OSCILLATIONS

Periodic motion :- If a particle moves such that it retraces its path regularly after equal intervals of time, its motion is said to be periodic. The interval of time, required to complete one cycle is called period.

The displacement of a particle in periodic motion can always be expressed in terms of sines and cosines. Thus they are also called harmonic motion.

Simple harmonic motion and harmonic oscillator.

A particle is said to execute simple harmonic motion when it vibrates periodically in such a way that at any instant the restoring force acting on it is proportional to its displacement from a fixed point in its path and is always directed towards that point. A system executing simple harmonic motion is called harmonic oscillator.

Consider a particle of mass 'm' executing SHM. If the displacement of the particle at any instant 't' be 'x', then its acceleration will be d^2x/dt^2.

We have restoring force $\propto$ - displacement

$$m\frac{d^2x}{dt^2} \propto -x$$

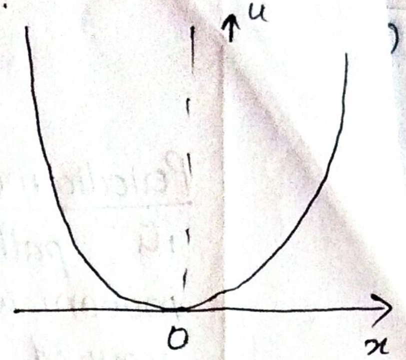
$$m\frac{d^2x}{dt^2} = -Cx \qquad C \rightarrow \text{const of proportionality}$$

$$\therefore m\frac{d^2x}{dt^2} + Cx = 0$$

put $\omega^2 = C/m$ $\Rightarrow$ $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ ——— (1)

This is the differential equation of motion for a simple harmonic oscillator.

Harmonic oscillator is a conservative system. At any instant its potential energy $U = \frac{1}{2}Cx^2$ and total energy $E = \frac{1}{2}mv^2 + \frac{1}{2}Cx^2$ is a constant. For harmonic oscillator the U-x curve will be a parabola.



Multiply (2) by $2dx/dt$

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + \omega^2 2x\frac{dx}{dt} = 0$$

On integrating,

$$\left(\frac{dx}{dt}\right)^2 + \omega^2x^2 = A$$

$A \rightarrow$ const of integration.

When displacement is maximum ie, $x = a$, $\frac{dx}{dt} = 0$

$$\therefore 0 + \omega^2a^2 = A \Rightarrow A = \omega^2a^2$$

$$\therefore \left(\frac{dx}{dt}\right)^2 + \omega^2x^2 = \omega^2a^2$$

$$\frac{dx}{dt} = \omega\sqrt{a^2-x^2}$$

$$\boxed{\frac{dx}{dt} = \omega\sqrt{a^2-x^2}} \quad \text{--- (2)}$$

This is the velocity of the particle at any time t.

$$\frac{dx}{\sqrt{a^2-x^2}} = \omega dt$$

On integrating,

$$\sin^{-1}(x/a) = \omega t + \phi$$

$$\boxed{x = a\sin(\omega t + \phi)} \quad \text{--- (3)}$$

$a \rightarrow$ Maximum value of displacement called Amplitude

$\phi \rightarrow$ initial phase / phase constant.

$(\omega t + \phi) \rightarrow$ phase of vibration

If the time is increased by $2\pi/\omega$, the function becomes,

$$x = a\sin\left[\omega\left(t+\frac{2\pi}{\omega}\right)+\phi\right]$$
$$= a\sin(\omega t+2\pi+\phi)$$
$$= a\sin(\omega t+\phi)$$

That is the displacement of the particle is the same after a time $2\pi/\omega$. Therefore $2\pi/\omega$ is the period(T) of the motion

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{c}}}$$

The no. of vibrations per second 'n' is called the frequency of the oscillator and is given by,

$$\boxed{n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{c}{m}}}$$

$$\boxed{\omega = 2\pi n = 2\pi/T}$$

ω is called the angular frequency.

Energy of harmonic oscillator

Harmonic oscillator possess two types of energy

(1) Potential energy 2) kinetic energy

The displacement of the harmonic oscillator at any instant t be given by,

$$x = a\sin(\omega t+\phi)$$

Its velocity $v = \frac{dx}{dt} = a\omega\cos(\omega t+\phi) = \omega\sqrt{a^2-x^2}$

Acceleration $\frac{d^2x}{dt^2} = -\omega^2 a\sin(\omega t+\phi) = -\omega^2 x$

Force $= -m\omega^2 x = -cx$

If the oscillator is displaced through dx distance, then work done on the oscillator is

$$dw = cx\,dx$$

If it is displaced from $x=0$ to $x=x$, then work done is

$$W = \int_0^x Cx\,dx = \frac{1}{2}Cx^2$$

This work done on the oscillator becomes its P.E,

$$U = \frac{1}{2}Cx^2. \quad —(1)$$

K.E of the oscillator at the displacement x is,

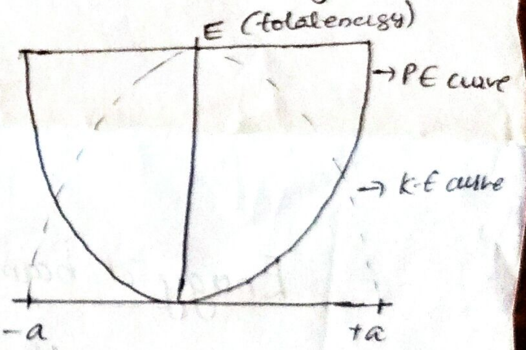
$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2-x^2) = \frac{1}{2}C(a^2-x^2) \quad —(2)$$

$\therefore$ Total energy $E = U+K$

$$E = \frac{1}{2}Cx^2 + \frac{1}{2}C(a^2-x^2) = \frac{1}{2}Ca^2$$

Thus total energy is a const. Hence if the system is once oscillated the motion will continue for indefinite period without any decrease in amplitude, provided no damping forces are acting on the system.

If we plot the energies, it will be as shown in figure.



The curve for total energy E is a horizontal line. At $x = \pm a$, the total energy of the oscillator is wholly potential but the K.E is zero ($\therefore E = U_{max} = \frac{1}{2}Ca^2 \Rightarrow a = \pm\sqrt{\frac{2E}{C}}$)

At equilibrium position, P.E is zero, but K.E is max

($E = K.E_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}m\omega^2a^2 = \frac{1}{2}Ca^2$) $\Rightarrow a = \pm\sqrt{2E}$

While at other intermediate points the energy is partly kinetic and partly potential but their sum is always $\frac{1}{2}Ca^2$.

Average values of K.E & P.E

Average K.E for a period T is given by

$$K_{av} = \frac{1}{T}\int_0^T K\,dt = \frac{1}{T}\int_0^T \frac{1}{2}mv^2\,dt$$

$$k_{av} = \frac{1}{T}\int_0^T \frac{1}{2} m a^2 \omega^2 \cos^2(\omega t+\phi)\,dt$$

$$= \frac{m a^2 \omega^2}{2T}\int_0^T \frac{1}{2}\left(1+\cos(2\omega t+2\phi)\right)dt$$

$$= \frac{m a^2 \omega^2}{2T\cdot 2}\left[t+\frac{\sin(2\omega t+2\phi)}{2\omega}\right]_0^T$$

$$= \frac{m a^2 \omega^2}{4T}\left[T+0\right] = \frac{m a^2 \omega^2}{4} = \frac{1}{4}Ca^2$$

Also $U_{av} = \frac{1}{T}\int_0^T U\,dt = \frac{1}{T}\int_0^T \frac{1}{2}Cx^2\,dt$

$$= \frac{1}{T}\int_0^T \frac{1}{2} C a^2 \sin^2(\omega t+\phi)\,dt = \frac{1}{4}Ca^2$$

Thus $k_{av} = U_{av}$

Examples of Harmonic oscillator

i) Simple pendulum

A simple pendulum consists of a heavy particle suspended by an inextensible weightless and flexible string from a point in a rigid support, about which the pendulum oscillates without friction.

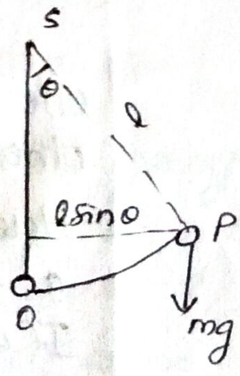


Figure represents a simple pendulum with S as the point of suspension O being the equilibrium position of the bob. If the bob is drawn to one side & then left free, it begins to oscillate about its mean position O. Let 'θ' be the angular displacement at any time 't'. If m be the mass of the bob, then due to its weight, the moment of force about the point S is $= -mgl\sin\theta$

If the moment of inertia of the bob about S is I then $I = ml^2$ and torque or moment of force $= I\,d^2\theta/dt^2$

where $d^2\theta/dt^2$ is the angular acceleration at the displacement θ.

$$\therefore I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$ml^2\frac{d^2\theta}{dt^2} + mgl\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

If θ is small, $\sin\theta \simeq \theta$

$$\therefore \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \quad \text{———} (1)$$

This equation represents a SHM, $\frac{d^2x}{dt^2} + \omega^2 x = 0$

$$\therefore \omega^2 = g/l$$

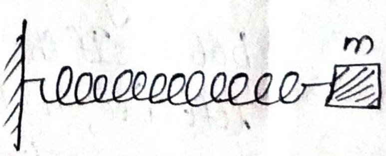
$$\omega = 2\pi/T$$

$$\therefore \boxed{T = 2\pi\sqrt{l/g}}$$

Here the displacement of the system is angular, hence such a ~~system~~ motion is called angular SHM.

2) Mass on a spring

The system consists of a massless spring one of its ends is connected to a mass 'm' and the other end is fixed to a rigid support.
Here the mass and spring are on a smooth horizontal surface. If a force is applied on the mass to stretch or compress the spring and then released, the motion of the mass is SHM. If 'x' is the displacement of mass 'm' at any time, then the restoring force exerted by the spring is



$$F = -Cx \quad , \; C \rightarrow \text{force constant}$$

Also $F = m(d^2x/dt^2)$

$$\therefore m\left(\frac{d^2x}{dt^2}\right) = -Cx$$

$$\frac{d^2x}{dt^2} + \frac{C}{m}x = 0$$

Hence the motion of mass is simple harmonic. Its periodic time is given by

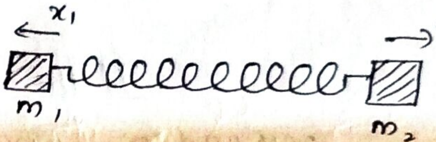
$$T = 2\pi\sqrt{m/c}$$

The solution of the eqn of motion is

$$x = a\sin(\omega t + \phi) \quad , \omega = \sqrt{c/m}$$

3) Oscillation of two particles connected by a spring.

Consider two masses m_1 and m_2 connected by a weightless spring. The masses are free to vibrate only along the axis of the spring. Such an oscillating system is called two body harmonic oscillator. When due to some reason one of the masses is slightly displaced from its equilibrium position, this produces an extension or compression in the spring. Due to spring action, it exerts a linear restoring force on the masses so that both of the masses vibrate about their mean position and the centre of mass of the system remains stationary.



If at any instant the length of the spring is increased by x, then the spring will exert an inward force Cx on each mass. Taking +ve outward, if x_1 and x_2 are the displacements of m_1 & m_2 from their mean positions then

$$x = x_1 + x_2 \quad \text{—— (1)}$$

The eqn of motion of m_1 & m_2 is,

$$m_1\frac{d^2x_1}{dt^2} = -Cx \quad \text{and} \quad m_2\frac{d^2x_2}{dt^2} = -Cx$$

from (1),

$$\frac{d^2x}{dt^2} = \frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) Cx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\left(\frac{m_1+m_2}{m_1 m_2}\right) Cx$$

$$\text{or} \quad \frac{d^2x}{dt^2} = -\frac{Cx}{\mu}$$

$$\frac{d^2x}{dt^2} + \frac{Cx}{\mu} = 0 \quad \text{——(2)}$$

$$\mu = \frac{m_1 m_2}{m_1+m_2} \quad \text{(reduced mass)}$$

(2) resembles the equation of SHM.

$$\therefore \omega^2 = C/\mu \quad , \quad \left(\frac{2\pi}{T}\right)^2 = \frac{C}{\mu}$$

$$\boxed{T = 2\pi\sqrt{\mu/C}}$$

Frequency of this oscillator $n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{C}{\mu}}$

The potential energy of the two body harmonic oscillator is $U = \frac{1}{2}Cx^2$.

If the distance b/w the two masses is r_0, when at rest and r, when the length of the spring changes by x, then $x = r - r_0$,

$$U = \frac{1}{2}C(r-r_0)^2$$

Compound pendulum

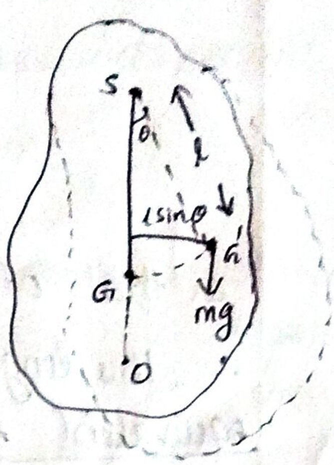
A compound pendulum is a rigid body capable of oscillating freely in a vertical plane about a horizontal axis passing through it.

The centre of suspension of the pendulum is the point in which the axis of rotation meets the vertical plane through the centre of gravity of the pendulum.

Let the figure represent the vertical section of a rigid body capable of oscillating about a horizontal axis through S, the centre of suspension.

Let G be the centre of gravity of the body lying vertically below S at a distance l, in its normal position of rest.

If the body is displaced through an angle θ, its CG takes the position G'. The weight of the body 'mg' and its reaction at the support constitute a couple



$$C = -mg \cdot G'A$$
$$= -mgl\sin\theta \quad \text{---(1)}$$

This is the restoring couple as it tends to bring the displaced body to its original position. If 'I' be the moment of inertia of the body about the horizontal axis through S, then the couple is

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$I\frac{d^2\theta}{dt^2} + mgl\sin\theta = 0$$

If θ is small, $\sin\theta \simeq \theta$

$$\frac{d^2\theta}{dt^2} + \frac{mgl}{I}\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad , \quad \omega = \sqrt{\frac{mgl}{I}}$$

$$\boxed{\therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}}$$

If 'I_g' be the moment of inertia of the body about a parallel axis through its centre of gravity, then from the theorem of parallel axes,

$$I = I_g + ml^2$$

If k be the radius of gyration of the body about a $\parallel^{el}$ axis passing through its centre of gravity, we have $I_g = mk^2$ Thus $I = mk^2 + ml^2$

$$\therefore T = 2\pi\sqrt{\frac{m(k^2+l^2)}{mgl}}$$

$$T = 2\pi\sqrt{\frac{k^2/l + l}{g}}$$

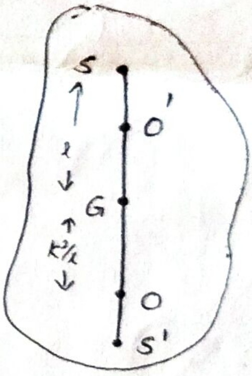
The time period of oscillation is the same as that of a simple pendulum of length $\left(\frac{k^2}{l}+l\right)$.

$\therefore$ this length $L = k^2/l + l$ is called length of an equivalent simple pendulum.

If we take a point 'O' on the line SG produced such that ($SO = k^2/l + l$), then this point O is called centre of oscillation.

Interchangeability of centres of suspension and oscillation

The distance of the centre of oscillation from the centre of gravity $= k^2/l = l'$



$\therefore$ The expression for the time period of the compound pendulum

$$T = 2\pi\sqrt{\frac{l'+l}{g}} \quad \text{---(1)}$$

If the pendulum is inverted and made to oscillate about the centre of oscillation, then its time period T' will be given by

$$T' = 2\pi\sqrt{\frac{k^2/l' + l'}{g}}$$

but $k^2/l = l' \Rightarrow k^2 = ll'$

$$\therefore T' = 2\pi\sqrt{\frac{ll'/l' + l'}{g}} = 2\pi\sqrt{\frac{l+l'}{g}} \quad \text{---(2)}$$

from (1) & (2) $T = T'$, Hence the time period of oscillation about the centre of suspension & centre of oscillation are equal. ie, they are interchangeable.

Four points collinear with CG, about which the time period is the same.

The time period of a compound pendulum is given by

$$T = 2\pi\sqrt{\frac{k^2+l^2}{lg}} \quad \text{—(1)}$$

$k \rightarrow$ radius of gyration of the compound pendulum about an axis passing through its centre of gravity and $\perp$ to the plane of oscillation and 'l' the distance between the centre of suspension and its centre of gravity.

Squaring (1) ; $T^2 = 4\pi^2\left(\frac{k^2+l^2}{lg}\right)$

$$l^2+k^2 = \frac{lgT^2}{4\pi^2}$$

$$l^2 - \frac{gT^2}{4\pi}l + k^2 = 0 \quad \text{—(2)}$$

This is a quadratic equation in l and gives two values of l. If l_1 and l_2 be two values, then

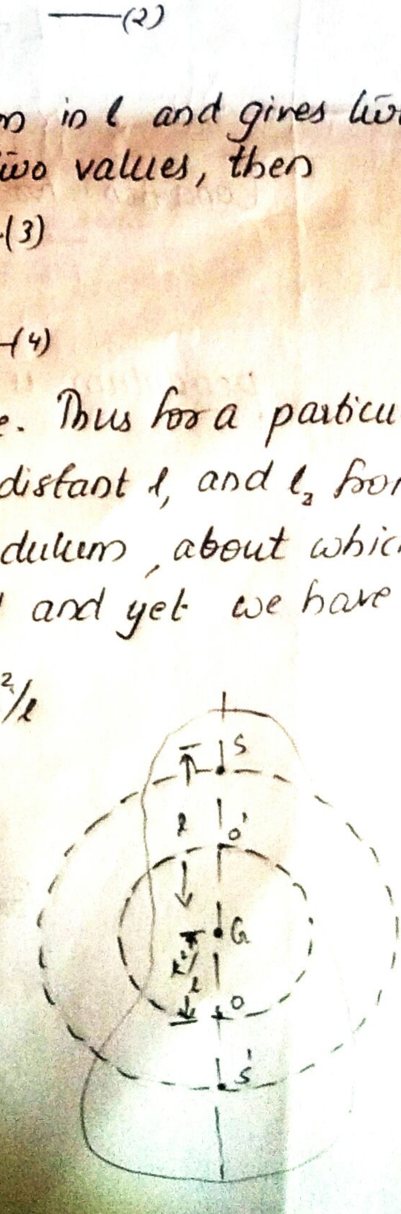
$$l_1 + l_2 = \frac{gT^2}{4\pi} \quad \text{—(3)}$$

$$l_1 l_2 = k^2 \quad \text{—(4)}$$

Both l_1 & l_2 are positive. Thus for a particular value of T there are two points distant l_1 and l_2 from the centre of gravity of the pendulum, about which the pendulum may be suspended and yet we have the same period.

If $l_1 = l \Rightarrow l_2 = k^2/l$

Now, if two circles are drawn with centre G and radii 'l' and k^2/l, the period of oscillation about any point of suspension on either circle is the same.



Hence there are infinite no. of points in a compound pendulum for which the time period has the same value. If a straight line through G is drawn, it will cut the circles in four points S, O', O, S'. Hence in all there are 4 points collinear with the CG of the pendulum, about which the time periods are equal.

Obviously if S is the centre of suspension, O is the centre of oscillation & vice versa. IIIly if S' is the centre of suspension, O' is the centre of oscillation or vice versa.

In the figure $SO = SG + GO = l + \frac{k^2}{l}$

IIIly $S'O' = l + k^2/l$.

$$\therefore T = 2\pi\sqrt{\frac{SO}{g}} = 2\pi\sqrt{\frac{S'O'}{g}} = 2\pi\sqrt{L/g}$$

$L \to$ length of equivalent pendulum.

Condition for maximum and minimum Time periods.

The period of oscillation of a compound pendulum is given by,

$$T = 2\pi\sqrt{\frac{k^2/l + l}{g}}$$

$$T^2 = \frac{4\pi^2}{g}\left(k^2/l + l\right)$$

Diff. w.r.t l,

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(\frac{-k^2}{l^2} + 1\right)$$

For minimum and maximum period,

$$\frac{dT}{dl} = 0$$

$$\frac{-k^2}{l}+l=0 \Rightarrow l=\pm k$$

If we calculate the value of $\frac{d^2T}{dl^2}$ and put $l=k$, the quantity $\frac{d^2T}{dl^2}$ comes out to be positive. Hence for this value of 'l', the time period of a compound pendulum has minimum value. Condition for the period to be minimum is,

$$l=k.$$

If we put $l=0$, the value of T becomes infinite

$l=k$	$\rightarrow T=$ minimum
$l=0$	$\rightarrow T=$ maximum

* Determination of the value of g (Refer text)

Damping force

The frictional force, acting on a body opposite to the direction of its motion, is called damping force. Such a force reduces the velocity and the kinetic energy of the moving body and so it is also called retarding or dissipative force.

If velocity is low, the damping force is proportional to v.

$$F=-\gamma v=-\gamma\frac{dx}{dt} \quad \text{—(1)}$$

$\gamma \rightarrow$ positive const; damping coefficient.

If it is the only external force acting on the moving particle then according to Newton's 2nd law, we have the eqn of motion as

$$\frac{m\,dv}{dt}=-\gamma v$$

$$\frac{dv}{dt}+\frac{\gamma}{m}v=0$$

Let $\tau = \frac{m}{\gamma}$ called relation time

Also $\frac{\gamma}{m} = 2k$, $k \to$ damping const

$$\therefore \frac{dV}{dt} + \frac{V}{\tau} = 0$$

$$\therefore \frac{dV}{V} = -\frac{dt}{\tau}$$

$$\int_{V_0}^{V} \frac{dV}{V} = -\frac{1}{\tau}\int_0^t dt$$

$V_0 \to$ velocity at $t=0$,

$$\ln V - \ln V_0 = -\frac{1}{\tau} t$$

$$\ln \frac{V}{V_0} = -\frac{t}{\tau}$$

$$\boxed{V = V_0 e^{-t/\tau}} \quad \text{——(2)}$$

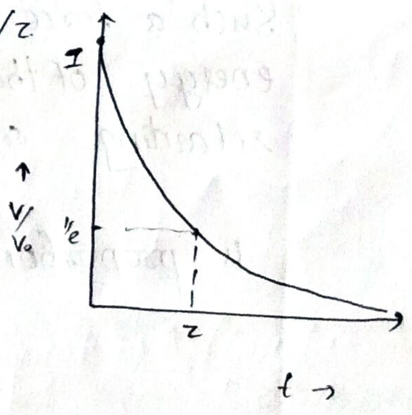
$\therefore$ Velocity decreases exponentially with time.

When $t=\tau$, $V = V_0/e$. Hence the ~~relatio~~ relaxation time or time const may be defined as the time in which the velocity becomes $1/e$ times the initial velocity.

The instantaneous K.E of the particle is

$$K = \frac{1}{2}mV^2 = \frac{1}{2}mV_0^2 e^{-2t/\tau}$$

$$K = K_0 e^{-2t/\tau} \quad \text{——(3)}$$



Put $V = \frac{dx}{dt}$ in (2)

$$\frac{dx}{dt} = V_0 e^{-t/\tau}$$

$$\therefore x = \int_0^t V_0 e^{-t/\tau} dt$$

$$x = V_0(-\tau) e^{-t/\tau} \Big|_0^t$$

$$x = V_0 \tau (1 - e^{-t/\tau}) \quad \text{——(4)}$$

Damped Harmonic oscillator

The motion of the oscillator is said to be damped if the amplitude of oscillations gradually decreases to zero as a result of frictional forces and the vibrating system is called damped harmonic oscillator.

If damping is considered, then a harmonic oscillator experiences (i) a restoring force $(-Cx)$ and (ii) a damping force proportional to the velocity, but opposite to it

∴ Equation of motion of the damped harmonic oscillator is

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - Cx$$

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + Cx = 0$$

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{C}{m}x = 0$$

$$\boxed{\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = 0} \quad \text{——(1)}$$

where $2k = \frac{\gamma}{m} = \frac{1}{\tau}$; k = damping const

$\omega_0 = \sqrt{\frac{C}{m}}$ natural angular frequency in absence of damping forces.

The above eqn is the differential eqn of damped harmonic oscillator.

To solve (1), consider the solution,

$$x = Ae^{\alpha t}$$

$$\frac{dx}{dt} = A\alpha e^{\alpha t} \; ; \; \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

$$\therefore (1) \Rightarrow \quad A\alpha^2 e^{\alpha t} + 2k \cdot A\alpha e^{\alpha t} + \omega_0^2 Ae^{\alpha t} = 0$$

$$\alpha^2 + 2k\alpha + \omega_0^2 = 0$$

$$\therefore \alpha = -k \pm \sqrt{k^2 - \omega_0^2}$$

$$\therefore x = A_1 e^{\left(-k + \sqrt{k^2 - \omega_0^2}\right)t} \quad \text{—(2)}$$

and $x = A_2 e^{\left(-k - \sqrt{k^2 - \omega_0^2}\right)t}$

$\therefore$ The most general soln is

$$x = A_1 e^{\left(-k + \sqrt{k^2 - \omega_0^2}\right)t} + A_2 e^{\left(-k - \sqrt{k^2 - \omega_0^2}\right)t}$$

$$x = e^{-kt}\left[A_1 e^{\left(\sqrt{k^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\sqrt{k^2 - \omega_0^2}\right)t}\right] \quad \text{—(3)}$$

A_1, A_2 are consts.

The quantity $\sqrt{k^2 - \omega_0^2}$ is imaginary, real or zero

(i) Under damped case :-

If $k < \omega_0$, $\sqrt{k^2 - \omega_0^2}$ becomes imaginary

$$\therefore \sqrt{k^2 - \omega_0^2} = \sqrt{-1(\omega_0^2 - k^2)} = i\sqrt{\omega_0^2 - k^2} = i\omega$$

where $\omega = \sqrt{\omega_0^2 - k^2}$

$\therefore$ (3) becomes

$$x = e^{-kt}\left[A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right]$$

$$= e^{-kt}\left[A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t + i\sin\omega t)\right]$$

$$x = e^{-kt}\left[(A_1 + A_2)\cos\omega t + i(A_1 - A_2)\sin\omega t\right] \quad \text{—(4)}$$

As x is a real quantity, $(A_1 + A_2)$ and $i(A_1 - A_2)$ must be real quantities. If $A_1 + A_2 = a_0 \sin\phi$

$i(A_1 - A_2) = a_0 \cos\phi$ then

$$\boxed{x = a_0 e^{-kt} \sin(\omega t + \phi)}$$

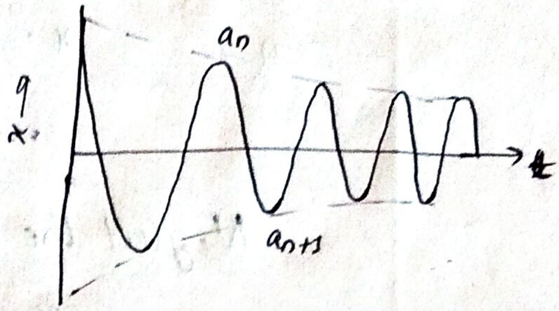
This eqn represents a damped harmonic motion.

$$\therefore T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - k^2}}$$

The amplitude of oscillatory motion is

$$a = a_0 e^{-kt} = a_0 e^{-t/2\tau}$$

where a_0 is the amplitude in the absence of damping. In the presence of damping, the amplitude decreases exponentially with time.



The time interval $6/\omega$ the successive maximum displacements of left & right is $T/2$, hence if a_n & a_{n+1} are the successive amplitudes then,

$$a_n = a_0 e^{-kt} \quad ; \quad a_{n+1} = a_0 e^{-k(t+T/2)}$$

$$\frac{a_n}{a_{n+1}} = \frac{e^{-kt}}{e^{-k(t+T/2)}} = e^{kT/2} = d = \text{const of motion}$$

$d \rightarrow$ decrement

$$\therefore \ln d = \frac{kT}{2} = \lambda \rightarrow \text{logarithmic decrement}$$

(ii) Overdamped case

If $k > \omega_0$, the $\sqrt{k^2 - \omega_0^2} = \beta$ is a real quantity

$$\therefore (3) \Rightarrow x = e^{-kt}\left[A_1 e^{\beta t} + A_2 e^{-\beta t}\right]$$

$$x = A_1 e^{-(k-\beta)t} + A_2 e^{-(k+\beta)t}$$

As $k > \beta$, both quantities of RHS decrease exponentially with time and the motion is non-oscillatory.

(iii) Critically damped case :

If $k = \omega_0$ then (3) $\Rightarrow$

$$x = (A_1 + A_2) e^{-kt} = C e^{-kt} \qquad C = A_1 + A_2$$

In this eqn, there is only one const; hence it doesnot provide us the solution of differential eqn.

Now suppose $\sqrt{k^2 - \omega_0^2} = h$ which is a small quantity

$$(3) \Rightarrow \quad \therefore \quad x = e^{-kt}\left(A_1 e^{ht} + A_2 e^{-ht}\right)$$

$$= e^{-kt}\left[A_1(1 + ht + \cdots) + A_2(1 - ht + \cdots)\right]$$

Neglect the other terms,

$$x = e^{-kt}\left[(A_1 + A_2) + h(A_1 - A_2)t\right]$$

$$x = e^{-kt}(P + Qt) \qquad \text{——— (4)}$$

where $P = A_1 + A_2$, $Q = h(A_1 - A_2)$

Also $\frac{dx}{dt} = e^{-kt} Q + (P + Qt) e^{-kt}(-k)$

$$v = e^{-kt} Q - k(P + Qt) e^{-kt} \qquad \text{——— (5)}$$

If initially at $t = 0$, the displacement of the particle is $x = x_0$ and there the velocity is v_0, then

$$x_0 = P \;, \quad v_0 = Q - kP = Q - kx_0 \Rightarrow Q = v_0 + kx_0$$

$$\therefore \quad x = \left[x_0 + (v_0 + kx_0)t\right] e^{-kt}$$