{dP_i}{dt} = \dot{P}_i$$

$$\therefore \quad F_i - \dot{P}_i = 0 \qquad , i = 1,2,..N$$

$\therefore$ for virtual displacements δr_i,

$$\sum_{i=1}^{N} (F_i - \dot{P}_i) \cdot \delta r_i = 0$$

but $\quad F_i = F_i^a + f_i$

$$\therefore \quad \sum_{i=1}^{N} (F_i^a + f_i - \dot{P}_i) \cdot \delta r_i = 0$$

again $\sum f_i \cdot \delta r_i = 0$

$$\therefore \boxed{\sum_{i=1}^{N} (F_i^a - \dot{P}_i) \cdot \delta r_i = 0}$$

Lagrange's equations from D'Alembert's principle

Consider a system of N particles. The transformation equations for the position vectors of the particles are

$$r_i = r_i(q_1, q_2, \cdots q_k \cdots q_n, t) \qquad \text{—— (1)}$$

where 't' is the time and q_k $(k = 1,2,..n)$ are generalised co-ordinates.

Diff (1) w.r.t t, we obtain the velocity of the ith particle.

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \cdots \frac{\partial r_i}{\partial q_k}\frac{dq_k}{dt} + \cdots \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

or $V_i = \dot{r}_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t}$ —— (2)

$\dot{q}_k \rightarrow$ generalised velocities.

The virtual displacement is given by,

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + \cdots \frac{\partial r_i}{\partial q_k}\delta q_k + \cdots + \frac{\partial r_i}{\partial q_n}\delta q_n$$

or $\delta r_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k$ —— (3)

since by definition, the virtual displacements do not depend on time.

According to D'Alembert's principle

$$\sum_{i=1}^{N} (F_i - \dot{P}_i)\cdot \delta r_i = 0 \qquad \text{—— (4)}$$

Here $\sum_{i=1}^{N} F_i \cdot \delta r_i = \sum_{i=1}^{N} F_i \cdot \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k$

$$= \sum_{i=1}^{N}\sum_{k=1}^{n}\left[F_i \cdot \frac{\partial r_i}{\partial q_k}\right]\delta q_k$$

$$= \sum_{k=1}^{n} G_k \delta q_k \qquad \text{—— (5)}$$

where $G_k = \sum_{i=1}^{N}\left[F_i \cdot \frac{\partial r_i}{\partial q_k}\right] = \sum_{i=1}^{N}\left(F_{x_i}\frac{\partial x_i}{\partial q_k} + F_{y_i}\frac{\partial y_i}{\partial q_k} + F_{z_i}\frac{\partial z_i}{\partial q_k}\right)$ —— (6)

are called the components of generalised force ~~accos~~ associated with the generalized co-ordinates q_k. The dimension of G_k can be different than those of force. However the dimension of $G_k \delta q_k$ are those of work.

Further $\sum_{i=1}^{N} \dot{p}_i \cdot \delta r_i = \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \delta q_k$

$$= \sum_{k=1}^{n} \left[\sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k} \right] \delta q_k \quad \text{—— (7)}$$

Now $\sum_{i=1}^{N} \frac{d}{dt} \left[m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_k} \right] = \sum_{i=1}^{N} \left[m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k} + m_i \dot{r}_i \cdot \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_k} \right] \right]$

$$\therefore \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_k} \right) - m_i \dot{r}_i \cdot \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_k} \right] \right] \quad \text{—— (8)}$$

It is easy to prove that

$$\frac{d}{dt} \left(\frac{\partial r_i}{\partial q_k} \right) = \frac{\partial}{\partial q_k} \left(\frac{d r_i}{dt} \right) = \frac{\partial V_i}{\partial q_k} \quad \text{—— (9)}$$

and $\frac{\partial r_i}{\partial q_k} = \frac{\partial V_i}{\partial \dot{q}_k}$ —— (10)

$$\therefore (8) \Rightarrow \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} \left[\frac{d}{dt} \left(m_i V_i \cdot \frac{\partial V_i}{\partial \dot{q}_k} \right) - m_i V_i \cdot \frac{\partial V_i}{\partial q_k} \right] \quad \text{—— (11)}$$

Sub this in (7)

$$\sum_{i=1}^{N} \dot{p}_i \cdot \delta r_i = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_k}\right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_k}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{N}\frac{1}{2}m_i(v_i\cdot v_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{N}\frac{1}{2}m_i(v_i\cdot v_i)\right\}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k \quad \text{——— (12)}$$

where $\sum_i \frac{1}{2}m_i(v_i\cdot v_i) = \sum_i \frac{1}{2}m v_i^2 = T$ is the kinetic energy of the system.

~~Sub for $\sum_i F_i \cdot \delta r_i$ from (5) and $\sum_i \dot{p}_i \cdot \delta r_i$ from (12)~~ in (4) the D'Alembert's principle becomes.

$$\sum_{k=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - G_k\right]\delta q_k = 0 \quad \text{——— (13)}$$

As the constraints are holonomic, it means that any virtual displacement δq_k is independent of δq_j. Therefore, the coefficient in the square bracket for each δq_k must be zero ie,

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right] - \frac{\partial T}{\partial q_k} - G_k = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k}$$

→ General form of Lagrange's eqn.

⟶ (14)

For a conservative system, the force is derivable from a scalar potential V

$$F_i = -\nabla_i V = -\hat{i}\frac{\partial V}{\partial x_i} - \hat{j}\frac{\partial V}{\partial y_i} - \hat{k}\frac{\partial V}{\partial z_i} \quad \text{---(15)}$$

Hence from (6), the generalized force components are

$$G_k = -\sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right] \quad \text{---(16)}$$

or $$G_k = -\frac{\partial V}{\partial q_k} \quad \text{---(17)}$$

$$\therefore (14) \Rightarrow \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right] - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k} = 0 \quad \text{---(18)}$$

~~Since the scalar potential~~ V is the function of generalized coordinate q_k only, we can write,

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_k} - \frac{\partial (T-V)}{\partial q_k} = 0 \quad \text{---(19)}$$

We define a new function $L = T - V$ —(20)

which is called Lagrangian of the system

$$\therefore (19) \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0} \quad k = 1, 2, \ldots n$$

These eqns are called Lagrange's eqns. for conservative system.

Q) Obtain eqn

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Q) Obtain Newton's equation of motion from Lagrange's eqns. by using cartesian co-ordinates as generalised co-ordinates.

The general form of the Lagrange's eqns is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \quad \text{---(1)}$$

Here $q_1 = x$, $q_2 = y$, $q_3 = z$.

$G_1 = F_x$, $G_2 = F_y$, $G_3 = F_z$.

The K.E T is $\frac{1}{2}m\left[\dot{x}^2+\dot{y}^2+\dot{z}^2\right]$

For x-co-ordinate, (1) $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x \quad \text{---(2)}$

$$\frac{\partial T}{\partial x} = 0 \quad ; \quad \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

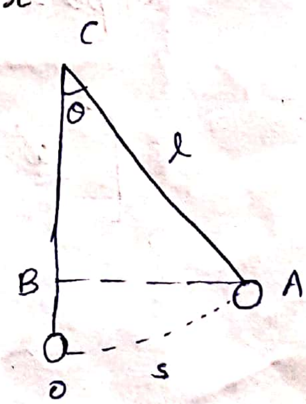
$\therefore$ (2) $\Rightarrow$ $\frac{d}{dt}(m\dot{x}) = F_x$ or $F_x = \frac{dP_x}{dt}$ $\quad$ $P_x \rightarrow$ x component of momentum

$|||^{ly}$ $F_y = \frac{dP_y}{dt}$, $F_z = \frac{dP_z}{dt}$

Thus $\bar{F} = \frac{d\bar{P}}{dt}$ which is Newton's eqn of motion

Q2) Obtain the equation of motion of a simple pendulum by using Lagrangian method and hence deduce the formula for its time period for small amplitude oscillations.

Let θ be the angular displacement of the simple pendulum from the equilibrium position. If 'l' be the effective length of the pendulum and 'm' be the mass of the bob, then the displacement along arc OA = S is given by



$$S = l\theta \quad \left[\theta = \frac{\text{Arc}}{\text{Radius}} = \frac{S}{l}\right]$$

K.E , $T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$ $\left[\because v = \frac{ds}{dt} = \frac{d(l\theta)}{dt}\right.$

If the potential energy of the system, when the bob is at O is zero, then the P.E, when the bob is at A is,

$$V = mg(OB) = mg(OC - BC) = mg(l - l\cos\theta)$$
$$= mgl(1 - \cos\theta)$$
$$\therefore L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta) \quad \text{---(1)}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

The Lagrange's eqn is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$
$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$
$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

This is the eqn of motion of a simple pendulum.

For small amplitude oscillations $\sin\theta \simeq \theta$

$$\therefore \ddot{\theta} + \frac{g}{l}\theta = 0$$

This ~~eq~~ represents a simple harmonic motion of period given by, $T = 2\pi\sqrt{l/g}$

$$\left[\omega^2 = \frac{g}{l} \Rightarrow \omega = \sqrt{g/l} \Rightarrow \frac{2\pi}{T} = \sqrt{g/l}\right.$$
$$\left.\Rightarrow T = 2\pi\sqrt{l/g}\right]$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad = \text{SHM}$$

(7)

Q3) Obtain the eqn of motion of a system of two masses, connected by an inextensible string passing over a small smooth pulley (Atwood's Machine).

The Atwood's machine is an example of a conservative system with holonomic constraint. The pulley is small, massless and frictionless. Let the two masses be m_1 & m_2 which are connected by an inextensible string of length l. Suppose x be the variable vertical distance from the pulley to the mass m_1. Then mass m_2 will be at a distance $(l-x)$ from the pulley.

Thus there is only one independent co-ordinate x. The velocities of the two masses are

$$V_1 = \frac{dx}{dt} = \dot{x} \quad \text{and} \quad V_2 = \frac{d(l-x)}{dt} = -\dot{x}$$

$$\therefore T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 = \frac{1}{2}(m_1+m_2)\dot{x}^2$$

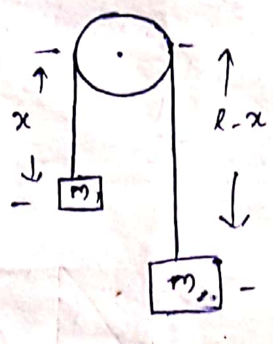
The P.E of the system w.r.t the pulley is,

$$V = -m_1 g x - m_2 g(l-x)$$

$$\therefore L = T - V = \frac{1}{2}(m_1+m_2)\dot{x}^2 + m_1 g x + m_2 g(l-x)$$

$$\therefore \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x}$$

$$\frac{\partial L}{\partial x} = m_1 g - m_2 g$$



$\therefore$ Lagrange's eqn is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left[(m_1+m_2)\dot{x}\right] - m_1 g + m_2 g = 0$$

$$(m_1+m_2)\ddot{x} - (m_1-m_2)g = 0$$

or $\boxed{\ddot{x} = \left(\frac{m_1-m_2}{m_1+m_2}\right)g}$ → Eqn of motion

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If $m_1 > m_2$, the mass m_1 descends with const acceleration and if $m_1 < m_2$, the mass m_1 ascends with const acceleration.

Q.) Calculate the acceleration of the system, in the above qns, if the pulley is a disc of radius R and moment of inertia I about an axis through its centre and $\perp^r$ to its plane.

Angular velocity of the pulley $\omega = \frac{v}{R} = \frac{\dot{x}}{R}$

Rotational kinetic energy of the pulley

$$= \frac{1}{2} I \omega^2 = \frac{1}{2} I \frac{\dot{x}^2}{R^2}$$

$$\therefore T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \frac{\dot{x}^2}{R^2}$$

Also $V = -m_1 g x - m_2 g (l - x)$

$$\therefore L = T - V = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} \frac{I \dot{x}^2}{R^2} + m_1 g x + m_2 g (l - x)$$

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2 \dot{x} + \frac{I \dot{x}}{R^2}$$

$$\frac{\partial L}{\partial x} = m_1 g - m_2 g$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

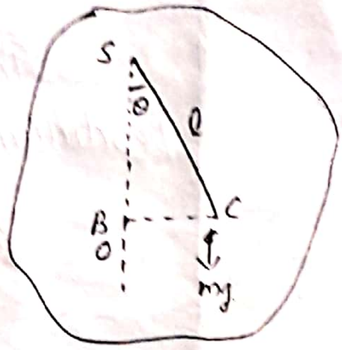
$$\frac{d}{dt}\left[m_1 \dot{x} + m_2 \dot{x} + \frac{I \dot{x}}{R^2}\right] - \left(m_1 g - m_2 g\right] = 0$$

$$\left(m_1 + m_2 + \frac{I}{R^2}\right) \ddot{x} - (m_1 - m_2) g = 0$$

$$\therefore \ddot{x} = \frac{(m_1 - m_2) g}{\left[m_1 + m_2 + \frac{I}{R^2}\right]}$$

05) Use Lagrange's eqns to find the eqn of motion of a compound pendulum in a vertical plane about a fixed horizontal axis. Hence find the period of small amplitude oscillations of the compound pendulum

Let the compound pendulum be suspended from S with C as centre of mass. It is oscillating in the vertical plane which is the plane of the paper.



M.I of the pendulum about the axis of rotation through S is given by

$$I = I_c + Ml^2$$

$$= Mk^2 + Ml^2 = M(k^2+l^2)$$

$M \rightarrow$ Mass of the pendulum.

$k \rightarrow$ radius of gyration

$l \rightarrow$ distance b/w centre of suspension ~~and centre of mass~~

If θ is the instantaneous angle which SC makes with the vertical axis through O, then the K.E of the oscillating system is

$$T = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} M(k^2+l^2)\dot{\theta}^2$$

P.E wrt horizontal plane through S is

$$V = -Mgl\cos\theta$$

$$L = T - V = \frac{1}{2} M(k^2+l^2)\dot{\theta}^2 + Mgl\cos\theta.$$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = M(k^2+l^2)\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -Mgl\sin\theta.$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0$$

$$\frac{d}{dt}\left[M(k^2+l^2)\dot{\theta}\right] + Mgl\sin\theta = 0$$

$$M(k^2+l^2)\ddot{\theta} + Mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{gl}{(k^2+l^2)}\sin\theta = 0$$

This is the eqn of motion of the compound pendulum. If θ is small $\sin\theta \simeq \theta$.

$$\therefore \ddot{\theta} + \frac{gl}{k^2+l^2}\theta = 0$$

This eqn represents a SHM whose period is

$$T = 2\pi\sqrt{\frac{k^2+l^2}{lg}} = 2\pi\sqrt{\frac{k^2/l + l}{g}}$$

Note:-

In Newtonian Mechanics, the eqn of motion involve vector quantities like force, momentum etc which increase complexity in solving the problems. This approach also cannot avoid constraints present in a problem. These forces of constraints, if not known, make the solution of the problem difficult and even if they are known, the use of rectangular or other commonly used co-ordinates may make the solution of the problem to be impossible. These drawbacks are removed in the Lagrangian mechanics, where the technique involved scalars, like potential & K.Es, instead of vectors. The use of generalized co-ordinates in the Lagrangian formulation often allows automatically for the constraints. In this formulation, the difficulty in solving the problem is many times much reduced, when any quantity like momentum, $(\text{length})^2$ is taken as the generalised co-ordinate instead of rectangular or other co-ordinates.