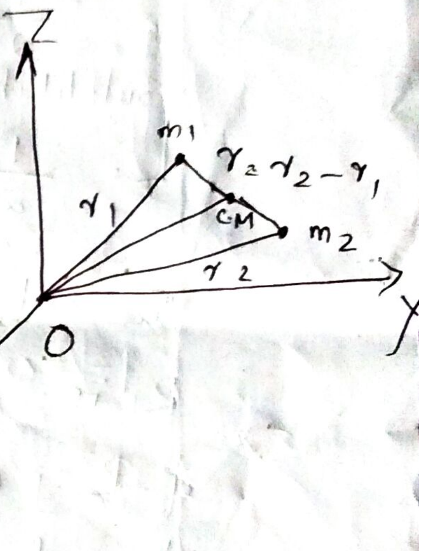
Motion in central Force field

Equivalent One body problem

Consider a system consisting of two bodies of masses m_1 and m_2 having position vectors r_1 and r_2 with respect to the origin O. The number of degrees of freedom is six



Distance of m_2 relative to m_1

$$\vec{r} = r_2 - r_1 \longrightarrow (1)$$

The position vector of centre of mass is defined as

$$\vec{R} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \longrightarrow (2)$$

Using (1) in (2)

$$r_1 = R - \frac{m_2 r}{m_1 + m_2}, \quad r_2 = R + \frac{m_1 r}{m_1 + m_2}$$

$$r_1 = r_2 - r$$

$$R = \frac{m_1(r_2 - r) + m_2 r_2}{m_1 + m_2}$$

$$R = \quad h \qquad = \frac{(m_1 + m_2) r_2}{m_1 + m_2} - \frac{m_1 r}{m_1 + m_2}$$

$$r_1 = R - \frac{m_2 r}{m_1 + m_2} \longrightarrow (3)$$

$$r_2 = R + \frac{m_1 r}{m_1 + m_2} \longrightarrow (4)$$

$$\dot{r}_1 = \dot{R} - \frac{m_2 \dot{r}}{m_1+m_2}$$

$$\dot{r}_2 = \dot{R} + \frac{m_1 \dot{r}}{m_1+m_2}$$

The Lagrangean of system under central force is

$$L = T - V$$

$$= \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - V(r) \longrightarrow (4)$$

Substituting values of $\dot{r}_1$ and $\dot{r}_2$

$$L = \frac{1}{2}(m_1+m_2)\dot{R}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\dot{r}^2 - V(r)$$

The leniar momentum corresponding to R is constant since the three co-ordinates of R are cyclic. Hence $\dot{R}$, the velocity of centre of mass is constant either it is at rest or moving with uniform velocity

∴ Legrange's equation of motion doesnot contain terms of R and $\dot{R}$

∴ Legrangean of a two particle system,

$$L = \frac{1}{2}\mu \dot{r}^2 - V(r) \longrightarrow (5)$$

where μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

The form of Legrangian is exactly

Same as that of a particle with mass μ moving at a vector distance r from centre O which exerts a central force. Thus a two body problem is reduced to equivalent one body problem.

Central force and its general features

If a force acts on a particle in such a way that it is always directed towards or away from a fixed centre and its magnitude depends only upon the distance r from the centre, then this force is called central force.

Central force is represented by

$$F = f(r)\,\hat{r} \qquad \cancel{f(r)} = \frac{F(r)}{}$$

If $f(r) < 0$ it is attractive

$f(r) > 0$ it is repulsive

Central force is always conservative

If $V(r)$ is the potential energy then central force can be represented as

$$F = -\frac{\partial V(r)}{\partial r} \longrightarrow (1)$$

The potential energy depends only on distance r. Hence the system under central force has spherical symmetry.

The angular momentum of a particle undergoing central force motion is a constant.

For a two particle system under central force motion angle of co-ordinate for rotation about fixed axis will have any effect on ~~the~~ motion. ie angle co-ordinate is cyclic resulting in conservation of angular momentum

Angular momentum, $J = \vec{r} \times \vec{p} =$ constant

$$r \cdot J = r \cdot (r \times p) = (r \times r) \cdot p = 0$$

$$p \neq 0 \quad r \times r = 0$$

$$r \cdot J = 0$$

ie $\vec{r}$ and J are perpendicular.

ie motion of particle under central force takes place in a plane

Motion under force obeying inverse square law

Forces obeying inverse square law are effective over large distances. $\therefore$ the distance involved in the interactions are also large

Gravitational and coulomb force between two particles are most important examples of ~~const~~ inverse square law force also central force.

The Gravitational force $f(r) = -\frac{G m_1 m_2}{r^2}$

Coulomb's law

$$f(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

The general force law governing these two forces are $f(r) = -k/r^2 \longrightarrow (1)$

If V is the potential,

$$f(r) = -\frac{\partial V}{\partial r} = -\frac{k}{r^2}$$

$$V = -k/r$$

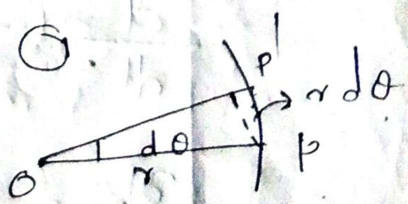
with constant of integration 0 by Assuming $V(r) = 0$ at infinite seperation

Equations of motion under central force and first integrals

Consider a particle of mass m moving about a fixed centre of force,

$$F(r) = f(r)\,\vec{r} \longrightarrow 0.$$

$$f(r)\,\hat{r} = -\frac{\partial V}{\partial r} \longrightarrow (2)$$



Acceleration of particle in terms of planar co-ordinates (r, θ)

$$a = \dot{r}^2 + r^2\dot{\theta}^2 \longrightarrow (3)$$

Legrangean $L = T - V$

$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$

L is independent of θ i.e. $\frac{\partial L}{\partial \theta} = 0$

θ is the cyclic co-ordinate

and corresponding angular momentum is a constant

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$

$\frac{1}{2} m (r^2 \times 2\dot{\theta})$
$m r^2 \dot{\theta} = \frac{\partial L}{\partial \dot{\theta}}$

Thus the angular momentum

$J = m r^2 \dot{\theta} \longrightarrow (4)$

Legrange's eqn for θ co-ordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$m r^2 \ddot{\theta} = 0$

$m r^2 \dot{\theta} = \text{Constant} \longrightarrow (5)$

i.e angular momentum is a constant

Legrange's equation for r co-ordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \longrightarrow (6)$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{\partial}{\partial \dot{r}}\left(\frac{1}{2} m \dot{r}^2\right) = m \dot{r}$$

$$\frac{\partial L}{\partial r} = \frac{d}{dr}\left(\frac{1}{2} m r^2 \dot{\theta}^2\right) - \frac{\partial V}{\partial r}$$

$$= m r \dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$\frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 - \frac{\partial V}{\partial r}\right) = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial r}$$

$$-\frac{\partial V}{\partial r} = f(r)$$

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

$$mr^2\dot{\theta} = J$$

$$\dot{\theta}^2 = \frac{J^2}{m^2 r^4}$$

$$m\ddot{r} - \frac{J^2}{mr^3} - \frac{\partial V}{\partial r} = 0$$

$$mr^2\dot{\theta}^2 = \cancel{mr^2} \times \frac{J^2}{m^{\cancel{2}} r^{\cancel{4}}}$$

$$\cancel{m\ddot{r}} \ \ \frac{\partial}{\partial r}\left(\cancel{\frac{J^2}{2mr^2}}\right)$$

$$m\ddot{r} - \frac{J^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\frac{\partial}{\partial r}\left(\frac{J^2}{mr^2}\right)$$

$$m\ddot{r} = \frac{J^2}{mr^3} + \frac{\partial V}{\partial r}$$

$$-\frac{J^2}{mr^3} = \frac{\partial}{\partial r}\left(\frac{J^2}{2mr^2}\right)$$

$$\frac{J^2}{mr^3} = -\frac{\partial}{\partial r}\left(\frac{J^2}{2mr^2}\right)$$

$$m\ddot{r} = -\frac{\partial}{\partial r}\left\{\frac{J^2}{2mr^2} + V\right\}$$

Xing throughout by $\dot{r} = \frac{\partial r}{\partial t}$

$$m\dot{r}\ddot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)$$

$$m\dot{r}\ddot{r} = -\frac{d}{dt}\left\{\frac{J^2}{2mr^2} + V\right\}$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left\{\frac{J^2}{2mr^2} + V\right\}$$

$$\frac{d}{dt}\left\{\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V\right\} = 0$$

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E = \text{constant}$$

Differential equation for an orbit

$$m\ddot{r} - \frac{J^2}{mr^3} = f(r) \longrightarrow (1)$$

$$\dot{r} = \frac{dr}{dt}, \quad \frac{dr}{d\theta} \times \frac{d\theta}{dt} = \frac{dr}{d\theta} \times \dot{\theta}$$

$$\boxed{\begin{matrix} J = mr^2\dot{\theta} \\ \dot{\theta} = \frac{J}{mr^2} \end{matrix}}$$

$$\dot{r} = \frac{dr}{d\theta} * \frac{J}{mr^2}$$

$$\ddot{r} = \frac{d(\dot{r})}{dt} = \frac{d}{dt}\left\{\frac{dr}{d\theta}\frac{J}{mr^2}\right\}$$

$$= \frac{d}{d\theta}\left\{\frac{J}{mr^2}\frac{dr}{d\theta}\right\}\frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left\{\frac{J}{mr^2}\frac{dr}{d\theta}\right\} \cdot \frac{J}{mr^2}$$

$$\frac{J}{mr^2}\frac{d}{d\theta}\left\{\frac{J}{mr^2} dr/d\theta\right\}.$$

$$\ddot{r} = \frac{J}{mr^2}\frac{d}{d\theta}\left\{\frac{J}{mr^2}\frac{dr}{d\theta}\right\}$$

If $u = 1/r$ $\quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$$\ddot{r} = \frac{J}{m}u^2 \frac{J}{m}\frac{d}{d\theta}\left\{\frac{-1}{r^2}\frac{dr}{d\theta}\right\}$$

$$\ddot{r} = -\frac{J^2}{m^2}u^2\frac{d}{d\theta}\left(\frac{du}{\theta}\right)$$

$$\ddot{r} = -\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

$\therefore$ eqn (1) becomes

$$-\left(\frac{J^2u^2}{m^2}\right) \times m\frac{d^2u}{d\theta^2} - \frac{J^2u^3}{m} = f(1/u)$$

$$\boxed{\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -f(1/u)}$$

This is the equation of orbit of a particle undergoing central force