14/2/19
Thursday

2

Vector Integration.

Line integrals.

Let C be a smooth cruve, in the xy plane and let $f(x,y)$ be continous and non-negative on C. Then the line integral of f w.r.t s along C is defined by

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta S_k.$$

Q Evaluate line integral.
If the curve C is represented parametic by th equa:

$$z = f(x,y)$$

$$C = \begin{array}{l} x = x(t) \\ y = y(t) \end{array}$$

$a \le t \le b$ then

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Evaluate.

$\int_C (1+xy^2)\,ds$ from $(0,0)$ to $(1,2)$ along the line segement C repersent parametic $x = t, y = 2t,$
$0 \le t \le 1.$

$$\int (1+xy^2)\,ds = \int_0^1 [1+t(2t)^2]\sqrt{1^2+2^2}\,dt.$$

$$= \int_0^1 (1+4t^3)\sqrt{5}\,dt$$

$$= \sqrt{5}\int_0^1 1+4t^3\,dt.$$

$$= \sqrt{5}\left[t+\frac{\cancel{4}t^4}{\cancel{4}}\right]_0^1 = \sqrt{5}\,[1+1] = 2\sqrt{5}$$

* $\int_C (x-y)\,ds$ where C is the curve $x=2t$, $y=3t^2$ $0\le t\le 1$; $\left(\frac{dx}{dt}\right) = \frac{\cancel{2t}}{6t}$

$$\int_S (x-y)\,ds = \int_0^1 (2t-3t^2)\sqrt{2^2+(6t)^2}\,dt.$$

$\sqrt{4}\cdot\sqrt{36}$

$2+6t$

$$= \int_0^1 (2t-3t^2)\sqrt{4+36t^2}\,dt.$$

$$= \int_0^1 (2t-3t^2)(\sqrt{4}+\sqrt{36t^2})\,dt$$

$$= \int_0^1 2t-3t^2\cdot 2\sqrt{1+9t^2}\,dt$$

$$= 2\int_0^1 (2t-3t^2)\sqrt{1+9t^2}\,dt$$

$$= -\frac{11}{108}\sqrt{10} - \frac{1}{36}\ln(\sqrt{10}-3) - \frac{4}{27}.$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

* $\int_C (xy+z^3)\,ds$ from $(1,0,0)$ to $(-1,0,\pi)$ along C given by $x = \cos t$, $y = \sin t$ $z = t$, $0 \le t \le \pi$.

$$\int_0^\pi (\cos t \sin t + t^3)\sqrt{(\sin t)^2 + (\cos t)^2 + 1}\,dt$$

$$= \int_0^\pi \cos t \sin t + t^3 \;\sqrt{1}\,dt \cdot = \int^\pi \cos t \sin t + t^3\,dt$$

$$= \left[\cos t \cos t + \sin t - \sin t + \frac{t^4}{4}\right]_0^\pi = \left[\cos^2 t - \sin^2 t + \frac{t^4}{4}\right]_0^\pi$$

6/6/19 Thursday

Line integral w.r.t x, y & z.

If C is the cruve represted parametrically by $x = x(t)$, $y = y(t)$, $z = z(t)$ $a \le t \le b$.

Then

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t), y(t), z(t))\,y'(t)\,dt$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$$

Q. Evaluate $\int_C 2xy\,dx + (x^2+y^2)\,dy$ along the circular arc C given by ~~exist~~ $x = \cos t, y = \sin t$, $(0 \le t \le \pi/2)$.

$$\int_C 2xy\,dx + (x^2+y^2)\,dy = \int_C 2xy\,dx + \int_C (x^2+y^2)\,dy.$$

$$= I_1 + I_2.$$

$x(t) = -\sin t.$
$y(t) = \cos t.$

$$I_1 = \int_0^{\pi/2} 2\cos t \sin t \cancel{dt} \cdot \bar{s}\text{in}\, t\, dt$$

$$= -\int_0^{\pi/2} 2\cos t \sin^2 t\, dt$$

Put $\sin t = u.$ $\cos t\, dt = du$

$$= -2\int_0^{\pi/2} \cancel{2}\, u^2 du = -2\left[\cancel{2}\, \frac{u^3}{3}\right]_0^{\pi/2}$$

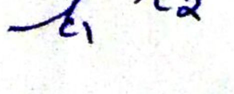
$$= -2\left[\frac{\sin^3 t}{3}\right]_0^{\pi/2} = \frac{-2}{3} //.$$

$$I_2 = \int_0^{\pi/2} (\sin^2 t + \cos^2 t) \cdot \cos t\, dt$$

$$= \int_0^{\pi/2} \cos t\, dt = [\sin t]_0^{\pi/2} = 1 //.$$

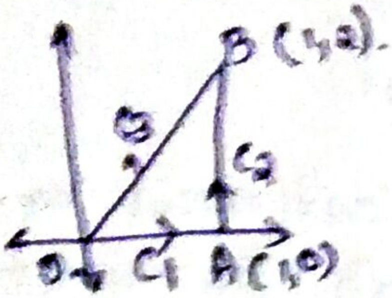
$$I_1 + I_2 = \frac{-2}{3} + 1 = \frac{1}{3} //.$$

Line integral along ~~pieces~~ piece-wise smooth curves



$$\int_C f = \int_{C_1} f + \int_{C_2} f$$

Q. Evaluate $\int_C x^2y\,dx + x\,dy$ in a counter clock wise direction as shown in the fig.



Note: The vector eqn of the line segment joining the points $\vec{r_0}$ & $\vec{r_1}$ is $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$; $0 \le t \le 1$

Note:

Let $I = \int_C x^2y\,dx + x\,dy$

$= \int_{C_1} x^2y\,dx + x\,dy + \int_{C_2} x^2y\,dx + x\,dy + \int_{C_3} x^2y\,dx + x\,dy$

$= I_1 + I_2 + I_3$

$I_1 =$ Line Segment join (0,0) & (1,0)

$r(t) = (1-t)<0,0> + t<1,0> = <t,0>$ $0 \le t \le 1$

$x = t,\ y = 0,\ 0 \le t \le 1.$

$I_1 = \int_{C_1} x^2y\,dx + \int_{C_1} x\,dy = \int_0^1 t^2 \times 0 \times dt + \int_0^1 t\,0\,dt$

$= 0$ //.

$(1-t)<1,0> + t<1,2>$

$I_2 = \int_{C_2} x^2y\,dx + x\,dy = \int_0^1 2t \times 0 + 1 \times 2dt$ $(1-t)<1,0> + <t,2t> = <1,2t>$

$x = 1, y = 2t$

$= \int_0^1 2\,dt = 2$ //.

$(1-t)<1,2> + t<0,0>$

$<1-t,\ 2-2t>$

$I_3 =$

$(\frac{7}{2})$

$I_1 + I_2 + I_3 = 1/2$ //.

7/6/19 Friday.

Work as a Line Integral.

If F is a continous vector field $ C is a smooth parametric curve in 2 space or 3 space with unit tangent vector $\vec{T}$, then the work performed by the vector field on a particle that moves along C in the direction of increasing parameter is

$$W = \int_C \vec{F} \cdot \vec{T}\, ds. \qquad \text{Since, } \vec{T} = \frac{d\vec{r}}{ds}.$$

$$= \int_C F \cdot \frac{d\vec{r}}{\cancel{ds}}\, \cancel{ds} = \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} \cancel{+ dz\,\hat{k}}$$

$$\vec{F} \cdot d\vec{r} = f(x,y)\,dx + g(x,y)\,dy$$

$$W = \int_C f(x,y)\,dx + g(x,y)\,dy.$$

Q. Find the workdone by the force due to $\vec{F}(x,y) = x^3 y\,\hat{i} + (x-y)\,\hat{j}$ on a particle that moves along the parabola $y = x^2$ from $(-2,4)$ to $(1,1)$

The parabola $y=x^2$ is parametrized as by taking $x=t$,
$y=t^2$, $-2\le t\le 1$

$y=f(x)$
Put $x=t$,
$y=f(t)$.

$$\vec{F} = x^3y\,\hat{i} + (x-y)\,\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\vec{F}\cdot d\vec{r} = x^3y\,dx + (x-y)\,dy$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C x^3y\,dx + (x-y)\,dy$$

$$= \int_{-2}^{1} t^3\cdot t^2\,dt + \int_{-2}^{1} (t-t^2)\,\cancel{dy}\,2t\,dt$$

$$= \int_{-2}^{1} t^5\,dt + 2\int_{-2}^{1} t^2 - t^3\,dt$$

$$= \left[\frac{t^6}{6}\right]_{-2}^{1} + 2\left[\frac{t^3}{3} - \frac{t^4}{4}\right]_{-2}^{1}$$

$$= \frac{1}{6} - \frac{64}{6} + 2\left[\left(\frac{1}{3}-\frac{1}{4}\right) - \left(-\frac{8}{3} - \frac{16}{4}\right)\right]$$

$$= -\frac{63}{6} + 2\left[\frac{1}{12} - \left(\frac{-80}{12}\right)\right] \qquad -32-48$$

$$= -\frac{63}{6} + \frac{1}{6} + \frac{80}{6} = \cancel{\frac{-62}{6} + 24} = \frac{18}{6}$$

$$\cancel{= \frac{-1302+288}{144} = \frac{-1014}{144} = \frac{507}{72} = \frac{169}{24}}$$

$= 3$ //.

2011 In each part evalute the integral $\int_C (3x+2y)dx + (2x-y)\,dy$ along the started curves

a) The line segment from (0,0) to (1,1)

b). The parabolic arc $y=x^2$ from (0,0) to (1,1)

c) The curve $x=y^3$ from (0,0) to (1,1)

Let C be the curve represented by the eq^ns. $x=2t$, $y=3t^2$, $0 \le t \le 1$.

In each part evaluate the line integral along C.

a). $\int_C (x-y)\,ds$

b) $\int_C (x-y)\,dx$

c) $\int_C (x-y)\,dy$

2. Find the workdone by the force-field $\vec{F}$ on a particle that moves along the curve C where $\vec{F}(x,y) = xy\hat{i} + x^2\hat{j}$ & $C: x=y^2$ from (0,0) to (1,1).

Green's theorem

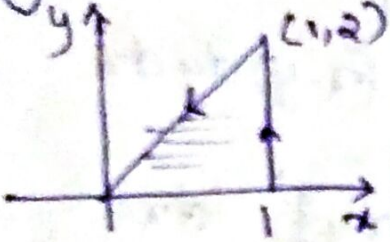
Let R be a simply connected plane region whoes boundary is a single simple, closed, piece wise, smooth curved

C oriented counter clockwise. If $f(x,y)$ & $g(x,y)$ are continous & have continous first partial derivaties on some openset containing R, then

Workdone

$$\boxed{\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.}$$

Q use green's theroem evaluate $\int x^2y\,dx + x\,dy$ along the triangular path shown in the fig.



$f(x,y) = x^2y$

$g(x,y) = x$.

$$\begin{pmatrix} 0 \le y \le 2x \\ 0 \le x \le 1 \end{pmatrix} \qquad \frac{\partial g}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = x^2$$

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

$$= \iint_R (1 - x^2)\,dA$$

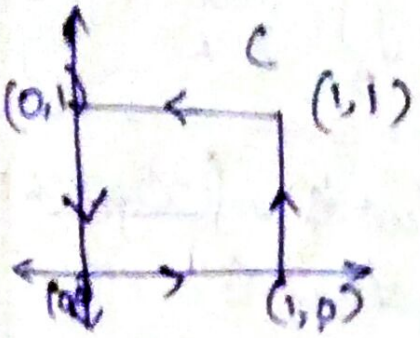
$$= \int_0^1 \int_0^{2x} (1 - x^2)\,dy\,dx$$

$$= \int_0^1 (y - x^2y)_0^{2x}\,dx = \int_0^1 2x - 2x^3\,dx = \cancel{[2 - 2x]'}$$

$$= \left[\frac{2x^2}{2} - \frac{2x^4}{4}\right]_0^1 = \cancel{\frac{14}{8} - \frac{1}{3}} = \frac{1}{2}$$

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Q. use Green's theorem to evaluate $\oint_C y^2 dx + x^2 dy$ where C is the square with vertices (0,0) (1,0), (1,1) & (0,1) oriented counter clockwise.



$f(x,y) = y^2$.

$g(x,y) = x^2$.

$\frac{\partial f}{\partial y} = 2y \quad \frac{\partial g}{\partial x} = 2x$.

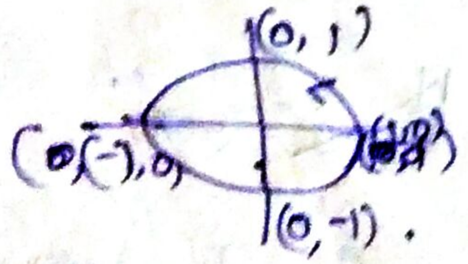
$$\int_0^1\int_0^1 (2x-2y)\,dy\,dx = 2\int_0^1\int_0^1 (x-y)\,dy\,dx$$

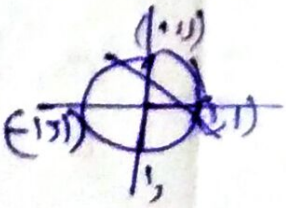
$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^1 dx$$

$$= 2\int_0^1 \left[x - \frac{1}{2}\right] dx = 2\left[\frac{x^2}{2} - \frac{1}{2}x\right]_0^1$$

$$= \left[x^2 - x\right]_0^1 = 1 - 1 = 0 //$$

Q. Use green's theroem to evalute $\int y\,dx + x\,dy$ where C is the unit circle oriented counter clockwise.





$f(x,y) = y \quad g(x,y) = x$

$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = 1$.

$$\oint_C \iint_R (1-1)\, dA = \iint_R 0\, dA = 0 //$$

$x = \cos t,\ y = \sin t \qquad 0 \le t \le 2\pi$

$$= \int_0^{2\pi} \sin t \times -\sin t\, dt + \cos t . \cos t\, dt$$

$$= \int_0^{2\pi} -\sin^2 t\, dt + \cos^2 t\, dt = \int_0^{2\pi} \cos^2 t\, dt - \sin^2 t\, dt$$

$$= \int_0^{2\pi} (\cos^2 t - \sin^2 t)\, dt = \int_0^{2\pi} \cos 2t\, dt = \left[\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}(\sin 2t)_0^{2\pi} = \frac{1}{2}(\sin 4\pi - \sin 0) = \frac{1}{2}(0-0) = 0 //$$

Q) Find the workdone by the force field $\vec{F}(x,y) = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the anticlockwise.

$$W = \oint_C (e^x - y^3)\, dx + (\cos y + x^3)\, dy$$

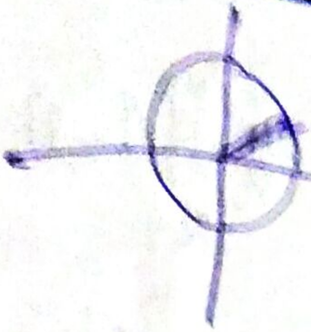
$f(x,y) = e^x - y^3 \qquad g(x,y) = \cos y + x^3.$

$$\frac{\partial f}{\partial y} = -3y^2 \qquad \frac{\partial g}{\partial x} = 3x^2$$

$$\int_{-1}^{1}\int_{-1}^{1} (-3y^2 - 3x^2)\, dx\, dy = \int_{-1}^{1} \left[-3y^2 x - \frac{3x^3}{3}\right]_{-1}^{1} dy$$

$-3y^2 \quad 3x^2$

$$= \iint_R 3x^2 + 3y^2 \, dA$$



$x = r\cos t$
$y = r\sin t$
$0 \le t \le 2\pi$

$$= 3\iint_R (x^2+y^2)\, dA$$

$$= 3\int_0^{2\pi}\int_0^1 r^2 . r \, dr\, d\theta.$$

$$= 3\int_0^{2\pi} \left(\frac{r^4}{4}\right)_0^1 d\theta = \frac{3}{4}\int_0^{2\pi} d\theta$$

$$= \frac{3}{4} \times [\theta]_0^{2\pi} = \frac{3}{4} \times 2\pi = \frac{3\pi}{2} //.$$

16/9 Tuesday

If $f(x,y,z)$ is continous on σ, then

$$\iint_\sigma f(x,y,z)\, ds = \iint_R f(x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA.$$

Q Evaluate the surface integral $\iint_\sigma z^2 ds$ over the sphere $x^2+y^2+z^2=1$

The sphere $x^2+y^2+z^2=1$ can be represented using spherical cordinate system as

$$\vec{r}(\rho,\phi,\theta) = \sin\phi\cos\theta\,\hat{i} + \sin\phi\sin\theta\,\hat{j} + \cos\phi\,\hat{k}$$

$0 \le \phi \le \pi, 0 \le \theta \le 2\pi$

$$\frac{\partial \vec{r}}{\partial \phi} = \cos\phi \cos\theta\, \hat{i} + \cos\phi \sin\theta\, \hat{j} - \sin\phi\, \hat{k}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -\sin\phi \sin\theta\, \hat{i} + \sin\phi \cos\theta\, \hat{j} + 0\, \hat{k}$$

$$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta\sin\phi & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \hat{i}\left[\sin^2\phi \cos\theta\right] - \hat{j}\left(-\sin\theta \sin^2\phi\right) + \hat{k}\left(\sin\phi \cos\phi \cos^2\theta + \sin\phi \cos\phi \sin^2\theta\right)$$

$$= \hat{i}(\sin^2\phi \cos\theta) + \hat{j}(\sin^2\phi \sin\theta)\, \hat{k}(\sin\phi \cos\phi).$$

$$\left\|\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}\right\| = \sqrt{\sin^4\phi \cos^2\theta + \sin^4\phi \sin^2\theta + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^4\phi + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^2\phi\left[\sin^2\phi + \cos^2\phi\right]} = \sqrt{\sin^2\phi} = \underline{\underline{\sin\phi}}$$

$$\iint_\sigma x^2\, dS = \iint_R (\sin\phi \cos\theta)^2 \sin\phi\, dA.$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi \cos^2\theta\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{3\sin\phi - \sin 3\phi}{4}\right) \cos^2\theta\, d\phi\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left[-3\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi}\cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left[\left(3-\frac{1}{3}\right)-\left(-3+\frac{1}{3}\right)\right]\cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}(6-2/3)\cos^2\theta\, d\theta$$

$$= \frac{1}{4}\times\frac{16}{3}\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= \frac{4}{3}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{4}{3}\left(\theta+\frac{\sin 2\theta}{2}\right)_0^{2\pi} = \frac{2}{3}\times 2\pi = 4\pi/3$$

Surface integrals over $z = g(x,y)$, $y = g(x,z)$ & $x = g(y,z)$.

Theorem

A). Let σ be a surface with eqn $z = g(x,y)$ & let R be its projection on the x-y plane. If g has continous first partial derivatives on R & $f(x,y,z)$ is continous on σ then

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

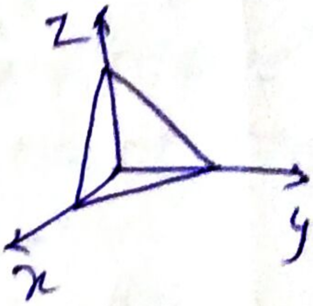
||| ly

b). $$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x,g(x,z),z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2+1}\,dA$$

~~c) $\iint_\sigma f(x,y,z)\,ds = \iint_R f(g(y,z),y,z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2+1}\,dA$~~

c) $\iint_\sigma f(x,y,z)\,ds = \iint_R f(g(y,z),y,z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2+1}\,dA.$

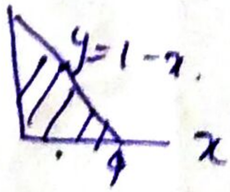
Q Evaluate the surface integral $\iint_\sigma xz\,ds$ where σ is the part of the plane $x+y+z=1$ that lies in the first ~~quadrant~~ octant.



$\sigma = x+y+z=1$

$z = 1-x-y$

$\frac{\partial z}{\partial x} = -1 \quad \frac{\partial z}{\partial y} = -1$



$$\iint_\sigma xz\,ds = \iint_R x(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\,dA$$

$R: 0 \le x \le 1,$
$0 \le y \le 1-x$

$$= \int_0^1\int_0^{1-x} x(1-x-y)\sqrt{3}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}(x-x^2-xy)\sqrt{3}\,dy\,dz = \sqrt{3}\int_0^1\left[xy-x^2y-\frac{xy^2}{2}\right]_0^{1-x}dx$$

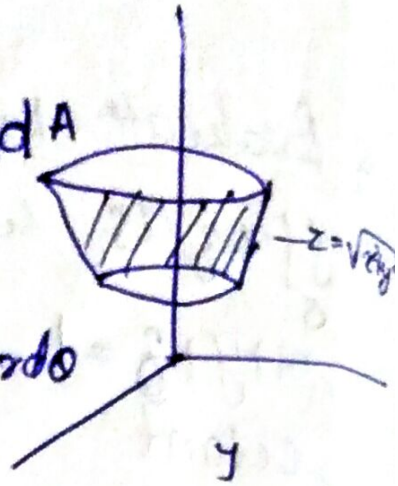
$$= \sqrt{3}\int_0^1 x-x^2-x^2+x^3-\ldots(1-x) \qquad \frac{1}{12}\times\frac{1}{12}.$$

$$= \sqrt{3}/24$$

/6/19 Wednesday

Q) Evaluate the surface Integral $\iint_\sigma y^2 z^2 \, ds$, where σ is the part of the $z=\sqrt{x^2+y^2}$ ~~that~~ lies b/w the phase $z=1$ & $z=2$.

$$\iint y^2 z^2 ds = \iint_R y^2(x^2+y^2)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$



$$=\sqrt{2}\int_0^{2\pi}\int_1^2 r^2\sin^2\theta\,(r^2\cos^2\theta + r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\int_1^2 r^2\sin^2\theta \; r^2(\cos^2\theta+\sin^2\theta)\, r\, dr\, d\theta$$

Polar form

$x = r\cos\theta$

$y = r\sin\theta$

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$dA = r\, dr\, d\theta.$

$$=\sqrt{2}\int_0^{2\pi}\int_1^2 r^5 \sin^2\theta\, dr\, d\theta.$$

$$=\sqrt{2}\int_0^{2\pi}\left[\frac{r^6}{6}\right]_1^2 \sin^2\theta\, d\theta$$

$$=\sqrt{2}\times\frac{1}{6}\int_0^{2\pi}(64-1)\sin^2\theta\, d\theta$$

$\frac{21}{4}$

$$=\sqrt{2}\,\frac{63}{6}\int_0^{2\pi}\sin^2\theta\, d\theta = \frac{63}{6}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\, d\theta$$

$$=\sqrt{2}\,\frac{21}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$2\cdot\sqrt{2}:2\cdot$

$$=\sqrt{2}\,\frac{21}{4}\times 2\pi \;\; \frac{21}{\sqrt{2}\cdot\sqrt{2}\cdot 2}\times 2\pi \;.\; \frac{21\pi}{\sqrt{2}}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \times 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial x}{\partial y}\right)^2 + 1} = \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} = \sqrt{2}$$

Evaluate the flux integral.

Flux (ϕ).

The net volume of fluid that passes through a surface (σ) per unit of time is called flux.

$$\text{Flux } \phi = \iint_{\sigma} \vec{F}.\vec{n}\, ds = \iint_{R} \vec{F}.\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) dA$$

(projection)

where σ is given by the vector eq^n $\vec{r} = \vec{r}(u,v)$ & $\vec{n}$ is the normal to σ (+ve orientation outward).

Q. Find the flux of the vector field $\vec{F}(x,y,z) = z\hat{k}$ across the outward oriented sphere. $x^2+y^2+z^2 = a^2$.

vector form

$$r(\rho, \phi, \theta) = a\sin\phi\cos\theta\,\hat{i} + a\sin\phi\sin\theta\,\hat{j} + a\cos\phi$$

$$0 \leq \phi \leq \pi$$

$$0 \leq \theta \leq 2\pi$$

8/9/19

Q) $\vec{r}(\phi\ \theta) = a\sin\phi\cos\theta\,\hat{i} + a\sin\phi\sin\theta\,\hat{j} + a\cos\phi\,\hat{k}$

$0 \le \phi \le \pi$

$0 \le \phi\ 2\pi$.

$\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{d\theta} = a^2\sin^2\phi\cos\theta\,\hat{i} + a^2\sin^2\theta\sin\theta\,\hat{j} + a^2\sin\phi\cos\phi\,\hat{k}$

$\vec{F} \cdot \left[\frac{\partial r}{\partial\phi} \times \frac{\partial r}{\partial\theta}\right] = a^2\sin\phi\cos^2\phi$

is Flux $\phi = \iint_\sigma \vec{F}\cdot\vec{r}\,ds$.

$= \iint_{\sigma} \vec{F}\cdot n\,ds = \iint_R \vec{F}\cdot\left(\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta}\right) ds$

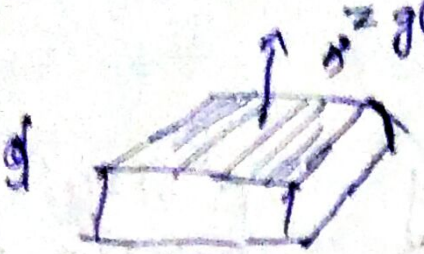
$= \int_0^{2\pi}\int_0^{\pi} a^3\sin\theta\cos^2\phi\,d\phi\,d\theta$

$= a^3\int_0^{2\pi}\left[\frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta$

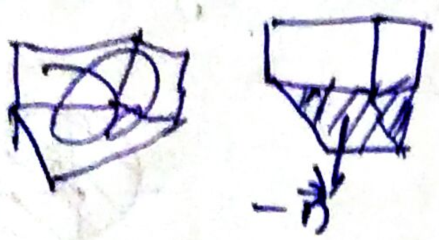
$= \frac{2a^3}{3}\int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}$ //.

Orientation of non-parmateric surface.

$z = g(x,y)$.



normal $\vec{n} = \dfrac{-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$



$-\vec{n} = \dfrac{\frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} - k}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$

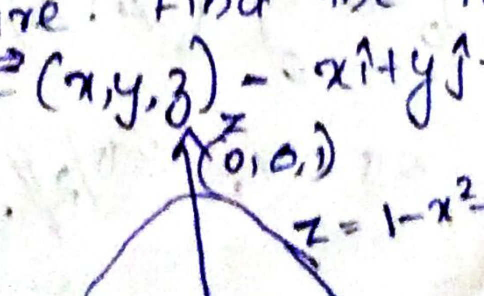
$$\iint_{\sigma} \vec{F}\cdot\vec{n}\, ds = \iint_{R} \vec{F}\left(-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}\right) dA.$$

(oriented up).

$$\iint_{\sigma} \vec{F}\cdot\vec{n}\, ds = \iint_{R} \vec{F}\cdot\left(\frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} - \hat{k}\right) dA$$

(oriented down)

Q Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy plane, & suppose that σ is oriented up, as shown in the figure. Find the flux of the vector field $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$ across σ.



$$\text{Flux } \phi = \iint F \cdot n \, ds \quad = \iint_R F \cdot \left(-\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k \right) dA$$

$$z = 1 - x^2 - y^2$$

$$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y, \quad \vec{F} = xi + yj + zk$$

$$= 2xi + 2yj + k$$

$$\iint F \cdot \left(-\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k \right)$$

$$= (xi + yj + zk) \cdot (2xi + 2yj + k)$$

$$= 2x^2 + 2y^2 + z.$$

$$\text{Flux } \Phi = \iint_R (2x^2 + 2y^2 + z) \, dA$$

$$\iint_R (2x^2 + 2y^2 + 1 - x^2 - y^2) \, dA.$$

polar form

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \int_0^1 (r^2 + 1) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^3 + r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^4}{4} + r^2 \right)_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{2} \right) d\theta = \frac{3}{4} \int_0^{2\pi} d\theta$$

$$\frac{3}{4} [\theta]_0^{2\pi} = 2\pi \times \frac{3}{4} = \frac{3\pi}{2}$$

9

Divergence theorem

Let G be a soild cohoes surface σ is oriented out coard. If
$$\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$$
where f, g & h have continous first partial derivative on same openset conting of G, & if $\hat{n}$ is the outward unit normal on σ, then
$$\iint_\sigma \vec{F}\cdot\hat{n}\, ds = \iiint_G \text{div}\, \vec{F}\, dv. \quad \text{(flux)}.$$
$$\boxed{V = \iiint_G 1 \cdot dv}$$

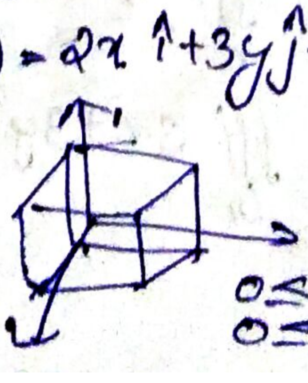
Q Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = z\hat{k}$ across the sphere $x^2+y^2+z^2=a^2$.

$$\text{div} F = \vec{\nabla}\cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot zk$$
$$= \frac{\partial z}{\partial z} = 1$$

$\phi = \iint_\sigma \vec{F}\cdot\hat{n}\, ds$ by divergence theorem

$$\iiint_G \text{div}\vec{F}\, dv = \iiint_G 1\cdot dv = \text{Volume of } G = \frac{4}{3}\pi a^3$$

Q Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = 2x\hat{i} + 3y\hat{j} + z^2\hat{k}$ across the unit cube:



$0 \le z \le 1$

$0 \le x \le 1$

$0 \le y \le 1$

$$\text{div}\cdot F = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}2x + \frac{\partial}{\partial y}3y + \frac{\partial}{\partial z}z^2$$

$$= 2+3+2z = \underline{5+2z.}$$

by divergence

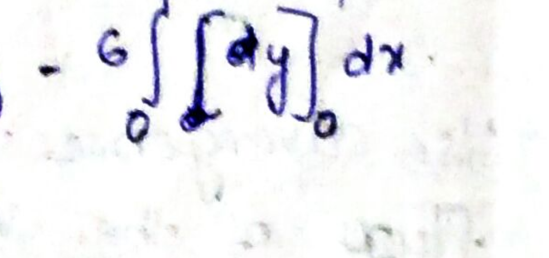
$$\iiint_G \text{div} F = \int_0^1\int_0^1\int_0^1 5+2z \, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left[5z + z^2\right]_0^1 dy\, dz.$$

$$= \int_0^1\int_0^1 (5+1)\, dy\, dz = 6\int_0^1 [y]_0^1 dx$$

$$= 6\int_0^1 [x]_0^1 = 6 //.$$

Q) Use divergence theorem to find the outward flux across of the vector field $\vec{F}(x,y,z) = x^3\hat{i} + y^3\hat{j} + z^2\hat{k}$ across the surface of the region that is enclosed by the cylinder $x^2+y^2=9$ & the planes $z=0$, & $z=2$.



$$\text{div} F = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^2$$

$$= 3x^2 + 3y^2 + 2z.$$

$$\oint = \iint_\sigma \vec{F}\cdot\hat{n}\, dS = \iiint_G \text{div} F\, dV$$

$$= \iiint_G 3x^2+3y^2+2z\, dV$$

cylindrical co-ordinates

$x = r\cos\theta$
$y = r\sin\theta.$
$z = z.$
$0 \le r \le 3$
$0 \le \theta \le 2\pi$
$0 \le z \le 2.$

$$= \int_0^2 \int_0^{2\pi} \int_0^3 (3r^2 + 2z) r \, dr \, d\theta \, dz$$

$3r^2\cos^2\theta + 3r^2\sin^2\theta + 2z$

$3r^2(\cos^2\theta + \sin^2\theta) + 2z$

$$= \int_0^2 \int_0^{2\pi} \int_0^3 (3r^3 + 2zr) \, dr \, d\theta \, dz$$

$3 \times 3 \times 3 \times 3$

$$= \int_0^2 \int_0^{2\pi} \left[\frac{3r^4}{4} + 2z\frac{r^2}{2} \right]_0^3 d\theta \, dz$$

$\frac{81}{4}$

243

$$\frac{243}{4} \left[\int_0^2 \int_0^{2\pi} 9z \, d\theta \, dz\right] = \frac{243}{4} \int_0^2 \left[9z\theta \right]_0^{2\pi} dz$$

$$= \int_0^{2\pi} \int_0^3 \int_0^2 (3r^3 + 2zr) \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \left[3r^3 z + 2r\frac{z^2}{2} \right]_0^2 dr \, d\theta = \int_0^{2\pi} \int_0^3 (6r^3 + 4r) \, dr \, d\theta$$

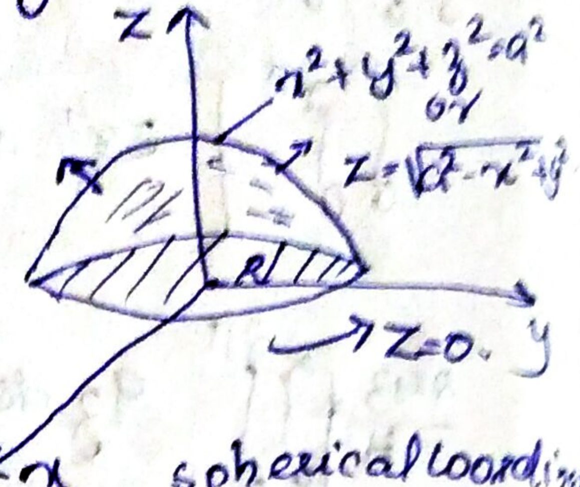
$$= \int_0^{2\pi} \left[6\frac{r^4}{4} + 4\frac{r^2}{2} \right]_0^3 d\theta = \int_0^{2\pi} \left(\frac{3r^4}{2} + 2r^2 \right)_0^3 d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{2} \times 81 + 2 \times 9 \right) d\theta = \left(\frac{243}{2} + 18 \right) \int_0^{2\pi} d\theta$$

$$= \left(\frac{243}{2} + \frac{18}{1} \right) \times 2\pi = \frac{279}{\cancel{2}} \times \cancel{2}\pi = \underline{\underline{279\pi}}$$

1/7/19 Monday

use divergence theorem find the outward flux of the vector field $F(x,y,z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ across the surface of the reigon that is enclosed by the hemisphere $x^2+y^2+z^2=a^2$ and the plane $z=0$.



$$\text{div}\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2+3y^2+3z^2 = 3(x^2+y^2+z^2)$$

spherical coordinate system

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$0 \le \phi \le \pi/2$

$0 \le \theta \le 2\pi$

$0 \le \rho \le a$

$\boxed{dv = \rho^2\sin\phi \, d\rho\, d\phi\, d\theta}$

Flux $\Phi = \iint_S \vec{F}\cdot\hat{n}\, ds$

$$= \iiint_G \text{div } F \cdot dv$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} (3\rho^2)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi/2}\left[\frac{\rho^5}{5}\sin\phi\right]_0^a d\phi\, d\theta$$

$$\frac{3a^5}{5}\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/2} d\theta = \frac{3a^5}{5}\int_0^{2\pi} d\theta$$

$$= \frac{3a^5}{5} \times (\theta)_0^{2\pi} = \frac{3a^5}{5} \times 2\pi = \frac{6\pi a^5}{5}$$

Stoke's theorem

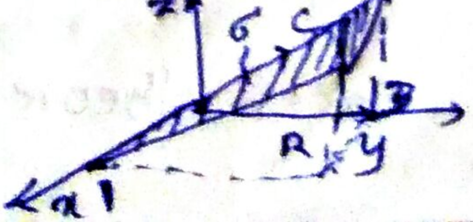
Let σ be a piece wise smooth oritened surface that is bounded by a simple closed piece wise smooth curve C with positive orientation. If the components of the vector field $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$ are continous & have continous first partial derivative on some opeset containg σ, & if $\vec{T}$ is the unit tangent vector C, then (works)

$$\oint_C \vec{F}\cdot\vec{T}\, ds = \iint_\sigma (\text{curl}\, \vec{F})\cdot\hat{n}\, dS$$

workdone using stoke's theorem.

$$W = \oint_C \vec{F}\cdot d\vec{r} = \iint_\sigma (\text{curl}\, \vec{F})\,\hat{n}\, dS$$

Q) Find work performed by the vector field $\vec{F}(x,y,z) = x^2\hat{i} + 4xy^3\hat{j} + y^2x\hat{k}$ for a particle that traverses the retangle C in the plane $z=y$ shown in the fig.



$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 4xy^3 & y^2x \end{vmatrix}$$

$\hat{n}\,ds \left(\frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} \ldots \right) dA$

$$= \hat{i}\left(\frac{\partial}{\partial y}(y^2x) - \partial/\partial z(4xy^3)\right) - \hat{j}\left(\partial/\partial x(y^2x) - \partial/\partial z\, x^2\right) + \hat{k}\left(\partial/\partial x(4xy^3) - \partial/\partial y\, x^2\right).$$

$$= 2yx\hat{i} - y^2\hat{j} + 4y^3\hat{k}$$

$$= \iint_R (\text{curl}\,\vec{F}) \cdot \left(\frac{\partial z}{\partial x}\hat{i} + \frac{\partial}{\partial y}z\hat{j} - \hat{k}\right) dA.$$

$$= \iint_R (2yx\hat{i} - y^2\hat{j} + 4y^3\hat{k}) \cdot (\hat{j} - \hat{k})\, dA$$

$z = y$

$\frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial y} = 1$

$$= \iint_R (-y^2 - 4y^3)\, dA = \int_0^1 \int_0^3 (-y^2 - 4y^3)\, dy\, dx$$

$$= \int_0^1 \left[\frac{-y^3}{3} - \frac{4y^4}{4}\right]_0^3 dx \quad \int_0^1 \frac{-27}{3} - 81\, dx.$$

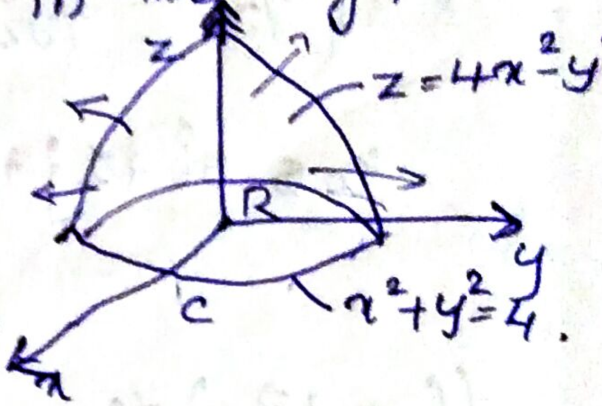
$$= -9 - 81 \int_0^1 dx$$

$$= -90.$$

Q. Verify stokes theorem by the vector field

$\vec{F}(x,y,z) = \cancel{2z\hat{i} + 3x\hat{j} + 5y\hat{k}}\ 2z\hat{i} + 3x\hat{j} + 5y\hat{k}$

taking σ to be the portion of the parabola in $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward oriation & C to be truly oriented circale $x^2+y^2=4$ that ~~fall~~ forms the boundray of σ in the xy plane.



An:- Parametric eq^n of circle

$x = 2\cos\theta, \quad y = 2\sin\theta$

$0 \leq \theta \leq 2\pi$.

$\vec{F} = 2z\hat{i} + 3x\hat{j} + 5y\hat{k} \qquad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\vec{F}\cdot d\vec{r} = 2z\,dx + 3x\,dy + 5y\,dz$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 2z\,dx + 3x\,dy + 5y\,dz$$

$$= \int_0^{2\pi} 2\times 0 + 3\times 2\cos\theta \times 2\cos\theta\, d\theta + 5\times 0.$$

$$= \int_0^{2\pi} \cancel{6\cos\theta\, d\theta} = \cancel{6\sin\theta} \int_0^{2\pi} 12\cos^2\theta\, d\theta$$

$$= 12\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta = 6\int_0^{2\pi} 1 + \cos 2\theta\, d\theta$$

$$= 6\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 12\pi$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & -3x & 5y \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial 5y}{\partial y} - \frac{\partial 3x}{\partial z}\right) - \hat{j}\left(\partial/\partial x\, 5y - \partial/\partial z\, 2z\right) + \hat{k}$$

$$= \hat{k}\left(\partial/\partial x\, 3x - \partial/\partial z\, 2z\right)$$

$$= \cancel{tos}\; 5\hat{i} + 2\hat{j} + 3k$$

$$Z = 4 - x^2 - y^2$$

$$\frac{\partial z}{\partial x} = -2x.$$

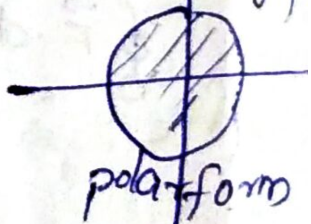
$$\frac{\partial z}{\partial y} = -2y.$$

$$\iint_S \text{curl}\vec{F} \cdot \hat{n}\, ds$$

$$= \iint_R (5i + 2\hat{j} + 3k) \cdot \left(-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + k\right) dA$$

$$= \iint_R (5\hat{i} + 2\hat{j} + 3k) \cdot (2x\hat{i} + 2y\hat{j} - \hat{k})\, dA$$

$$= \iint_R 10x + 4y + 3\; dA.$$



$x^2+y^2=4$

polar form

$$= \int_0^{2\pi}\int_0^2 (10r\cos\theta + 4r\sin\theta + 3)\, r\, dr\, d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \left[10\frac{r^3}{3}\cos\theta + \frac{4r^3}{3}\sin\theta + \frac{3r^2}{2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 8\; d\theta.$$

=

$$\cancel{\frac{80}{3}\left[\cos\theta\right]_0^{2\pi} + \frac{32}{3}\left[\sin\theta\right]_0^{2\pi}}$$

$$= \frac{80}{3}(\sin\theta)_0^{2\pi} + \frac{32}{3}(-\cos\theta)_0^{2\pi} + (6\theta)_0^{2\pi}$$

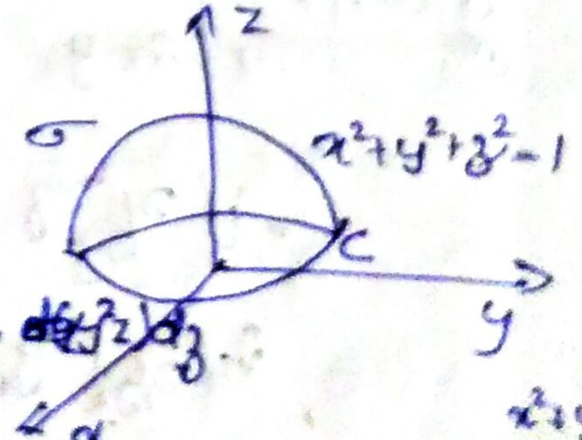
$\cos 0 = 1$
$\cos\pi = -1$
$\cos 2\pi = 1$
$\cos 3\pi = -1$

$$= \frac{80}{3}\times 0 - \frac{32}{3}(1-1) + 12\pi$$

$$= 12\pi$$

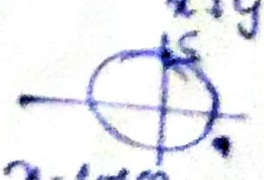
$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \oint_C \vec{F}\cdot dr$. Hence its verified.

Q Verify stokes theorem when $\vec{F} = (2x-y)\hat{i} - (yz^2)\hat{j} - (y^2z)\hat{k}$ where S is the upper half surface of the unit sphere $x^2+y^2+z^2=1$ and C is it boundary.



$\oint_C \vec{F}\cdot dr$

$\vec{F}\cdot dr = (2x-y)\,dx - (yz^2)\,dy - (y^2z)\,dz.$



$$= \int_0^{2\pi} (2x-y)\,dx - (yz^2)\,dy - (y^2z)\,dz.$$

$x = \cos\theta$
$y = \sin\theta$
$0 \le \theta \le 2\pi$.

$$= \int_0^{2\pi} \cancel{2\cos\theta - \sin^2\theta \times 0 - 0.}$$

$$\cancel{= \int_0^{2\pi} 2\cos\theta \, d\theta = \left[2\sin\theta\right]_0^{2\pi} =}$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta) - \sin\theta \, d\theta \qquad -\sin\theta \times 0 - y_{z=0}$$

$$\sin^2\theta$$

$$= \int_0^{2\pi} -2\cos\theta\sin\theta + \sin^2\theta \, d\theta$$

$$= \int_0^{2\pi} -\sin 2\theta + \frac{1-\cos 2\theta}{2} \, d\theta = \left(\frac{\cos 2\theta}{2}\right)_0^{2\pi} + \frac{1}{2}(\theta - \sin$$

$$= \frac{1}{2}[\cos 4\pi - \cos 0] + \frac{1}{2}[2\pi - 0]$$

$$= \frac{1}{2}(1-1) + \frac{2\pi}{2} = \pi //$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= i\left(\partial/\partial y (-y^2 z) - \partial/\partial z(-yz^2)\right) - j\left(\partial/\partial x(-y^2 z) + \partial/\partial z(2x-y)\right)$$

$$k\left(\partial/\partial x(-yz^2) - \partial/\partial y(2x-y)\right)$$

$$= -2yz + 2yz\,\hat{i} - j \times 0 + k(1)$$

$$= \hat{k} //$$

$$\sigma : z = \sqrt{1-x^2-y^2} \qquad \partial z/\partial x = \frac{1}{2\sqrt{1-x^2-y^2}} - 2x.$$

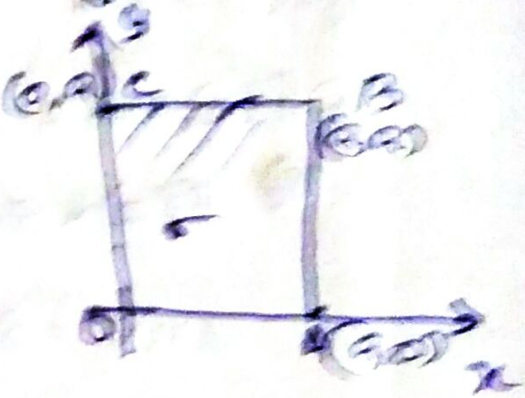
$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{1-x^2-y^2}} \times -2y$$

$$\iint_S (\text{curl}\,\vec{F} \cdot \hat{n}\, ds = \iint_R \hat{k} \cdot \left(\frac{-\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}\right) dA$$

$$= \iint_R dA = \text{Area of } \text{circle} = \pi r^2 = \pi \times 1 = \pi //$$

Hence verified. $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, dS$.

Q Verify stoke theorem for the function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated around the square in the plane $z=0$ whose sides are along the lines $x=0, y=0, x=a, y=a$.



$\hat{n} = \hat{k}$

$\text{curl}\,\vec{F} = yk$,

$$\iint_R \text{curl}\,\vec{F}\, n\, dA = \int_0^a\int_0^a y\, dy\, dx$$

$=$

7/6/19
Monday

If $f(z)$ is a complex varied function it real part & imangary parts are function of x & y.

eg: $f(z) = z^2$, $z = x+iy$.

$= (x+iy)^2 = (x^2 - y^2) + i2xy$ $f(u+iv)$

equ. real & imanigary parts we get

$u = x^2 - y^2$ $\quad v = 2xy$.

Q) Separate the real & imaginary parts of

a) $f(z) = z^3$, ~~$z=x+iy$~~ $z = x+iy$. $\qquad \cos iy = \cosh y$

b) $f(z) \sin z$, $z = x+iy$. $\qquad \sin iy = i \sinh y$

b) $f(z) = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$u + iv = \sin x \cosh y + i \cos x \sinh y$.

$u = \sin x \cosh y$; $v = \cos x \sinh y$.

c). $f(z) = \cos z$.

a) $f(z) = z^3$. $(x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$= x^3 + 3x^2 y i - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + i(3x^2 y - y^3$.

complex integeration

The complex integral of a given function $f(z)$ along a given oriented curve C the complex z plane is denoted by $\int f(z)dz$. $f(z)$ may be analytic or not and ~~C not~~ may be either a closed curve or arc, but in either case we shall assume that C is piece wise smooth and simple. the curve C is called the path of integration. or coutour (path of integration)

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx - vdy) + i\int_C udy + vdx.$$

Thus we can ~~evalu~~ evalute $\int_C f(z)dz$ by evaluating 2 real line integrals mainly $\int_C udx - vdy$ \$ $\int_C udy + vdx$.

Evaluate $\int_C |z|^2 dz$ where C is a straight line from $z=0$ to $z=1+i$ $z = (0,0)$ to $(1,1)$

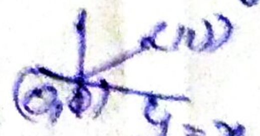
$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$|z| = x^2+y^2$.

$$\int_C |z|^2 dz = \int_C (x^2+y^2)(dx+idy)$$

$$= \int_C (x^2+y^2)dx + i\int (x^2+y^2)dy.$$



Along $C \Rightarrow y = x$.

$\therefore dy = dx$ & x varies from $(0,1)$

$\therefore \int_C |z|^2 dz = \int_0^1 2x^2 dx + \int_0^1 i\,(2y^2\,dx$.

$= \left[\frac{2x^3}{3}\right]_0^1 + i\left[\frac{2x^3}{3}\right]_0^1$

$= \underline{\underline{\frac{2}{3} + i\,2/3}}$.

Q) Evaluate $\int_C z^2 dz$ where c is given by the line ~~given~~ $x = 2y$ from $(0,0)$ to $(2,1)$.

$\int_C (x^2 - y^2 + i\,2xy)\,(dx + i dy)$

$= \int_C (x^2 - y^2)\,dx - 2xy\,dy + i\int (x^2 - y^2)$
$+ 2xy\,dx$ —

Along C $dx = 2dy$. & x varies from
$dy = dx/2$. 0 to 2.

$= \int_0^2 \left[\left(x^2 - \frac{x^2}{4}\right)dx - 2x \cdot x/2 \cdot \frac{dx}{2}\right] + i\int_0^2 \left(x^2 - \frac{x^2}{4}\right) dx/2 + (2x \cdot x/2$

$= \int_0^2 \left(\frac{3x^2}{4} - x^2/2\right) dx + i\int_0^2 \left(\frac{3x^2}{4 \cdot 2} + \frac{x^2}{2}\right) dx$

$= \int_0^2 \frac{x^2}{4} dx + i\int \frac{11}{8} x^2 dx = \left[\frac{x^3}{12}\right]_0^2 + i\left[\frac{11}{8}\frac{x^3}{3}\right]_0^2$

$= \frac{2}{3} + i\,\frac{11}{3}$

Q Evaluate ~~[illegible]~~ $\int_0^{1+i} (x^2+iy)\,dz$ along ~~[illegible]~~ $y = x$ & $y = x^2$.

$$\int_0^{1+i} (x^2+iy)(dx+idy) = \int_0^{1+i} x^2\,dx - y\,dy + i\int_0^{1+i} x^2 dy + y\,dx \quad \text{—①}$$

Along $y = x$
$dy = dx$. $\quad x = 0$ to 1

Independent path of integration

$$\int_0^{1+i} x^2 + iy\,dz = \int_0^1 (x^2 - x)\,dx + i\int_0^1 (x^2+x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 + i\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$$

dy

$$= \left[\frac{1}{3} - \frac{1}{2}\right] + i\left[\frac{1}{3} + \frac{1}{2}\right] = -\frac{1}{6} + \frac{5}{6}i$$

Along $y = x^2$
$dy = 2x\,dx$. $\quad x = 0$ to 1

$$\int_0^{1+i} (x^2+iy)\,dz = \int_0^1 (x^2 - x^2 \cdot 2x)\,dx + i\int_0^1 (x^2 \cdot 2x + x^2)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{2x^4}{4}\right]_0^1 + i\left[\frac{2x^4}{4} + \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{2}\right) + i\left(\frac{1}{2} + \frac{1}{3}\right) = -\frac{1}{6} + i\,\frac{5}{6}$$

Q. Evaluate $\int_C (2x+iy+1)\,dz$ where C is a straight line joining $1-i$ to $2+i$

$$= \int_C [(2x+1)+iy](dx+i\,dy)$$

$$= \int_C (2x+1)\,dx - y\,dy + i\int_C (2x+1)\,dy + y\,dx \quad \text{—①}$$

$y+1 = \frac{1+1}{2-1}(x-1)$

$(y+1) = 2(x-1)$

$y = 2x-3$

$dy = 2\,dx$

$x = 1$ to 2

Along C (1, −1) to (2, 1)

$$= \int_1^2 [(2x+1) - (2x-3)2]\,dx + i\int_0^2 [(2x+1)2 + 2x-3]\,dx$$

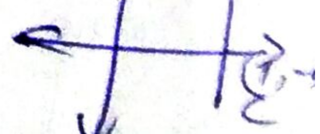
$$= \int_1^2 -2x+7\,dx + i\int_1^2 (6x-1)\,dx$$

$$= \left[-\frac{2x^2}{2}+7x\right]_1^2 + i\left[\frac{6x^2}{2}-x\right]_1^2$$

$$= (-4+14)-(-1+7) + i[9-1]$$

$$= 4+i8 = 4(1+i2)$$

* Evaluate $\int_C e^z\,dz$ where C consist of two straight line segments from $z=i$ to $z=1+i$ & then from $z=1+i$ to $z=1+2i$.

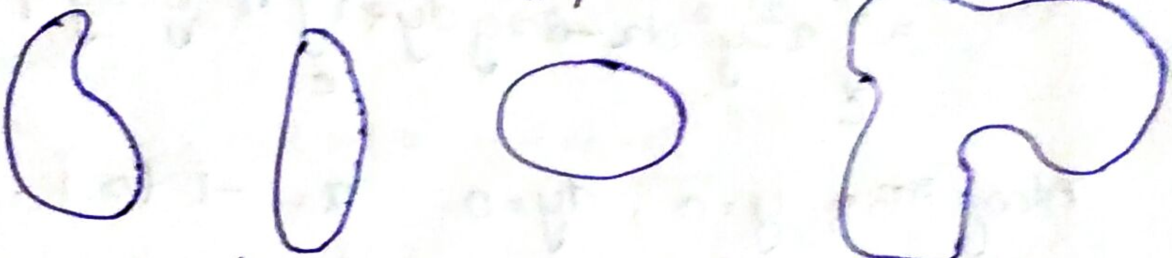


Cauchy's theorem

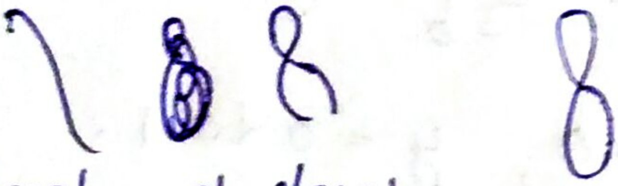
If $f(z)$ analytic in a simply connected domain D then for any simple closed path C in D

$$\int_C f(z)\,dz = 0.$$

A simple closed path is a closed path that doesnot intersept or touch it self



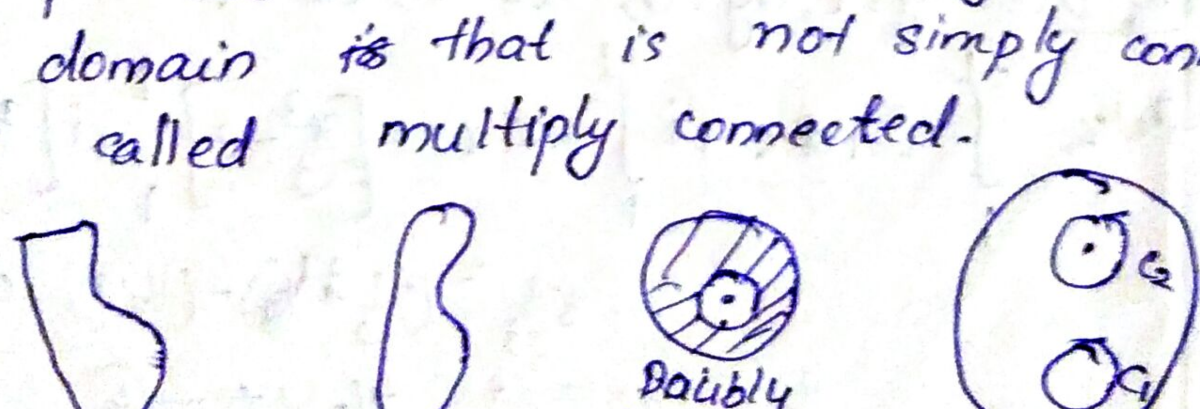
simple closed curves.



not simple.

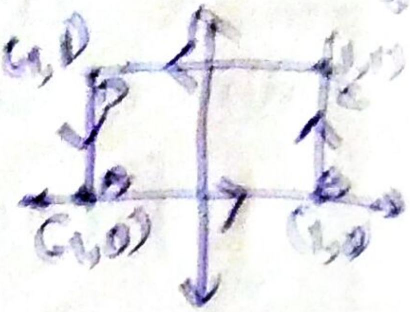
Simple not closed.

A simply connected domain D in the complex plane is a domain such that every simple closed path in D encloses only points of D. The interier of a circle, ellipse, or any simple closed curve are simpley connected.
A domain ~~is~~ that is not simply connected is called multiply connected.



simply connected Doubly connected Triply connected

9. Verify Cauchy's theorem for the integral of z^2 taken counter clockwise over boundary of a rectangle which vertices $-1, 1, 1+i, -1+i$.



$\int_C z^2\, dz = \int_C (x^2-y^2)(i2xy)\, dx + i dy.$

$= \int_C x^2-y^2\, dx - 2xy\, dy + i \int_C x^2-y^2\, dy + 2xy\, dx$

Along AB: $y=0$, $dy=0$. $x = -1$ to 1.

$\int_C z^2 dz = \int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 dx = 2\frac{x^3}{3}\Big|_0^1 = 2/3$ — ②.

Along BC, $x=1$, $dx=0$, $y=0$ to 1.

$\int_C z^2 dz = \int_0^1 -2y\,dy + i\int_0^1 1-y^2\,dy = \left[-\frac{2y^2}{2}\right]_0^1 + i\left[y-\frac{y^3}{3}\right]_0^1$

$= -1 + i\, 2/3$ — ③

Along DC $y=1$, $dy=0$, $x = 1$, to -1.

$\int_C z^2 dz = \int_1^{-1} x^2-1\, dx + i\int_1^{-1} 2x\, dx = \left[\frac{x^3}{3}-x\right]_1^{-1} + i\left[2\frac{x^2}{2}\right]_1^{-1}$

$= (-\frac{1}{3}+1) - (\frac{1}{3}-1) + i[1-1]$

$= 4/3$ — ④.

Along AD. $x = -1$, $dx = 0$ $y = 1$ to 0.

$$\int_C z^2 dz = \int_1^0 2y\,dy + i\int_1^0 1 - y^2 dy.$$

$0 - (\frac{1}{3})$ $\frac{1}{3}$
$-\frac{2}{3}$

$$= \left(2\frac{y^2}{2}\right)_1^0 + i\left[y - \frac{y^3}{3}\right]_1^0 = -1 + i\left[-\frac{2}{3}\right] \quad -⑤$$

$$\int_C z^2 dz = \frac{2}{3} + \left(-1 + i\frac{2}{3}\right) + \frac{4}{3} + \left(-1 - i\frac{2}{3}\right).$$

$$= \frac{6}{3} - 1 - 1 = 2 - 2 = \underline{0}.$$

Thus the Cauchy's theorem is verified.

Complex Analysis

Cauchy-Riceman Equations

The fauncation $w = f(z) = u(x,y) + iv(x,y)$ is analytic is a damain D if & only if the first partial derivatives & u & v satifys the two eq^n

$u_x = v_y$ & $u_y = -v_x$.

Then $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \frac{\partial v}{\partial x}$, $v_y = \frac{\partial v}{\partial y}$

Note.

In ploar form, $z = r(\cos\theta + i\sin\theta)$ &

$f(z) = u(r,\theta) + i\,v(r,\theta)$.

Then C-R Eq^n are

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Q. show that $f(z) = \bar{z}$ is not analytic.

$f(z) = x - iy$.

$u = x$, $v = -y$.

$u_x = \frac{\partial u}{\partial x} = 1$ $\quad v_x = \frac{\partial v}{\partial x} = 0$

$u_y = \frac{\partial u}{\partial y} = 0$ $\quad v_y = \frac{\partial v}{\partial y} = -1$.

The $u_x \neq v_y$, So not analytic.

Q. Show that $f(z) = z^3$ is analytic.

$f(z) = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$= x^3 + 3x^2yi - 3xy^2 - iy^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

then $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ $\quad$ $v_x = \cancel{3x^2 - 3y^2}\ 6xy$

$u_y = -6xy$ $\quad$ $v_y = 3x^2 - 3y^2$

there $u_x = v_y$ & $u_y = -v_x$. $\therefore f(z)$ is analytic

Imp Q. If $f(z)$ analytic in a domain D & $|f(z)|$ is a constant. Then show that $f(z)$ is constant

Given $|f(z)| = \sqrt{u^2+v^2}$ is constant.

$\therefore \sqrt{u^2+v^2} = k$, where k is a constant.

$\therefore u^2 + v^2 = k^2$

$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0$. (Diff. partially w.r.to x

$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$ (" to y

$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0$.

$u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = 0$.

Since $f(z)$ is analytic, C-R equation are satisfies.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \$ \ \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x}.$$

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \text{ --(1)}; \quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0. \text{ --(2)}.$$

multiple (1) by u \$ (2) by v

$$u^2\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} = 0;$$

$$uv\frac{\partial u}{\partial y} + v^2\frac{\partial u}{\partial x} = 0.$$

$$u^2\frac{\partial u}{\partial x} + v^2\frac{\partial u}{\partial x} = 0, \quad (u^2+v^2)\frac{\partial u}{\partial x} = 0. \text{ --(3)}.$$

Again multiple (1) by $-v$ \$ (2) by u and add.

$$-uv\frac{\partial u}{\partial x} + v^2\frac{\partial u}{\partial y} = 0.$$

$$u^2\frac{\partial u}{\partial y} + uv\frac{\partial u}{\partial x} = 0.$$

$$(u^2+v^2)\frac{\partial u}{\partial y} = 0 \text{ --(4)}$$

from (3) \$ (4).

$$u^2+v^2 = 0 \text{ or } \quad \text{or } \frac{\partial u}{\partial x} = 0. \ \$ \ \frac{\partial u}{\partial y} = 0.$$

If $u^2+v^2=0$, then $u=0$ \$ $V=0$.

$\therefore f(z)=0$, a constant.

If $\frac{\partial u}{\partial x}=0$, then u is constant.

By C-R eq^n, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ \$ $\frac{\partial v}{\partial y} = 0$ v is constant.

u is constant of v is constant $\Rightarrow f(z) = u + iv$ is constant.

Laplace's Equations

The equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is called Laplace's Equation of a function $f(z)$

$\nabla^2 f = 0$.

Theorem

If $f(z) = u(x,y) + iv(x,y)$ is analaytic in a domain D then u \$ v satify Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ \$ $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

~~If f(z) is analytic~~

(Harmonic fn) Proof Since $f(z)$ is analytic on D, f satisfies C-R equs ie $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ \$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \$ \quad \frac{\partial u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \$ \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\quad\therefore$ u ~~statis~~ satisfied Laplace eqn.

V satisfied Laplace.

Harmonic functions.

Solutions of Laplace's equation having continous second order partial dervatives are called harmonic functions. The real & imaginary parts of an analytic function are harmonic function.

Conjugate harmonic functions

If two harmonic function $u(x,y)$ & $v(x,y)$ satisfy the C-R eq^n in a domain D, then $v(x,y)$ is said to be a conjugate harmonic function of $u(x,y)$ in D.

5/7/16

Q) Find a conjugate harmonic function of the harmonic function $u = x^2 - y^2$.

$u = x^2 - y^2 \quad \frac{\partial u}{\partial x} = 2x \;;\; \frac{\partial u}{\partial y} = 2y.$

conjugate of u multisatisfy C-R equal

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} - \frac{-\partial v}{\partial x} \qquad \therefore \frac{\partial v}{\partial y} = 2x$ — ①

$\frac{\partial v}{\partial x} = 2y$ — ② since $\frac{\partial v}{\partial x} = 2y \Rightarrow v = \int 2y\, dx$

$= 2yx + f(y)$

Now $\frac{\partial v}{\partial y} = 2x + f'(y)$

from ① $\frac{\partial v}{\partial y} = \cancel{2x+f'(y)}\ 2x$

$\therefore 2x = 2x + f'(y) \qquad \therefore f'(y) = 0 \quad f(y) = k$, a constant

$\therefore V = 2yx + k.$

Analytic function; $f(z) = u + iv$

$(x^2 - y^2) + i(2yx + k) = x^2 - y^2 + i2xy + ki = (x+iy)^2 + ki$

$= z^2 + ki$

Q) Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane & find a conjugate harmonic function v of (u).

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y - 1$

$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2 + 2 = 0$

$\therefore u$ satisfy Laplace's eqn. $\therefore u$ is harmonic

The harmonic conjugate v, satisfies C-R h

$u_x = v_y \qquad u_y = -v_x.$

$v_y = 2x$ — ① $\quad v_x = 2y + 1$ — ②.

From :

$\frac{\partial v}{\partial y} = 2x. \qquad \therefore v = \int 2x\,dy = 2xy + f(x)$

$\therefore \frac{\partial v}{\partial x} = 2y + f'(x)$

ie $v_x = 2y + f'(x)$ — ④.

From ② ④. $\qquad 2y + 1 = 2y + f'(x)$

$\therefore f'(x) = 1$

$f(x) = \int 1\,dx = x + C$

$V = 2xy + x + C$ (from (3))

Analytic.

$f(z) = u+iv = (x^2-y^2-y)+i(2xy+x+c)$

$= x^2-y^2-y+i2xy+ix+ic.$

chaze $x \to y$ & $y=0$.

$f(z) = z^2+iz+ic = z^2+i(z+c).$

Q Are the following for analytic.

1. $f(z) = e^x(\cos y + i\sin y)$ (2) $f(z) = z+\bar{z}$

(2) Find the most general analytic for $f(z) = u+iv$ (1) $u = xy$ (2) $u = e^x\cos y$

(3) Are the far harmonic? If so find the corresponding analytic for $f(z) = u+iv$

(1) $u = x^3-3xy^2$ (2) $u = xy$ (3) $u = e^x\cos 2y$

(4) $u = \dfrac{x}{x^2+y^2}$.

• Three important Analytic form.

1. Exponential function.

$f(z) = e^z = e^{x+iy} = e^x.e^{iy}$

$= e^x[\cos y + i\sin y]$

ie, $e^z = e^x[\cos y + i\sin y]$.

Realpart $= e^x\cos y$, Imaginary part $= e^x\sin y$.

2). Trigonometric functions.

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{ix} + e^{-ix} = 2\cos x$

$\cos x \; \dfrac{e^{ix} + e^{-ix}}{2}$

$\cos z = \dfrac{e^{iz} + e^{iz}}{z}$

$\sin z = \dfrac{e^{iz} - e^{iz}}{2i}$

$\tan z = \dfrac{\sin z}{\cos x}$

$= \dfrac{e^{iz} - e^{iz}}{e^{iz} + e^{iz}}$

$\cot z = \dfrac{e^{iz} + e^{iz}}{e^{iz} - e^{iz}}$

$\sec z = \dfrac{1}{\cos z}$

$\operatorname{cosec} z = \dfrac{1}{\sin z}$

3). Hyperbolic function

$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

$$\boxed{\begin{aligned} \sinh z &= \frac{e^{z} - e^{-z}}{2} \\ \cosh z &= \frac{e^{z} + e^{-z}}{2} \end{aligned}}$$

$\tanh z = \dfrac{\sinh x}{\cosh z}$

$\operatorname{sech} z = \dfrac{1}{\cosh z}$

$\operatorname{cosech} z = \dfrac{1}{\sinh z}.$

$\coth z = \dfrac{\cosh z}{\sinh z}$

Complex Integrals.

Let C be a three space smooth path represented by $z = z(t)$ where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then.

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

→ If C is the unit circle, evaluate $\int_C \frac{dz}{z}$.

we can represent 'c' in the form.

$z(t) = \cos t + i \sin t$ $\qquad 0 \le t \le 2\pi.$

Now, $\int_C \frac{dz}{z} = \int_C \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\cos t + i\sin t} \times (-\sin t + i\cos t)\, dt$

$\cos t + i \sin t = e^{it}$

$\frac{1}{\cos t + i\sin t} = \frac{1}{e^{it}}$

$= e^{-it}$

$= \cos t - i \sin t$

$= \int_0^{2\pi} (\cos t - i\sin t) \times (i^2 \sin t + i \cos t)\, dt$

$= \int_0^{2\pi} (\cos t - i\sin t)\, i\, (\cos t + i \sin t)\, dt$

$= i \int_0^{2\pi} (\cos t - i\sin t)(\cos t + i\sin t)\, dt$

$= i \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = i \int_0^{2\pi} 1\, dt = i(t)_0^{2\pi} = 2\pi i$ //.

? Integrate z^2 along the line segment from 0 to 2+i

$f(z) = z^2$

C: line segment from 0 to 2+i

i.e. (0,0) to (2,1)

$r(t) = (1-t)(0,0) + t(2,1)$

$= (0,0) + (2t, t) = \langle 2t, t \rangle$

$x = 2t \quad y = t.$

$z(t) = 2t + ti, \quad 0 \le t \le 1$

$dz = 2dt + dt\, i = (2+i)\,dt$

$$\int_C z^2 dz = \int_0^1 (2t+ti)^2 (2+i)\, dt.$$

$$= (2+i)\int_0^1 (2t+ti)^2 dt = (2+i)\int_0^1 t^2 (2+i)^2 dt$$

$$= (2+i)^3 \int_0^1 t^2 dt = (2+i)^3 \left[\frac{t^3}{3}\right]_0^1$$

$$= (2+i)^3 \cdot \frac{1}{3} = \underline{\underline{\frac{(2+i)^3}{3}}}.$$

Q. Evaluate $\int_C e^z dz$ where C is the line segement from 0 to $3+4i$.

$f(z) = e^z$

C: line segement from 0 to $3+4i$

i.e. $(0,0)$ to $(3,4)$.

$r(t) = (1-t)(0,0) + t(3,4)$

$= (3t, 4t)$

$x = 3t, \quad y = 4t. \quad 0 \le t \le 1$

$z(t) = 3t + 4ti = t(3+4i)$

$z'(t) = (3+4i)\,dt.$

$$\int_C e^z dt = \int_0^1 e^{(3+4i)t} (3+4i)\,dt$$

$$= (3+4i)\int_0^1 e^{(3+4i)t}\,dt = \cancel{(3+4i)}\left.\frac{e^{(3+4i)t}}{\cancel{3+4i}}\right|_0^1$$

$$= \underline{\underline{e^{3+4i}-1}}$$

Q. Evaluate $\int_0^{1+i} z^2\,dz$.

$$= \left(\frac{z^3}{3}\right)_0^{1+i} = \frac{(1+i)^3}{3} //.$$

$$\boxed{\begin{array}{l}\sin ix = i\sinh x \\ \cos ix = \cosh x.\end{array}}$$

Q. $\int_{-\pi i}^{\pi i} \cos z\,dz$

$$= \left(\sin z\right)_{-\pi i}^{\pi i} = \sin \pi i - \sin(-\pi i)$$

$$= \sin \pi i + \sin \pi i = 2\sin(\pi i) = \underline{i\,2\sinh \pi.}$$

Q. $\int_{-i}^{i} \frac{1}{z}\,dz$

$$= (\ln z)_{-i}^{i} = \ln(i) - \ln(-i)$$

$$= i\pi/2 - -i\pi/2 = 2i\pi/2 = i\pi //.$$

$$\boxed{\begin{array}{l} e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i \\ \ln e^{i\pi/2} = \ln i \\ \underline{\underline{i\pi/2 = \ln i}} \end{array}}$$

Q.

9/7/19 Tuesday

Cauchy's integral ~~formal~~ formula.

Let $f(z)$ be analytic in a simply connected domain D $z_0 \in D$ then for any simple closed path C in D that encloses z_0 $\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

- eg: $\int_C \frac{e^z}{z-2} dz$ where $|z|=3$.

$f(z) = e^z$ is analytic for all z.

$z_0 = 2$ ~~belo~~ $\in C = |z| = 3$.

$\therefore \int_C \frac{f(z)}{z-2} dz = 2\pi i [e^z]_{z=2} = 2\pi i e^2$.

- Prove that e^z is analytic for all z.

$f(z) = e^z = e^{x+iy} = e^x . e^{iy} . e^x(\cos y + i \sin y)$

$= e^x \cos y + i e^x \sin y$

$\therefore u = e^x \cos y$, $v = e^x \sin y$.

$\frac{\partial u}{\partial x} = e^x \sin y$ $\frac{\partial v}{\partial y} = -e^x \sin y$.

$\frac{\partial v}{\partial x} = e^x \sin y$ $\frac{\partial u}{\partial y} = e^x \cos y$.

The ~~positive~~ partial derivates are continuous for all x, y. The C-R equations namely $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

$\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$ $\therefore f(z)$ is analytic for all z.

$\therefore f(z)$ is an entire function (Analytic for all z).

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos x + ie^x \sin y$

$= e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z$

Thus $\frac{d}{dz}(e^z) = e^z$

- $\int_C \left(\frac{z^3-6}{2z-1}\right) dz$ where $C : |z| = 1$.

$\int_C \frac{\frac{z^3}{2}-3}{z-\frac{1}{2}} dz$; $f(z) = z^3/2 - 3$ is analytic for all z.

$z_0 = i/2 \in C : |z| = 1$.

$= 2\pi i \left(z^3/2 - 3\right)_{i/2} = 2\pi i \left[\frac{(i/2)^3}{2} - 3\right] = 2\pi i \left(\frac{(i/8)}{2} - 3\right)$

$= \pi/8 - 6\pi i$

- $\int_{|z|=2} \frac{e^z}{z-1} dz$.

$f(z) = e^z$ is analytic for all z

$z_0 = 1 \in C : |z| = 2$. $\therefore$ cauchy's integral formula

$= 2\pi i \left(e^z\right)_1 = 2\pi i e^1 = 2\pi i e$

- $\int_C \frac{dz}{z^2-5z}$ where $C : |z| = 3$.

$\int_C \frac{dz}{z^2-5z} = \int_C \frac{dz}{z(z-5)} = \int_C \left(\frac{1/z-5}{z}\right) dz = \int_C \frac{f(z)}{z-0} dz$.

$f(z) = \frac{1}{z-5}$ is analytic in C.

$z_0 = 0 \in C$.

$\therefore$ Cauchy's integral formula

$\int_C \frac{\frac{1}{z-5}}{z} dz = 2\pi i \left[1/z-5\right]_{z=0} = -2/5 \pi i$

17/7/19
Wednesday

Matrice Transformation

Let A be an $m \times n$ matrices & let $T: \mathbb{R}^n \to \mathbb{R}^m$, defined by $T(x) = A(x)$ is called a matrice transformation.

? Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ & defined $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$,

Let $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ & $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

1) Find the image of u under T
2) Find all $x \in \mathbb{R}^2$ where image under T is V.

ans: 1). $T(u) = Au$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} //$$

2) $T(x) = V$

$\Rightarrow Ax = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad T(x) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

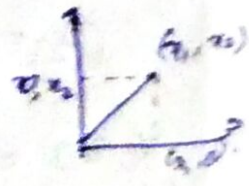
$$\begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad x_1 = 3/2 // \quad x_2 = 2 //$$

$\therefore x = \begin{bmatrix} 3/2 \\ 2 \end{bmatrix}$

⇒ Some special matrices transformation.

1) Projections

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = AX$ is the projection onto x_1 axis.



eg $T(x) = AX = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

|||ly $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects $\mathbb{R}^2$ on to x_2 axis.

eg. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ consider $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x) = AX$

For any $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$,

$$T(X) = AX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

This transformation projects $\mathbb{R}^3$ on to $x_1 x_2$ plane.

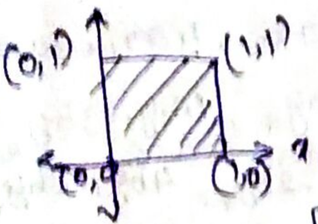
(2) Shears

Let $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ & $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = AX$

Now for $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, $T(x) = AX = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix}$

This matrix transformation represent a shear in the x_1 direction with factor k.

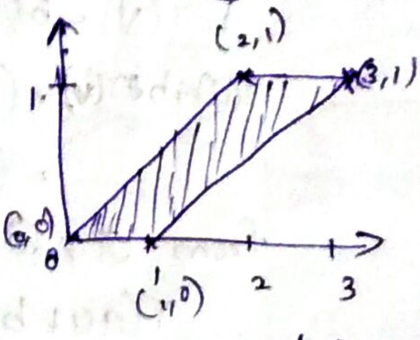
eg: $k = 2$ ~~the~~ take $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ find the image of the unit squire $[0,1] \times [0,1]$.



$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



HW

Q) Find the image of the unit squire with the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Linear transformations.

A transformation or mapping T is called a linear transformation if

(i) $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T.

(ii) $T(cu) = cT(u)$ for any scalar c & vector u in the domain of T.

Note: equivalent to the condition

$T(au+bv) = aT(u) + bT(v)$.

Proof-2

$T(0) = T(0u) = 0\,T(u)$

$= 0$//

2) If T is a linear transformation, then T(0)...
Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_1 - 2x_2, 4x_1 ...)$
Is T a linear transformation? Justify your answer.

Let $u(x_1, x_2)$ & $v = (y_1, y_2)$ & a, b are scalars.

$au + bv = a(x_1, x_2) + b(y_1, y_2) = (ax_1, ax_2) + (by_1, by_2)$

$= (ax_1 + by_1, ax_2 + by_2)$

$T(u) = T(x_1, x_2) = (x_1 - 2x_2, 4x_1 + 3x_2)$

$T(v) = T(y_1, y_2) = (y_1 - 2y_2, 4y_1 + 3y_2)$

$T(au+bv) = T(ax_1 + by_1, ax_2 + by_2)$

$= (ax_1 + by_1 - 2(ax_2 + by_2), 4(ax_1 + by_1) + 3(ax_2 + by_2))$

$= (ax_1 + by_1 - 2ax_2 + 2by_2, 4ax_1 + 4by_1 + 3ax_2 + 3by_2)$ ①

$aT(u) = a(x_1 - 2x_2, 4x_1 + 3x_2) = (ax_1 - 2ax_2, 4ax_1 + 3ax_2)$

$bT(v) = b(y_1 - 2y_2, 4y_1 + 3y_2) = (by_1 - 2by_2, 4by_1 + 3by_2)$

$aT(u) + bT(v) = (ax_1 - 2ax_2, 4ax_1 + 3ax_2) + (by_1 - 2by_2, 4by_1 + 3by_2)$

$= (ax_1 + by_1 - 2ax_2 + 2by_2, 4ax_1 + by_2 + 3ax_2 + 3by_2)$ ②

from ① & ②.

$T(au + bv) = aT(u) + bT(v)$. Hence proved.

$\therefore$ T is a linear transformation.

Q. Verify that $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}$ is linear.

Let $u = (x_1, x_2, x_3)$ $v = (y_1, y_2, y_3)$

$au + bv = (ax_1, ax_2, ax_3) + (by_1, by_2, by_3)$

$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

$T(au+bv) = T(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

$= [(ax_1 + by_1) - (ax_2 + by_2), (ax_1 + by_1)(ax_2 + by_2)]$

$= (ax_1 - ax_2 + by_1 - by_2, ax_1 + ax_2 + by_1 + by_2)$

$= (ax_1 - ax_2, ax_1 + ax_2) + (by_1 - by_2, by_1 + by_2)$

$= (a(x_1-x_2), a(x_1+x_2)) + (b(y_1-y_2), b(y_1+y_2))$

$= a(x_1-x_2, x_1+x_2) + b(y_1-y_2, y_1+y_2) = aT(x_1,x_2,x_3) + bT(y_1,y_2,y_3)$

$= aT(u) + b(T(v))$ $\therefore T$ is linear.

Q Let ~~map~~ $T: \mathbb{R}^3 \to \mathbb{R}$ be defined by $T(x_1,x_2,x_3) = x_1+2x_2+x_3$ ST T is a lineal map.

Q Is $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1+1, x_2)$ is linear. Justify

No $T(0,0) = (0+1, 0) = (1,0) \neq (0,0)$ $\therefore T$ is not linear.

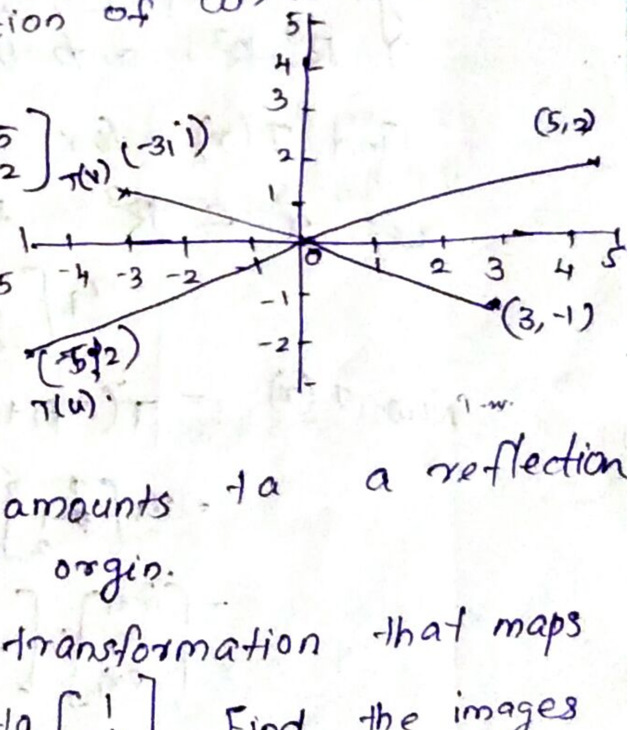
Q Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ Let $T(x) = Ax$ for $x \in \mathbb{R}^2$

a) plot $u, v, T(u)$ & $T(v)$ on a rectangular coordinate system

b) Give a geometric description of what T does to a vector $x \in \mathbb{R}^2$

a) $T(u) = Au = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$

$T(v) = Av = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$



b). This transformation T amounts to a reflection of the vectors about the orgin.

Q Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformation that maps $u = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ & $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ Find the images under T of $2u$, $2u+3v$, $2u-3v$.

$T(2u) = 2T(u) = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$2T(u) + 3(v) = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -12 \end{bmatrix} = \begin{bmatrix} 7 \\ -12 \end{bmatrix}$ //

18/7/19 Thursday

Matrix of a linear transformation.

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by action on the vectors in the columns of $n \times n$ identity matrix is usually denoted by I_n. The j^{th} column of I_n e_j & $e_j = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ where 1 is the j^{th} row

Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then their exist a unique matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$. The associated matrix relative to a standard basis $[e_1, e_2, e_3 \ldots e_n]$ is $A = [T(e_1) \ldots T(e_n)]$

eg:- Let $e_1 = [1, 0, 0]$ $e_2 = [0, 1, 0]$ $e_3 [0\ 0\ 1]$ Given $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(e_1) = [2, -1, 2]$ $T(e_2) = [-2, 0, 1]$ $T(e_3)$

Find $T(x)$ for $x \in \mathbb{R}^3$.

Let $x \in \mathbb{R}^3$ then $x = [x_1, x_2, x_3]$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

Now $T(x) = T(x_1 e_1 + x_2 e_2 + x_3 e_3) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$

$$= x_1 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ 2x_3 \\ 1x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 + x_3 \\ -x_1 + 2x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}$$

Note: The matrix $A = [T(e_1), T(e_2) \ldots T(e_n)]$ is called the <u>standard matrix</u> for the LT of T.

Q. Write the SM for the LT $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(e_1) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $T(e_3) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$= \cancel{x_1 + 2x_2 - x_3}$
$= 4x_1 + 5x_2 -$ $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 5 & 4 \end{bmatrix}$

Q. T be linear transformation $T: R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (3x_1 + 2x_2 - 4x_3,\ x_1 - 5x_2 + 3x_3)$

Find a matrix for T relative to the std Basic.

Ans: The std basic for R^3 $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now $T(e_1) = T[1\ 0\ 0]$ $= \begin{bmatrix} 3\times1 + 2\times0 - 4\times0 \\ 1 - 5\times0 + 3\times0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

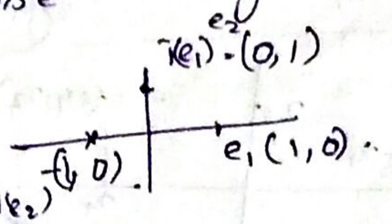
$T(e_2) = T[0\ 1\ 0] = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

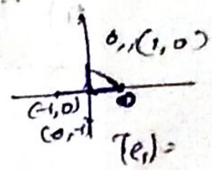
$T(e_3) = T[0\ 0\ 1] = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

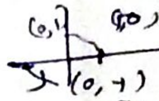
$\therefore$ Matrix for T is $\begin{bmatrix} 3 & 2 & -4 \\ 1 & -5 & 3 \end{bmatrix}$

Q. Let $T: R^2 \to R^2$ be a transformation that rotates each point in R^2, counter clockwise through $\pi/2$ rad. Find matrix T relative to std basics?

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.







Matrix = $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Q. $T: R^3 \to R^3$ be a LT which projects each point (x_1, x_2, x_3) on to the $x_1 - x_2$ plane. Find std matrix of T.

$T(x_1\ x_2\ x_3) = (x_1\ x_2\ 0)$

$e_1 = (1\ 0\ 0)$ $\quad T(e_1) = (1\ 0\ 0)$

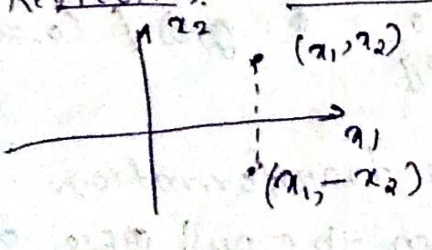
$e_2 = (0\ 1\ 0)$ $\quad T(e_2) = (0\ 1\ 0)$

$e_3 = (0\ 0\ 1)$ $\quad T(e_3) = (0\ 0\ 0)$

matrix of $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Geometric linear transformations of R^2

(1) Reflection about x_1 axis.



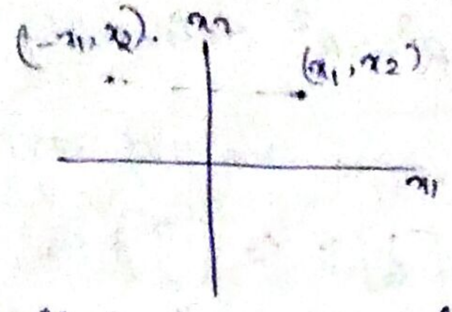
$T(x_1, x_2) = (x_1, -x_2)$

$T(e_1) = T(1,0) = (1,0)$

$T(e_2) = T(0,1) = (0,-1)$

$\therefore$ matrix for $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection about x_2 axis



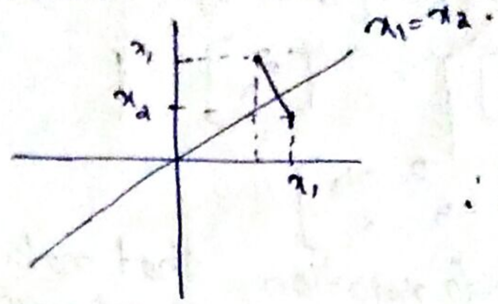
$T(x_1, x_2) = (-x_1, x_2)$

$T(e_2) = T(1,0) = (0, 1)$

$T(e_1) = T(0\,1) = (-1, 0)$

$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$

3) Reflection in the line $x_1 = x_2$.



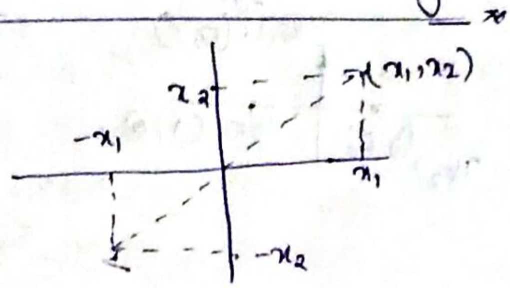
$T(x_1, x_2) = (x_2, x_1)$

$T(e_1) = T(1,0) = (0,1)$

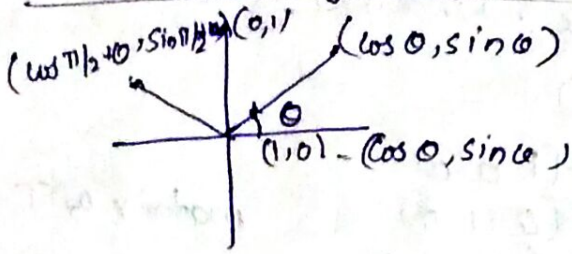
$T(e_2) = T(0,1) = (1,0)$

$\therefore$ matrix for $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

4) Reflection in the Orgin.



3) Rotation through a +ve angle θ.



$T(e_1) = T(1,0) - T(\cos\theta, \sin\theta)$
$= (\cos\theta, \sin\theta)$

$T(e_2) = T(0,1) = T(\cos \pi/2, \sin \pi/2)$
$- \cos(\pi/2 + \theta), \sin(\pi/2 + \theta)$
$= (-\sin\theta, \cos\theta)$

$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Q. Test whether $T: R^2 \to R^3$ defined by $T(x,y) = (x+c, y, 0)$ where c is a non zero constant. Is linear or not?

$T(a\alpha) = \alpha T(a)$ $\quad T(0,0,0) = (x+c, y, 0)$

$\cancel{x+c \neq 0}$ $= (c, 0, 0) \neq (0,0,0)$ not

$\cancel{T \neq}$ //.

a linear.

Q. Define the nullspace of a linear transformation.

Let $T: V \to W$ is a LT. Then the null space of T denoted by $N(T) = \{\alpha \in V ; T(\alpha) = 0\}$.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (3x+4y, 2x-5y)$
Find the matrix of T w.r.t the basis $S = \{(1,2), (2,3)\}$

$T(1,2) = \cancel{3...} (11, -8)$

$T(2,3) = (18, -11)$

$$T = \begin{bmatrix} 11 & 18 \\ -8 & -11 \end{bmatrix}.$$

Q. Use cauchy's integral formula evaluate $\int_C \frac{z^2}{z-2} dz$ where C is the circle $|z|=3$

$f(z)$ analytic with in C & $z=2$ is interior to C

$\therefore$ By cauchy's integration

$$\int \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\therefore \int \frac{z^2}{z-2} dz = 2\pi i \; f(2) = 2\pi i \times 2^2 = \underline{8\pi i}$$