Special theory of relativity.

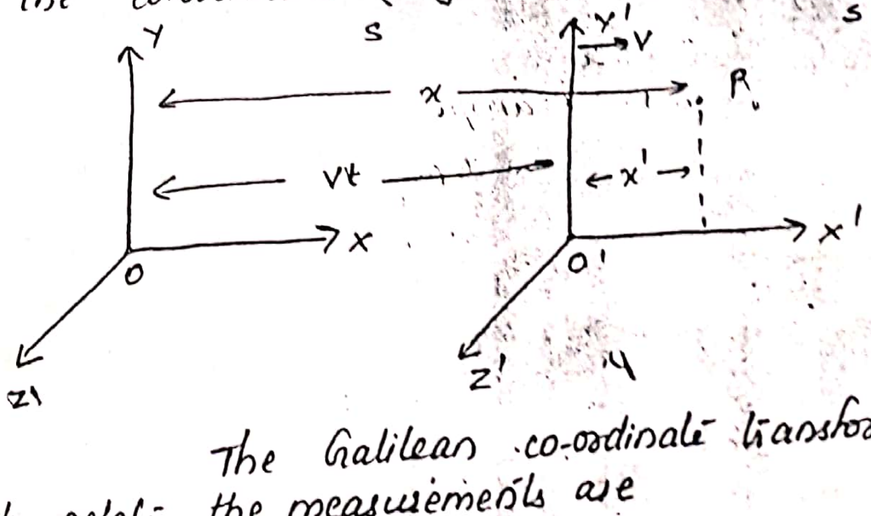
A parti

Inertial frame of reference.

→ is one in which Newton's 1st law of motion holds. In such a frame, an object at rest remains at rest and an object in motion continues to move at constant velocity if no force acts on it. Any frame of reference that moves at const velocity relative to an inertial frame is itself an inertial frame. All inertial frames are equally valid.

Galilean transformations

Let s & s' be two inertial frames. Let s be at rest and s' move with uniform velocity v along the +ve x direction. Let the origins of the 2 frames coincide at t=0. Suppose some event occurs at the point P. The observer O in the frame s specifies the co-ordinates (x,y,z,t) and the co-ordinates (x',y',z',t') by the observer in s'.



The Galilean co-ordinate transformations which relate the measurements are

$x' = x - vt$ $\quad$ $y' = y$

$z' = z$ $\quad$ $t' = t$

Inverse Galilean transformations can be written by changing primed into unprimed quantities & replacing v by −v

$x = x' + vt'$ $\quad$ $y = y'$

$z = z'$ $\quad$ $t = t'$

The transformation of velocities from one system to the other is obtained by taking time derivatives

(2)

$$\frac{dx'}{dt} = \frac{dx}{dt} - V \Rightarrow u' = u - V$$

To obtain acceleration, diff. again w.r.t time

$$\frac{du_x'}{dt} = \frac{du_x}{dt} \Rightarrow a_x' = a_x, \quad a_y' = a_y, \quad a_z' = a_z$$

This implies acceleration remains invariant when passing from one inertial frame to another that is in uniform relative translational motion.

Since $F = ma \Rightarrow F = F'$

Q) Consider a ship moving with a uniform velocity of 18 m/s relative to the earth. Let a ball be rolled at a speed of 2 m/s relative to the ship, in the direction of motion of the ship. Find the speed of the ball relative to the earth.

$V = 18\ m/s$

$u' = 2\ m/s$

$u = u' + V = 20\ m/s.$

All inertial ~~accelerating~~ frame of references are non accelerating

Non inertial frame and fictitious forces

Accelerating frame of reference are called non-inertial frames. Examples are.

1) Reference frame with translational acceleration.

Consider 2 non-inertial frames S & S' such that the frame S' is moving with acceleration a_0 w.r.t S. Let a particle have an acceleration a w.r.t S. Then to the observer in S', it will appear to have acceleration a' given by,

$$a' = a - a_0$$

If m is the mass of the particle, then force on the particle in S' is

$$F' = ma' = m(a - a_0)$$

$$F' = F - F_0$$

$F \rightarrow$ force seen by an observer in S

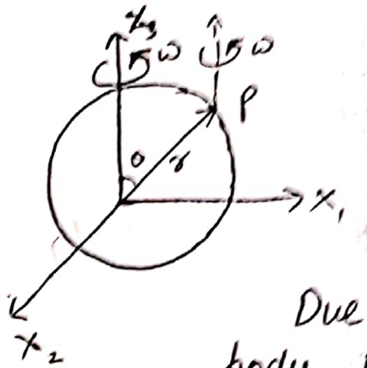
$F_0 \rightarrow$ Force due to relative acceleration a_0 b/w the 2 frames

When $F = 0$, $F' = -F_0$

Thus particle seems to experience a force $-F_0$ when viewed from s' even when there is no force of it to s. Thus F_0 is called fictitious or pseudoforce. These arise from the acceleration of the reference frame and go on increasing with enhanced acceleration.

2) Uniformly rotating frame : Coriolis & centrifugal force.

Let $x_1 x_2 x_3$ be an inertial frame s fixed in space and $x_1' x_2' x_3'$ be a reference frame s' that is fixed to a rigid body & is uniformly rotating in space w.r.t s with angular velocity ω. The unit vectors i_1, i_2, i_3 refer to the reference frame s and i_1', i_2', i_3' to the frame s'.



The position vector r of the point P is given by

$$r = x_1 i_1 + x_2 i_2 + x_3 i_3$$

$$r = x_1' i_1' + x_2' i_2' + x_3' i_3'$$

Due to rotational motion of the rigid body, the unit base vectors i_1', i_2', i_3' are continuously changing and in taking time derivatives, the unit vectors are treated as variables

$$\therefore \frac{dr}{dt} = \frac{d}{dt}\left(x_1' i_1' + x_2' i_2' + x_3' i_3'\right)$$

$$= \frac{dx_1'}{dt} i_1' + \frac{dx_2'}{dt} i_2' + \frac{dx_3'}{dt} i_3' +$$

$$x_1' \frac{di_1'}{dt} + x_2' \frac{di_2'}{dt} + x_3' \frac{di_3'}{dt}$$

$$\dot{r} = \dot{x}_1' i_1' + \dot{x}_2' i_2' + \dot{x}_3' i_3' + x_1' \frac{di_1'}{dt} + x_2' \frac{di_2'}{dt} + x_3' \frac{di_3'}{dt}$$

The linear velocity V of a particle is expressed as

$$\frac{d\bar{r}}{dt} = \bar{\omega} \times \bar{r}$$

$$\therefore \frac{di_1'}{dt} = \omega \times i_1' \qquad \frac{di_2'}{dt} = \omega \times i_2'$$

$$\frac{di_3'}{dt} = \omega \times i_3'$$

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This can be written as,

$$\left(\frac{dr}{dt}\right)_{space} = \left(\frac{dr}{dt}\right)_{body} + \omega \times r \quad \text{——(1)}$$

$\left(\frac{dr}{dt}\right)_{space}$ → velocity of rigid body w.r.t S

$\left(\frac{dr}{dt}\right)_{body}$ → velocity w.r.t S'

It can express as operator form,

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + (\omega \times\ ) \quad \text{——(2)}$$

(1) ⇒ $V_{space} = V_{body} + \omega \times r$ ——(3)

Applying (3) in (2)

$$\left(\frac{dV_{space}}{dt}\right)_{space} = \left(\frac{dV_{space}}{dt}\right)_{body} + \omega \times V_{space}$$

$$a_{space} = \frac{d}{dt}\left(V_{body} + (\omega \times r)\right)_{body} + \omega \times \left(V_{body} + \omega \times r\right)$$

$$a_{space} = a_{body} + \frac{d}{dt}(\omega \times r)_{body} + \omega \times V_{body} + \omega \times (\omega \times r)$$

$$a_{space} = a_{body} + \omega \times V_{body} + \frac{d\omega}{dt} \times r + \omega \times V_{body} + \omega \times (\omega \times r)$$

$$a_{space} = a_{body} + 2\left(\omega \times V_{body}\right) + \omega \times (\omega \times r) + \frac{d\omega}{dt} \times r$$

Eqn of motion in fixed space axis,

$$F_{space} = m a_{space}$$

$$F_{body} = m a_{body}$$

$$F_{body} = m\left(a_{space} - 2(\omega \times V_{body}) - \omega \times (\omega \times r) - \frac{d\omega}{dt} \times r\right)$$

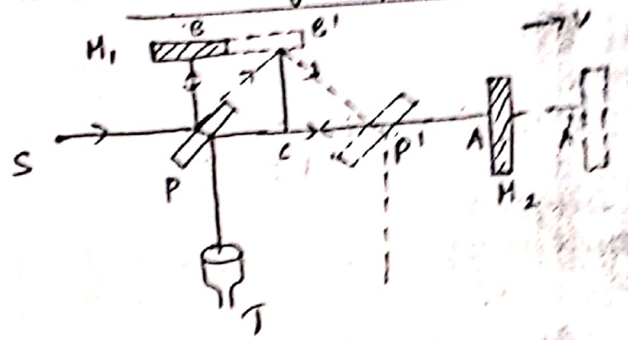
$$F_{body} = F_{space} - 2m(\omega \times V_{body}) - m\omega \times (\omega \times r)$$
$$- m \frac{d\omega}{dt} \times r$$

This is the effective force in which the body appears to be moving to an observer in the rotating frame. $m\omega \times (\omega \times r)$ is the ordinary centrifugal force and is $\perp^r$ to ω. $-2m(\omega \times V_{body})$ is the coriolis force and is $\perp^r$ to both ω and V_{body}. Last term $(d\omega/dt) \times r$ is non zero only when $(d\omega/dt) \neq 0$ and will vanish when ω is const.

Thus the fictitious force is given by

$$F_0 = -2m(\omega \times V_{body}) - m\omega \times (\omega \times r)$$

Michelson-Morley experiment



A material medium is a necessity for the propagation of waves. It was considered that light propagates through ether as the sound waves propagate through air. Ether pervades all space. Then we can consider the relative velocity of earth w.r.t ether. If such a motion can be detected, we can choose a fixed frame of reference in a stationary ether. Michelson & Morley conducted an expt to find the existence of ether.

A beam of light from a monochromatic light source S falls on a half-silvered glass plate P, placed at an angle of 45° to the beam. The incident beam is split up into two parts by P. The reflected portion travels in a direction at right angles to the incident beam, falls normally at B on the plane mirror M_1 and is reflected back to P. It gets refracted through P enters the telescope T. The transmitted portion travels along the direction of the initial beam, falls normally on mirror M_2 at A and is reflected back at P.

... surface of P, it enters telescope T. The two reflected beams interfere and the interference fringes are viewed with the help of T. One arm (PA) points in the direction of earth's motion round the sun and the other (PB) points ⊥° to this motion.

Assume that the velocity of the apparatus or Earth relative to fixed ether is v in the direction PA. Let PA = PB = d.

∴ Time taken by light to travel from P to A $= \frac{d}{(c-v)}$

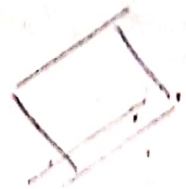
" " " A to P $= \frac{d}{c+v}$

∴ Total time, $t = \frac{d}{c-v} + \frac{d}{c+v} = \frac{d(c+v)+d(c-v)}{c^2-v^2}$

$$t = \frac{2cd}{c^2-v^2} = \frac{2d}{c}\left(1-\frac{v^2}{c^2}\right)^{-1} = \frac{2d}{c}\left(1+\frac{v^2}{c^2}\right) \quad \text{---(1)}$$

Now consider the ray moving upwards from P to B. It will strike the mirror M_1 not at B but at B' due to the motion of the earth. If t_1 is the time taken by the ray starting from P to reach M_1, then $PB' = ct_1$ & $BB' = vt_1$

$$\therefore PB'^2 = BB'^2 + B'C^2$$



$$(ct_1)^2 = (vt_1)^2 + d^2$$

$$t_1^2(c^2-v^2) = d^2$$

$$t_1 = \frac{d}{\sqrt{c^2-v^2}}$$

∴ Total time taken by the ray to travel the whole path of PB'P

$$t' = 2t_1 = \frac{2d}{\sqrt{c^2-v^2}} = \frac{2d}{c\sqrt{1-v^2/c^2}} = \frac{2d}{c}\left(1+\frac{v^2}{2c^2}\right) \quad \text{---(2)}$$

$$\Delta t = t - t'$$

$$\Delta t = \frac{2d}{c}\left[1+\frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2}\right] = \frac{2d}{c}\frac{v^2}{2c^2} = \frac{dv^2}{c^3} \quad \text{---(3)}$$

The distance travelled by light in time $\Delta t = c \times \Delta t = \frac{dv^2}{c^2}$ ---(4)

This path difference may occur even when PB ≠ PA. To elliminate such as error the apparatus is turned through 90° and the expt is repeated. Michelson and Morley expected a fringe shift of 0.4 in their apparatus, but they found nothing. The negative result shows that the ether hypothesis was wrong and thus no absolute space can be considered.
0.4

Postulates of the special theory of relativity.

1) The principle of relativity.
The laws of Physics are the same in all inertial systems so that there is no preferred inertial frame and all the inertial frames are equivalent. Thus there is no such thing as absolute rest; there is no physical reasoning to prefer one inertial frame over the other.

2) The postulate of constancy of velocity of light.
The light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the source, intervening medium or observer.

Consequences of Einstein's postulates

The -ve result of the Michelson-Morley expt forced Einstein to conclude that the E.M laws hold in all inertial systems, with the value of the velocity of light, which is the same in all directions and is independent of the relative motion of the observer, medium and source. This invariance of the velocity of light c is embodied in the relationship as

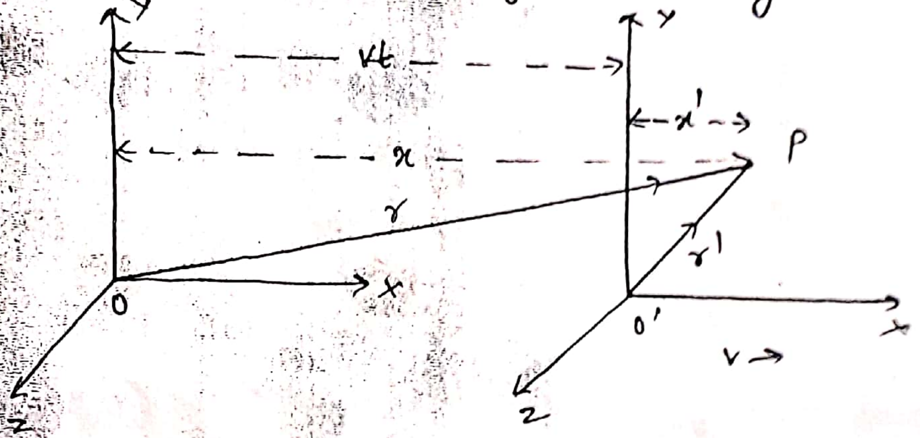
$$x^2+y^2+z^2-c^2t^2 = x'^2+y'^2+z'^2-c^2t'^2$$

where (x,y,z,t) refer to the termini of the light path in the unprimed system 's' and (x',y',z',t') to the termini in the primed system s' which is moving with velocity v relative to s. When x' is different from x, on account of relative motion in that direction, it will inevitably imply that t is different from t'. The new transformations, time will be no longer considered absolute for all observers in relative motion.

- Simultaneity of event
- Length contraction
- Time dilation.

Lorentz transformations

Consider two observers O and O' in two systems S and S'. System S' is moving with a constant velocity v relative to system S along the +ve x-axis.



Suppose we make measurements of time from the instant when the origins of S & S' just coincide ie, $t=0$ when O & O' coincide. Suppose a light pulse is emitted when O & O' coincide. The light pulse produced at $t=0$ will spread out as a growing sphere. The radius of the wavefront produced in this way will grow with speed c. After a time t, the observer O will note that the light has reached a point $P(x,y,z)$. For him, the distance of the point P is given by $r=ct$. From figure $r^2=x^2+y^2+z^2$

$$\therefore x^2+y^2+z^2=c^2t^2 \quad \text{——— (1)}$$

III'y the observer O' will note that the light has reached the same point P in a time t' with same velocity.

$\therefore r' = ct'$

$\therefore x'^2 + y'^2 + z'^2 = c^2 t'^2$ ——— (2)

(1) & (2) must be equal since both the observers are at the centre of the same expanding wavefront

$x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$ ——— (3)

Since there is no motion in Y & Z directions

$y' = y$ and $z' = z$

$\therefore (3) \Rightarrow x^2 - c^2 t^2 = x'^2 - c^2 t'^2$ ——— (4)

The transformation eqn relating to x & x' can be written as

$x' = k(x - vt)$ ——— (5)

The reason to take x' in this form is that the transformation must reduce to Galilean transformation when $v << c$.

$|||^{ly}$ $t' = a(t - bx)$ ——— (6) $k, a, b \rightarrow$ consts

Sub. these values in (4)

$x^2 - c^2 t^2 = k^2 (x - vt)^2 - c^2 a^2 (t - bx)^2$

$x^2 - c^2 t^2 = k^2 (x^2 - 2xvt + v^2 t^2) - a^2 c^2 (t^2 - 2tbx + b^2 x^2)$

$x^2 - c^2 t^2 = x^2 (k^2 - a^2 b^2 c^2) - 2(k^2 v - a^2 b c^2) xt - \left(a^2 - \frac{k^2 v^2}{c^2}\right) c^2 t^2$

Equating coefficients of corresponding terms.

$1 = k^2 - a^2 b^2 c^2$

$1 = a^2 - \frac{k^2 v^2}{c^2}$

$0 = k^2 v - a^2 b c^2$

Solving above eqns. for k, a & b,

$k = a = \frac{1}{\sqrt{1 - v^2/c^2}}$ & $b = v/c^2$

$$\therefore \boxed{x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad y' = y \qquad z' = z}$$

→ Lorentz transformation eqns.

The inverse eqns are

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \quad y = y' \quad z = z' \quad t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

Q) S.T for values of $v \ll c$, Lorentz transformation reduces to the Galilean transformation.

$v \ll c$ When $v \ll c$ $\frac{v}{c} \approx 0$

$$\therefore \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1$$

$x' = x - vt$ $\therefore x' = x - vt$ $y' = y$ $z' = z$ $t = t'$

kinematic consequences of Lorentz transformations

1) Lorentz - Fitzgerald contraction / length contraction

Suppose there is a rod at rest in the system s and let the co-ordinates of its ends be x_1 and x_2 so that its length as measured by an observer in s is given by $l = x_2 - x_1$. The same rod in measured by an observer in s' at time t_1' to whom it appears to have length l'.

$$\therefore x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}} \qquad x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}}$$

$l' = l\sqrt{1 - v^2/c^2}$

$$\therefore x_2 - x_1 = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} \qquad \therefore l = \frac{l'}{\sqrt{1 - v^2/c^2}}$$

$$\boxed{\therefore l' = l\sqrt{1 - v^2/c^2}} \quad \text{or} \quad l = l_0\sqrt{1 - v^2/c^2}$$

The length of an object in its rest frame is called its proper length so that the proper length is always the greatest and to any other observer who is moving with velocity v, the rod appears to be constracted in the ratio $\sqrt{1 - v^2/c^2}$.