14/2/19
thursday

2

Vector Integration.

Line integrals.

Let C be a smooth cruve, in the xy plane and let $f(x,y)$ be continous and non-negative on C. Then the line integral of f w.r.t S along C is defined by

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta S_k.$$

Q Evaluate line integral.

If the curve C is represented parametic by th equa

$z = f(x,y)$

$C = x = x(t)$

$y = y(t)$

$a \le t \le b$ then

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Evaluate.

$\int_C (1+xy^2)\,ds$ from $(0,0)$ to $(1,2)$ along the line segement C represent parametic $x = t, y = 2t$, $0 \le t \le 1$.

$$\int_C (1+xy^2)\,ds = \int_0^1 [1+t(2t)^2]\sqrt{1^2+2^2}\,dt.$$

$$= \int_0^1 (1+4t^3)\sqrt{5}\,dt$$

$$= \sqrt{5}\int_0^1 1+4t^3\,dt.$$

$$= \sqrt{5}\left[t+\frac{4t^4}{4}\right]_0^1 = \sqrt{5}\,[1+1] = 2\sqrt{5}$$

* $\int_C (x-y)\,ds$ where C is the curve $x=2t$, $y=3t^2$ $0\le t\le 1$; $\left(\frac{dx}{dt}\right) = 6t$

$$\int_S (x-y)\,ds = \int_0^1 (2t-3t^2)\sqrt{2^2+(6t)^2}\,dt.$$

$\sqrt{4}$ $2.6t$

$$= \int_0^1 (2t-3t^2)\sqrt{4+36t^2}\,dt.$$

$$= \int_0^1 (2t-3t^2)(\sqrt{4}+\sqrt{36t^2})\,dt$$

$$= \int_0^1 2t-3t^2 \cdot 2\sqrt{1+9t^2}\,dt$$

$$= 2\int_0^1 (2t-3t^2)\sqrt{1+9t^2}\,dt$$

$$= -\frac{11}{108}\sqrt{10} - \frac{1}{36}\ln(\sqrt{10}-3) - \frac{4}{27}.$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

* $\int_C (xy+z^3)\,ds$ from $(1,0,0)$ to $(-1,0,\pi)$ along C given by $x = \cos t$, $y = \sin t$ $z = t$, $0 \le t \le \pi$.

$2\sqrt{3}$//.

$$\int_0^\pi (\cos t\sin t + t^3) \sqrt{(-\sin t)^2+(\cos t)^2+1}\,dt$$

$$= \int_0^\pi \cos t\sin t + t^3 \sqrt{1}\,dt \cdot = \int_0^\pi \cos t\sin t + t^3\,dt$$

$$\left[\cos t \cos t + \sin t - \sin t + \frac{t^4}{4}\right]_0^\pi = \left[\cos^2 t - \sin^2 t + \frac{t^4}{4}\right]_0^\pi$$

6/6/19 thursday

Line integral w.r.t x, y & z.

If C is the cruve represted parametrically by $x = x(t)$, $y = y(t)$, $z = z(t)$ $a \le t \le b$.

Then

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\,dt$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\,dt$$

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\,dt.$$

Q. Evaluate $\int_C 2xy\,dx + (x^2+y^2)\,dy$ along the

circular arc C given by ~~exist~~ $x = \cos t, y = \ldots$
$(0 \le t \le \pi/2)$.

$$\int_C 2xy\,dx + (x^2+y^2)\,dy = \int_C 2xy\,dx + \int_C (x^2+y^2)\,dy$$
$$= I_1 + I_2 .$$

$$I_1 = \int_0^{\pi/2} 2\cos t \sin t \,\cancel{dt} \cdot -\sin t \, dt \qquad x'(t) = -\sin t, \; y'(t) = \cos t.$$

$$= -\int_0^{\pi/2} 2\cos t \sin^2 t \, dt$$

Put $\sin t = u$, $\cos t\,dt = du$

$$= -2\int_0^{\pi/2} u^2\,du = -2\left[\frac{u^3}{3}\right]_0^{\pi/2}$$

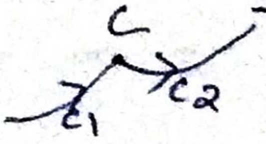
$$= -2\left[\frac{\sin^3 t}{3}\right]_0^{\pi/2} = \frac{-2}{3} //.$$

$$I_2 = \int_0^{\pi/2} (\sin^2 t + \cos^2 t) \cos t \, dt$$

$$= \int_0^{\pi/2} \cos t \, dt = [\sin t]_0^{\pi/2} = 1 //.$$

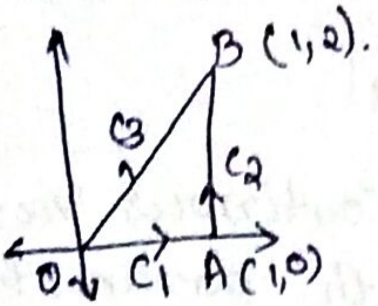
$$I_1 + I_2 = \frac{-2}{3} + 1 = \frac{1}{3} //.$$

Line integral along ~~pices~~ piece-wise smooth curves



$$\int_C f = \int_{C_1} f + \int_{C_2} f$$

Q Evaluate $\oint_C x^2y\,dx + x\,dy$ in a counter clock wise direction as shown in the fig.



Note: The vector eqn of the line segment joining the points $\vec{r}_0$ & $\vec{r}_1$ is $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$; $0 \le t \le 1$

Note:

Let $I = \int_C x^2y\,dx + x\,dy$

$= \int_{C_1} x^2y\,dx + x\,dy + \int_{C_2} x^2y\,dx + x\,dy + \int_{C_3} x^2y\,dx + x\,dy$

$= I_1 + I_2 + I_3$

C_1 = Line segment join (0,0) & (1,0)

$r(t) = (1-t)\langle 0,0\rangle + t\langle 1,0\rangle = \langle t,0\rangle$ $0 \le t \le 1$.

$x = t,\ y = 0,\ 0 \le t \le 1.$

$I_1 = \int_{C_1} x^2y\,dx + \int_{C_1} x\,dy = \int_0^1 t^2 \times 0 \times dt + \int_0^1 t\,0\,dt$

$= 0$

$(1-t)\langle 1,0\rangle + t\langle 1,2\rangle$

$I_2 = \int_{C_2} x^2y\,dx + x\,dy = \int_0^1 2t \times 0\,dt + 1 \times 2dt$

$\langle 1-t, 0\rangle + \langle t, 2t\rangle = \langle 1, 2t\rangle$

$x = 1, y = 2t$

$= \int_0^1 2\,dt = 2$

$(1-t)\langle 1,2\rangle + t\langle 0,0\rangle$

$\langle 1-t, 2-2t\rangle$

$I_3 =$

$(-3/2)$.

$I_1 + I_2 + I_3 = 1/2$ //.

7/6/19 Friday.

Work as a Line Integral.

If F is a continous vector field $ C is a smooth parametric curve in 2 space or 3 space with unit tangent vector $\vec{T}$, then the work performed by the vector field on a particle that moves along C in the direction of increasing parameter is

$$W = \int_C \vec{F} \cdot \vec{T} \, ds. \qquad \text{Since, } \vec{T} = \frac{d\vec{r}}{ds}.$$

$$= \int_C F \cdot \frac{d\vec{r}}{\cancel{ds}} \cancel{ds} = \int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} \cancel{+ dz\,\hat{k}}$$

$$\vec{F} \cdot d\vec{r} = f(x,y)\,dx + g(x,y)\,dy$$

$$W = \int_C f(x,y)\,dx + g(x,y)\,dy.$$

Q. Find the workdone by the force due to $\vec{F}(x,y) = x^3 y\,\hat{i} + (x-y)\,\hat{j}$ on a particle that moves along the parabola $y = x^2$ from $(-2,4)$ to $(1,1)$

The parabola $y = x^2$ is parametrized as by taking $x = t$

$y = t^2, \ -2 \le t \le 1$

$y = f(x)$ Put $x = t$, $y = f(t)$

$\vec{F} = x^3y\,\hat{i} + (x-y)\,\hat{j}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$\vec{F}\cdot d\vec{r} = x^3y\,dx + (x-y)\,dy$

$W = \int_C \vec{F}\cdot d\vec{r} = \int_C x^3y\,dx + (x-y)\,dy$

$= \int_{-2}^{1} t^3 \cdot t^2\,dt + \int_{-2}^{1} (t-t^2)\,\cancel{dy}\,2t\,dt$

$= \int_{-2}^{1} t^5\,dt + 2\int_{-2}^{1} t^2 - t^3\,dt$

$= \left[\frac{t^6}{6}\right]_{-2}^{1} + 2\left[\frac{t^3}{3} - \frac{t^4}{4}\right]_{-2}^{1}$

$= \frac{1}{6} - \frac{64}{6} + 2\left[\left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{-8}{3} - \frac{16}{4}\right)\right]$

$= -\frac{63}{6} + 2\left[\frac{1}{12} - \left(\frac{-80}{12}\right)\right]$ $\quad -32 - 48$

$= -\frac{63}{6} + \frac{1}{6} + \frac{80}{6} = \cancel{\frac{-62+1+27}{6}}$ $\quad 48$ $\quad = \frac{18}{6}$

$= \cancel{\frac{-1302+288}{144} = \frac{-1014}{144} = \frac{-507}{72} = \frac{169}{24}}$

$= 3$ //.

20/1/1 In each part evaluate the integral $\int_{\gamma} (3x+2y)dx + (2x-y)\,dy$ along the stated curves

a) The line segment from (0,0) to (1,1)

b). The parabolic arc $y=x^2$ from (0,0) to (1,1)

c) The curve $x=y^3$ from (0,0) to (1,1)

Q Let C be the curve represented by the eq^n s. $x=2t$, $y=3t^2$, $0\le t\le 1$.

In each part evaluate the line integral along C.

a). $\int_C (x-y)\,ds$

b) $\int_C (x-y)\,dx$

c) $\int_C (x-y)\,dy$

2 Find the workdone by the force-field $\vec{F}$ on a particle that moves along the curve C where $\vec{F}(x,y) = xy\hat{i} + x^2\hat{j}$ & $C: x=y^2$ from (0,0) to (1,1).

Green's theorem

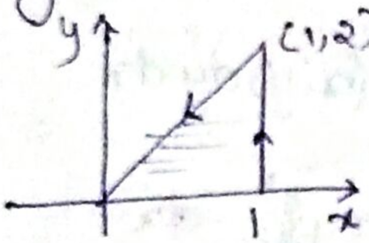
Let R be a simply connected plane region whose boundary is a single simple, closed, piece wise, smooth curve

C ~~an~~ oriented counter clockwise. If $f(x,y)$ & $g(x,y)$ are continous & have continous first partial derivaties on some openset containing R, then

Workdone

$$\boxed{\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.}$$

Q use Green's theroem evaluate $\int x^2y\,dx + x\,dy$ along the triangular path shown in the fig.



$\begin{vmatrix} 0 \le y \le 2x \\ 0 \le x \le 1 \end{vmatrix}$

$f(x,y) = x^2y$

$g(x,y) = x$.

$\frac{\partial g}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = x^2$

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

$$= \iint_R (1 - x^2)\,dA.$$

$$= \int_0^1 \int_0^{2x} (1 - x^2)\,dy\,dx$$

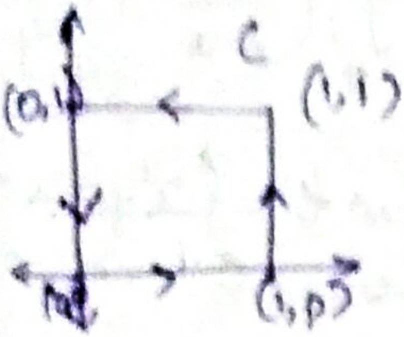
$$= \int_0^1 \left(y - x^2y\right)_0^{2x} dx = \int_0^1 2x - 2x^3\,dx = \cancel{\left(2 - 2x\right]^1}$$

$$= \left[\frac{2x^2}{2} - \frac{2x^4}{4}\right]_0^1 = \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} = \frac{1}{2}$$

2

$$\boxed{\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA}$$

Q) use Green's theorem to evaluate $\oint_C y^2 dx + x^2 dy$ where C is the square with vertices (0,0) (1,0), (1,1) & (0,1) oriented counter clockwise.



$f(x,y) = y^2$

$g(x,y) = x^2$

$\frac{\partial f}{\partial y} = 2y \quad \frac{\partial g}{\partial x} = 2x$.

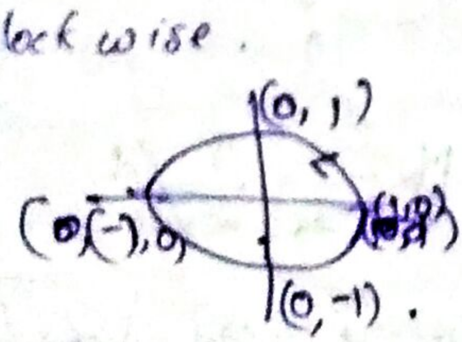
$$\int_0^1\int_0^1 (2x-2y)\,dy\,dx = 2\int_0^1\int_0^1 (x-y)\,dy\,dx$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^1 dx$$

$$= 2\int_0^1 \left[x - \frac{1}{2}\right] dx = 2\left[\frac{x^2}{2} - \frac{1}{2}x\right]_0^1$$

$$= \left[x^2 - x\right]_0^1 = 1 - 1 = 0$$

Q) Use green's theorem to evalute $\int y\,dx + x\,dy$ where C is the unit circle oriented counter clockwise.



$f(x,y) = y \quad g(x,y) = x$

$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = 1$.

contd

$$\oint_C \iint_R (1-1)\, dA = \iint_R 0\, dA = 0$$

$x = \cos t$, $y \sin t$ $\quad 0 \le t \le 2\pi$

$$= \int_0^{2\pi} \sin t \times -\sin t\, dt + \cos t . \cos t\, dt$$

$$= \int_0^{2\pi} -\sin^2 t\, dt + \cos^2 t\, dt = \int_0^{2\pi} \cos^2 t\, dt - \sin^2 t\, dt$$

$$= \int_0^{2\pi} (\cos^2 t - \sin^2 t)\, dt = \int_0^{2\pi} \cos 2t\, dt = \left[\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}(\sin 2t)_0^{2\pi} = \frac{1}{2}(\sin 4\pi - \sin 0) = \frac{1}{2}(0-0) = 0$$

Q. Find the Workdone by the force field $\vec{F}(x,y) = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the anticlockwise.

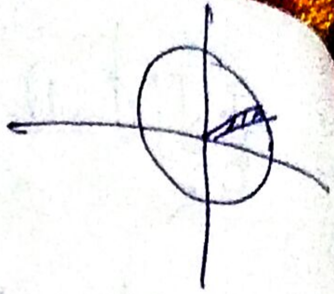
$$W = \oint_C (e^x - y^3)\, dx + (\cos y + x^3)\, dy$$

$f(x,y) = e^x - y^3 \quad g(x,y) = \cos y + x^3$

$$\frac{\partial f}{\partial y} = -3y^2 \quad \frac{\partial g}{\partial x} = 3x^2$$

$$= \int_{-1}^{1}\int_{-1}^{1} (-3y^2 - 3x^2)\, dx\, dy = \int_{-1}^{1} \left[-3y^2 x - \frac{3x^3}{3}\right]_{-1}^{1} dy$$

$\cancel{-3y^2 - 3x^2}$



$= \iint_R 3x^2 + 3y^2 \, dA$

$x = r\cos t$
$y = r\sin t$
$0 \le t \le 2\pi$

$= 3\iint_R (x^2 + y^2) \, dA$

$= 3 \int_0^{2\pi} \int_0^1 r^2 . r \, dr \, d\theta .$

$= 3 \int_0^{2\pi} \left(\frac{r^4}{4} \right)_0^1 d\theta = \frac{3}{4} \int_0^{2\pi} d\theta$

$= \frac{3}{4} \times [\theta]_0^{2\pi} = \frac{3}{4} \times 2\pi = \frac{3\pi}{2}$ //.

25/6/19 (Tuesday) 8

If $f(x,y,z)$ is continous on σ, then

$$\iint_\sigma f(x,y,z) \, ds = \iint_R f(x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA .$$

Q. Evaluate the surface integral $\iint_\sigma x^2 ds$ over the sphere $x^2+y^2+z^2=1$

The sphere $x^2+y^2+z^2=1$ can be represented using spherical coordinate system as

$$\vec{r}(\rho, \phi, \theta) = \sin\phi\cos\theta \, \hat{i} + \sin\phi \sin\theta \, \hat{j} + \cos\phi \, \hat{k}$$

$0 \le \phi \le \pi, 0 \le \theta \le 2\pi$

$$\frac{\partial \vec{r}}{\partial \phi} = \cos\phi \cos\theta\, \hat{i} + \cos\phi \sin\theta\, \hat{j} - \sin\phi\, \hat{k}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -\sin\phi \sin\theta\, \hat{i} + \sin\phi \cos\theta\, \hat{j} + 0\, \hat{k}$$

$$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta\sin\phi & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \hat{i}\left[\sin^2\phi \cos\theta\right] - \hat{j}\left(-\sin\theta \sin^2\phi\right) + \hat{k}\left(\sin\phi \cos\phi \cos^2\theta + \sin\phi \cos\phi \sin^2\theta\right)$$

$$= \hat{i}(\sin^2\phi \cos\theta) + \hat{j}(\sin^2\phi \sin\theta)\, \hat{k}(\sin\phi \cos\phi).$$

$$\left\|\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}\right\| = \sqrt{\sin^4\phi \cos^2\theta + \sin^4\phi \sin^2\theta + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^4\phi\,(\cos^2\theta + \sin^2\theta) + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^4\phi + \sin^2\phi \cos^2\phi}$$

$$= \sqrt{\sin^2\phi\left[\sin^2\phi + \cos^2\phi\right]} = \sqrt{\sin^2\phi} = \underline{\underline{\sin\phi}}$$

$$\iint_\sigma x^2\, dS = \iint_R (\sin\phi \cos\theta)^2 \sin\phi\, dA.$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi \cos^2\theta\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{3\sin\phi - \sin 3\phi}{4}\right) \cos^2\theta\, d\phi\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left[-3\cos\phi + \frac{\cos 3\phi}{3}\right]_0^{\pi} \cos^2\theta \, d\theta.$$

$$= 1/4\int_0^{2\pi}\left[\left(3-\frac{1}{3}\right) - \left(-3+1/3\right)\right] \cos^2\theta \, d\theta.$$

$$= \frac{1}{4}\int_0^{\pi}(6 - 2/3)\cos^2\theta \, d\theta$$

$$= \frac{1}{4}\times\frac{16}{3}\int_0^{2\pi}\cos^2\theta \, d\theta$$

$$= \frac{4}{3}\int_0^{2\pi} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{4}{3}\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{2\pi} = \frac{2}{3}\times 2\pi = 4\pi/3$$

Surface integrals over $z = g(x,y)$, $y = g(x,z)$ & $x = g(y,z)$.

Therem

A). Let σ be a surface with eqⁿ $z = g(x,y)$ & let R be its projection on the x-y plane. If g has continous first partial derivatives on R & $f(x,y,z)$ is continous on σ then

$$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x, y, g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

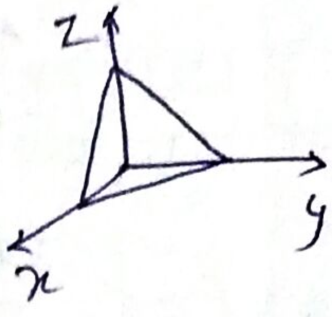
$|||^{ly}$

b). $$\iint_\sigma f(x,y,z)\,ds = \iint_R f(x, g(x,z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

$\cancel{c)\ \iint_\sigma f(x,y,z)\,dS = \iint_R f(x(y,z),y,z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2+1}\,dA}$

c) $\iint_\sigma f(x,y,z)\,dS = \iint_R f(x(y,z),y,z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2+1}\,dA$.

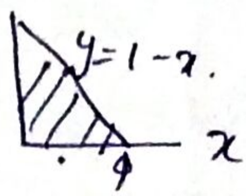
Q Evaluate the surface integral $\iint_\sigma xz\,dS$ where σ is the part of the plane $x+y+z=1$ that lies in the first ~~opposited~~ octant.



$\sigma = x+y+z=1$

$z = 1-x-y$

$\frac{\partial z}{\partial x} = -1 \quad \frac{\partial z}{\partial y} = -1$



$\iint_\sigma xz\,dS = \iint_R x(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\,dA$

$R: 0 \le x \le 1$

$0 \le y \le 1-x$

$= \int_0^1 \int_0^{1-x} x(1-x-y)\sqrt{3}\,dy\,dx$

$= \int_0^1 \int_0^{1-x} (x-x^2-xy)\sqrt{3}\,dy\,dz = \sqrt{3}\int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_0^{1-x} dx$

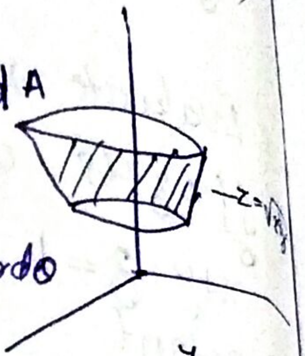
$= \sqrt{3}\int_0^1 x-x^2-x^2+x^3 - \ldots 1-x)$ $\quad \frac{1}{12}\times\frac{1}{12}.$

$= \sqrt{3}/24$

26/6/19 Wednesday

Q. Evaluate the surface Integral $\iint_\sigma y^2z^2\,ds$, where σ is the part of the $z=\sqrt{x^2+y^2}$ that lies b/w the phase $z=1$ \$ $z=2$.

$$\iint y^2z^2\,ds = \iint_R y^2(x^2+y^2)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$



$$=\sqrt{2}\int_0^{2\pi}\int_1^2 r^2\sin^2\theta(r^2\cos^2\theta+r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\int_1^2 r^2\sin^2\theta\; r^2(\cos^2\theta+\sin^2\theta)\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\int_1^2 r^5\sin^2\theta\,dr\,d\theta.$$

Polar form
$x=r\cos\theta$
$y=r\sin\theta$
$1\le r\le 2$
$0\le\theta\le 2\pi$
$dA=r\,dr\,d\theta$

$$=\sqrt{2}\int_0^{2\pi}\left[\frac{r^6}{6}\right]_1^2 d\sin^2\theta\,d\theta$$

$$=\sqrt{2}\times\frac{1}{6}\int_0^{2\pi}(64-1)\sin^2\theta\,d\theta$$

$\frac{21}{4}$

$$=\sqrt{2}\,\frac{63}{6}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{63}{6}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta$$

$$=\sqrt{2}\,\frac{21}{4}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$2\cdot\sqrt{2}\cdot 2$

$$=\sqrt{2}\,\frac{21}{4}\times 2\pi = \frac{\sqrt{2}\cdot 21}{\sqrt{2}\cdot\sqrt{2}\cdot 2}\times 2\pi = \frac{21\pi}{\sqrt{2}}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1} = \sqrt{2}$$

Evaluate the flux integral

Flux (ϕ).

The net volume of fluid that passes through a surface (σ) per unit of time is called flux.

$$\text{Flux } \phi = \iint_{\sigma} \vec{F}.\vec{n}\, ds = \iint_{R} \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) dA$$

(projection)

where σ is given by the vector eq^n $\vec{r} = \vec{r}(u,v)$ & $\vec{n}$ is the normal to σ (+ve orientation outward).

Q. Find the flux of the vector field $\vec{F}(x,y,z) = z.\hat{k}$ across the outward oriented sphere. $x^2+y^2+z^2 = a^2$.

vector form

$$r(\rho, \phi, \theta) = a\sin\phi\cos\theta\hat{i} + a\sin\phi\sin\theta\hat{j} + a\cos\phi$$

$$0 \le \phi \le \pi$$

$$0 \le \theta \le 2\pi$$

28/9/19

Q. $\vec{r}(\phi\ \theta) = a\sin\phi\cos\theta\ \hat{i} + a\sin\phi\sin\theta\ \hat{j} + a\cos\phi\ \hat{k}$

$0 \le \phi \le \pi$

$0 \le \phi\ 2\pi.$

$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = a^2\sin^2\phi\cos\theta\,\hat{i} + a^2\sin^2\theta\sin\theta\,\hat{j} + a^2\sin\phi\cos\phi\,\hat{k}$

$\vec{F} \cdot \left[\frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta}\right] = a^2\sin\phi\cos^2\phi$

is Flux $\phi = \iint_{\sigma} \vec{F}\cdot\vec{r}\,ds.$

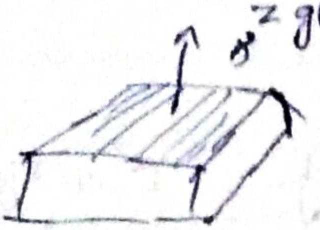
$= \iint_{\sigma} \vec{F}\cdot n\,ds = \iint_{R} \vec{F}\cdot\left(\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}\right) ds$

$= \int_0^{2\pi}\int_0^{\pi} a^3\sin\theta\cos^2\phi\,d\phi\,d\theta$

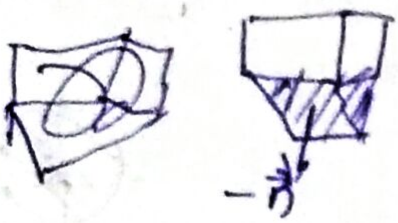
$= a^3 \int_0^{2\pi} \left[\frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta$

$= \frac{2a^3}{3}\int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}$ //

Orientation of non-parmatenic surface.



normal $\vec{n} = \dfrac{-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$

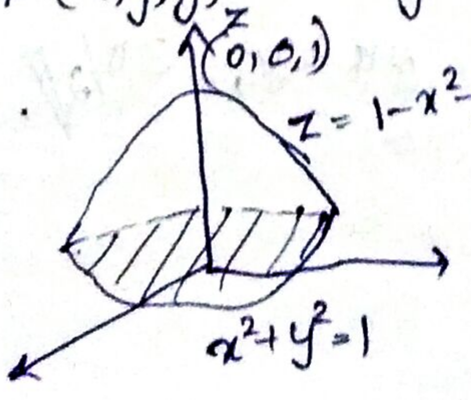


$$-\vec{n} = \frac{\frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} - k}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

$$\iint_\sigma \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\left(-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}\right) dA.$$
(oriented up).

$$\iint_\sigma \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\left(\frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} - \hat{k}\right) dA$$
(oriented down).

Q Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy plane, & suppose that σ is oriented up, as shown in the figure. Find the flux of the vector field $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$ across σ.



Flux $\phi = \iint \vec{F}\cdot n\, ds \cdot \quad = \iint_R \vec{F}\cdot\left(-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}\right) dA$

$z = 1 - x^2 - y^2$

$\frac{\partial z}{\partial x} = -2x \quad , \frac{\partial z}{\partial y} = -2y \quad . \quad \vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 2x\hat{i} + 2y\hat{j} + \hat{k}$

$\iint F\cdot\left(-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y} + k\right).$

$= (x\hat{i} + y\hat{j} + z\hat{k})\cdot(2x\hat{i} + 2y\hat{j} + \hat{k})$

$= 2x^2 + 2y^2 + z.$

Flux $\Phi = \iint_R (2x^2 + 2y^2 + z)\, dA$

polar form

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$\iint_R (2x^2 + 2y^2 + 1 - x^2 - y^2)\, dA.$

$= \int_0^{2\pi}\int_0^1 (r^2 + 1)\, r\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 r^3 + r\, dr\, d\theta$

$= \int_0^{2\pi}\left(\frac{r^4}{4} + r^2\right)_0^1 d\theta = \int_0^{2\pi}\left(1/4 + 1/2\right) d\theta = 3/4\int_0^{2\pi} d\theta$

$3/4\,[\theta]_0^{2\pi} = 2\pi \times 3/4 = 3\pi/2$ //.

a.

Divergence theorem

Let G be a soild whoes surface σ is oriented out ward. If

$$\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$$

where f, g & h have continous first partial derivative on same openset conting of G, & if $\hat{n}$ is the outward unit normal on σ, then $\iint_\sigma \vec{F}\cdot\hat{n}\, ds = \iiint_G \text{div}\, \vec{F}\, dv$.

$\boxed{V = \iiint_G 1 \cdot dv}$

Q Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = z\hat{k}$ across the sphere $x^2+y^2+z^2=a^2$.

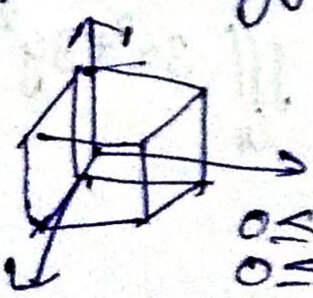
$$\text{div}\, F = \vec{\nabla}\cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot zk$$

$$= \frac{\partial z}{\partial z} = 1$$

$\phi = \iint_\sigma \vec{F}\cdot\hat{n}\, ds$ by divergence theorem

$\iiint_G \text{div}\vec{F}\, dv = \iiint_G 1 \cdot dv =$ Volume of $G = \frac{4}{3}\pi a^3$ //.

Q Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = 2x\hat{i} + 3y\hat{j} + z^2\hat{k}$ across the unit cube:



$0 \le z \le 1$

$0 \le x \le 1$

$0 \le y \le 1$

$$\operatorname{div} F = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}2x + \frac{\partial}{\partial y}3y + \frac{\partial}{\partial z}z^2$$

$$= 2+3+2z = \underline{5+2z.}$$

by divergence

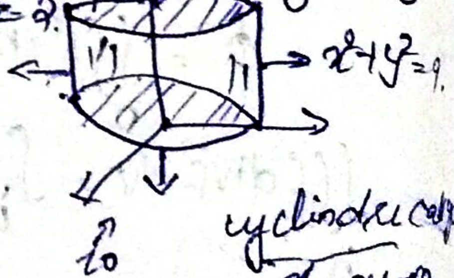
$$\iiint_G \operatorname{div} F = \int_0^1\int_0^1\int_0^1 5+2z\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left[5z + z^2\right]_0^1 dy\, dz.$$

$$= \int_0^1\int_0^1 (5+1)\, dy\, dz - 6\int_0^1 \left[y\right]_0^1 dx.$$

$$= 6\int_0^1 [x]_0^1 = 6 //.$$

Q) Use divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = x^3\hat{i} + y^3\hat{j} + z^2\hat{k}$ across the surface of the region that is enclosed by the cylinder $x^2+y^2=9$ & the planes $z=0$, & $z=2$.



$$\operatorname{div} F = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^2$$

$$= 3x^2 + 3y^2 + 2z.$$

cylindrical coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$.

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$0 \le z \le 2$.

$$\Phi = \iint_\sigma \vec{F}\cdot\hat{n}\, dS = \iiint_G \operatorname{div} F\, dv$$

$$= \iiint_G 3x^2 + 3y^2 + 2z\, dv$$

$$= \int_0^2\int_0^{2\pi}\int_0^3 ((3r^2+2z)r\,dr\,d\theta\,dz.$$

$3r^2\cos^2\theta + 3r^2\sin^2\theta + 2z$

$3r^2(\cos^2+\sin^2\theta)$
$+2z$

$$= \int_0^2\int_0^{2\pi}\int_0^3 (3r^3+2zr)\,dr\,d\theta\,dz$$

$3\times3\times3\times3$

$\frac{81}{4}$

243

$$= \int_0^2\int_0^{2\pi}\left[\frac{3r^4}{4}+2z\frac{r^2}{2}\right]_0^3 d\theta\,dz =$$

$$\frac{243}{4}+\left[\int_0^2\int_0^{2\pi} 9z\,d\theta\,dz\right] = \frac{243}{4}\int_0^2\left[9z\theta\right]_0^{2\pi} dz.$$

$$= \int_0^{2\pi}\int_0^3\int_0^2 (3r^3+2zr)\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3\left[3r^3z+2r\frac{z^2}{2}\right)_0^2 dr\,d\theta = \int_0^{2\pi}\int_0^3 (6r^3+4r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{6r^4}{4}+\frac{4r^2}{2}\right]_0^3 d\theta = \int_0^{2\pi}\left(\frac{3r^4}{2}+2r^2\right)_0^3 d\theta$$

$$= \int_0^{2\pi}\left(\frac{3}{2}\times 81+2\times 9\right)d\theta = \left(\frac{243}{2}+18\right)\int_0^{2\pi} d\theta$$

$$= \left(\frac{243}{2}+\frac{18}{1}\right)\times 2\pi = \frac{279}{2}\times 2\pi = \underline{\underline{279\pi}}$$