Kindergarten model: Approximating Time Delays in Evolutionary Games

Supplementary Material

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Model description

We consider a large, but finite, population is considered. Individuals in the population take part in games and receive payoff depending on their own strategy and the strategy of their co-player. Two strategies are present in the population: C - cooperation and D - defection. The payoffs are represented by the following matrix, where the strategy of the focal player is represented rows and of the co-player by the columns:

```
In[1]:= matrix = {{R, S}, {T, P}} // MatrixForm
```

Out[1]//MatrixForm=

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix}$$

The average payoff of a cooperator and a defector is defined as:

```
In[2]:= U_C[x_] := R x + S (1 - x);
        U_D[x_] := T x + P (1 - x);
```

The following quantities are of interest:

- $p(t)$ - size of the population at time t
- $p_C(t)$ - number of Cooperators at time t
- $x(t) = \dfrac{p_C(t)}{p(t)}$ - fraction of Cooperators in the population at time t

Now, we introduce a delay between the creation of an offspring, as an effect of the game played by the parent, and it's addition in the adult population. Notably, the delays depend on the strategy on the parent and are represented as

- τ_C - delay (maturation time) experienced by cooperators
- τ_D - delay (maturation time) experienced by defectors

In the Kindergarten model time delays are represented by the inverse of rates at which an offspring matures . During maturation, juveniles are placed in a strategy-dependent kindergarten. The sizes of kindergartens are represented as

- $k_C(t)$ - size of the cooperator kindergarten at time t
- $k_D(t)$ - size of the defectors kindergarten at time t

In the model, we track the size of the kindergartens relative to the population size:

- $y_C(t) = \dfrac{k_C(t)}{p(t)}$ - relative size of the cooperator kindergarten at time t
- $y_D(t) = \dfrac{k_D(t)}{p(t)}$ - relative size of the defector kindergarten at time t

The change in the fraction of cooperators in the adult population and the relative sizes of kindergarten is defined by the following system of ODEs, as derived in the manuscript:

In[4]:=

$$\texttt{dx[x\_, yc\_, yd\_] :=} \frac{yc\ (1-x)}{\tau_C} - \frac{yd\ x}{\tau_D};$$

$$\texttt{dy}_C\texttt{[x\_, yc\_, yd\_] :=}\ yc \left(\frac{\tau_C - 1}{\tau_C} - \frac{yc}{\tau_C} - \frac{yd}{\tau_D}\right) + x\, U_C[x];$$

$$\texttt{dy}_D\texttt{[x\_, yc\_, yd\_] :=}\ yd \left(\frac{\tau_D - 1}{\tau_D} - \frac{yc}{\tau_C} - \frac{yd}{\tau_D}\right) + (1-x)\, U_D[x];$$

In the case of only one delay present, the system takes the following form, for $\tau_C = 0$:

In[225]:=

$$\texttt{dx}_{\tau_C 0}\texttt{[x\_, yc\_, yd\_] :=}\ x \left(U_C[x]\ (1-x) - \frac{yd}{\tau_D}\right);$$

$$\texttt{dy}_{C,\tau_C 0}\texttt{[x\_, yc\_, yd\_] := 0;}$$

$$\texttt{dy}_{D,\tau_C 0}\texttt{[x\_, yc\_, yd\_] :=}\ yd \left(\frac{\tau_D - 1}{\tau_D} - x\, U_C[x] - \frac{yd}{\tau_D}\right) + (1-x)\, U_D[x];$$

for $\tau_D = 0$:

In[727]:=

$$\texttt{dx}_{\tau_D 0}\texttt{[x\_, yc\_, yd\_] :=}\ \frac{yc}{\tau_C} - x \left(U_D[x]\ (1-x) + \frac{yc}{\tau_C}\right);$$

$$\texttt{dy}_{C,\tau_D 0}\texttt{[x\_, yc\_, yd\_] :=}\ yc \left(\frac{\tau_C - 1}{\tau_C} - (1-x)\, U_D[x] - \frac{yc}{\tau_C}\right) + x\, U_C[x];$$

$$\texttt{dy}_{D,\tau_D 0}\texttt{[x\_, yc\_, yd\_] := 0;}$$

Change in the size of population is described by the following function: $\frac{\mathrm{dp}(t)}{\mathrm{dt}} = p(t)\left(-1 + \frac{\mathrm{yc}}{\tau_C} + \frac{\mathrm{yd}}{\tau_D}\right)$.

Then, for the population to grow we need the following to hold true: $\frac{\mathrm{yc}}{\tau_C} + \frac{\mathrm{yd}}{\tau_D} > 1$

In the following, the analysis of the behaviour of this systems in different games is presented.

Results

Stag Hunt game

The Stag Hunt game is characterised by the following payoff matrix:

In[57]:=

```
R = a;
S = 0;
T = b;
P = b;
matrixSH = {{R, S}, {T, P}} // MatrixForm
```

Out[61]//MatrixForm=

$$\begin{pmatrix} a & 0 \\ b & b \end{pmatrix}$$

where $\boxed{a>b>0}$

In this game, the system characterizing the Kindergarten model becomes:

In[62]:=
```
FullSimplify[dx[x, y_C, y_D]]
FullSimplify[dy_C[x, y_C, y_D]]
FullSimplify[dy_D[x, y_C, y_D]]
```

Out[62]= $-\frac{(-1+x)\ y_C}{\tau_C}-\frac{x\ y_D}{\tau_D}$

Out[63]= $a\ x^2+y_C\left(1-\frac{1+y_C}{\tau_C}-\frac{y_D}{\tau_D}\right)$

Out[64]= $b-b\ x+y_D\left(1-\frac{y_C}{\tau_C}-\frac{1+y_D}{\tau_D}\right)$

Homogenous stationary states

First, we analyse the full defection stationary state. We know, that the fraction of cooperators is equal to 0. Then, we determine the relative sizes of kindergartens in the stationary states

In[65]:=
```
x_0 = 0;
y_C,0 = y_C /. Solve[dx[x_0, y_C, y_D] == 0 && dy_C[x_0, y_C, y_D] == 0, y_C][[1]]
```

Out[66]= 0

With no cooperators present in the population, the cooperator kindergarten is also empty.

In[67]:=
```
y_D,0 = y_D /. Solve[dy_D[x_0, y_C,0, y_D] == 0 && τ_C > 0 && τ_D > 0 && a > b > 0 && y_D > 0, y_D]
```

Out[67]= $\left\{\frac{1}{2}\ (-1+\tau_D)+\frac{1}{2}\ \sqrt{1-2\ \tau_D+4\ b\ \tau_D+\tau_D^2}\ \ \text{if}\ \ \tau_C>0\ \&\&\ \tau_D>0\ \&\&\ b>0\ \&\&\ a>b\right\}$

Full defection stationary state is of the following form: $e_0=\{0,0,\frac{1}{2}\ (-1+\tau_D)+\frac{1}{2}\ \sqrt{1-2\ \tau_D+4\ b\ \tau_D+\tau_D^2}\}$

Next, we check for which parameter values the population grows exponentially and does not go extinct in e_0

In[68]:=
```
Reduce[Normal[y_C,0/τ_C + y_D,0/τ_D] > 1 && τ_C > 0 && τ_D > 0 && a > b > 0]
```

Out[68]= $\tau_C>0\ \&\&\ \tau_D>0\ \&\&\ b>1\ \&\&\ a>b$

In the full defection stationary state the population grows when $b>1$.

Next, we investigate the full cooperation ($x_1=1$) stationary state:

In[69]:=
```
x_1 = 1;
y_D,1 = y_D /. Solve[dx[x_1, y_C, y_D] == 0 && dy_D[x_1, y_C, y_D] == 0, y_D][[1]]
```

Out[70]= 0

With no defectors present in the population, the defector kindergarten is also empty.

In[71]:=
```
y_C,1 = y_C /. Solve[dy_C[x_1, y_C, y_D,1] == 0 && τ_C > 0 && τ_D > 0 && a > b > 1 && y_C > 0, y_C]
```

Out[71]= $\left\{\frac{1}{2}\ (-1+\tau_C)+\frac{1}{2}\ \sqrt{1-2\ \tau_C+4\ a\ \tau_C+\tau_C^2}\ \ \text{if}\ \ \tau_D>0\ \&\&\ \tau_C>0\ \&\&\ a>1\ \&\&\ 1<b<a\right\}$

Full cooperation equilibrium is of the following form: $e_1=\{1,\frac{1}{2}\ (-1+\tau_C)+\frac{1}{2}\ \sqrt{1-2\ \tau_C+4\ a\ \tau_C+\tau_C^2},0\}$

The population grow exponentially in this stationary state if:

In[72]:= $\text{Reduce}\left[\text{Normal}\left[\frac{y_{C,1}}{\tau_C}+\frac{y_{D,1}}{\tau_D}\right]>1\;\&\&\;\tau_C>0\;\&\&\;\tau_D>0\;\&\&\;a>b>1\right]$

Out[72]= $\tau_D>0\;\&\&\;\tau_C>0\;\&\&\;a>1\;\&\&\;1<b<a$

In the full cooperation stationary state the population grows if $a>1$.

Heterogenous stationary states

Next, we investigate the existence of internal stationary states. First, we find the value of y_C depending on x and y_D

In[73]:= $\text{Simplify}[\text{Solve}[dx[x, y_C, y_D] == 0, y_C]]$

Out[73]= $\left\{\left\{y_C\to\frac{x\,y_D\,\tau_C}{\tau_D-x\,\tau_D}\right\}\right\}$

In[74]:= $y_{C,2}[x\_, yd\_] := \frac{x\,yd\,\tau_C}{\tau_D-x\,\tau_D};$

Next, we determine the value of $y_D$depending on x

In[75]:= $\text{Simplify}\left[\text{Solve}\left[dy_C\left[x, y_{C,2}[x, y_D], y_D\right] == 0\;\&\&\;\tau_C>0\;\&\&\;\tau_D>0\;\&\&\;a>b>1\;\&\&\;0<x<1\;\&\&\;y_D>0, y_D\right]\right]$

Out[75]= $\left\{\left\{y_D\to\begin{cases}-\frac{(-1+x)\left(-1+\tau_C+\sqrt{1+(-2+4\,a\,x)\,\tau_C+\tau_C^2}\right)\tau_D}{2\,\tau_C} & \text{if } \tau_C>0\;\&\&\;a>1\;\&\&\;1<b<a\;\&\&\;0<x<1\;\&\&\;\tau_D>0\end{cases}\right\}\right\}$

In[76]:= $y_{D,2}[x\_] := -\frac{(-1+x)\left(-1+\tau_C+\sqrt{1+(-2+4\,a\,x)\,\tau_C+\tau_C^2}\right)\tau_D}{2\,\tau_C};$

Then, $y_C$becomes:

In[77]:= $y_{C,2,x}[x\_] := y_{C,2}\left[x, y_{D,2}[x]\right]$
$y_{C,2,x}[x]$

Out[78]= $-\frac{(-1+x)\,x\left(-1+\tau_C+\sqrt{1+(-2+4\,a\,x)\,\tau_C+\tau_C^2}\right)\tau_D}{2\,(\tau_D-x\,\tau_D)}$

Finally, we find the possible values of x

In[79]:= $\text{solutions} = \text{Simplify}\left[\text{Solve}\left[\text{FullSimplify}\left[dy_D\left[x, y_{C,2,x}[x], y_{D,2}[x]\right]\right] == 0\;\&\&\;\tau_C>0\;\&\&\;\tau_D>0\;\&\&\;a>b>1\;\&\&\;0<x<1, x, \text{Reals}\right], \tau_C>0\;\&\&\;\tau_D>0\;\&\&\;a>b>0\right]$

Out[79]= $\left\{\left\{x\to\begin{cases}\frac{\tau_C-\tau_D-\tau_C\,\tau_D+2\,b\,\tau_C\,\tau_D+\tau_D^2+(-\tau_C+\tau_D)\sqrt{1+(-2+4\,b)\,\tau_D+\tau_D^2}}{2\,a\,\tau_D^2} & \text{if condition}\end{cases}\right\},\left\{x\to\begin{cases}\frac{\tau_C-\tau_D-\tau_C\,\tau_D+2\,b\,\tau_C\,\tau_D+\tau_D^2+(-\tau_C+\tau_D)\sqrt{1+(-2+4\,b)\,\tau_D+\tau_D^2}}{2\,a\,\tau_D^2} & \text{if } b>1\;\&\&\;\tau_D>\tau_C\end{cases}\right\}\right\}$

In[80]:= $x_2 = \frac{\tau_C-\tau_D-\tau_C\,\tau_D+2\,b\,\tau_C\,\tau_D+\tau_D^2+(-\tau_C+\tau_D)\sqrt{1+(-2+4\,b)\,\tau_D+\tau_D^2}}{2\,a\,\tau_D^2};$

We see that there are two possible internal equilibria. However, both are characterized with the same functional form of x a coexist. Hence, at any point in the parameter space at most one internal equilibrium can exist with $x_2 = \frac{\tau_C-\tau_D-\tau_C\,\tau_D+2\,b\,\tau_C\,\tau_D+\tau_D^2+(-\tau_C+\tau_D)\,\sqrt{1+(-2+4\,b)\,\tau_D+\tau_D^2}}{2\,a\,\tau_D^2}$

The possible internal equilibrium takes the following form: $e_2 = \left\{x_2,\ y_D,\ \frac{x_2\,y_D\,\tau_C}{\tau_D-x_2\,\tau_D}\right\}$ where $x_2 = \frac{\tau_C-\tau_D-\tau_C\,\tau_D+2\,b\,\tau_C\,\tau_D+\tau_D^2+(-\tau_C+\tau_D)\,\sqrt{1+(-2+4\,b)\,\tau_D+\tau_D^2}}{2\,a\,\tau_D^2}$ and $y_D = -\frac{(-1+x_2)\left(-1+\tau_C+\sqrt{1+(-2+4\,a\,x_2)\,\tau_C+\tau_C^2}\right)\tau_D}{2\,\tau_C}$.

Lastly, we check when does the population grow in this stationary state and is not threatened by extinction

In[81]:= $\text{Reduce}\left[\text{Normal}\left[\frac{y_{C,2,x}[x]}{\tau_C}+\frac{y_{D,2}[x]}{\tau_D}\right]>1\ \&\&\ \tau_C>0\ \&\&\ \tau_D>0\ \&\&\ a>b>1\right]$

Out[81]= $b>1\ \&\&\ a>b\ \&\&\ x>\frac{1}{a}\ \&\&\ \tau_C>0\ \&\&\ \tau_D>0$

In theinternal stationary state the population grows if $x_2 > 1/a$.

Stability analysis

Full defection stationary state

We perform the stability analysis of full defection stationary state e_0.

First, we determine the eigenvalues of the system of ODEs:

In[ ]:=
```
system = {dx[x, yC, yD], dyC[x, yC, yD] , dyD[x, yC, yD]};
J = D[system, {{x, yC, yD}}] ;
J // MatrixForm
Jstar = J /. {x → Normal[x0], yD → Normal[yD,0][[1]], yC → Normal[yC,0]};
Jstar // MatrixForm
eigens = Eigenvalues[Jstar];
eigens // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{y_C}{\tau_C}-\frac{y_D}{\tau_D} & \frac{1-x}{\tau_C} & -\frac{x}{\tau_D} \\ 2\,a\,x & -\frac{2\,y_C}{\tau_C}+\frac{-1+\tau_C}{\tau_C}-\frac{y_D}{\tau_D} & -\frac{y_C}{\tau_D} \\ -b\,(1-x)-b\,x & -\frac{y_D}{\tau_C} & -\frac{y_C}{\tau_C}-\frac{2\,y_D}{\tau_D}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}}{\tau_D} & \frac{1}{\tau_C} & 0 \\ 0 & \frac{-1+\tau_C}{\tau_C}-\frac{\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}}{\tau_D} & 0 \\ -b & -\frac{\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}}{\tau_C} & \frac{-1+\tau_D}{\tau_D}-\frac{2\left(\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}\right)}{\tau_D} \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}}{\tau_D} \\ -\frac{-1+\tau_D+\sqrt{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}}{2\,\tau_D} \\ \frac{\tau_C-2\,\tau_D+\tau_C\,\tau_D-\tau_C\,\sqrt{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}}{2\,\tau_C\,\tau_D} \end{pmatrix}$$

Next, we determine when the stationary state is stable

In[ ]:= $\text{Reduce}\left[\text{eigens}[[1]]<0\ \&\&\ \text{eigens}[[2]]<0\ \&\&\ \text{eigens}[[3]]<0\ \&\&\ \tau_C>0\ \&\&\ \tau_D>0\ \&\&\ \tau_C\neq\tau_D\ \&\&\ a>b>1\right]$

Out[ ]= $\tau_C>0\ \&\&\ \left(\left(0<\tau_D<\tau_C\ \&\&\ b>1\ \&\&\ a>b\right)\ ||\ \left(\tau_D>\tau_C\ \&\&\ b>1\ \&\&\ a>b\right)\right)$

Full defection stationary state is always stable.

Full cooperation stationary state

We perform the stability analysis of full cooperation stationary state e_1.

First, we determine the eigenvalues of the system of ODEs:

In[ ]:=
```
system = {dx[x, yC, yD], dyC[x, yC, yD] , dyD[x, yC, yD]};
J = D[system, {{x, yC, yD}}] ;
J // MatrixForm
Jstar = J /. {x → Normal[x1], yD → Normal[yD,1], yC → Normal[yC,1][[1]]};
Jstar // MatrixForm
eigens = Eigenvalues[Jstar];
eigens // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{y_C}{\tau_C}-\frac{y_D}{\tau_D} & \frac{1-x}{\tau_C} & -\frac{x}{\tau_D} \\ 2\,a\,x & -\frac{2\,y_C}{\tau_C}+\frac{-1+\tau_C}{\tau_C}-\frac{y_D}{\tau_D} & -\frac{y_C}{\tau_D} \\ -b\,(1-x)-b\,x & -\frac{y_D}{\tau_C} & -\frac{y_C}{\tau_C}-\frac{2\,y_D}{\tau_D}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2}}{\tau_C} & 0 & -\frac{1}{\tau_D} \\ 2\,a & \frac{-1+\tau_C}{\tau_C}-\frac{2\left(\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2}\right)}{\tau_C} & -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2}}{\tau_D} \\ -b & 0 & -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2}}{\tau_C}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2}}{\tau_C} \\ \frac{-\tau_C+\tau_D-\sqrt{1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2}\,\tau_D-\sqrt{\tau_C^2-2\,\tau_C^2\,\tau_D+4\,b\,\tau_C^2\,\tau_D-\tau_D^2+2\,\tau_C\,\tau_D^2-4\,a\,\tau_C\,\tau_D^2+\left(1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2\right)\tau_D^2}}{2\,\tau_C\,\tau_D} \\ \frac{-\tau_C+\tau_D-\sqrt{1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2}\,\tau_D+\sqrt{\tau_C^2-2\,\tau_C^2\,\tau_D+4\,b\,\tau_C^2\,\tau_D-\tau_D^2+2\,\tau_C\,\tau_D^2-4\,a\,\tau_C\,\tau_D^2+\left(1-2\,\tau_C+4\,a\,\tau_C+\tau_C^2\right)\tau_D^2}}{2\,\tau_C\,\tau_D} \end{pmatrix}$$

Next, we determine when the stationary state is stable

In[ ]:=
```
Reduce[eigens[[1]] < 0 && eigens[[2]] < 0 && eigens[[3]] < 0 && τC > 0 && τD > 0 &&
  τC ≠ τD && a > b > 1, τC]
```

Out[ ]=

$$\tau_D > 0\ \&\&\ b > 1\ \&\&\ a > b\ \&\&\ \left(0 < \tau_C < \tau_D \,||\, \tau_D < \tau_C < \frac{a-b-a\,\tau_D-b\,\tau_D+2\,a\,b\,\tau_D}{2\,(-1+b)\,b}+\frac{1}{2}\sqrt{\left(\frac{1}{(-1+b)^2\,b^2}\left(a^2-2\,a\,b+b^2-2\,a^2\,\tau_D+4\,a\,b\,\tau_D+4\,a^2\,b\,\tau_D-2\,b^2\,\tau_D-8\,a\,b^2\,\tau_D+4\,b^3\,\tau_D+a^2\,\tau_D^2-2\,a\,b\,\tau_D^2+b^2\,\tau_D^2\right)\right)}\right)$$

In[ ]:=
$$m = \frac{a - b - a\,\tau_D - b\,\tau_D + 2\,a\,b\,\tau_D}{2\,(-1+b)\,b} + \frac{1}{2}\sqrt{\left(\frac{1}{(-1+b)^2\,b^2}\left(a^2 - 2\,a\,b + b^2 - 2\,a^2\,\tau_D + 4\,a\,b\,\tau_D + 4\,a^2\,b\,\tau_D - 2\,b^2\,\tau_D - 8\,a\,b^2\,\tau_D + 4\,b^3\,\tau_D + a^2\,\tau_D^2 - 2\,a\,b\,\tau_D^2 + b^2\,\tau_D^2\right)\right)}$$

Out[ ]=
$$\frac{a - b - a\,\tau_D - b\,\tau_D + 2\,a\,b\,\tau_D}{2\,(-1+b)\,b} + \frac{1}{2}\sqrt{\left(\frac{1}{(-1+b)^2\,b^2}\left(a^2 - 2\,a\,b + b^2 - 2\,a^2\,\tau_D + 4\,a\,b\,\tau_D + 4\,a^2\,b\,\tau_D - 2\,b^2\,\tau_D - 8\,a\,b^2\,\tau_D + 4\,b^3\,\tau_D + a^2\,\tau_D^2 - 2\,a\,b\,\tau_D^2 + b^2\,\tau_D^2\right)\right)}$$

Full cooperation stationary state is stable when $\tau_C < m$, where

$$m = \frac{a-b-a\,\tau_D-b\,\tau_D+2\,a\,b\,\tau_D}{2\,(-1+b)\,b} + \frac{1}{2}\sqrt{\frac{a^2-2\,a\,b+b^2-2\,a^2\,\tau_D+4\,a\,b\,\tau_D+4\,a^2\,b\,\tau_D-2\,b^2\,\tau_D-8\,a\,b^2\,\tau_D+4\,b^3\,\tau_D+a^2\,\tau_D^2-2\,a\,b\,\tau_D^2+b^2\,\tau_D^2}{(-1+b)^2\,b^2}}$$

The effect of delays on the internal stationary state

Now, we analyse the effect of each of the delays on the value of x in the internal stationary state e_2.

First, we check the effect of cooperator delay:

In[ ]:=
```
Reduce[D[x2, τC] > 0 && τC > 0 && τD > 0 && a > b > 1]
```

Out[ ]= $\tau_C > 0\ \&\&\ \tau_D > 0\ \&\&\ b > 1\ \&\&\ a > b$

The value of x_2 always increases with τ_C. We check what happens in the limit of $\tau_C \to \infty$

In[ ]:=
```
Simplify[Limit[x2, τC → Infinity], a > b > 1 && τD > 0]
```

Out[ ]= ∞

In the limit of $\tau_C \to \infty$ the value of x in e_2 goes to infinity. As we are only interested in $x \in (0, 1)$ we heck when $x = 1$, that is, when the two stationary states collide.

In[ ]:=
```
Reduce[x2 == 1 && τC > 0 && τD > 0 && τC ≠ τD && a > b > 1, τC]
```

Out[ ]= $a > 1\ \&\&\ 1 < b < a\ \&\&\ \tau_D > 0\ \&\&$
$$\tau_C == \frac{a - b - a\,\tau_D - b\,\tau_D + 2\,a\,b\,\tau_D}{2\,(-1+b)\,b} + \frac{1}{2}\sqrt{\left(\frac{1}{(-1+b)^2\,b^2}\left(a^2 - 2\,a\,b + b^2 - 2\,a^2\,\tau_D + 4\,a\,b\,\tau_D + 4\,a^2\,b\,\tau_D - 2\,b^2\,\tau_D - 8\,a\,b^2\,\tau_D + 4\,b^3\,\tau_D + a^2\,\tau_D^2 - 2\,a\,b\,\tau_D^2 + b^2\,\tau_D^2\right)\right)}$$

In[ ]:=

$$\text{Reduce}\Big[\frac{a-b-a\,\tau_D-b\,\tau_D+2\,a\,b\,\tau_D}{2\,(-1+b)\,b}+\frac{1}{2}\sqrt{\left(\frac{1}{(-1+b)^2\,b^2}\left(a^2-2\,a\,b+b^2-2\,a^2\,\tau_D+4\,a\,b\,\tau_D+4\,a^2\,b\,\tau_D-2\,b^2\,\tau_D-8\,a\,b^2\,\tau_D+4\,b^3\,\tau_D+a^2\,\tau_D^2-2\,a\,b\,\tau_D^2+b^2\,\tau_D^2\right)\right)} ==\frac{a-b-a\,\tau_D-b\,\tau_D+2\,a\,b\,\tau_D}{2\,(-1+b)\,b}+\frac{1}{2}\sqrt{\left(\frac{1}{(-1+b)^2\,b^2}\left(a^2-2\,a\,b+b^2-2\,a^2\,\tau_D+4\,a\,b\,\tau_D+4\,a^2\,b\,\tau_D-2\,b^2\,\tau_D-8\,a\,b^2\,\tau_D+4\,b^3\,\tau_D+a^2\,\tau_D^2-2\,a\,b\,\tau_D^2+b^2\,\tau_D^2\right)\right)}\Big]$$

Out[ ]= True

At the point $\tau_C = m$ the internal stationary state reaches the full cooperation and disappears. At the same time, the full cooperation stationary state changes stability and becomes unstable.
Next, we investigate the effects of defector delay:

```
In[ ]:= Reduce[D[x2, τD] > 0 && τC > 0 && τD > 0 && a > b > 1]
Out[ ]= False
```

```
In[ ]:= Reduce[D[x2, τD] < 0 && τC > 0 && τD > 0 && a > b > 1]
Out[ ]= τD > 0 && b > 1 && τC > 0 && a > b
```

The value of x_2 always decreases with τ_2.

Now, we check what happens in the limit of $\tau_D \to \infty$

```
In[ ]:= Simplify[Limit[x2, τD → Infinity], a > b > 1 && τD > 0]
```

Out[ ]= $\frac{1}{a}$

With the value of defector delay growing the value of x in e_2 approaches $1/a$. This ensures that the population always exists in the internal stationary state and confirms that full defector stationary state is always stable (never collides with e_2).

One delay present

No cooperator delay (τ_C=0)

First, we determine the values of the possible internal stationary state by solving the system of ODEs.

First, we notice that the cooperator kindergarten is empty, hence we have:

```
In[ ]:= yC,τC0 = 0;
```

Next, we obtain the value of y_D

```
In[ ]:= yD,τC0[x_] := yD /. Solve[dxτC0[x, yC,τC0, yD] == 0, yD][[1]]
```

Then, we calculate x

In[ ]:=
```
x_{τC0} =
 x /.
  Solve[dy_{D,τC 0}[x, y_{C,τC 0}, y_{D,τC0}[x]] == 0 && τC > 0 && τD > 0 && a > b > 1 && τD ≠ τC && 0 < x < 1,
    x][[1]]
```

Out[ ]= $\frac{-1+\tau_D}{2\,a\,\tau_D}+\frac{1}{2}\sqrt{\frac{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}{a^2\,\tau_D^2}}$ if $a>1\ \&\&\ 1<b<a\ \&\&\ \tau_D>0\ \&\&\ \tau_C>0$

and y_D takes the form:

In[ ]:=
```
Simplify[y_{D,τC0}[Normal[x_{τC0}]]]
```

Out[ ]= $$-\frac{\left(-1+\tau_D\left(1+a\sqrt{\frac{1+(-2+4\,b)\,\tau_D+\tau_D^2}{a^2\,\tau_D^2}}\right)\right)\left(-1+\tau_D\left(1+a\left(-2+\sqrt{\frac{1+(-2+4\,b)\,\tau_D+\tau_D^2}{a^2\,\tau_D^2}}\right)\right)\right)}{4\,a\,\tau_D}$$

If it exists the internal stationary state takes the following form:

$$\tilde{e}_2=\left(\frac{-1+\tau_D}{2\,a\,\tau_D}+\frac{1}{2}\sqrt{\frac{1-2\,\tau_D+4\,b\,\tau_D+\tau_D^2}{a^2\,\tau_D^2}},\ 0,\ -\frac{\left(-1+\tau_D\left(1+a\sqrt{\frac{1+(-2+4\,b)\,\tau_D+\tau_D^2}{a^2\,\tau_D^2}}\right)\right)\left(-1+\tau_D\left(1+a\left(-2+\sqrt{\frac{1+(-2+4\,b)\,\tau_D+\tau_D^2}{a^2\,\tau_D^2}}\right)\right)\right)}{4\,a\,\tau_D}\right)$$

Now we show, that in the limit of $\tau_C \to 0$ the internal stationary state e_2 goes to $\tilde{e}_2$.

In[ ]:=
```
Reduce[Limit[x_2, τC → 0] == x_{τC0} && τC > 0 && τD > 0 && a > b > 1 && τD ≠ τC]
```

Out[ ]= $\tau_D>0\ \&\&\ b>1\ \&\&\ a>b\ \&\&\ (0<\tau_C<\tau_D\ ||\ \tau_C>\tau_D)$

In[ ]:=
```
Reduce[Limit[y_{C,2,x}[x_2], τC → 0] == y_{C,τC 0} && τC > 0 && τD > 0 && a > b > 1 && τD ≠ τC]
```

Out[ ]= $a>1\ \&\&\ 1<b<a\ \&\&\ \tau_C>0\ \&\&\ (0<\tau_D<\tau_C\ ||\ \tau_D>\tau_C)$

In[ ]:=
```
Reduce[Limit[y_{D,2}[x_2], τC → 0] == y_{D,τC0}[x_{τC0}] && τC > 0 && τD > 0 && a > b > 1 && τD ≠ τC]
```

Out[ ]= $\tau_D>0\ \&\&\ b>1\ \&\&\ a>b\ \&\&\ (0<\tau_C<\tau_D\ ||\ \tau_C>\tau_D)$

Furthermore, in the limit of $\tau_D \to 0$ $\tilde{e}_2$ goes to the solution of the Stag hunt game with no delays:

In[ ]:=
```
Limit[x_{τC0}, τD → 0, Direction → "FromAbove"]
```

Out[ ]= $\frac{b}{a}$ if $a>1\ \&\&\ 1<b<a\ \&\&\ \tau_C>0$

In[ ]:=
```
Limit[y_{C,τC 0}, τD → 0]
```

Out[ ]= 0

In[ ]:=
```
Limit[y_{D,τC0}[x], τD → 0]
```

Out[ ]= 0

No defector delay (τ_D=0)

First, we determine the values of the possible internal stationary state by solving the system of ODEs.

First, we notice that the defector kindergarten is empty, hence we have:

```
In[ ]:= y_{D,τ_D 0} = 0;
```

Next, we obtain the value of y_C

```
In[ ]:= y_{C,τ_{D0}}[x_] := y_C /. Solve[dx_{τ_D 0}[x, y_C, y_{D,τ_D 0}] == 0, y_C][[1]]
```

Then, we calculate x

```
In[ ]:= x_{τ_{D0}} =
         x /.
          Solve[dy_{C,τ_D 0}[x, y_{C,τ_{D0}}[x], y_{D,τ_D 0}] == 0 && τ_C > 0 && τ_D > 0 && a > b > 1 && τ_D ≠ τ_C && 0 < x < 1,
            x][[1]]
```

Out[ ]= $\frac{b - b\,\tau_C + b^2\,\tau_C}{a}$ if condition (Head: And, Byte count: 976, Uniconize)

and y_C takes the form:

```
In[ ]:= Simplify[y_{C,τ_{D0}}[Normal[x_{τ_{D0}}]]]
```

Out[ ]= $\frac{b^2\,\tau_C\,\left(1 + \left(-1 + b\right)\,\tau_C\right)}{a}$

If it exists the internal stationary state takes the following form:

$$\overline{e}_2 = \left(\frac{b-b\,\tau_C+b^2\,\tau_C}{a}, \frac{b^2\,\tau_C\,(1+(-1+b)\,\tau_C)}{a}, 0\right)$$

Now we show, that in the limit of $\tau_D \to 0$ the internal stationary state e_2 goes to $\overline{e}_2$.

```
In[ ]:= Reduce[Limit[Normal[x_2], τ_D → 0] == Normal[x_{τ_{D0}}] && τ_C > 0 && a > b > 1]
```

Out[ ]= $\tau_C > 0$ && $a > 1$ && $1 < b < a$

```
In[ ]:= Reduce[Limit[y_{C,2,x}[x_2], τ_D → 0] == y_{C,τ_{D0}}[Normal[x_{τ_{D0}}]] && τ_C > 0 && a > b > 1]
```

Out[ ]= $a > 1$ && $1 < b < a$ && $\tau_C > 0$

```
In[ ]:= Reduce[Limit[y_{D,2}[x_2], τ_D → 0] == y_{D,τ_D 0} && τ_C > 0 && a > b > 1]
```

Out[ ]= $\tau_C > 0$ && $a > 1$ && $1 < b < a$

Furthermore, in the limit of $\tau_C \to 0$ $\overline{e}_2$ goes to the solution of the Stag hunt game with no delays:

```
In[ ]:= Limit[x_{τ_{D0}}, τ_C → 0, Direction → "FromAbove"]
```

Out[ ]= $\frac{b}{a}$ if $a > 1$ && $1 < b < a$ && $\tau_D > 0$

```
In[ ]:= Limit[y_{C,τ_{D0}}[x], τ_C → 0]
```

Out[ ]= 0

In[ ]:=
```
Limit[y_{D,τ_D 0}, τ_C → 0]
```
Out[ ]=
```
0
```

Example

Next, we present an example of the Stag Hunt game and show the analysis of the Kindergarten model in that game.

We will analyse the following game:

In[ ]:=
```
a = 5;
b = 3;
matrix
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 5 & 0 \\ 3 & 3 \end{pmatrix}$$

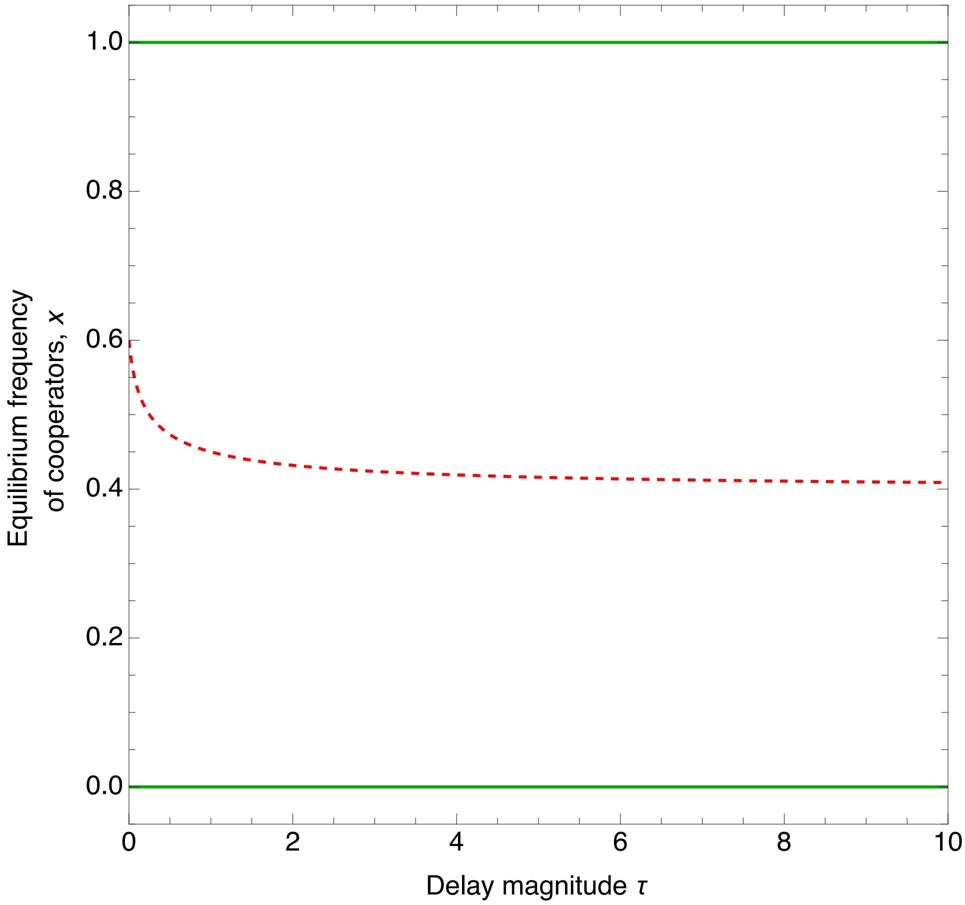
We will plot the value of x_2 for different values of delays

In[ ]:=
```
xstar = Plot[b / a, {tc, 0, 10}, PlotRange → {{-0.9, 10}, {-0.1, 1}},
   PlotStyle → {Darker[Gray, 1.2], Dashed, Thickness[0.002]}];
Plotx[tc_, td_, linestyle_] :=
 Plot[If[1 > x2 /. {τC → tc, τD → td}, {x2 /. {τC → tc, τD → td}}], {τ, 0, 10},
  PlotRange → {{0, 10}, {-0.05, 1.05}}, PlotStyle → {{Red, linestyle}},
  Ticks → {Automatic, {0, 0.2, 0.4, 0.8, 1}}, Frame → True,
  FrameLabel → {"Delay magnitude τ",
    "Equilibrium frequency \n of cooperators, x" },
  LabelStyle → {14, Black, FontFamily → "Helvetica"}, AspectRatio → 1,
  ImageSize → 500]
Plot0[tc_, td_] := Plot[{0}, {τ, 0, 10}, PlotStyle → Darker[Green]]
Plot1st[tc_, td_] := Plot[If[tc < m /. {τC → tc, τD → td}, 1], {τ, 0, 10},
  PlotStyle → Darker[Green]]
Plot1unst[tc_, td_, linestyle_] :=
 Plot[If[tc ≥ m /. {τC → tc, τD → td}, 1], {τ, 0, 10}, PlotStyle → {Red, linestyle}]
```

$\tau_C = \tau, \tau_D = 2\ \tau$

In[ ]:=
```
panela = Show[Plotx[τ, 2 τ, Dashed], Plot0[τ, 2 τ], Plot1st[τ, 2 τ],
  Plot1unst[τ, 2 τ,  Dashed]]
```

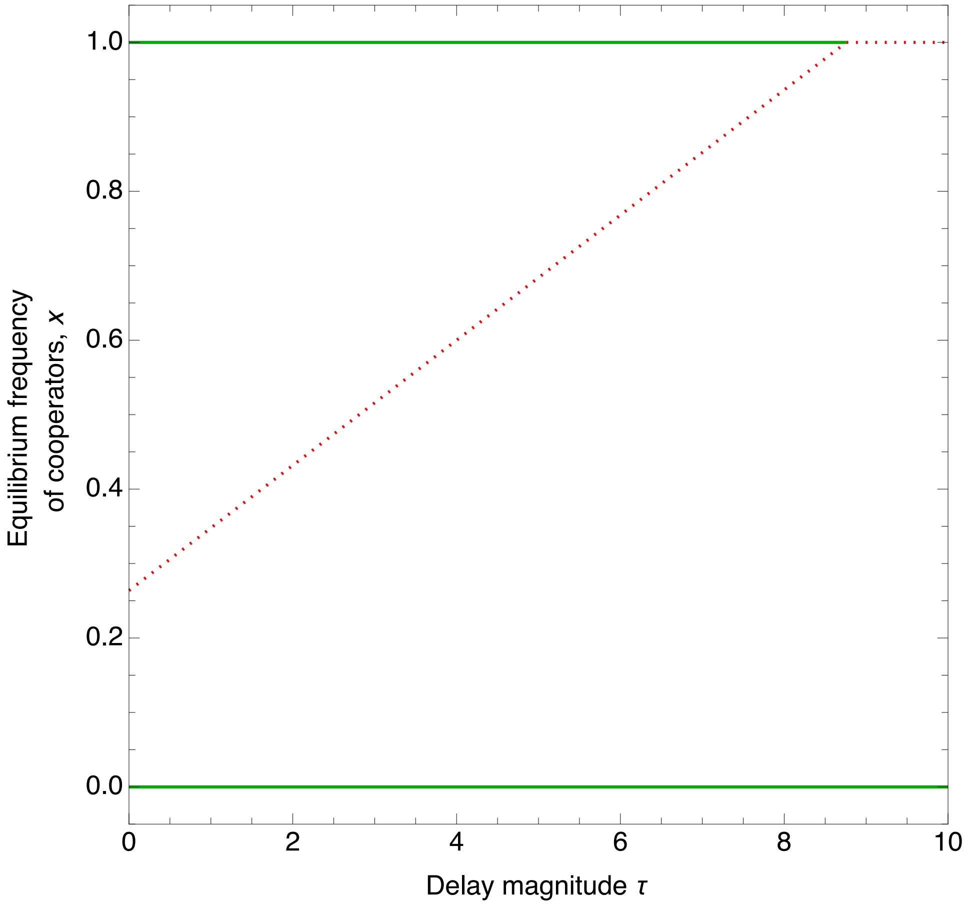
Out[ ]=



$\tau_C = \tau, \tau_D = 4$

In[ ]:=
```
panelb = Show[Plotx[τ, 4, Dotted], Plot0[τ, 4], Plot1st[τ, 4],
  Plot1unst[τ, 4, Dotted]]
```

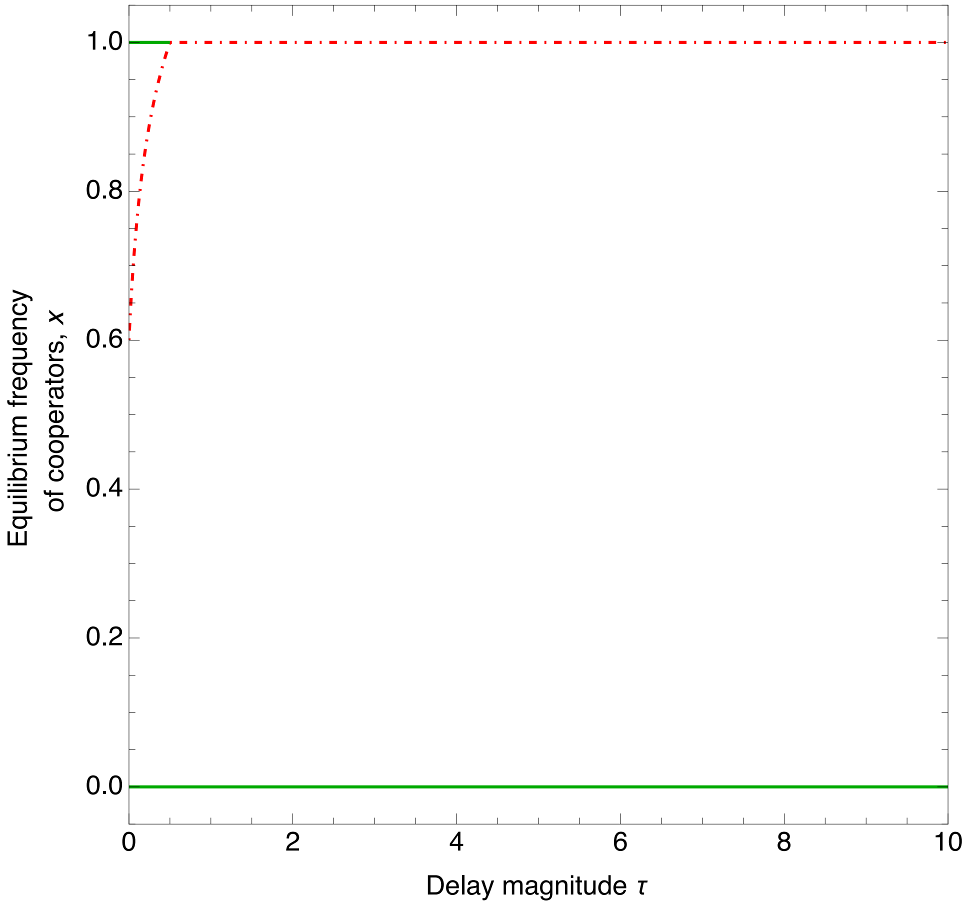
Out[ ]=



$\tau_C = 3\,\tau, \tau_D = \tau$

In[ ]:=
```
panelc = Show[Plotx[3 τ, τ, DotDashed], Plot0[3 τ, τ], Plot1st[3 τ, τ],
  Plot1unst[3 τ, τ, DotDashed]]
```

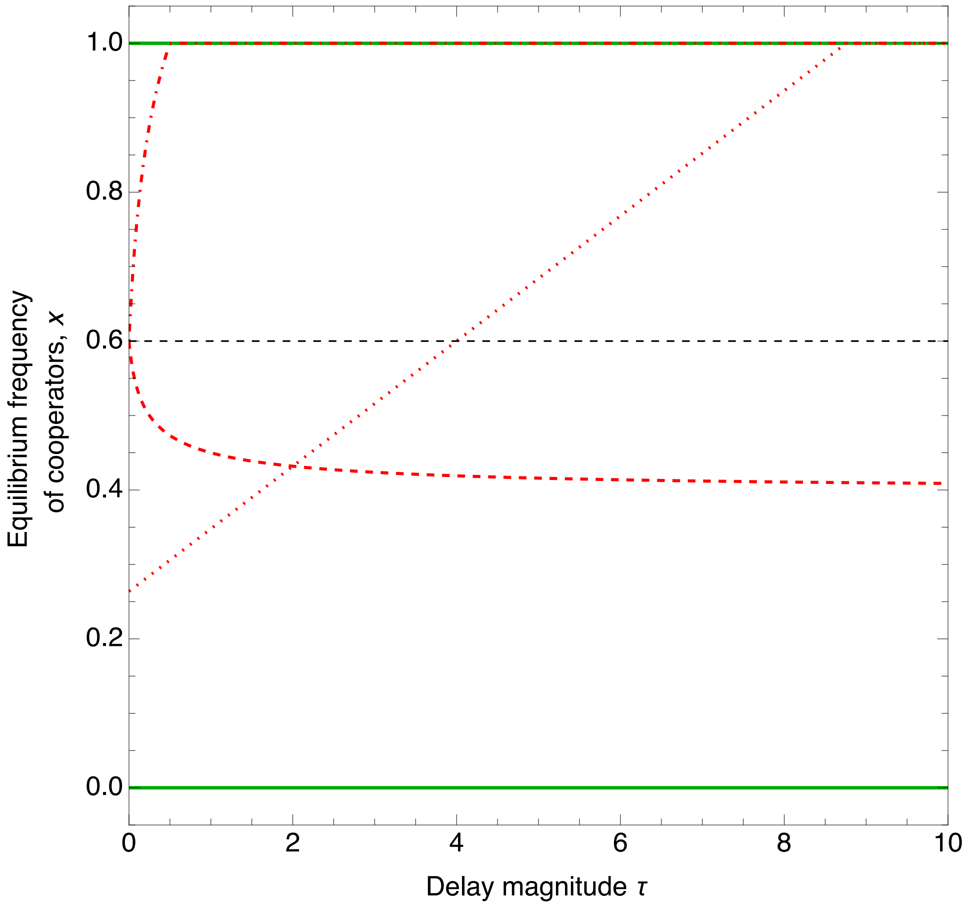
Out[ ]=



Figure

In[◦]:=
```
Show[panela, panelb, panelc, xstar]
```

Out[◦]=



Stability

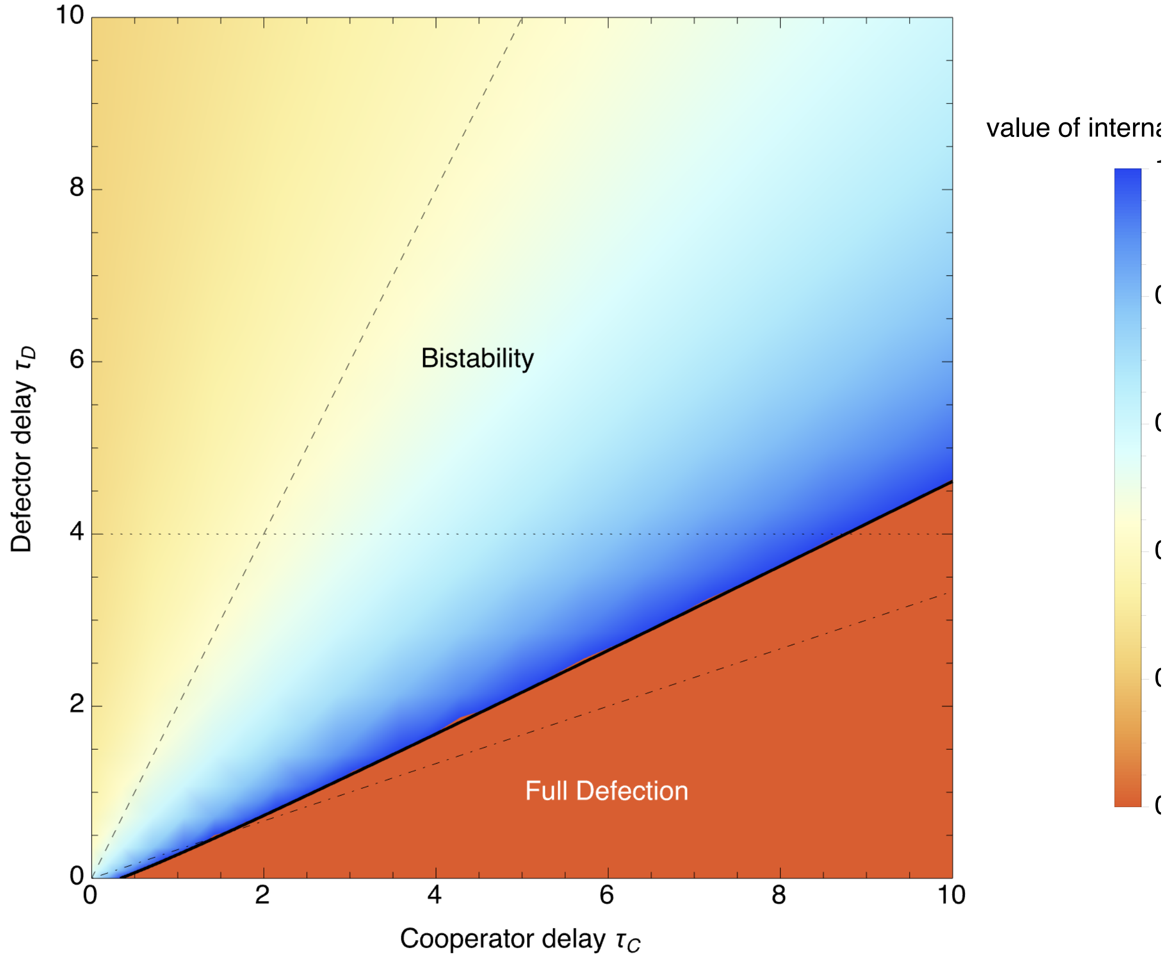
Lastly, we plot the value of x in the internal stationary state depending on the values of delays.

In[◦]:=
```
boundary = Solve[ τC == m && τC > 0 && τD ≠ τC && 0 < x < 1, τD, Reals][[1]];
bound = Plot[τD /. Normal[boundary], {τC, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → Black];
```

In[ ]:=
```
rainbow = DensityPlot[x₂, {τC, 0, 10}, {τD, 0, 10},
   PlotRange → {{0, 10}, {0, 10}, {0, 1} },
   ColorFunction → (ColorData["LightTemperatureMap", (1 - #)] &),
   FrameLabel → {"Cooperator delay τC", "Defector delay τD"}, Frame → True,
   LabelStyle → {14, Black, FontFamily → "Helvetica"},
   ColorFunctionScaling → False, ColorFunctionScaling → False];
alldefect = DensityPlot[0, {tc, 0, 10}, {td, 0, 10},
   PlotRange → {{0, 10}, {0, 10}, {0, 1} },
   ColorFunction → (ColorData["LightTemperatureMap", (1 - #)] &),
   PlotLegends → Placed[BarLegend[Automatic, LegendLayout → "Column",
      LegendLabel → "value of internal equilibrium"], Right],
   FrameLabel → {"Cooperator delay τC", "Defector delay τD"}, Frame → True,
   LabelStyle → {14, Black, FontFamily → "Helvetica"}, ImageSize → Medium];
text =
  Graphics[{Text[Style["Bistability", 14, Black, FontFamily → "Helvetica"],
     {4.5, 6} ],
    Text[Style["Full Defection", 14, White, FontFamily → "Helvetica"], {6, 1}]}];
otherpanelsa = Plot[2 τC,  {τC, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → {Black, Dashed}];
otherpanelsb = Plot[4,  {τC, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → {Black, Dotted}];
otherpanelsc = Plot[1 / 3 τC,  {τC, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → {Black, DotDashed}];
```

In[∘]:=
```
paneld = Show[alldefect, rainbow, bound, text, otherpanelsa, otherpanelsb,
  otherpanelsc, ImageSize → 500]
```

Out[∘]=



Snowdrift game

The Snowdrift game is characterised by the following payoff matrix:

In[286]:=
```
Clear[b, c]
R = b - c / 2;
S = b - c;
T = b;
P = 0;
matrixSG = {{R, S}, {T, P}} // MatrixForm
```

Out[291]//MatrixForm=

$$\begin{pmatrix} b - \frac{c}{2} & b - c \\ b & 0 \end{pmatrix}$$

where $b > c > 0$

In this game, the system characterizing the Kindergarten model becomes:

In[292]:=
```
FullSimplify[dx[x, yC, yD]]
FullSimplify[dyC[x, yC, yD]]
FullSimplify[dyD[x, yC, yD]]
```

Out[292]= $-\frac{(-1+x)\, y_C}{\tau_C} - \frac{x\, y_D}{\tau_D}$

Out[293]= $b\, x + \frac{1}{2}\, c\, (-2+x)\, x + y_C \left(1 - \frac{1+y_C}{\tau_C} - \frac{y_D}{\tau_D}\right)$

Out[294]= $-b\, (-1+x)\, x + y_D \left(1 - \frac{y_C}{\tau_C} - \frac{1+y_D}{\tau_D}\right)$

Homogenous stationary states

First, we analyse the full defection stationary state. We know, that the fraction of cooperators is equal to 0. Then, we determine the relative sizes of kindergartens in the stationary states

In[295]:=
```
x0 = 0;
yC,0 = yC /. Solve[dx[x0, yC, yD] == 0 && dyC[x0, yC, yD] == 0, yC][[1]]
```

Out[296]= 0

With no cooperators present in the population, the cooperator kindergarten is also empty.

In[297]:=
```
yD,0 = yD /. Solve[dyD[x0, yC,0, yD] == 0 && τC > 0 && τD > 0 && b > c > 0 && yD > 0, yD]
```

Out[297]= $\{ -1+\tau_D \text{ if } \tau_C > 0 \,\&\&\, b > 0 \,\&\&\, 0 < c < b \,\&\&\, \tau_D > 1 \}$

Full defection stationary state is of the following form: $e_0 = \{0, 0, -1+\tau_D\}$ when $\tau_D > 1$, then, it grows if

In[298]:= $\text{Reduce}\left[\text{Normal}\left[\frac{y_{C,0}}{\tau_C} + \frac{y_{D,0}}{\tau_D}\right] > 1 \,\&\&\, \tau_C > 0 \,\&\&\, \tau_D > 0 \,\&\&\, b > c > 0\right]$

Out[298]= False

If $\tau_D \le 1$ the full defection stationary state take the form $e_0 = \{0, 0, 0\}$.

In both of those cases the population goes extinct if it ends up in this stationary state.

Next, we investigate the full cooperation ($x_1 = 1$) stationary state:

In[299]:=
```
x1 = 1;
yD,1 = yD /. Solve[dx[x1, yC, yD] == 0 && dyD[x1, yC, yD] == 0, yD][[1]]
```

Out[300]= 0

With no defectors present in the population, the defector kindergarten is also empty.

In[302]:=
```
yC,1 = yC /. Solve[dyC[x1, yC, yD,1] == 0 && τC > 0 && τD > 0 && b > c > 0 && yC > 0, yC]
```

Out[302]= $\left\{ \frac{1}{2}\,(-1+\tau_C) + \frac{1}{2}\sqrt{1 - 2\,\tau_C + 4\,b\,\tau_C - 2\,c\,\tau_C + \tau_C^2} \text{ if } \tau_D > 0 \,\&\&\, c > 0 \,\&\&\, b > c \,\&\&\, \tau_C > 0 \right\}$

Full cooperation equilibrium is of the following form: $e_1 = \{1, \frac{1}{2}\,(-1+\tau_C) + \frac{1}{2}\sqrt{1 - 2\,\tau_C + 4\,b\,\tau_C - 2\,c\,\tau_C + \tau_C^2}, 0\}$

The population grow exponentially in this stationary state if:

In[303]:= $\text{Reduce}\left[\text{Normal}\left[\frac{y_{C,1}}{\tau_C} + \frac{y_{D,1}}{\tau_D}\right] > 1 \,\&\&\, \tau_C > 0 \,\&\&\, \tau_D > 0 \,\&\&\, b > c > 0\right]$

Out[303]= $\tau_D > 0 \,\&\&\, \left(\left(1 < b \le 2 \,\&\&\, 0 < c < -2+2\,b \,\&\&\, \tau_C > 0\right) \,||\, \left(b > 2 \,\&\&\, 0 < c < b \,\&\&\, \tau_C > 0\right)\right)$

In the full cooperation stationary state the population grows if $b > 2$ or $(1 < b < \, = 2 \wedge 0 < c < -2 + 2\, b)$.

Heterogenous stationary states

Next, we investigate the existence of internal stationary states. First, we find the value of y_C depending on x and y_D

In[304]:=
```
Simplify[ Solve[dx[x, yC, yD] == 0, yC]]
```

Out[304]= $\left\{\left\{y_C \to \frac{x\, y_D\, \tau_C}{\tau_D - x\, \tau_D}\right\}\right\}$

In[305]:=
$$y_{C,2}[x\_, yd\_] := \frac{x\, yd\, \tau_C}{\tau_D - x\, \tau_D};$$

Next, we determine the value of $y_D$depending on x

In[321]:=
```
Simplify[
 Solve[dyD[x, yC,2[x, yD], yD] == 0 && yD > 0 && b > c > 0 && τC > 0 && τD > 0 && 0 < x < 1,
  yD]]
```

Out[321]= $\left\{\left\{y_D \to \begin{cases} -\frac{1}{2}\,(-1 + x)\,\left(-1 + \tau_D + \sqrt{1 + (-2 + 4\, b\, x)\, \tau_D + \tau_D^2}\right) & \text{if } \tau_C > 0\, \&\&\, \tau_D > 0\, \&\&\, 0 < x < 1\, \&\&\, b > 0\, \&\&\, 0 < c < b \end{cases}\right\}\right\}$

In[334]:=
$$y_{D,2}[x\_] := -\frac{1}{2}\,(-1 + x)\,\left(-1 + \tau_D + \sqrt{1 + (-2 + 4\, b\, x)\, \tau_D + \tau_D^2}\right);$$

Then, $y_C$becomes:

In[335]:=
```
yC,2,x[x_] := yC,2[x, yD,2[x]]
Simplify[yC,2,x[x]]
```

Out[336]=
$$\frac{x\, \tau_C\,\left(-1 + \tau_D + \sqrt{1 + (-2 + 4\, b\, x)\, \tau_D + \tau_D^2}\right)}{2\, \tau_D}$$

Finally, we find the possible values of x

In[326]:= solutions = Solve[FullSimplify[dy_C[x, $y_{C,2,x}$[x], $y_{D,2}$[x]]] == 0, x, Reals]

Out[326]=

$$\left\{\left\{x \to 0 \text{ if } \boxed{condition}\right\},\right.$$

$$\left\{x \to \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}\left(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4\,b\,c\,\tau_C\,\tau_D + c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2\right) - \sqrt{\left(\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^4}\left(4\,b^2\,\tau_C^2 - 4\,b\,c\,\tau_C^2 + c^2\,\tau_C^2 - 8\,b^2\,\tau_C^3 + 16\,b^3\,\tau_C^3 + 4\,b\,c\,\tau_C^3 - 16\,b^2\,c\,\tau_C^3 + 4\,b^2\,\tau_C^4 - 8\,b^2\,\tau_C\,\tau_D + 8\,b\,c\,\tau_C\,\tau_D - 2\,c^2\,\tau_C\,\tau_D + 16\,b^2\,\tau_C^2\,\tau_D - 32\,b^3\,\tau_C^2\,\tau_D - 4\,b\,c\,\tau_C^2\,\tau_D + 24\,b^2\,c\,\tau_C^2\,\tau_D - 2\,c^2\,\tau_C^2\,\tau_D + 8\,b\,c^2\,\tau_C^2\,\tau_D - 8\,b^2\,\tau_C^3\,\tau_D - 4\,b\,c\,\tau_C^3\,\tau_D + 4\,b^2\,\tau_D^2 - 4\,b\,c\,\tau_D^2 + c^2\,\tau_D^2 - 8\,b^2\,\tau_C\,\tau_D^2 + 16\,b^3\,\tau_C\,\tau_D^2 - 4\,b\,c\,\tau_C\,\tau_D^2 + 4\,c^2\,\tau_C\,\tau_D^2 - 16\,b\,c^2\,\tau_C\,\tau_D^2 + 4\,b^2\,\tau_C^2\,\tau_D^2 + 8\,b\,c\,\tau_C^2\,\tau_D^2 + c^2\,\tau_C^2\,\tau_D^2 + 4\,b\,c\,\tau_D^3 - 8\,b^2\,c\,\tau_D^3 - 2\,c^2\,\tau_D^3 + 8\,b\,c^2\,\tau_D^3 - 4\,b\,c\,\tau_C\,\tau_D^3 - 2\,c^2\,\tau_C\,\tau_D^3 + c^2\,\tau_D^4\right)\right)} \text{ if } \boxed{condition}\right\},$$

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Byte count: 1 797 856
Uniconize

$$\left\{x \to \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}\left(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4\,b\,c\,\tau_C\,\tau_D + c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2\right) + \sqrt{\left(\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^4}\left(4\,b^2\,\tau_C^2 - 4\,b\,c\,\tau_C^2 + c^2\,\tau_C^2 - 8\,b^2\,\tau_C^3 + 16\,b^3\,\tau_C^3 + 4\,b\,c\,\tau_C^3 - 16\,b^2\,c\,\tau_C^3 + 4\,b^2\,\tau_C^4 - 8\,b^2\,\tau_C\,\tau_D + 8\,b\,c\,\tau_C\,\tau_D - 2\,c^2\,\tau_C\,\tau_D + 16\,b^2\,\tau_C^2\,\tau_D - 32\,b^3\,\tau_C^2\,\tau_D - 4\,b\,c\,\tau_C^2\,\tau_D + 24\,b^2\,c\,\tau_C^2\,\tau_D - 2\,c^2\,\tau_C^2\,\tau_D + 8\,b\,c^2\,\tau_C^2\,\tau_D - 8\,b^2\,\tau_C^3\,\tau_D - 4\,b\,c\,\tau_C^3\,\tau_D + 4\,b^2\,\tau_D^2 - 4\,b\,c\,\tau_D^2 + c^2\,\tau_D^2 - 8\,b^2\,\tau_C\,\tau_D^2 + 16\,b^3\,\tau_C\,\tau_D^2 - 4\,b\,c\,\tau_C\,\tau_D^2 + 4\,c^2\,\tau_C\,\tau_D^2 - 16\,b\,c^2\,\tau_C\,\tau_D^2 + 4\,b^2\,\tau_C^2\,\tau_D^2 + 8\,b\,c\,\tau_C^2\,\tau_D^2 + c^2\,\tau_C^2\,\tau_D^2 + 4\,b\,c\,\tau_D^3 - 8\,b^2\,c\,\tau_D^3 - 2\,c^2\,\tau_D^3 + 8\,b\,c^2\,\tau_D^3 - 4\,b\,c\,\tau_C\,\tau_D^3 - 2\,c^2\,\tau_C\,\tau_D^3 + c^2\,\tau_D^4\right)\right)} \text{ if } \boxed{condition}\right\}\right\}$$

In[413]:=

```
FullSimplify[
 1/(-2 b τC + c τD)^2
  (-2 b τC + c τC + 2 b τC^2 + 2 b τD - c τD - 2 b τC τD + 4 b^2 τC τD - c τC τD - 4 b c τC τD +
    c τD^2 - 2 b c τD^2 + 2 c^2 τD^2) -
  Sqrt[(1/(-2 b τC + c τD)^4
     (4 b^2 τC^2 - 4 b c τC^2 + c^2 τC^2 - 8 b^2 τC^3 + 16 b^3 τC^3 + 4 b c τC^3 - 16 b^2 c τC^3 + 4 b^2 τC^4 -
       8 b^2 τC τD + 8 b c τC τD - 2 c^2 τC τD + 16 b^2 τC^2 τD - 32 b^3 τC^2 τD - 4 b c τC^2 τD +
       24 b^2 c τC^2 τD - 2 c^2 τC^2 τD + 8 b c^2 τC^2 τD - 8 b^2 τC^3 τD - 4 b c τC^3 τD + 4 b^2 τD^2 - 4 b c τD^2 +
       c^2 τD^2 - 8 b^2 τC τD^2 + 16 b^3 τC τD^2 - 4 b c τC τD^2 + 4 c^2 τC τD^2 - 16 b c^2 τC τD^2 + 4 b^2 τC^2 τD^2 +
       8 b c τC^2 τD^2 + c^2 τC^2 τD^2 + 4 b c τD^3 - 8 b^2 c τD^3 - 2 c^2 τD^3 + 8 b c^2 τD^3 - 4 b c τC τD^3 -
       2 c^2 τC τD^3 + c^2 τD^4))], b > c > 0 && τC > 0 && τD > 0]

FullSimplify[
 1/(-2 b τC + c τD)^2
  (-2 b τC + c τC + 2 b τC^2 + 2 b τD - c τD - 2 b τC τD + 4 b^2 τC τD - c τC τD - 4 b c τC τD +
    c τD^2 - 2 b c τD^2 + 2 c^2 τD^2) +
  Sqrt[(1/(-2 b τC + c τD)^4
     (4 b^2 τC^2 - 4 b c τC^2 + c^2 τC^2 - 8 b^2 τC^3 + 16 b^3 τC^3 + 4 b c τC^3 - 16 b^2 c τC^3 + 4 b^2 τC^4 -
       8 b^2 τC τD + 8 b c τC τD - 2 c^2 τC τD + 16 b^2 τC^2 τD - 32 b^3 τC^2 τD - 4 b c τC^2 τD +
       24 b^2 c τC^2 τD - 2 c^2 τC^2 τD + 8 b c^2 τC^2 τD - 8 b^2 τC^3 τD - 4 b c τC^3 τD + 4 b^2 τD^2 - 4 b c τD^2 +
       c^2 τD^2 - 8 b^2 τC τD^2 + 16 b^3 τC τD^2 - 4 b c τC τD^2 + 4 c^2 τC τD^2 - 16 b c^2 τC τD^2 + 4 b^2 τC^2 τD^2 +
       8 b c τC^2 τD^2 + c^2 τC^2 τD^2 + 4 b c τD^3 - 8 b^2 c τD^3 - 2 c^2 τD^3 + 8 b c^2 τD^3 - 4 b c τC τD^3 -
       2 c^2 τC τD^3 + c^2 τD^4))], b > c > 0 && τC > 0 && τD > 0]
```

$$\frac{1}{(-2 b \tau_C + c \tau_D)^2}\Big(2 b \tau_C^2 + \tau_C (-2 b + c + (2 b (-1 + 2 b - 2 c) - c) \tau_D) + \tau_D (2 b - c + c (1 - 2 b + 2 c) \tau_D) + \mathrm{Abs}[\tau_C - \tau_D] \sqrt{(-2 b + c)^2 + (2 b \tau_C - c \tau_D) (4 b (-1 + 2 b - 2 c) + 2 c + 2 b \tau_C - c \tau_D)}\Big)$$

Out[414]=

$$\frac{1}{(-2 b \tau_C + c \tau_D)^2}\Big(2 b \tau_C^2 + \tau_C (-2 b + c + (2 b (-1 + 2 b - 2 c) - c) \tau_D) + \tau_D (2 b - c + c (1 - 2 b + 2 c) \tau_D) + \mathrm{Abs}[\tau_C - \tau_D] \sqrt{(-2 b + c)^2 + (2 b \tau_C - c \tau_D) (4 b (-1 + 2 b - 2 c) + 2 c + 2 b \tau_C - c \tau_D)}\Big)$$

In[441]:= $$x_2 = \frac{1}{(-2\, b\, \tau_C + c\, \tau_D)^2} \left(2\, b\, \tau_C^2 + \tau_C\, (-2\, b + c + (2\, b\, (-1 + 2\, b - 2\, c) - c)\, \tau_D) + \tau_D\, (2\, b - c + c\, (1 - 2\, b + 2\, c)\, \tau_D) + (\tau_C - \tau_D)\, \sqrt{(-2\, b + c)^2 + (2\, b\, \tau_C - c\, \tau_D)\, (4\, b\, (-1 + 2\, b - 2\, c) + 2\, c + 2\, b\, \tau_C - c\, \tau_D)}\right);$$

$$x_3 = \frac{1}{(-2\, b\, \tau_C + c\, \tau_D)^2} \left(2\, b\, \tau_C^2 + \tau_C\, (-2\, b + c + (2\, b\, (-1 + 2\, b - 2\, c) - c)\, \tau_D) + \tau_D\, (2\, b - c + c\, (1 - 2\, b + 2\, c)\, \tau_D) + (\tau_D - \tau_C)\, \sqrt{(-2\, b + c)^2 + (2\, b\, \tau_C - c\, \tau_D)\, (4\, b\, (-1 + 2\, b - 2\, c) + 2\, c + 2\, b\, \tau_C - c\, \tau_D)}\right);$$

The system has two possible internal stationary states:

$e_2 = \left(x,\ \frac{x\, \tau_C\, \left(-1+\tau_D+\sqrt{1+(-2+4\, b\, x)\, \tau_D+\tau_D^2}\right)}{2\, \tau_D},\ -\frac{1}{2}\, (-1+x)\left(-1+\tau_D+\sqrt{1+(-2+4\, b\, x)\, \tau_D+\tau_D^2}\right)\right)$ where

$$x_2 = \frac{2\, b\, \tau_C^2+\tau_C\, (-2\, b+c+(2\, b\, (-1+2\, b-2\, c)-c)\, \tau_D)+\tau_D\, (2\, b-c+c\, (1-2\, b+2\, c)\, \tau_D)+(\tau_C-\tau_D)\, \sqrt{(-2\, b+c)^2+(2\, b\, \tau_C-c\, \tau_D)\, (4\, b\, (-1+2\, b-2\, c)+2\, c+2\, b\, \tau_C-c\, \tau_D)}}{(-2\, b\, \tau_C+c\, \tau_D)^2}$$

or

$$e_3 = \left(x,\ \frac{x\, \tau_C\, \left(-1+\tau_D+\sqrt{1+(-2+4\, b\, x)\, \tau_D+\tau_D^2}\right)}{2\, \tau_D},\ -\frac{1}{2}\, (-1+x)\left(-1+\tau_D+\sqrt{1+(-2+4\, b\, x)\, \tau_D+\tau_D^2}\right)\right)$$

where

$$x_3 = \frac{2\, b\, \tau_C^2+\tau_C\, (-2\, b+c+(2\, b\, (-1+2\, b-2\, c)-c)\, \tau_D)+\tau_D\, (2\, b-c+c\, (1-2\, b+2\, c)\, \tau_D)+(\tau_D-\tau_C)\, \sqrt{(-2\, b+c)^2+(2\, b\, \tau_C-c\, \tau_D)\, (4\, b\, (-1+2\, b-2\, c)+2\, c+2\, b\, \tau_C-c\, \tau_D)}}{(-2\, b\, \tau_C+c\, \tau_D)^2}$$

where

In[331]:= $$o = (-2\, b + c)^2 + (2\, b\, \tau_C - c\, \tau_D)\, (4\, b\, (-1 + 2\, b - 2\, c) + 2\, c + 2\, b\, \tau_C - c\, \tau_D);$$

Lastly, we check when does the population grow in this stationary state and is not threatened by extinction

In[338]:= $$\text{Reduce}\left[\text{Normal}\left[\frac{y_{C,2,x}[x]}{\tau_C} + \frac{y_{D,2}[x]}{\tau_D}\right] \geq 1\ \&\&\ \tau_D > 0\ \&\&\ ((1 < b \leq 2\ \&\&\ 0 < c < -2 + 2\, b\ \&\&\ \tau_C > 0)\ ||\ (b > 2\ \&\&\ 0 < c < b\ \&\&\ \tau_C > 0))\right]$$

Out[338]= $$\left(\tau_C > 0\ \&\&\ 0 < c < 2\ \&\&\ \frac{2 + c}{2} < b \leq 2\ \&\&\ x \geq \frac{1}{b}\ \&\&\ \tau_D > 0\right)\ ||\ \left(\tau_C > 0\ \&\&\ \left(\left(b > 2\ \&\&\ 0 < c \leq 2\ \&\&\ x \geq \frac{1}{b}\ \&\&\ \tau_D > 0\right)\ ||\ \left(c > 2\ \&\&\ b > c\ \&\&\ x \geq \frac{1}{b}\ \&\&\ \tau_D > 0\right)\right)\right)$$

In the internal stationary states the population grows when $x >\ = 1/b$

Stability analysis

Full defection stationary state

We perform the stability analysis of full defection stationary state e_0.

First, we determine the eigenvalues of the system of ODEs:

```
In[374]:= system = {dx[x, yC, yD], dyC[x, yC, yD] , dyD[x, yC, yD]};
          J = D[system, {{x, yC, yD}}] ;
          J // MatrixForm
          Jstar = J /. {x → Normal[x0], yD → Normal[yD,0][[1]], yC → Normal[yC,0]};
          Jstar // MatrixForm
          eigens = Eigenvalues[Jstar];
          eigens // MatrixForm
```

Out[376]//MatrixForm=

$$\begin{pmatrix} -\frac{y_C}{\tau_C}-\frac{y_D}{\tau_D} & \frac{1-x}{\tau_C} & -\frac{x}{\tau_D} \\ (b-c)\ (1-x)+\left(b-\frac{c}{2}\right)x+\frac{c\,x}{2} & -\frac{2\,y_C}{\tau_C}+\frac{-1+\tau_C}{\tau_C}-\frac{y_D}{\tau_D} & -\frac{y_C}{\tau_D} \\ b\ (1-x)-b\,x & -\frac{y_D}{\tau_C} & -\frac{y_C}{\tau_C}-\frac{2\,y_D}{\tau_D}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[378]//MatrixForm=

$$\begin{pmatrix} -\frac{-1+\tau_D}{\tau_D} & \frac{1}{\tau_C} & 0 \\ b-c & \frac{-1+\tau_C}{\tau_C}-\frac{-1+\tau_D}{\tau_D} & 0 \\ b & -\frac{-1+\tau_D}{\tau_C} & -\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[380]//MatrixForm=

$$\begin{pmatrix} -\frac{-1+\tau_D}{\tau_D} \\ \frac{2\,\tau_C^2\,\tau_D-\tau_C\,\tau_D^2-\tau_C^2\,\tau_D^2-\tau_C\,\sqrt{1-2\,\tau_C+4\,b\,\tau_C-4\,c\,\tau_C+\tau_C^2}\,\tau_D^2}{2\,\tau_C^2\,\tau_D^2} \\ \frac{2\,\tau_C^2\,\tau_D-\tau_C\,\tau_D^2-\tau_C^2\,\tau_D^2+\tau_C\,\sqrt{1-2\,\tau_C+4\,b\,\tau_C-4\,c\,\tau_C+\tau_C^2}\,\tau_D^2}{2\,\tau_C^2\,\tau_D^2} \end{pmatrix}$$

Next, we determine when the stationary state is stable

```
In[382]:= Reduce[eigens[[1]] < 0 && eigens[[2]] < 0 && eigens[[3]] < 0 && τC > 0 && τD > 1 &&
             τC ≠ τD && b > c > 0 &&
             ((1 < b < 2 && 0 < c ≤ -2 + 2 b && tc > 0) || (b ≥ 2 && 0 < c < b && tc > 0))]
```

Out[382]=

$$\left(0<c<2\ \&\&\ \frac{2+c}{2}\le b<1+c\ \&\&\ \tau_C>0\ \&\&\ \tau_D>\frac{-1-\tau_C}{2\left(-1+b-c\right)}+\frac{1}{2}\sqrt{\frac{1-2\,\tau_C+4\,b\,\tau_C-4\,c\,\tau_C+\tau_C^2}{\left(-1+b-c\right)^2}}\ \&\&\ tc>0\right)\ ||\ \left(c\ge 2\ \&\&\ c<b<1+c\ \&\&\ \tau_C>0\ \&\&\ \tau_D>\frac{-1-\tau_C}{2\left(-1+b-c\right)}+\frac{1}{2}\sqrt{\frac{1-2\,\tau_C+4\,b\,\tau_C-4\,c\,\tau_C+\tau_C^2}{\left(-1+b-c\right)^2}}\ \&\&\ tc>0\right)$$

Since the population always goes extinct in the full defection stationary state,we only consider case when it is unstable. Hence, we only consider games when $\boxed{b \geq c+1}$

Full cooperation stationary state

We perform the stability analysis of full cooperation stationary state e_1.

First, we determine the eigenvalues of the system of ODEs:

In[349]:=
```
system = {dx[x, yC, yD], dyC[x, yC, yD] , dyD[x, yC, yD]};
J = D[system, {{x, yC, yD}}] ;
J // MatrixForm
Jstar = J /. {x → Normal[x1], yD → Normal[yD,1], yC → Normal[yC,1][[1]]};
Jstar // MatrixForm
eigens = Eigenvalues[Jstar];
eigens // MatrixForm
```

Out[351]//MatrixForm=

$$\begin{pmatrix} -\frac{y_C}{\tau_C}-\frac{y_D}{\tau_D} & \frac{1-x}{\tau_C} & -\frac{x}{\tau_D} \\ (b-c)\ (1-x)+\left(b-\frac{c}{2}\right)x+\frac{c\,x}{2} & -\frac{2\,y_C}{\tau_C}+\frac{-1+\tau_C}{\tau_C}-\frac{y_D}{\tau_D} & -\frac{y_C}{\tau_D} \\ b\ (1-x)-b\,x & -\frac{y_D}{\tau_C} & -\frac{y_C}{\tau_C}-\frac{2\,y_D}{\tau_D}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[353]//MatrixForm=

$$\begin{pmatrix} -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C-2\,c\,\tau_C+\tau_C^2}}{\tau_C} & 0 & -\frac{1}{\tau_D} \\ b & \frac{-1+\tau_C}{\tau_C}-\frac{2\left(\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C-2\,c\,\tau_C+\tau_C^2}\right)}{\tau_C} & -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C-}}{\tau_D} \\ -b & 0 & -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C-2\,c\,\tau}}{\tau_C} \end{pmatrix}$$

Out[355]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{1-2\,\tau_C+4\,b\,\tau_C-2\,c\,\tau_C+\tau_C^2}}{\tau_C} \\ \frac{-2\,\tau_C+2\,\tau_D-2\sqrt{1-2\,\tau_C+4\,b\,\tau_C-2\,c\,\tau_C+\tau_C^2}\,\tau_D-\sqrt{4\,\tau_C^2-8\,\tau_C^2\,\tau_D+16\,b\,\tau_C^2\,\tau_D+4\,\tau_C^2\,\tau_D^2}}{4\,\tau_C\,\tau_D} \\ \frac{-2\,\tau_C+2\,\tau_D-2\sqrt{1-2\,\tau_C+4\,b\,\tau_C-2\,c\,\tau_C+\tau_C^2}\,\tau_D+\sqrt{4\,\tau_C^2-8\,\tau_C^2\,\tau_D+16\,b\,\tau_C^2\,\tau_D+4\,\tau_C^2\,\tau_D^2}}{4\,\tau_C\,\tau_D} \end{pmatrix}$$

Next, we determine when the stationary state is stable

In[384]:=
```
Reduce[eigens[[1]] < 0 && eigens[[2]] < 0 && eigens[[3]] < 0 && τD > 0 &&
  ((1 < b ≤ 2 && 0 < c < -2 + 2 b && τC > 0) || (b > 2 && 0 < c < b && τC > 0)), τD]
```

Out[384]=

$$\left(0<c\leq 2\ \&\&\ \frac{2+c}{2}<b<1+c\ \&\&\ \tau_C>0\ \&\&\ \tau_D>\frac{-1-\tau_C}{2\,(-1+b-c)}+\frac{1}{2}\sqrt{\frac{1-2\,\tau_C+4\,b\,\tau_C-4\,c\,\tau_C+\tau_C^2}{(-1+b-c)^2}}\right)\ ||$$
$$\left(c>2\ \&\&\ c<b<1+c\ \&\&\ \tau_C>0\ \&\&\ \tau_D>\frac{-1-\tau_C}{2\,(-1+b-c)}+\frac{1}{2}\sqrt{\frac{1-2\,\tau_C+4\,b\,\tau_C-4\,c\,\tau_C+\tau_C^2}{(-1+b-c)^2}}\right)$$

In[359]:=

$$n = \text{FullSimplify}\left[\frac{c-4\,b\,\tau_C+4\,b^2\,\tau_C+c\,\tau_C-2\,b\,c\,\tau_C}{-4\,b+4\,b^2+2\,c-4\,b\,c+c^2}+\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\,b\,c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{(-4\,b+4\,b^2+2\,c-4\,b\,c+c^2)^2}}\right]$$

Out[359]=

$$\frac{c+\left(4\,b^2+c-2\,b\,(2+c)\right)\tau_C}{4\,b^2-4\,b\,(1+c)+c\,(2+c)}+\sqrt{\frac{c^2\left(1+\tau_C\left(4\,b-2\,(1+c)+\tau_C\right)\right)}{(-2\,b+c)^2\,(2-2\,b+c)^2}}$$

Full cooperation stationary state is stable when $\tau_D > n$ where

$$n=\frac{c+\left(4\,b^2+c-2\,b\,(2+c)\right)\tau_C}{4\,b^2-4\,b\,(1+c)+c\,(2+c)}+\sqrt{\frac{c^2\left(1+\tau_C\left(4\,b-2\,(1+c)+\tau_C\right)\right)}{(-2\,b+c)^2\,(2-2\,b+c)^2}}$$

Internal stationary states

Now, we analyse the two possible internal stationary states e_2 and e_3.

First, we check for internal stationary states, considering the additional constraint of $\boxed{b \geq c+1}$ and different delay values

$\tau_C > \tau_D$

In[400]:=
```
solutions =
  Solve[FullSimplify[dyC[x, yC,2,x[x], yD,2[x]]] == 0 && b ≥ 1 + c && τC > τD > 0 && c > 0,
   x, Reals]
```

$$\Big\{\Big\{x \to \boxed{0 \text{ if } \boxed{condition}}\Big\}, \Big\{x \to \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}$$

$$\big(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C$$
$$c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2\big) -$$
$$\sqrt{\Big(\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^4}}$$
$$\big(4\,b^2\,\tau_C^2 - 4\,b\,c\,\tau_C^2 + c^2\,\tau_C^2 - 8\,b^2\,\tau_C^3 + 16\,b^3\,\tau_C^3 + 4\,b\,c\,\tau_C^3 - 16\,b^2$$
$$8\,b^2\,\tau_C\,\tau_D + 8\,b\,c\,\tau_C\,\tau_D - 2\,c^2\,\tau_C\,\tau_D + 16\,b^2\,\tau_C^2\,\tau_D - 32\,b^3\,\tau_C^2\,\tau_D$$
$$24\,b^2\,c\,\tau_C^2\,\tau_D - 2\,c^2\,\tau_C^2\,\tau_D + 8\,b\,c^2\,\tau_C^2\,\tau_D - 8\,b^2\,\tau_C^3\,\tau_D - 4\,b\,c\,\tau_C^3$$
$$4\,b\,c\,\tau_D^2 + c^2\,\tau_D^2 - 8\,b^2\,\tau_C\,\tau_D^2 + 16\,b^3\,\tau_C\,\tau_D^2 - 4\,b\,c\,\tau_C\,\tau_D^2 + 4\,c^2$$
$$4\,b^2\,\tau_C^2\,\tau_D^2 + 8\,b\,c\,\tau_C^2\,\tau_D^2 + c^2\,\tau_C^2\,\tau_D^2 + 4\,b\,c\,\tau_D^3 - 8\,b^2\,c\,\tau_D^3 - 2\,c^2$$
$$4\,b\,c\,\tau_C\,\tau_D^3 - 2\,c^2\,\tau_C\,\tau_D^3 + c^2\,\tau_D^4\big)\Big) \text{ if}$$
$$0 < c < -1 + b\ \&\&\ 0 < \tau_D < 1\ \&\&\ b > 1\ \&\&\ \tau_C > \frac{-4\,b\,\tau_D + c\,\tau_D + 4\,b\,\tau_D^2}{-}$$

$$\Big\{x \to \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}$$
$$\big(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4\,b\,c\,\tau_C\,\tau_D +$$
$$c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2\big) +$$
$$\sqrt{\Big(\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^4}}$$
$$\big(4\,b^2\,\tau_C^2 - 4\,b\,c\,\tau_C^2 + c^2\,\tau_C^2 - 8\,b^2\,\tau_C^3 + 16\,b^3\,\tau_C^3 + 4\,b\,c\,\tau_C^3 - 16\,b^2\,c\,\tau_C^3 + 4\,b^2\,\tau_C^4 -$$
$$8\,b^2\,\tau_C\,\tau_D + 8\,b\,c\,\tau_C\,\tau_D - 2\,c^2\,\tau_C\,\tau_D + 16\,b^2\,\tau_C^2\,\tau_D - 32\,b^3\,\tau_C^2\,\tau_D - 4\,b\,c\,\tau_C^2\,\tau_D +$$
$$24\,b^2\,c\,\tau_C^2\,\tau_D - 2\,c^2\,\tau_C^2\,\tau_D + 8\,b\,c^2\,\tau_C^2\,\tau_D - 8\,b^2\,\tau_C^3\,\tau_D - 4\,b\,c\,\tau_C^3\,\tau_D + 4\,b^2\,\tau_D^2 -$$
$$4\,b\,c\,\tau_D^2 + c^2\,\tau_D^2 - 8\,b^2\,\tau_C\,\tau_D^2 + 16\,b^3\,\tau_C\,\tau_D^2 - 4\,b\,c\,\tau_C\,\tau_D^2 + 4\,c^2\,\tau_C\,\tau_D^2 - 16\,b\,c^2\,\tau_C\,\tau_D^2 +$$
$$4\,b^2\,\tau_C^2\,\tau_D^2 + 8\,b\,c\,\tau_C^2\,\tau_D^2 + c^2\,\tau_C^2\,\tau_D^2 + 4\,b\,c\,\tau_D^3 - 8\,b^2\,c\,\tau_D^3 - 2\,c^2\,\tau_D^3 + 8\,b\,c^2\,\tau_D^3 -$$
$$4\,b\,c\,\tau_C\,\tau_D^3 - 2\,c^2\,\tau_C\,\tau_D^3 + c^2\,\tau_D^4\big)\Big) \text{ if}$$
$$(0 < c < -1 + b\ \&\&\ b > 1\ \&\&\ \tau_D > 1\ \&\&\ \tau_C > \tau_D)\ ||$$
$$\Big(0 < c < -1 + b\ \&\&\ 0 < \tau_D < 1\ \&\&\ b > 1\ \&\&$$
$$\tau_C > \frac{-4\,b\,\tau_D + c\,\tau_D + 4\,b\,\tau_D^2 - 8\,b^2\,\tau_D^2 - 2\,c\,\tau_D^2 + 8\,b\,c\,\tau_D^2 + c\,\tau_D^3}{-2\,b + 2\,b\,\tau_D^2}\Big)\ ||$$
$$\Big(0 < c < -1 + b\ \&\&\ 0 < \tau_D < 1\ \&\&$$
$$\tau_D < \tau_C < \frac{-4\,b\,\tau_D + c\,\tau_D + 4\,b\,\tau_D^2 - 8\,b^2\,\tau_D^2 - 2\,c\,\tau_D^2 + 8\,b\,c\,\tau_D^2 + c\,\tau_D^3}{-2\,b + 2\,b\,\tau_D^2}\ \&\&\ b > 1\Big)\Big\}\Big\}$$

We start with compering the two possible solutions with x_2 and x_3

In[447]:=
```
Reduce[Normal[x /. solutions[[2]]] == x2 && b ≥ 1 + c && τC > τD > 0 && c > 0]
Reduce[Normal[x /. solutions[[2]]] == x3 && b ≥ 1 + c && τC > τD > 0 && c > 0]
```

Out[447]= False

Out[448]= $b > 1\ \&\&\ 0 < c \le -1 + b\ \&\&\ \tau_D > 0\ \&\&\ \tau_C > \tau_D$

Now we check, id x_3 exists in the interesting interval of (0,1):

In[451]:=
```
Reduce[0 < x3 < 1 && b ≥ 1 + c && τC > τD > 0 && c > 0 && 0 < c < -1 + b && 0 < τD < 1 &&
    b > 1 && τC > (-4 b τD + c τD + 4 b τD^2 - 8 b^2 τD^2 - 2 c τD^2 + 8 b c τD^2 + c τD^3)/(-2 b + 2 b τD^2)]
```

Out[451]= False

Then, we investigate the second solution

In[452]:=
```
Reduce[Normal[x /. solutions[[3]]] == x2 && b ≥ 1 + c && τC > τD > 0 && c > 0]
Reduce[Normal[x /. solutions[[3]]] == x3 && b ≥ 1 + c && τC > τD > 0 && c > 0]
```

Out[452]= $b > 1 \,\&\&\, 0 < c \leq -1 + b \,\&\&\, \tau_D > 0 \,\&\&\, \tau_C > \tau_D$

Out[453]= False

And check if x_2 exists:

```
Reduce[0 < x2 < 1 && (0 < c < -1 + b && b > 1 && τD > 1 && τC > τD) ||
   (0 < c < -1 + b && 0 < τD < 1 && b > 1 &&
     τC > (-4 b τD + c τD + 4 b τD^2 - 8 b^2 τD^2 - 2 c τD^2 + 8 b c τD^2 + c τD^3)/(-2 b + 2 b τD^2)) ||
   (0 < c < -1 + b && 0 < τD < 1 &&
     τD < τC < (-4 b τD + c τD + 4 b τD^2 - 8 b^2 τD^2 - 2 c τD^2 + 8 b c τD^2 + c τD^3)/(-2 b + 2 b τD^2) && b > 1)]
```

Out[454]= $b > 1 \,\&\&\, 0 < c < -1 + b \,\&\&\, \left(\left(0 < \tau_D < 1 \,\&\&\, \left(\tau_D < \tau_C < \frac{-4 b \tau_D + c \tau_D + 4 b \tau_D^2 - 8 b^2 \tau_D^2 - 2 c \tau_D^2 + 8 b c \tau_D^2 + c \tau_D^3}{-2 b + 2 b \tau_D^2} \,||\, \tau_C > \frac{-4 b \tau_D + c \tau_D + 4 b \tau_D^2 - 8 b^2 \tau_D^2 - 2 c \tau_D^2 + 8 b c \tau_D^2 + c \tau_D^3}{-2 b + 2 b \tau_D^2}\right)\right) \,||\, (\tau_D > 1 \,\&\&\, \tau_C > \tau_D)\right)$

Then, we check the effects of delays on x_2 :

In[455]:=
```
Reduce[D[x2, τC] < 0 && b ≥ 1 + c && τC > τD > 0 && c > 0]
```

Out[455]= $b > 1 \,\&\&\, 0 < c \leq -1 + b \,\&\&\, \tau_D > 0 \,\&\&\, \tau_C > \tau_D$

The value of internal stationary state always decreases with the increase o cooperator delay. Then, we check the limit of $\tau_C \to \infty$

In[457]:=
```
Simplify[Limit[x2, τC → ∞], b ≥ 1 + c && τC > τD > 0 && c > 0]
```

Out[457]= $\frac{1}{b}$

With the cooperator delay approaching infinity, x_2 approaches $1/b$, ensuring that the population always grows in this stationary state. Next, we investigate the effect of defector delay

In[458]:=
```
Reduce[D[x2, τD] < 0 && b ≥ 1 + c && τC > τD > 0 && c > 0]
```

Out[458]= False

In[459]:=
```
Reduce[D[x2, τD] > 0 && b ≥ 1 + c && τC > τD > 0 && c > 0]
```

Out[459]= $b > 1 \&\& 0 < c \leq -1 + b \&\& \tau_C > 0 \&\& 0 < \tau_D < \tau_C$

The increase in defector delay leads to an increase in value of x_2. Then, we investigate the limit of $\tau_D - > \tau_C$.

In[460]:=
```
Simplify[Limit[x2, τD → τC ], b ≥ 1 + c && c > 0]
```

Out[460]= $\frac{2\,(b - c)}{2\,b - c}$

The limit coincides with the internal stationary state of the Snowdrift game with no delays.

$\tau_C < \tau_D$ *and* $b < 2 + \sqrt{5}$

Again, we solve for possible solutions of the system in this parameter space:

In[557]:=
```
solutions =
  Solve[FullSimplify[dyC[x, yC,2,x[x], yD,2[x]]] == 0 && b ≥ 1 + c && 0 < τC < τD &&
    c > 0 && b < 2 + √5 && x > 0, x, Reals]
```

$$\Big\{\Big\{x \to \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}$$

$$\Big(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4\,b\,c\,\tau_C\,\tau_D + c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2\Big) -$$

$$\sqrt{\Big(\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^4}}$$

$$\Big(4\,b^2\,\tau_C^2 - 4\,b\,c\,\tau_C^2 + c^2\,\tau_C^2 - 8\,b^2\,\tau_C^3 + 16\,b^3\,\tau_C^3 + 4\,b\,c\,\tau_C^3 - 16\,b^2\,c\,\tau_C^3 + 4\,b^2\,\tau_C^4 -$$
$$8\,b^2\,\tau_C\,\tau_D + 8\,b\,c\,\tau_C\,\tau_D - 2\,c^2\,\tau_C\,\tau_D + 16\,b^2\,\tau_C^2\,\tau_D - 32\,b^3\,\tau_C^2\,\tau_D - 4\,b\,c\,\tau_C^2\,\tau_D +$$
$$24\,b^2\,c\,\tau_C^2\,\tau_D - 2\,c^2\,\tau_C^2\,\tau_D + 8\,b\,c^2\,\tau_C^2\,\tau_D - 8\,b^2\,\tau_C^3\,\tau_D - 4\,b\,c\,\tau_C^3\,\tau_D + 4\,b^2\,\tau_D^2 -$$
$$4\,b\,c\,\tau_D^2 + c^2\,\tau_D^2 - 8\,b^2\,\tau_C\,\tau_D^2 + 16\,b^3\,\tau_C\,\tau_D^2 - 4\,b\,c\,\tau_C\,\tau_D^2 + 4\,c^2\,\tau_C\,\tau_D^2 - 16\,b\,c^2\,\tau_C\,\tau_D^2 +$$
$$4\,b^2\,\tau_C^2\,\tau_D^2 + 8\,b\,c\,\tau_C^2\,\tau_D^2 + c^2\,\tau_C^2\,\tau_D^2 + 4\,b\,c\,\tau_D^3 - 8\,b^2\,c\,\tau_D^3 - 2\,c^2\,\tau_D^3 + 8\,b\,c^2\,\tau_D^3 -$$
$$4\,b\,c\,\tau_C\,\tau_D^3 - 2\,c^2\,\tau_C\,\tau_D^3 + c^2\,\tau_D^4\Big)\Big) \text{ if}$$

$$\Big(0 < c < -1 + b \,\&\&$$
$$0 < \tau_D < \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \,\&\&$$
$$0 < \tau_C < \frac{c\,\tau_D}{2\,b} \,\&\&\, 1 < b < 2 + \sqrt{5}\Big) \,||$$

$$\Big(0 < c < -1 + b \,\&\&$$
$$0 < \tau_D < \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \,\&\&$$
$$1 < b < 2 + \sqrt{5} \,\&\&\, \frac{c\,\tau_D}{2\,b} < \tau_C < \tau_D\Big) \,||$$

$$\Big(0 < c < -1 + b \,\&\&\, 1 < b < 2 + \sqrt{5} \,\&\&\, \frac{c\,\tau_D}{2\,b} < \tau_C < \tau_D \,\&\&$$
$$\tau_D > \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}}\Big) \,||$$

$$\Big(0 < c < -1 + b \,\&\&\, 1 < b < 2 + \sqrt{5} \,\&\&$$
$$\sqrt{2}\,\sqrt{\frac{-2\,b^2 + 2\,b^3 + 3\,b\,c - 4\,b^2\,c - c^2 + 2\,b\,c^2}{b}} + \frac{2\,b - 4\,b^2 - c + 4\,b\,c + c\,\tau_D}{2\,b} < \tau_C < \frac{c\,\tau_D}{2\,b} \,\&$$
$$\tau_D > \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}}\Big)$$

We analyse the first possible stationary state

In[560]:=
```
Simplify[x /. Normal[solutions[[1]]], b ≥ 1 + c && 0 < τC < τD && c > 0 && b < 2 + √5]
```

Out[560]=
$$\frac{1}{\left(-2\,b\,\tau_C+c\,\tau_D\right)^2}\Big(-2\,b\,\tau_C+c\,\tau_C+2\,b\,\tau_C^2+2\,b\,\tau_D-c\,\tau_D-2\,b\,\tau_C\,\tau_D+4\,b^2\,\tau_C\,\tau_D-c\,\tau_C\,\tau_D-4\,b\,c\,\tau_C\,\tau_D+c\,\tau_D^2-2\,b\,c\,\tau_D^2+2\,c^2\,\tau_D^2+\tau_C\sqrt{\left(\left(-2\,b+c\right)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}-\tau_D\sqrt{\left(\left(-2\,b+c\right)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}\Big)$$

In[561]:=
```
x2
```

Out[561]=
$$\frac{1}{\left(-2\,b\,\tau_C+c\,\tau_D\right)^2}\Big(2\,b\,\tau_C^2+\tau_C\left(-2\,b+c+\left(2\,b\left(-1+2\,b-2\,c\right)-c\right)\tau_D\right)+\tau_D\left(2\,b-c+c\left(1-2\,b+2\,c\right)\tau_D\right)+\left(\tau_C-\tau_D\right)\sqrt{\left(-2\,b+c\right)^2+\left(2\,b\,\tau_C-c\,\tau_D\right)\left(4\,b\left(-1+2\,b-2\,c\right)+2\,c+2\,b\,\tau_C-c\,\tau_D\right)}\Big)$$

In[562]:=
$$\mathrm{Reduce}\Big[\frac{1}{(-2\,b\,\tau_C+c\,\tau_D)^2}\Big(-2\,b\,\tau_C+c\,\tau_C+2\,b\,\tau_C^2+2\,b\,\tau_D-c\,\tau_D-2\,b\,\tau_C\,\tau_D+4\,b^2\,\tau_C\,\tau_D-c\,\tau_C\,\tau_D-4\,b\,c\,\tau_C\,\tau_D+c\,\tau_D^2-2\,b\,c\,\tau_D^2+2\,c^2\,\tau_D^2+\tau_C\sqrt{\left((-2\,b+c)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}-\tau_D\sqrt{\left((-2\,b+c)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}\Big)==\frac{1}{(-2\,b\,\tau_C+c\,\tau_D)^2}\Big(2\,b\,\tau_C^2+\tau_C\,(-2\,b+c+(2\,b\,(-1+2\,b-2\,c)-c)\,\tau_D)+\tau_D\,(2\,b-c+c\,(1-2\,b+2\,c)\,\tau_D)+(\tau_C-\tau_D)\sqrt{(-2\,b+c)^2+(2\,b\,\tau_C-c\,\tau_D)\,(4\,b\,(-1+2\,b-2\,c)+2\,c+2\,b\,\tau_C-c\,\tau_D)}\Big)\Big]$$

Out[562]=
```
True
```

Then, we check when $x_2 < 1$

In[563]:= `Reduce[Normal[x /. solutions[[1]] ] < 1 && 0 < τC < τD && b ≥ c + 1 && b ≤ 2 + √5 && c > 0, τD ]`

Out[563]=
$$1 < b \leq 2 + \sqrt{5} \,\&\&\, 0 < c \leq -1 + b \,\&\& \left(\left(0 < \tau_C < \frac{2 b c^2 - c^3}{-8 b^3 + 8 b^4 + 12 b^2 c - 16 b^3 c - 4 b c^2 + 8 b^2 c^2} \,\&\& \left(\tau_C < \tau_D < \frac{2 b \tau_C}{c} \,||\, \frac{2 b \tau_C}{c} < \tau_D < \frac{c - 4 b \tau_C + 4 b^2 \tau_C + c \tau_C - 2 b c \tau_C}{-4 b + 4 b^2 + 2 c - 4 b c + c^2} + \sqrt{\frac{c^2 - 2 c^2 \tau_C + 4 b c^2 \tau_C - 2 c^3 \tau_C + c^2 \tau_C^2}{\left(-4 b + 4 b^2 + 2 c - 4 b c + c^2\right)^2}} \,||\, \tau_D \geq 2 \sqrt{2} \sqrt{\frac{-2 b^3 + 2 b^4 + 3 b^2 c - 4 b^3 c - b c^2 + 2 b^2 c^2}{c^2}} + \frac{-2 b + 4 b^2 + c - 4 b c + 2 b \tau_C}{c}\right)\right) \,||\, \left(\tau_C == \frac{2 b c^2 - c^3}{-8 b^3 + 8 b^4 + 12 b^2 c - 16 b^3 c - 4 b c^2 + 8 b^2 c^2} \,\&\& \left(\tau_C < \tau_D < \frac{2 b \tau_C}{c} \,||\, \tau_D \geq 2 \sqrt{2} \sqrt{\frac{-2 b^3 + 2 b^4 + 3 b^2 c - 4 b^3 c - b c^2 + 2 b^2 c^2}{c^2}} + \frac{-2 b + 4 b^2 + c - 4 b c + 2 b \tau_C}{c}\right)\right) \,||\, \left(\tau_C > \frac{2 b c^2 - c^3}{-8 b^3 + 8 b^4 + 12 b^2 c - 16 b^3 c - 4 b c^2 + 8 b^2 c^2} \,\&\& \left(\tau_C < \tau_D < \frac{c - 4 b \tau_C + 4 b^2 \tau_C + c \tau_C - 2 b c \tau_C}{-4 b + 4 b^2 + 2 c - 4 b c + c^2} + \sqrt{\frac{c^2 - 2 c^2 \tau_C + 4 b c^2 \tau_C - 2 c^3 \tau_C + c^2 \tau_C^2}{\left(-4 b + 4 b^2 + 2 c - 4 b c + c^2\right)^2}} \,||\, \tau_D \geq 2 \sqrt{2} \sqrt{\frac{-2 b^3 + 2 b^4 + 3 b^2 c - 4 b^3 c - b c^2 + 2 b^2 c^2}{c^2}} + \frac{-2 b + 4 b^2 + c - 4 b c + 2 b \tau_C}{c}\right)\right)\right)$$

The existence of e_2 depends only on values of delays and not on values of game parameters.

Then, we check the effects of delays on x_2 :

In[567]:= `Reduce[D[x2, τC] < 0 && 0 < τC < τD && b ≥ c + 1 && b ≤ 2 + √5 && c > 0]`

Out[567]=
$$1 < b \le 2+\sqrt{5} \,\&\&\, 0 < c \le -1+b \,\&\& \left(\left(0<\tau_D\le \frac{-2b+4b^2+c-4bc}{c}-2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}} \,\&\& \left(0<\tau_C<\frac{c\,\tau_D}{2b} \,||\, \frac{c\,\tau_D}{2b}<\tau_C<\tau_D\right)\right) \,||\right.$$
$$\left(\frac{-2b+4b^2+c-4bc}{c}-2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}<\tau_D\le \frac{-2b+4b^2+c-4bc}{c}+2\sqrt{\frac{b^2-4b^3+4b^4+6b^2c-8b^3c-2bc^2+4b^2c^2}{c^2}} \,\&\&\right.$$
$$\left.\left(\sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}}+\frac{2b-4b^2-c+4bc+c\,\tau_D}{2b}<\tau_C<\frac{c\,\tau_D}{2b} \,||\, \frac{c\,\tau_D}{2b}<\tau_C<\tau_D\right)\right) \,||$$
$$\left(\tau_D>\frac{-2b+4b^2+c-4bc}{c}+2\sqrt{\frac{b^2-4b^3+4b^4+6b^2c-8b^3c-2bc^2+4b^2c^2}{c^2}} \,\&\& \left(0<\tau_C<\frac{-4b\,\tau_D+c\,\tau_D+4b\,\tau_D^2-8b^2\,\tau_D^2-2c\,\tau_D^2+8bc\,\tau_D^2+c\,\tau_D^3}{-2b+2b\,\tau_D^2} \,||\right.\right.$$
$$\left.\left.\left.\sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}}+\frac{2b-4b^2-c+4bc+c\,\tau_D}{2b}<\tau_C<\frac{c\,\tau_D}{2b} \,||\, \frac{c\,\tau_D}{2b}<\tau_C<\tau_D\right)\right)\right)$$

In[572]:= `Reduce[D[x2, τC] ≥ 0 && 0 < τC < τD && b ≥ c + 1 && b ≤ 2 + √5 && c > 0 && x2 > 0]`

Out[572]= `False`

The value of internal stationary state always decreases with the increase of cooperator delay. Then, we check the limit of $\tau_C \to \tau_D$

In[568]:= `Simplify[Limit[x2, τC → τD ], b ≥ c + 1 && b ≤ 2 + √5 && c > 0]`

Out[568]=
$$\frac{2\,(b-c)}{2b-c}$$

The limit coincides with the internal stationary state of the Snowdrift game with no delays.

Next, we investigate the effect of defector delay

In[578]:= `Reduce[D[x2, τD] < 0 && 0 < τC < τD && b ≥ c + 1 && b ≤ 2 + √5 && c > 0 && x2 > 0]`

Out[578]= `False`

In[579]:= Reduce[D[x2, τD] > 0 && 0 < τC < τD && b ≥ c + 1 && b ≤ 2 + √5 && c > 0 && x2 > 0]

Out[579]= $1 < b \leq 2 + \sqrt{5}$ && $0 < c \leq -1 + b$ && $\tau_C > 0$ && $\left(\tau_C < \tau_D < \frac{2\, b\, \tau_C}{c} \,||\, \frac{2\, b\, \tau_C}{c} < \tau_D < -2\sqrt{2}\sqrt{\frac{-2 b^3 + 2 b^4 + 3 b^2 c - 4 b^3 c - b c^2 + 2 b^2 c^2}{c^2}} + \frac{-2 b + 4 b^2 + c - 4 b c + 2 b \tau_C}{c}\right)$

The increase in defector delay leads to an increase in value of x_2. Then, we investigate the limit of $\tau_D \to \infty$

In[580]:= Simplify[Limit[x2, τD → Infinity], 0 < τC < τD && b ≥ c + 1 && b ≤ 2 + √5 && c > 0]

Out[580]= $2 - \frac{2\, b}{c}$

In[583]:= Reduce[$2 - \frac{2\, b}{c} \leq 1$ && b ≥ c + 1 && b ≤ 2 + √5 && c > 0]

Out[583]= $1 < b \leq 2 + \sqrt{5}$ && $0 < c \leq -1 + b$

In the limit of $\tau_D \to \infty$ the value of x_2 goes to a limiting value $2 - \frac{2\, b}{c}$.

Next, we investigate the second possible internal stationary state

In[584]:= Simplify[x /. Normal[solutions⟦2⟧], b ≥ 1 + c && 0 < τC < τD && c > 0 && b < 2 + √5]

Out[584]=
$$\frac{1}{(-2\, b\, \tau_C + c\, \tau_D)^2}\Big(-2\, b\, \tau_C + c\, \tau_C + 2\, b\, \tau_C^2 + 2\, b\, \tau_D - c\, \tau_D - 2\, b\, \tau_C\, \tau_D + 4\, b^2\, \tau_C\, \tau_D - c\, \tau_C\, \tau_D - 4\, b\, c\, \tau_C\, \tau_D + c\, \tau_D^2 - 2\, b\, c\, \tau_D^2 + 2\, c^2\, \tau_D^2 - \tau_C \sqrt{\left((-2\, b + c)^2 + 4\, b^2\, \tau_C^2 - 2\, c\, (-2\, b + 4\, b^2 + c - 4\, b\, c)\, \tau_D + c^2\, \tau_D^2 + 4\, b\, \tau_C\, (-2\, b + 4\, b^2 + c - 4\, b\, c - c\, \tau_D)\right)} + \tau_D \sqrt{\left((-2\, b + c)^2 + 4\, b^2\, \tau_C^2 - 2\, c\, (-2\, b + 4\, b^2 + c - 4\, b\, c)\, \tau_D + c^2\, \tau_D^2 + 4\, b\, \tau_C\, (-2\, b + 4\, b^2 + c - 4\, b\, c - c\, \tau_D)\right)}\Big)$$

In[585]:= x3

Out[585]=
$$\frac{1}{(-2\, b\, \tau_C + c\, \tau_D)^2}\Big(2\, b\, \tau_C^2 + \tau_C\, (-2\, b + c + (2\, b\, (-1 + 2\, b - 2\, c) - c)\, \tau_D) + \tau_D\, (2\, b - c + c\, (1 - 2\, b + 2\, c)\, \tau_D) + (-\tau_C + \tau_D)\sqrt{(-2\, b + c)^2 + (2\, b\, \tau_C - c\, \tau_D)\, (4\, b\, (-1 + 2\, b - 2\, c) + 2\, c + 2\, b\, \tau_C - c\, \tau_D)}\Big)$$

In[586]:= Reduce[

$$\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}\Big(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4\,b\,c\,\tau_C\,\tau_D + c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2 - \tau_C\sqrt{\Big((-2\,b + c)^2 + 4\,b^2\,\tau_C^2 - 2\,c\,\big(-2\,b + 4\,b^2 + c - 4\,b\,c\big)\,\tau_D + c^2\,\tau_D^2 + 4\,b\,\tau_C\,\big(-2\,b + 4\,b^2 + c - 4\,b\,c - c\,\tau_D\big)\Big)} + \tau_D\sqrt{\Big((-2\,b + c)^2 + 4\,b^2\,\tau_C^2 - 2\,c\,\big(-2\,b + 4\,b^2 + c - 4\,b\,c\big)\,\tau_D + c^2\,\tau_D^2 + 4\,b\,\tau_C\,\big(-2\,b + 4\,b^2 + c - 4\,b\,c - c\,\tau_D\big)\Big)}\Big) == \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}\Big(2\,b\,\tau_C^2 + \tau_C\,(-2\,b + c + (2\,b\,(-1 + 2\,b - 2\,c) - c)\,\tau_D) + \tau_D\,(2\,b - c + c\,(1 - 2\,b + 2\,c)\,\tau_D) + (-\tau_C + \tau_D)\,\sqrt{(-2\,b + c)^2 + (2\,b\,\tau_C - c\,\tau_D)\,(4\,b\,(-1 + 2\,b - 2\,c) + 2\,c + 2\,b\,\tau_C - c\,\tau_D)}\Big)\Big]$$

Out[586]= True

Then, we check when $x_3 < 1$

In[590]:= Reduce[x_3 < 1 && 0 < τ_C < τ_D && b ≥ c + 1 && b ≤ 2 + $\sqrt{5}$ && c > 0 , τ_C]

Out[590]=

$$1 < b \le 2 + \sqrt{5}\ \&\&\ 0 < c \le -1 + b\ \&\&$$
$$\Bigg(\Bigg(0 < \tau_D \le \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} + 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}}\ \&\&$$
$$\frac{-c - 4\,b\,\tau_D + 4\,b^2\,\tau_D + c\,\tau_D - 2\,b\,c\,\tau_D}{4\,(-1 + b)\,b} + \frac{1}{4}\sqrt{\frac{c^2 - 2\,c^2\,\tau_D + 4\,b\,c^2\,\tau_D + c^2\,\tau_D^2}{(-1 + b)^2\,b^2}} < \tau_C < \tau_D\Bigg)\ ||$$
$$\Bigg(\tau_D > \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} + 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}}\ \&\&$$
$$\Bigg(0 < \tau_C \le -\sqrt{2}\,\sqrt{\frac{-2\,b^2 + 2\,b^3 + 3\,b\,c - 4\,b^2\,c - c^2 + 2\,b\,c^2}{b}} + \frac{2\,b - 4\,b^2 - c + 4\,b\,c + c\,\tau_D}{2\,b}\ ||$$
$$\frac{-c - 4\,b\,\tau_D + 4\,b^2\,\tau_D + c\,\tau_D - 2\,b\,c\,\tau_D}{4\,(-1 + b)\,b} + \frac{1}{4}\sqrt{\frac{c^2 - 2\,c^2\,\tau_D + 4\,b\,c^2\,\tau_D + c^2\,\tau_D^2}{(-1 + b)^2\,b^2}} < \tau_C < \tau_D\Bigg)\Bigg)\Bigg)$$

and check if it exists in each of the possible parameter intervals:

In[594]:= solutions⟦2⟧

$$\left\{x \to \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}\left(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4\,b\,c\,\tau_C\,\tau_D + c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2\right) + \sqrt{\left(\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^4}\left(4\,b^2\,\tau_C^2 - 4\,b\,c\,\tau_C^2 + c^2\,\tau_C^2 - 8\,b^2\,\tau_C^3 + 16\,b^3\,\tau_C^3 + 4\,b\,c\,\tau_C^3 - 16\,b^2\,c\,\tau_C^3 + 4\,b^2\,\tau_C^4 - 8\,b^2\,\tau_C\,\tau_D + 8\,b\,c\,\tau_C\,\tau_D - 2\,c^2\,\tau_C\,\tau_D + 16\,b^2\,\tau_C^2\,\tau_D - 32\,b^3\,\tau_C^2\,\tau_D - 4\,b\,c\,\tau_C^2\,\tau_D + 24\,b^2\,c\,\tau_C^2\,\tau_D - 2\,c^2\,\tau_C^2\,\tau_D + 8\,b\,c^2\,\tau_C^2\,\tau_D - 8\,b^2\,\tau_C^3\,\tau_D - 4\,b\,c\,\tau_C^3\,\tau_D + 4\,b^2\,\tau_D^2 - 4\,b\,c\,\tau_D^2 + c^2\,\tau_D^2 - 8\,b^2\,\tau_C\,\tau_D^2 + 16\,b^3\,\tau_C\,\tau_D^2 - 4\,b\,c\,\tau_C\,\tau_D^2 + 4\,c^2\,\tau_C\,\tau_D^2 - 16\,b\,c^2\,\tau_C\,\tau_D^2 + 4\,b^2\,\tau_C^2\,\tau_D^2 + 8\,b\,c\,\tau_C^2\,\tau_D^2 + c^2\,\tau_C^2\,\tau_D^2 + 4\,b\,c\,\tau_D^3 - 8\,b^2\,c\,\tau_D^3 - 2\,c^2\,\tau_D^3 + 8\,b\,c^2\,\tau_D^3 - 4\,b\,c\,\tau_C\,\tau_D^3 - 2\,c^2\,\tau_C\,\tau_D^3 + c^2\,\tau_D^4\right)\right)} \text{ if }$$

$$\left(0 < c < -1 + b \;\&\&\; 0 < \tau_D < \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \;\&\&\; 0 < \tau_C < \frac{c\,\tau_D}{2\,b} \;\&\&\; 1 < b < 2 + \sqrt{5}\right) \;||$$

$$\left(0 < c < -1 + b \;\&\&\; 1 < b < 2 + \sqrt{5} \;\&\&\; \sqrt{2}\,\sqrt{\frac{-2\,b^2 + 2\,b^3 + 3\,b\,c - 4\,b^2\,c - c^2 + 2\,b\,c^2}{b}} + \frac{2\,b - 4\,b^2 - c + 4\,b\,c + c\,\tau_D}{2\,b} < \tau_C < \frac{c\,\tau_D}{2\,b} \;\&\&\; \tau_D > \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}}\right)$$

In[595]:=

$$\text{Reduce}\left[0 < \tau_D \le \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} + 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \;\&\&\; 0 < \tau_D < \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \;\&\&\; 0 < \tau_C < \tau_D \;\&\&\; b \ge c + 1 \;\&\&\; b \le 2 + \sqrt{5} \;\&\&\; c > 0\right]$$

Out[595]=

$$1 < b \le 2 + \sqrt{5} \;\&\&\; 0 < c \le -1 + b \;\&\&\; 0 < \tau_D < \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \;\&\&\; 0 < \tau_C < \tau_D$$

In[597]:= $\text{Reduce}\left[\frac{-c-4 b \tau_D+4 b^2 \tau_D+c \tau_D-2 b c \tau_D}{4(-1+b) b}+\frac{1}{4} \sqrt{\frac{c^2-2 c^2 \tau_D+4 b c^2 \tau_D+c^2 \tau_D^2}{(-1+b)^2 b^2}}<\tau_C \,\&\&\, 0<\tau_C<\frac{c \tau_D}{2 b} \,\&\&\, b \geq c+1 \,\&\&\, b \leq 2+\sqrt{5} \,\&\&\, c>0\right]$

Out[597]= False

x_3 does not exist if $0 < \tau_D \leq \frac{-2 b+4 b^2+c-4 b c}{c}+2 \sqrt{2} \sqrt{\frac{-2 b^3+2 b^4+3 b^2 c-4 b^3 c-b c^2+2 b^2 c^2}{c^2}}$.

In[598]:= $\text{Reduce}\left[\tau_D>\frac{-2 b+4 b^2+c-4 b c}{c}-2 \sqrt{2} \sqrt{\frac{-2 b^3+2 b^4+3 b^2 c-4 b^3 c-b c^2+2 b^2 c^2}{c^2}} \,\&\&\, \tau_D>\frac{-2 b+4 b^2+c-4 b c}{c}+2 \sqrt{2} \sqrt{\frac{-2 b^3+2 b^4+3 b^2 c-4 b^3 c-b c^2+2 b^2 c^2}{c^2}} \,\&\&\, b \geq c+1 \,\&\&\, b \leq 2+\sqrt{5} \,\&\&\, c>0\right]$

Out[598]= $1<b \leq 2+\sqrt{5} \,\&\&\, 0<c \leq -1+b \,\&\&\, \tau_D>\frac{-2 b+4 b^2+c-4 b c}{c}+2 \sqrt{2} \sqrt{\frac{-2 b^3+2 b^4+3 b^2 c-4 b^3 c-b c^2+2 b^2 c^2}{c^2}}$

In[599]:= $\text{Reduce}\left[0<\tau_C \leq -\sqrt{2} \sqrt{\frac{-2 b^2+2 b^3+3 b c-4 b^2 c-c^2+2 b c^2}{b}}+\frac{2 b-4 b^2-c+4 b c+c \tau_D}{2 b} \,\&\&\, \sqrt{2} \sqrt{\frac{-2 b^2+2 b^3+3 b c-4 b^2 c-c^2+2 b c^2}{b}}+\frac{2 b-4 b^2-c+4 b c+c \tau_D}{2 b}<\tau_C \,\&\&\, b \geq c+1 \,\&\&\, b \leq 2+\sqrt{5} \,\&\&\, c>0\right]$

Out[599]= False

In[602]:= $\text{Reduce}\left[\frac{-c-4 b \tau_D+4 b^2 \tau_D+c \tau_D-2 b c \tau_D}{4(-1+b) b}+\frac{1}{4} \sqrt{\frac{c^2-2 c^2 \tau_D+4 b c^2 \tau_D+c^2 \tau_D^2}{(-1+b)^2 b^2}}<\tau_C<\tau_D \,\&\&\, 0<c<-1+b \,\&\&\, 1<b<2+\sqrt{5} \,\&\&\, \sqrt{2} \sqrt{\frac{-2 b^2+2 b^3+3 b c-4 b^2 c-c^2+2 b c^2}{b}}+\frac{2 b-4 b^2-c+4 b c+c \tau_D}{2 b}<\tau_C<\frac{c \tau_D}{2 b} \,\&\&\, \tau_D>\frac{-2 b+4 b^2+c-4 b c}{c}+2 \sqrt{2} \sqrt{\frac{-2 b^3+2 b^4+3 b^2 c-4 b^3 c-b c^2+2 b^2 c^2}{c^2}}\right]$

Out[602]= False

x_3 does not exist if $\tau_C < \tau_D$ and $b < 2+\sqrt{5}$

$\tau_C < \tau_D$ and $b \geq 2+\sqrt{5}$

Again, we solve for possible solutions of the system in this parameter space:

In[603]:=
```
solutions =
  Solve[FullSimplify[dy_C[x, y_{C,2,x}[x], y_{D,2}[x]]] == 0 && b ≥ 1 + c && 0 < τ_C < τ_D &&
    c > 0 && b ≥ 2 + √5 && x > 0, x, Reals]
```

$$\Big\{\Big\{x \to \frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^2}\big(-2\,b\,\tau_C + c\,\tau_C + 2\,b\,\tau_C^2 + 2\,b\,\tau_D - c\,\tau_D - 2\,b\,\tau_C\,\tau_D + 4\,b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4\,b\,c\,\tau_C\,\tau_D + c\,\tau_D^2 - 2\,b\,c\,\tau_D^2 + 2\,c^2\,\tau_D^2\big) -$$

$$\sqrt{\Big(\frac{1}{(-2\,b\,\tau_C + c\,\tau_D)^4}\big(4\,b^2\,\tau_C^2 - 4\,b\,c\,\tau_C^2 + c^2\,\tau_C^2 - 8\,b^2\,\tau_C^3 + 16\,b^3\,\tau_C^3 + 4\,b\,c\,\tau_C^3 - 16\,b^2\,c\,\tau_C^3 + 4\,b^2\,\tau_C^4 - 8\,b^2\,\tau_C\,\tau_D + 8\,b\,c\,\tau_C\,\tau_D - 2\,c^2\,\tau_C\,\tau_D + 16\,b^2\,\tau_C^2\,\tau_D - 32\,b^3\,\tau_C^2\,\tau_D - 4\,b\,c\,\tau_C^2\,\tau_D + 24\,b^2\,c\,\tau_C^2\,\tau_D - 2\,c^2\,\tau_C^2\,\tau_D + 8\,b\,c^2\,\tau_C^2\,\tau_D - 8\,b^2\,\tau_C^3\,\tau_D - 4\,b\,c\,\tau_C^3\,\tau_D + 4\,b^2\,\tau_D^2 - 4\,b\,c\,\tau_D^2 + c^2\,\tau_D^2 - 8\,b^2\,\tau_C\,\tau_D^2 + 16\,b^3\,\tau_C\,\tau_D^2 - 4\,b\,c\,\tau_C\,\tau_D^2 + 4\,c^2\,\tau_C\,\tau_D^2 - 16\,b\,c^2\,\tau_C\,\tau_D^2 + 4\,b^2\,\tau_C^2\,\tau_D^2 + 8\,b\,c\,\tau_C^2\,\tau_D^2 + c^2\,\tau_C^2\,\tau_D^2 + 4\,b\,c\,\tau_D^3 - 8\,b^2\,c\,\tau_D^3 - 2\,c^2\,\tau_D^3 + 8\,b\,c^2\,\tau_D^3 - 4\,b\,c\,\tau_C\,\tau_D^3 - 2\,c^2\,\tau_C\,\tau_D^3 + c^2\,\tau_D^4\big)\Big)} \quad \text{if}$$

$$\Big(0 < c < -1 + b \;\&\&\; 0 < \tau_D < \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \;\&\&\; 0 < \tau_C < \frac{c\,\tau_D}{2\,b} \;\&\&\; b > 2 + \sqrt{5}\Big) \;||$$

$$\Big(0 < c < -1 + b \;\&\&\; 0 < \tau_D < \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} \;\&\&\; \frac{c\,\tau_D}{2\,b} < \tau_C < \tau_D \;\&\&\; b > 2 + \sqrt{5}\Big) \;||$$

$$\Big(0 < c < -1 + b \;\&\&\; \frac{c\,\tau_D}{2\,b} < \tau_C < \tau_D \;\&\&\; b > 2 + \sqrt{5} \;\&\&\; \tau_D > \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}}\Big) \;||$$

$$\Big(0 < c < -1 + b \;\&\&\; \sqrt{2}\,\sqrt{\frac{-2\,b^2 + 2\,b^3 + 3\,b\,c - 4\,b^2\,c - c^2 + 2\,b\,c^2}{b}} + \frac{2\,b - 4\,b^2 - c + 4\,b\,c + c\,\tau}{2\,b}$$

$$\tau_C < \frac{c\,\tau_D}{2\,b} \;\&\&\; b > 2 + \sqrt{5} \;\&\&\; \tau_D > \frac{-2\,b + 4\,b^2 + c - 4\,b\,c}{c} - 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}}\Big)$$

We analyse the first possible stationary state

In[604]:=
```
Simplify[x /. Normal[solutions[[1]]], b ≥ 1 + c && 0 < τC < τD && c > 0 && b ≥ 2 + √5]
```

Out[604]=
$$\frac{1}{\left(-2\,b\,\tau_C+c\,\tau_D\right)^2}\Big(-2\,b\,\tau_C+c\,\tau_C+2\,b\,\tau_C^2+2\,b\,\tau_D-c\,\tau_D-2\,b\,\tau_C\,\tau_D+4\,b^2\,\tau_C\,\tau_D-c\,\tau_C\,\tau_D-4\,b\,c\,\tau_C\,\tau_D+c\,\tau_D^2-2\,b\,c\,\tau_D^2+2\,c^2\,\tau_D^2+\tau_C\sqrt{\left(\left(-2\,b+c\right)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}-\tau_D\sqrt{\left(\left(-2\,b+c\right)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}\Big)$$

In[605]:= x_2

Out[605]=
$$\frac{1}{\left(-2\,b\,\tau_C+c\,\tau_D\right)^2}\Big(2\,b\,\tau_C^2+\tau_C\left(-2\,b+c+\left(2\,b\left(-1+2\,b-2\,c\right)-c\right)\tau_D\right)+\tau_D\left(2\,b-c+c\left(1-2\,b+2\,c\right)\tau_D\right)+\left(\tau_C-\tau_D\right)\sqrt{\left(-2\,b+c\right)^2+\left(2\,b\,\tau_C-c\,\tau_D\right)\left(4\,b\left(-1+2\,b-2\,c\right)+2\,c+2\,b\,\tau_C-c\,\tau_D\right)}\Big)$$

In[606]:= Reduce[

$$\frac{1}{\left(-2\,b\,\tau_C+c\,\tau_D\right)^2}\Big(-2\,b\,\tau_C+c\,\tau_C+2\,b\,\tau_C^2+2\,b\,\tau_D-c\,\tau_D-2\,b\,\tau_C\,\tau_D+4\,b^2\,\tau_C\,\tau_D-c\,\tau_C\,\tau_D-4\,b\,c\,\tau_C\,\tau_D+c\,\tau_D^2-2\,b\,c\,\tau_D^2+2\,c^2\,\tau_D^2+\tau_C\sqrt{\left(\left(-2\,b+c\right)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}-\tau_D\sqrt{\left(\left(-2\,b+c\right)^2+4\,b^2\,\tau_C^2-2\,c\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}\Big)=\frac{1}{\left(-2\,b\,\tau_C+c\,\tau_D\right)^2}\Big(2\,b\,\tau_C^2+\tau_C\left(-2\,b+c+\left(2\,b\left(-1+2\,b-2\,c\right)-c\right)\tau_D\right)+\tau_D\left(2\,b-c+c\left(1-2\,b+2\,c\right)\tau_D\right)+\left(\tau_C-\tau_D\right)\sqrt{\left(-2\,b+c\right)^2+\left(2\,b\,\tau_C-c\,\tau_D\right)\left(4\,b\left(-1+2\,b-2\,c\right)+2\,c+2\,b\,\tau_C-c\,\tau_D\right)}\Big)$$
]

Out[606]= True

Then, we check when $x_2 < 1$

In[607]:=
```
Reduce[Normal[x /. solutions[[1]] ] < 1 && 0 < τC < τD && b ≥ c + 1 && b ≥ 2 + √5 && c > 0, τD ]
```

Out[607]=
$$\Bigg(b=2+\sqrt{5}\ \&\&\ 0<c\le 1+\sqrt{5}\ \&\&\ \Bigg(\Bigg(\Bigg(0<\tau_C<\left(2\left(2+\sqrt{5}\right)c^2-c^3\right)\Big/\Big(-8\left(2+\sqrt{5}\right)^3+8\left(2+\sqrt{5}\right)^4+12\left(2+\sqrt{5}\right)^2 c-16\left(2+\sqrt{5}\right)^3 c-4\left(2+\sqrt{5}\right)c^2+8\left(2+\sqrt{5}\right)^2 c^2\Big)\ \&\&\ \Bigg(\tau_C<\tau_D<\frac{2\left(2+\sqrt{5}\right)\tau_C}{c}\ ||$$

$$\frac{2\left(2+\sqrt{5}\right)\tau_C}{c} < \tau_D < \frac{c-4\left(2+\sqrt{5}\right)\tau_C+4\left(2+\sqrt{5}\right)^2\tau_C+c\,\tau_C-2\left(2+\sqrt{5}\right)c\,\tau_C}{-4\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+2\,c-4\left(2+\sqrt{5}\right)c+c^2} +$$

$$\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\left(2+\sqrt{5}\right)c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{\left(-4\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+2\,c-4\left(2+\sqrt{5}\right)c+c^2\right)^2}} \;||$$

$$\tau_D \geq 2\,\sqrt{2}\,\sqrt{\left(\frac{1}{c^2}\left(-2\left(2+\sqrt{5}\right)^3+2\left(2+\sqrt{5}\right)^4+3\left(2+\sqrt{5}\right)^2c-\right.\right.}$$

$$\left.4\left(2+\sqrt{5}\right)^3c-\left(2+\sqrt{5}\right)c^2+2\left(2+\sqrt{5}\right)^2c^2\right)+$$

$$\left.\left.\frac{-2\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+c-4\left(2+\sqrt{5}\right)c+2\left(2+\sqrt{5}\right)\tau_C}{c}\right)\right) \;||$$

$$\left(\tau_C == \left(2\left(2+\sqrt{5}\right)c^2-c^3\right)\Big/\left(-8\left(2+\sqrt{5}\right)^3+8\left(2+\sqrt{5}\right)^4+12\left(2+\sqrt{5}\right)^2c-\right.\right.$$

$$\left.16\left(2+\sqrt{5}\right)^3c-4\left(2+\sqrt{5}\right)c^2+8\left(2+\sqrt{5}\right)^2c^2\right)\;\&\&$$

$$\left(\tau_C < \tau_D < \frac{2\left(2+\sqrt{5}\right)\tau_C}{c}\;||\;\tau_D \geq 2\,\sqrt{2}\,\sqrt{\left(\frac{1}{c^2}\left(-2\left(2+\sqrt{5}\right)^3+2\left(2+\sqrt{5}\right)^4+\right.\right.}\right.$$

$$\left.3\left(2+\sqrt{5}\right)^2c-4\left(2+\sqrt{5}\right)^3c-\left(2+\sqrt{5}\right)c^2+2\left(2+\sqrt{5}\right)^2c^2\right)+$$

$$\left.\left.\frac{-2\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+c-4\left(2+\sqrt{5}\right)c+2\left(2+\sqrt{5}\right)\tau_C}{c}\right)\right) \;||$$

$$\left(\tau_C > \left(2\left(2+\sqrt{5}\right)c^2-c^3\right)\Big/\left(-8\left(2+\sqrt{5}\right)^3+8\left(2+\sqrt{5}\right)^4+12\left(2+\sqrt{5}\right)^2c-\right.\right.$$

$$\left.16\left(2+\sqrt{5}\right)^3c-4\left(2+\sqrt{5}\right)c^2+8\left(2+\sqrt{5}\right)^2c^2\right)\;\&\&$$

$$\left(\tau_C < \tau_D < \frac{c-4\left(2+\sqrt{5}\right)\tau_C+4\left(2+\sqrt{5}\right)^2\tau_C+c\,\tau_C-2\left(2+\sqrt{5}\right)c\,\tau_C}{-4\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+2\,c-4\left(2+\sqrt{5}\right)c+c^2} +\right.$$

$$\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\left(2+\sqrt{5}\right)c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{\left(-4\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+2\,c-4\left(2+\sqrt{5}\right)c+c^2\right)^2}} \;||$$

$$\tau_D \geq 2\,\sqrt{2}\,\sqrt{\left(\frac{1}{c^2}\left(-2\left(2+\sqrt{5}\right)^3+2\left(2+\sqrt{5}\right)^4+3\left(2+\sqrt{5}\right)^2c-\right.\right.}$$

$$\left.4\left(2+\sqrt{5}\right)^3c-\left(2+\sqrt{5}\right)c^2+2\left(2+\sqrt{5}\right)^2c^2\right)+$$

$$\left.\left.\left.\left.\frac{-2\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+c-4\left(2+\sqrt{5}\right)c+2\left(2+\sqrt{5}\right)\tau_C}{c}\right)\right)\right)\right) \;||$$

$$\Bigg(b > 2 + \sqrt{5} \;\&\&\; \Bigg(\Bigg(0 < c \le \text{Root}\left[-8\,b^2 + 8\,b^3 + \left(8\,b - 12\,b^2\right) \#1 + 2\,b\,\#1^2 + \#1^3 \;\&,\, 2\right] \;\&\&$$

$$\Bigg(\Bigg(0 < \tau_C < \frac{2\,b\,c^2 - c^3}{-8\,b^3 + 8\,b^4 + 12\,b^2\,c - 16\,b^3\,c - 4\,b\,c^2 + 8\,b^2\,c^2} \;\&\&$$

$$\Bigg(\tau_C < \tau_D < \frac{2\,b\,\tau_C}{c} \;||\; \frac{2\,b\,\tau_C}{c} < \tau_D <$$

$$\frac{c - 4\,b\,\tau_C + 4\,b^2\,\tau_C + c\,\tau_C - 2\,b\,c\,\tau_C}{-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2} + \sqrt{\frac{c^2 - 2\,c^2\,\tau_C + 4\,b\,c^2\,\tau_C - 2\,c^3\,\tau_C + c^2\,\tau_C^2}{\left(-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2\right)^2}} \;||$$

$$\tau_D \ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} +$$

$$\frac{-2\,b + 4\,b^2 + c - 4\,b\,c + 2\,b\,\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(\tau_C == \frac{2\,b\,c^2 - c^3}{-8\,b^3 + 8\,b^4 + 12\,b^2\,c - 16\,b^3\,c - 4\,b\,c^2 + 8\,b^2\,c^2} \;\&\&$$

$$\Bigg(\tau_C < \tau_D < \frac{2\,b\,\tau_C}{c} \;||\; \tau_D \ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} +$$

$$\frac{-2\,b + 4\,b^2 + c - 4\,b\,c + 2\,b\,\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(\tau_C > \frac{2\,b\,c^2 - c^3}{-8\,b^3 + 8\,b^4 + 12\,b^2\,c - 16\,b^3\,c - 4\,b\,c^2 + 8\,b^2\,c^2} \;\&\&\; \Bigg(\tau_C < \tau_D <$$

$$\frac{c - 4\,b\,\tau_C + 4\,b^2\,\tau_C + c\,\tau_C - 2\,b\,c\,\tau_C}{-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2} + \sqrt{\frac{c^2 - 2\,c^2\,\tau_C + 4\,b\,c^2\,\tau_C - 2\,c^3\,\tau_C + c^2\,\tau_C^2}{\left(-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2\right)^2}} \;||$$

$$\tau_D \ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} +$$

$$\frac{-2\,b + 4\,b^2 + c - 4\,b\,c + 2\,b\,\tau_C}{c}\Bigg)\Bigg)\Bigg)\Bigg) \;||$$

$$\Bigg(\text{Root}\left[-8\,b^2 + 8\,b^3 + \left(8\,b - 12\,b^2\right) \#1 + 2\,b\,\#1^2 + \#1^3 \;\&,\, 2\right] < c \le -1 + b \;\&\&$$

$$\Bigg(\Bigg(0 < \tau_C < 1 - 2\,b + c + \frac{1}{2\,\sqrt{2}}\Big(\sqrt{}\Big(\Big(64\,b^4 - 128\,b^5 + 64\,b^6 - 128\,b^3\,c + 320\,b^4\,c - 192\,b^5\,c +$$

$$80\,b^2\,c^2 - 272\,b^3\,c^2 + 208\,b^4\,c^2 - 16\,b\,c^3 + 88\,b^2\,c^3 - 96\,b^3\,c^3 + c^4 - 8\,b\,c^4 +$$

$$16\,b^2\,c^4\Big) \Big/ \left(b\,\left(-2\,b^2 + 2\,b^3 + 3\,b\,c - 4\,b^2\,c - c^2 + 2\,b\,c^2\right)\right)\Big)\Big) \;\&\&\; \Big(\tau_C < \tau_D <$$

$$\frac{2\,b\,\tau_C}{c} \;||\; \frac{2\,b\,\tau_C}{c} < \tau_D \le -2\sqrt{2}\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}} + \frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c} \;||$$

$$\tau_D \ge 2\sqrt{2}\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}} + \frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(\tau_C == 1-2\,b+c+\frac{1}{2\sqrt{2}}\Big(\sqrt{\Big(\Big(64\,b^4-128\,b^5+64\,b^6-128\,b^3\,c+320\,b^4\,c-192\,b^5\,c+80\,b^2\,c^2-272\,b^3\,c^2+208\,b^4\,c^2-16\,b\,c^3+88\,b^2\,c^3-96\,b^3\,c^3+c^4-8\,b\,c^4+16\,b^2\,c^4\Big)\Big/\Big(b\,\big(-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2\big)\Big)\Big)}\Big)\;\&\&\;\Bigg(\tau_C < \tau_D <$$

$$\frac{2\,b\,\tau_C}{c} \;||\; \frac{2\,b\,\tau_C}{c} < \tau_D < -2\sqrt{2}\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}} + \frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c} \;||$$

$$\tau_D \ge 2\sqrt{2}\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}} + \frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(1-2\,b+c+\frac{1}{2\sqrt{2}}\Big(\sqrt{\Big(\Big(64\,b^4-128\,b^5+64\,b^6-128\,b^3\,c+320\,b^4\,c-192\,b^5\,c+80\,b^2\,c^2-272\,b^3\,c^2+208\,b^4\,c^2-16\,b\,c^3+88\,b^2\,c^3-96\,b^3\,c^3+c^4-8\,b\,c^4+16\,b^2\,c^4\Big)\Big/\Big(b\,\big(-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2\big)\Big)\Big)}\Big) < \tau_C <$$

$$\frac{2\,b\,c^2-c^3}{-8\,b^3+8\,b^4+12\,b^2\,c-16\,b^3\,c-4\,b\,c^2+8\,b^2\,c^2}\;\&\&\;\Bigg(\tau_C < \tau_D < \frac{2\,b\,\tau_C}{c} \;||\; \frac{2\,b\,\tau_C}{c} < \tau_D <$$

$$\frac{c-4\,b\,\tau_C+4\,b^2\,\tau_C+c\,\tau_C-2\,b\,c\,\tau_C}{-4\,b+4\,b^2+2\,c-4\,b\,c+c^2} + \sqrt{\frac{c^2-2\,c^2\,\tau_C+4\,b\,c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{\left(-4\,b+4\,b^2+2\,c-4\,b\,c+c^2\right)^2}} \;||$$

$$\tau_D \ge 2\sqrt{2}\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}} + \frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(\tau_C == \frac{2\,b\,c^2-c^3}{-8\,b^3+8\,b^4+12\,b^2\,c-16\,b^3\,c-4\,b\,c^2+8\,b^2\,c^2}\;\&\&$$

$$\Bigg(\tau_C < \tau_D < \frac{2\,b\,\tau_C}{c} \;||\; \tau_D \ge 2\sqrt{2}\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}} +$$

$$\left.\left.\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\right)\right)\ ||$$

$$\left(\tau_C>\frac{2\,b\,c^2-c^3}{-8\,b^3+8\,b^4+12\,b^2\,c-16\,b^3\,c-4\,b\,c^2+8\,b^2\,c^2}\ \&\&\ \left(\tau_C<\tau_D<\frac{c-4\,b\,\tau_C+4\,b^2\,\tau_C+c\,\tau_C-2\,b\,c\,\tau_C}{-4\,b+4\,b^2+2\,c-4\,b\,c+c^2}+\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\,b\,c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{\left(-4\,b+4\,b^2+2\,c-4\,b\,c+c^2\right)^2}}\ ||\ \tau_D\geq 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\right)\right)\left.\right)\left.\right)\left.\right)\left.\right)$$

The existence of e_2 depends only on values of delays and not on values of game parameters.

Then, we check the effects of delays on x_2 :

In[608]:=
```
Reduce[D[x₂, τC] < 0 && 0 < τC < τD && b ≥ c + 1 && b ≥ 2 + √5 && c > 0]
```

Out[608]=

$$b\geq 2+\sqrt{5}\ \&\&\ 0<c\leq -1+b\ \&\&\ \left(\left(0<\tau_D\leq\frac{-2\,b+4\,b^2+c-4\,b\,c}{c}-2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}\ \&\&\ \left(0<\tau_C<\frac{c\,\tau_D}{2\,b}\ ||\ \frac{c\,\tau_D}{2\,b}<\tau_C<\tau_D\right)\right)\ ||\right.$$

$$\left(\frac{-2\,b+4\,b^2+c-4\,b\,c}{c}-2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}<\tau_D\leq\frac{-2\,b+4\,b^2+c-4\,b\,c}{c}+2\,\sqrt{\frac{b^2-4\,b^3+4\,b^4+6\,b^2\,c-8\,b^3\,c-2\,b\,c^2+4\,b^2\,c^2}{c^2}}\ \&\&\ \left(\sqrt{2}\,\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}+\frac{2\,b-4\,b^2-c+4\,b\,c+c\,\tau_D}{2\,b}<\tau_C<\frac{c\,\tau_D}{2\,b}\ ||\ \frac{c\,\tau_D}{2\,b}<\tau_C<\tau_D\right)\right)\ ||$$

$$\left.\left(\tau_D>\frac{-2\,b+4\,b^2+c-4\,b\,c}{c}+2\,\sqrt{\frac{b^2-4\,b^3+4\,b^4+6\,b^2\,c-8\,b^3\,c-2\,b\,c^2+4\,b^2\,c^2}{c^2}}\ \&\&\ \left(0<\tau_C<\frac{-4\,b\,\tau_D+c\,\tau_D+4\,b\,\tau_D^2-8\,b^2\,\tau_D^2-2\,c\,\tau_D^2+8\,b\,c\,\tau_D^2+c\,\tau_D^3}{-2\,b+2\,b\,\tau_D^2}\ ||\ \sqrt{2}\,\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}+\frac{2\,b-4\,b^2-c+4\,b\,c+c\,\tau_D}{2\,b}<\tau_C<\frac{c\,\tau_D}{2\,b}\ ||\ \frac{c\,\tau_D}{2\,b}<\tau_C<\tau_D\right)\right)\right)$$

In[609]:= Reduce[D[x_2, τ_C] ≥ 0 && 0 < τ_C < τ_D && b ≥ c + 1 && b ≥ 2 + $\sqrt{5}$ && c > 0 && x_2 > 0]

Out[609]= False

The value of internal stationary state always decreases with the increase of cooperator delay. Then, we check the limit of $\tau_C \to \tau_D$

In[610]:= Simplify[Limit[x_2, $\tau_C \to \tau_D$], b ≥ c + 1 && b ≥ 2 + $\sqrt{5}$ && c > 0]

Out[610]= $$\frac{2\,(b-c)}{2\,b-c}$$

The limit coincides with the internal stationary state of the Snowdrift game with no delays.

Next, we investigate the effect of defector delay

In[611]:= Reduce[D[x_2, τ_D] < 0 && 0 < τ_C < τ_D && b ≥ c + 1 && b ≥ 2 + $\sqrt{5}$ && c > 0 && x_2 > 0]

Out[611]= False

In[612]:= Reduce[D[x_2, τ_D] > 0 && 0 < τ_C < τ_D && b ≥ c + 1 && b ≥ 2 + $\sqrt{5}$ && c > 0 && x_2 > 0]

Out[612]= $$b \geq 2+\sqrt{5}\ \&\&\ 0 < c \leq -1+b\ \&\&\ \tau_C > 0\ \&\&\ \left(\tau_C < \tau_D < \frac{2\,b\,\tau_C}{c}\ ||\ \frac{2\,b\,\tau_C}{c} < \tau_D < -2\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\right)$$

The increase in defector delay leads to an increase in value of x_2. Then, we investigate the limit of $\tau_D \to \infty$

In[613]:= Simplify[Limit[x_2, τ_D → Infinity], 0 < τ_C < τ_D && b ≥ c + 1 && b ≥ 2 + $\sqrt{5}$ && c > 0]

Out[613]= $$2-\frac{2\,b}{c}$$

In[614]:= Reduce[$2-\frac{2\,b}{c}$ ≤ 1 && b ≥ c + 1 && b ≥ 2 + $\sqrt{5}$ && c > 0]

Out[614]= $b \geq 2+\sqrt{5}$ && $0 < c \leq -1+b$

In the limit of $\tau_D \to \infty$ the value of x_2 goes to a limiting value $2-\frac{2\,b}{c}$.

Next, we investigate the second possible internal stationary state

In[615]:= Simplify[x /. Normal[solutions⟦2⟧], b ≥ 1 + c && 0 < τ_C < τ_D && c > 0 && b ≥ 2 + $\sqrt{5}$]

Out[615]= $$\frac{1}{\left(-2\,b\,\tau_C+c\,\tau_D\right)^2}\Big(-2\,b\,\tau_C+c\,\tau_C+2\,b\,\tau_C^2+2\,b\,\tau_D-c\,\tau_D-2\,b\,\tau_C\,\tau_D+4\,b^2\,\tau_C\,\tau_D-c\,\tau_C\,\tau_D-4\,b\,c\,\tau_C\,\tau_D+c\,\tau_D^2-2\,b\,c\,\tau_D^2+2\,c^2\,\tau_D^2-\tau_C\sqrt{\left((-2\,b+c)^2+4\,b^2\,\tau_C^2-2\,c\,\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}+\tau_D\sqrt{\left((-2\,b+c)^2+4\,b^2\,\tau_C^2-2\,c\,\left(-2\,b+4\,b^2+c-4\,b\,c\right)\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\left(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D\right)\right)}\Big)$$

In[ ]:= x_3

In[616]:=
$$\frac{1}{(-2\,b\,\tau_C+c\,\tau_D)^2}\left(2\,b\,\tau_C^2+\tau_C\,(-2\,b+c+(2\,b\,(-1+2\,b-2\,c)-c)\,\tau_D)+\tau_D\,(2\,b-c+c\,(1-2\,b+2\,c)\,\tau_D)+(-\tau_C+\tau_D)\,\sqrt{(-2\,b+c)^2+(2\,b\,\tau_C-c\,\tau_D)\,(4\,b\,(-1+2\,b-2\,c)+2\,c+2\,b\,\tau_C-c\,\tau_D)}\right)$$

Out[616]=
$$\frac{1}{(-2\,b\,\tau_C+c\,\tau_D)^2}\left(2\,b\,\tau_C^2+\tau_C\,(-2\,b+c+(2\,b\,(-1+2\,b-2\,c)-c)\,\tau_D)+\tau_D\,(2\,b-c+c\,(1-2\,b+2\,c)\,\tau_D)+(-\tau_C+\tau_D)\,\sqrt{(-2\,b+c)^2+(2\,b\,\tau_C-c\,\tau_D)\,(4\,b\,(-1+2\,b-2\,c)+2\,c+2\,b\,\tau_C-c\,\tau_D)}\right)$$

In[617]:=
$$\text{Reduce}\Big[\frac{1}{(-2\,b\,\tau_C+c\,\tau_D)^2}\Big(-2\,b\,\tau_C+c\,\tau_C+2\,b\,\tau_C^2+2\,b\,\tau_D-c\,\tau_D-2\,b\,\tau_C\,\tau_D+4\,b^2\,\tau_C\,\tau_D-c\,\tau_C\,\tau_D-4\,b\,c\,\tau_C\,\tau_D+c\,\tau_D^2-2\,b\,c\,\tau_D^2+2\,c^2\,\tau_D^2-\tau_C\sqrt{\left((-2\,b+c)^2+4\,b^2\,\tau_C^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\,(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D)\right)}+\tau_D\sqrt{\left((-2\,b+c)^2+4\,b^2\,\tau_C^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2+4\,b\,\tau_C\,(-2\,b+4\,b^2+c-4\,b\,c-c\,\tau_D)\right)}\Big)==\frac{1}{(-2\,b\,\tau_C+c\,\tau_D)^2}\left(2\,b\,\tau_C^2+\tau_C\,(-2\,b+c+(2\,b\,(-1+2\,b-2\,c)-c)\,\tau_D)+\tau_D\,(2\,b-c+c\,(1-2\,b+2\,c)\,\tau_D)+(-\tau_C+\tau_D)\,\sqrt{(-2\,b+c)^2+(2\,b\,\tau_C-c\,\tau_D)\,(4\,b\,(-1+2\,b-2\,c)+2\,c+2\,b\,\tau_C-c\,\tau_D)}\right)\Big]$$

Out[617]= True

Then, we check when $x_3 < 1$

In[644]:=
$$\text{Reduce}\left[x_3<1\ \&\&\ 0<\tau_C<\tau_D\ \&\&\ b\geq c+1\ \&\&\ b\geq 2+\sqrt{5}\ \&\&\ c>0\ ,\ \tau_D\right]$$

Out[644]=
$$\Bigg(b==2+\sqrt{5}\ \&\&\ 0<c\leq 1+\sqrt{5}\ \&\&\ \tau_C>0\ \&\&$$
$$\Bigg(\tau_C<\tau_D<\frac{c-4\left(2+\sqrt{5}\right)\tau_C+4\left(2+\sqrt{5}\right)^2\tau_C+c\,\tau_C-2\left(2+\sqrt{5}\right)c\,\tau_C}{-4\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+2\,c-4\left(2+\sqrt{5}\right)c+c^2}-\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\left(2+\sqrt{5}\right)c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{\left(-4\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+2\,c-4\left(2+\sqrt{5}\right)c+c^2\right)^2}}\ ||$$
$$\tau_D\geq 2\,\sqrt{2}\,\sqrt{\left(\frac{1}{c^2}\left(-2\left(2+\sqrt{5}\right)^3+2\left(2+\sqrt{5}\right)^4+3\left(2+\sqrt{5}\right)^2c-4\left(2+\sqrt{5}\right)^3c-\left(2+\sqrt{5}\right)c^2+2\left(2+\sqrt{5}\right)^2c^2\right)\right)}+\frac{-2\left(2+\sqrt{5}\right)+4\left(2+\sqrt{5}\right)^2+c-4\left(2+\sqrt{5}\right)c+2\left(2+\sqrt{5}\right)\tau_C}{c}\Bigg)\Bigg)\ ||$$

$$\Bigg(b > 2 + \sqrt{5} \;\&\&\; \Bigg(\Bigg(0 < c \le \text{Root}\big[-8\,b^2 + 8\,b^3 + \left(8\,b - 12\,b^2\right) \#1 + 2\,b\,\#1^2 + \#1^3 \;\&,\; 2\big] \;\&\&\; \tau_C > 0 \;\&\&$$

$$\Bigg(\tau_C < \tau_D < \frac{c - 4\,b\,\tau_C + 4\,b^2\,\tau_C + c\,\tau_C - 2\,b\,c\,\tau_C}{-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2} - \sqrt{\frac{c^2 - 2\,c^2\,\tau_C + 4\,b\,c^2\,\tau_C - 2\,c^3\,\tau_C + c^2\,\tau_C^2}{\left(-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2\right)^2}} \;||$$

$$\tau_D \ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} +$$

$$\frac{-2\,b + 4\,b^2 + c - 4\,b\,c + 2\,b\,\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(\text{Root}\big[-8\,b^2 + 8\,b^3 + \left(8\,b - 12\,b^2\right) \#1 + 2\,b\,\#1^2 + \#1^3 \;\&,\; 2\big] < c \le -1 + b \;\&\&$$

$$\Bigg(\Bigg(0 < \tau_C < 1 - 2\,b + c + \frac{1}{2\,\sqrt{2}}\Big(\sqrt{}\Big(\Big(64\,b^4 - 128\,b^5 + 64\,b^6 - 128\,b^3\,c + 320\,b^4\,c - 192\,b^5\,c +$$

$$80\,b^2\,c^2 - 272\,b^3\,c^2 + 208\,b^4\,c^2 - 16\,b\,c^3 + 88\,b^2\,c^3 - 96\,b^3\,c^3 + c^4 - 8\,b\,c^4 +$$

$$16\,b^2\,c^4\Big) \Big/ \left(b\,\left(-2\,b^2 + 2\,b^3 + 3\,b\,c - 4\,b^2\,c - c^2 + 2\,b\,c^2\right)\right)\Big)\Big)\Big) \;\&\&\; \Bigg(\tau_C < \tau_D <$$

$$\frac{c - 4\,b\,\tau_C + 4\,b^2\,\tau_C + c\,\tau_C - 2\,b\,c\,\tau_C}{-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2} - \sqrt{\frac{c^2 - 2\,c^2\,\tau_C + 4\,b\,c^2\,\tau_C - 2\,c^3\,\tau_C + c^2\,\tau_C^2}{\left(-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2\right)^2}} \;||$$

$$\frac{c - 4\,b\,\tau_C + 4\,b^2\,\tau_C + c\,\tau_C - 2\,b\,c\,\tau_C}{-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2} + \sqrt{\frac{c^2 - 2\,c^2\,\tau_C + 4\,b\,c^2\,\tau_C - 2\,c^3\,\tau_C + c^2\,\tau_C^2}{\left(-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2\right)^2}} <$$

$$\tau_D \le -2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} +$$

$$\frac{-2\,b + 4\,b^2 + c - 4\,b\,c + 2\,b\,\tau_C}{c} \;||$$

$$\tau_D \ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3 + 2\,b^4 + 3\,b^2\,c - 4\,b^3\,c - b\,c^2 + 2\,b^2\,c^2}{c^2}} +$$

$$\frac{-2\,b + 4\,b^2 + c - 4\,b\,c + 2\,b\,\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(\tau_C \ge 1 - 2\,b + c + \frac{1}{2\,\sqrt{2}}\Big(\sqrt{}\Big(\Big(64\,b^4 - 128\,b^5 + 64\,b^6 - 128\,b^3\,c + 320\,b^4\,c - 192\,b^5\,c +$$

$$80\,b^2\,c^2 - 272\,b^3\,c^2 + 208\,b^4\,c^2 - 16\,b\,c^3 + 88\,b^2\,c^3 - 96\,b^3\,c^3 + c^4 - 8\,b\,c^4 +$$

$$16\,b^2\,c^4\Big) \Big/ \left(b\,\left(-2\,b^2 + 2\,b^3 + 3\,b\,c - 4\,b^2\,c - c^2 + 2\,b\,c^2\right)\right)\Big)\Big)\Big) \;\&\&\; \Bigg(\tau_C < \tau_D <$$

$$\frac{c - 4\,b\,\tau_C + 4\,b^2\,\tau_C + c\,\tau_C - 2\,b\,c\,\tau_C}{-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2} - \sqrt{\frac{c^2 - 2\,c^2\,\tau_C + 4\,b\,c^2\,\tau_C - 2\,c^3\,\tau_C + c^2\,\tau_C^2}{\left(-4\,b + 4\,b^2 + 2\,c - 4\,b\,c + c^2\right)^2}} \;||$$

$$\tau_D \geq 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}+\frac{-2b+4b^2+c-4bc+2b\,\tau_C}{c}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)$$

and check if it exists in each of the possible parameter intervals:

In[623]:=
```
solutions[[2]]
```

$$\Big\{x \to \frac{1}{(-2b\,\tau_C + c\,\tau_D)^2}\left(-2b\,\tau_C + c\,\tau_C + 2b\,\tau_C^2 + 2b\,\tau_D - c\,\tau_D - 2b\,\tau_C\,\tau_D + 4b^2\,\tau_C\,\tau_D - c\,\tau_C\,\tau_D - 4bc\,\tau_C\,\tau_D + c\,\tau_D^2 - 2bc\,\tau_D^2 + 2c^2\,\tau_D^2\right) + \sqrt{\left(\frac{1}{(-2b\,\tau_C + c\,\tau_D)^4}\left(4b^2\tau_C^2 - 4bc\,\tau_C^2 + c^2\tau_C^2 - 8b^2\tau_C^3 + 16b^3\tau_C^3 + 4bc\,\tau_C^3 - 16b^2c\,\tau_C^3 + 4b^2\tau_C^4 - 8b^2\tau_C\tau_D + 8bc\,\tau_C\tau_D - 2c^2\tau_C\tau_D + 16b^2\tau_C^2\tau_D - 32b^3\tau_C^2\tau_D - 4bc\,\tau_C^2\tau_D + 24b^2c\,\tau_C^2\tau_D - 2c^2\tau_C^2\tau_D + 8bc^2\tau_C^2\tau_D - 8b^2\tau_C^3\tau_D - 4bc\,\tau_C^3\tau_D + 4b^2\tau_D^2 - 4bc\,\tau_D^2 + c^2\tau_D^2 - 8b^2\tau_C\tau_D^2 + 16b^3\tau_C\tau_D^2 - 4bc\,\tau_C\tau_D^2 + 4c^2\tau_C\tau_D^2 - 16bc^2\tau_C\tau_D^2 + 4b^2\tau_C^2\tau_D^2 + 8bc\,\tau_C^2\tau_D^2 + c^2\tau_C^2\tau_D^2 + 4bc\,\tau_D^3 - 8b^2c\,\tau_D^3 - 2c^2\tau_D^3 + 8bc^2\tau_D^3 - 4bc\,\tau_C\tau_D^3 - 2c^2\tau_C\tau_D^3 + c^2\tau_D^4\right)\right)} \text{ if }$$

$$\left(0 < c < -1+b \;\&\&\; 0 < \tau_D < \frac{-2b+4b^2+c-4bc}{c} - 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}} \;\&\&\; 0 < \tau_C < \frac{c\,\tau_D}{2b} \;\&\&\; b > 2+\sqrt{5}\right) \;||$$

$$\left(0 < c < -1+b \;\&\&\; \sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}} + \frac{2b-4b^2-c+4bc+c\,\tau_D}{2b}\right.$$

$$\left.\tau_C < \frac{c\,\tau_D}{2b} \;\&\&\; b > 2+\sqrt{5} \;\&\&\; \tau_D > \frac{-2b+4b^2+c-4bc}{c} - 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}\right)$$

First, we see that x_3 does not exists when $b = 2 + \sqrt{5}$

$$\text{Reduce}\Big[0<\tau_D<\frac{-2\,b+4\,b^2+c-4\,b\,c}{c}-2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}\ \&\&\ 0<\tau_C<\frac{c\,\tau_D}{2\,b}\ \&\&\ 0<c<-1+b\ \&\&\ b>2+\sqrt{5},\ \tau_D\Big]$$

Out[643]=

$$b>2+\sqrt{5}\ \&\&\ 0<c<-1+b\ \&\&\ 0<\tau_C<\frac{-2\,b+4\,b^2+c-4\,b\,c}{2\,b}-\sqrt{2}\,\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}\ \&\&\ \frac{2\,b\,\tau_C}{c}<\tau_D<\frac{-2\,b+4\,b^2+c-4\,b\,c}{c}-2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}$$

In[652]:=

$$\text{Reduce}\Big[x_3<1\ \&\&\ 0<\tau_C<\tau_D\ \&\&\ b\geq c+1\ \&\&\ b>2+\sqrt{5}\ \&\&\ c>0\ \&\&\ 0<\tau_C<\frac{-2\,b+4\,b^2+c-4\,b\,c}{2\,b}-\sqrt{2}\,\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}\ \&\&\ \frac{2\,b\,\tau_C}{c}<\tau_D,\ \tau_D\Big]$$

Out[652]=

$$\Bigg(2+\sqrt{5}<b<5.92\ldots\ \&\&\ \Bigg(\Bigg(0<c\leq\text{Root}\big[-8\,b^2+8\,b^3+(8\,b-12\,b^2)\,\#1+2\,b\,\#1^2+\#1^3\,\&,\,2\big]\ \&\&\ 0<\tau_C<\frac{-2\,b+4\,b^2+c-4\,b\,c}{2\,b}-\sqrt{2}\,\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}\ \&\&\ \tau_D\geq 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\Bigg)\ ||$$

$$\Bigg(\text{Root}\big[-8\,b^2+8\,b^3+(8\,b-12\,b^2)\,\#1+2\,b\,\#1^2+\#1^3\,\&,\,2\big]<c\leq -1+b\ \&\&$$

$$\Bigg(\Bigg(0<\tau_C<1-2\,b+c+\frac{1}{2\,\sqrt{2}}\Big(\sqrt{\big(\big(64\,b^4-128\,b^5+64\,b^6-128\,b^3\,c+320\,b^4\,c-192\,b^5\,c+80\,b^2\,c^2-272\,b^3\,c^2+208\,b^4\,c^2-16\,b\,c^3+88\,b^2\,c^3-96\,b^3\,c^3+c^4-8\,b\,c^4+16\,b^2\,c^4\big)\big/\big(b\,(-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2)\big)\big)}\Big)\ \&\&$$

$$\left(\frac{c-4\,b\,\tau_C+4\,b^2\,\tau_C+c\,\tau_C-2\,b\,c\,\tau_C}{-4\,b+4\,b^2+2\,c-4\,b\,c+c^2}+\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\,b\,c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{(-4\,b+4\,b^2+2\,c-4\,b\,c+c^2)^2}}<\right.$$

$$\tau_D\leq -2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\ ||$$

$$\left.\tau_D\geq 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\right)\Bigg)\ ||$$

$$\left(1-2b+c+\frac{1}{2\sqrt{2}}\left(\sqrt{\left(\left(64b^4-128b^5+64b^6-128b^3c+320b^4c-192b^5c+80b^2c^2-272b^3c^2+208b^4c^2-16bc^3+88b^2c^3-96b^3c^3+c^4-8bc^4+16b^2c^4\right)/\left(b\left(-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2\right)\right)\right)}\right)\le\tau_C<\right.$$

$$\frac{-2b+4b^2+c-4bc}{2b}-\sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}}\ \&\&$$

$$\left.\tau_D\ge 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}+\frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\right)\Bigg)\Bigg)\Bigg)\Bigg)\ ||$$

$$\left(b==\boxed{5.92\ldots}\ \&\&\ \left(\left(0<c\le\mathrm{Root}\left[-8b^2+8b^3+\left(8b-12b^2\right)\#1+2b\#1^2+\#1^3\&,2\right]\ \&\&\right.\right.\right.$$

$$0<\tau_C<\frac{-2b+4b^2+c-4bc}{2b}-\sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}}\ \&\&$$

$$\left.\tau_D\ge 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}+\frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\right)\ ||$$

$$\left(\mathrm{Root}\left[-8b^2+8b^3+\left(8b-12b^2\right)\#1+2b\#1^2+\#1^3\&,2\right]<c<-1+b\ \&\&\right.$$

$$\left(\left(0<\tau_C<1-2b+c+\frac{1}{2\sqrt{2}}\left(\sqrt{\left(\left(64b^4-128b^5+64b^6-128b^3c+320b^4c-192b^5c+80b^2c^2-272b^3c^2+208b^4c^2-16bc^3+88b^2c^3-96b^3c^3+c^4-8bc^4+16b^2c^4\right)/\left(b\left(-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2\right)\right)\right)}\right)\ \&\&\right.\right.$$

$$\left(\frac{c-4b\tau_C+4b^2\tau_C+c\tau_C-2bc\tau_C}{-4b+4b^2+2c-4bc+c^2}+\sqrt{\frac{c^2-2c^2\tau_C+4bc^2\tau_C-2c^3\tau_C+c^2\tau_C^2}{\left(-4b+4b^2+2c-4bc+c^2\right)^2}}<\right.$$

$$\tau_D\le -2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}+\frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\ ||$$

$$\left.\left.\tau_D\ge 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}+\frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\right)\right)\ ||$$

$$\left(1-2b+c+\frac{1}{2\sqrt{2}}\left(\sqrt{\left(\left(64b^4-128b^5+64b^6-128b^3c+320b^4c-192b^5c+80b^2c^2-272b^3c^2+208b^4c^2-16bc^3+88b^2c^3-96b^3c^3+c^4-\right.\right.}\right.\right.$$

$$8\,b\,c^4+16\,b^2\,c^4\big)\big/\left(b\left(-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2\right)\right)\big)\big)\big)\le\tau_C<$$

$$\frac{-2\,b+4\,b^2+c-4\,b\,c}{2\,b}-\sqrt{2}\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}\;\&\&$$

$$\tau_D\ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+$$

$$\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\Bigg)\Bigg)\Bigg)\Bigg)\;||\;\Bigg(c==-1+b\;\&\&$$

$$0<\tau_C<\frac{-2\,b+4\,b^2+c-4\,b\,c}{2\,b}-\sqrt{2}\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}\;\&\&$$

$$\Bigg(\frac{c-4\,b\,\tau_C+4\,b^2\,\tau_C+c\,\tau_C-2\,b\,c\,\tau_C}{-4\,b+4\,b^2+2\,c-4\,b\,c+c^2}+\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\,b\,c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{\left(-4\,b+4\,b^2+2\,c-4\,b\,c+c^2\right)^2}}<$$

$$\tau_D\le -2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+$$

$$\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\;||$$

$$\tau_D\ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+$$

$$\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\;||$$

$$\Bigg(b>5.92\ldots\;\&\&\;\Bigg(\Bigg(\Bigg(0<c\le\mathrm{Root}\left[-8\,b^2+8\,b^3+\left(8\,b-12\,b^2\right)\#1+2\,b\,\#1^2+\#1^3\,\&,\,2\right]\;\&\&$$

$$0<\tau_C<\frac{-2\,b+4\,b^2+c-4\,b\,c}{2\,b}-\sqrt{2}\sqrt{\frac{-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2}{b}}\;\&\&$$

$$\tau_D\ge 2\,\sqrt{2}\,\sqrt{\frac{-2\,b^3+2\,b^4+3\,b^2\,c-4\,b^3\,c-b\,c^2+2\,b^2\,c^2}{c^2}}+$$

$$\frac{-2\,b+4\,b^2+c-4\,b\,c+2\,b\,\tau_C}{c}\Bigg)\;||$$

$$\Bigg(\mathrm{Root}\left[-8\,b^2+8\,b^3+\left(8\,b-12\,b^2\right)\#1+2\,b\,\#1^2+\#1^3\,\&,\,2\right]<c<$$

$$\mathrm{Root}\left[32\,b^3-32\,b^4+\left(-48\,b^2+64\,b^3\right)\#1+\left(18\,b-34\,b^2\right)\#1^2+\left(-2+3\,b\right)\#1^3\,\&,\,1\right]\;\&\&$$

$$\Bigg(\Bigg(0<\tau_C<1-2\,b+c+\frac{1}{2\,\sqrt{2}}\Big(\sqrt{}\Big(\Big(64\,b^4-128\,b^5+64\,b^6-128\,b^3\,c+320\,b^4\,c-192\,b^5\,c+$$

$$80\,b^2\,c^2-272\,b^3\,c^2+208\,b^4\,c^2-16\,b\,c^3+88\,b^2\,c^3-96\,b^3\,c^3+c^4-$$

$$8\,b\,c^4+16\,b^2\,c^4\big)\big/\left(b\left(-2\,b^2+2\,b^3+3\,b\,c-4\,b^2\,c-c^2+2\,b\,c^2\right)\right)\big)\big)\big)\;\&\&$$

$$\Bigg(\frac{c-4\,b\,\tau_C+4\,b^2\,\tau_C+c\,\tau_C-2\,b\,c\,\tau_C}{-4\,b+4\,b^2+2\,c-4\,b\,c+c^2}+\sqrt{\frac{c^2-2\,c^2\,\tau_C+4\,b\,c^2\,\tau_C-2\,c^3\,\tau_C+c^2\,\tau_C^2}{\left(-4\,b+4\,b^2+2\,c-4\,b\,c+c^2\right)^2}}<$$

$$\tau_D \le -2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}} + \frac{-2b+4b^2+c-4bc+2b\tau_C}{c} \;||$$

$$\tau_D \ge 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}} + \frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\Bigg)\Bigg) \;||$$

$$\Bigg(1-2b+c+\frac{1}{2\sqrt{2}}\Big(\sqrt{}\Big(\Big(64b^4-128b^5+64b^6-128b^3c+320b^4c-192b^5c+80b^2c^2-272b^3c^2+208b^4c^2-16bc^3+88b^2c^3-96b^3c^3+c^4-8bc^4+16b^2c^4\Big)\Big/\Big(b\left(-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2\right)\Big)\Big)\Big)\Big) \le \tau_C <$$

$$\frac{-2b+4b^2+c-4bc}{2b}-\sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}} \;\&\&$$

$$\tau_D \ge 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}} + \frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\Bigg)\Bigg)\Bigg)\Bigg) \;||$$

$$\Big(\mathrm{Root}\left[32b^3-32b^4+\left(-48b^2+64b^3\right)\#1+\left(18b-34b^2\right)\#1^2+\left(-2+3b\right)\#1^3\,\&,\,1\right] \le$$

$$c \le -1+b \;\&\&$$

$$0<\tau_C<\frac{-2b+4b^2+c-4bc}{2b}-\sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}} \;\&\&$$

$$\left(\frac{c-4b\tau_C+4b^2\tau_C+c\tau_C-2bc\tau_C}{-4b+4b^2+2c-4bc+c^2}+\sqrt{\frac{c^2-2c^2\tau_C+4bc^2\tau_C-2c^3\tau_C+c^2\tau_C^2}{\left(-4b+4b^2+2c-4bc+c^2\right)^2}} <\right.$$

$$\tau_D \le -2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}} + \frac{-2b+4b^2+c-4bc+2b\tau_C}{c} \;||$$

$$\tau_D \ge 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}} + \frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)$$

In[653]:=

$$\mathrm{Reduce}\Big[\tau_D \geq 2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}+\frac{-2b+4b^2+c-4bc+2b\tau_C}{c}\ \&\&\ \tau_D < \frac{-2b+4b^2+c-4bc}{c}-2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}\ \&\&\ 0<\tau_C<\tau_D\ \&\&\ b\geq c+1\ \&\&\ b>2+\sqrt{5}\ \&\&\ c>0\Big]$$

Out[653]= False

In[660]:=

$$\mathrm{Reduce}\Big[0<c<-1+b\ \&\&\ \sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}}+\frac{2b-4b^2-c+4bc+c\tau_D}{2b}<\tau_C<\frac{c\,\tau_D}{2b}\ \&\&\ b>2+\sqrt{5}\ \&\&\ \tau_D>\frac{-2b+4b^2+c-4bc}{c}-2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}\Big]$$

Out[660]=

$$b>2+\sqrt{5}\ \&\&\ 0<c<-1+b\ \&\&\ \tau_D>\frac{-2b+4b^2+c-4bc}{c}-2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}\ \&\&\ \sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}}+\frac{2b-4b^2-c+4bc+c\tau_D}{2b}<\tau_C<\frac{c\,\tau_D}{2b}$$

In[663]:=

$$\mathrm{Reduce}\Big[x_3<1\ \&\&\ 0<\tau_C<\tau_D\ \&\&\ b\geq c+1\ \&\&\ b>2+\sqrt{5}\ \&\&\ c>0\ \&\&\ \sqrt{2}\sqrt{\frac{-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2}{b}}+\frac{2b-4b^2-c+4bc+c\tau_D}{2b}<\tau_C<\frac{c\,\tau_D}{2b},\ \tau_D\Big]$$

Out[663]=

$$b>2+\sqrt{5}\ \&\&\ \mathrm{Root}\big[-8b^2+8b^3+(8b-12b^2)\#1+2b\#1^2+\#1^3\ \&,\ 2\big]<c\leq -1+b\ \&\&\ 0<\tau_C<1-2b+c+\frac{\sqrt{\frac{64b^4-128b^5+64b^6-128b^3c+320b^4c-192b^5c+80b^2c^2-272b^3c^2+208b^4c^2-16bc^3+88b^2c^3-96b^3c^3+c^4-8bc^4+16b^2c^4}{b\left(-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2\right)}}}{2\sqrt{2}}\ \&\&$$

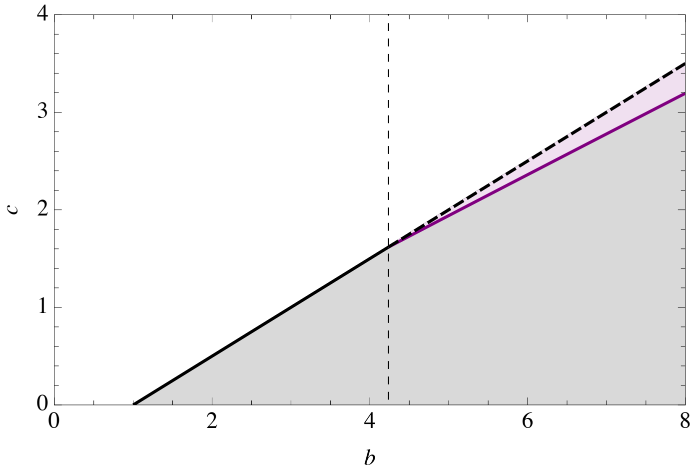
$$\frac{c-4b\tau_C+4b^2\tau_C+c\tau_C-2bc\tau_C}{-4b+4b^2+2c-4bc+c^2}+\sqrt{\frac{c^2-2c^2\tau_C+4bc^2\tau_C-2c^3\tau_C+c^2\tau_C^2}{\left(-4b+4b^2+2c-4bc+c^2\right)^2}}<\tau_D<-2\sqrt{2}\sqrt{\frac{-2b^3+2b^4+3b^2c-4b^3c-bc^2+2b^2c^2}{c^2}}+\frac{-2b+4b^2+c-4bc+2b\tau_C}{c}$$

We see that e_3 can only exist if $\mathrm{Root}\big[-8b^2+8b^3+(8b-12b^2)\#1+2b\#1^2+\#1^3\ \&,\ 2\big]<c\leq -1+b$ and $0<\tau_C<1-2b+c+\frac{\sqrt{\frac{64b^4-128b^5+64b^6-128b^3c+320b^4c-192b^5c+80b^2c^2-272b^3c^2+208b^4c^2-16bc^3+88b^2c^3-96b^3c^3+c^4-8bc^4+16b^2c^4}{b\left(-2b^2+2b^3+3bc-4b^2c-c^2+2bc^2\right)}}}{2\sqrt{2}}$

Then, hence, we can divide the game parameter space into 3 regions: One where the population is in danger of extinction, one where only e_2 can exist and one where two internal stationary states can exist, depending on values of delays

```
In[654]:= bc1 = Plot[{Root[-b^2 + b^3 + (2 b - 3 b^2) #1 + b #1^2 + #1^3 &, 2], 1/2 (-1 + b)},
    {b, 2 + √5 , 8}, PlotStyle → {Purple, Black}, Filling → {1 → {2}},
    FillingStyle → LightPurple];
  bc = Plot[{1/2 (-1 + b)}, {b, 0, 8}, PlotStyle → {Black},
    Filling → {1 → {0, {Transparent, LightGray}}}, Frame → True,
    PlotRange → {{0, 8}, {0, 4}}, FrameLabel → {b, c},
    LabelStyle → {Medium, Black}];
  linebc =
   Graphics[{Black, Dashed, Thickness[0.002], Line[{{2 + √5 , 0}, {2 + √5 , 10}}]}];
  plot = Show[bc, bc1, linebc]
```

Out[657]=



One delay present

No cooperator delay (τ_C=0), $b < 2 + \sqrt{5}$

First, we determine the values of the possible internal stationary state by solving the system of ODEs.

First, we notice that the cooperator kindergarten is empty, hence we have:

In[666]:= $y_{C,\tau_C 0} = 0;$

Next, we obtain the value of y_D

In[673]:= $y_{D,\tau_{C0}}$[x_] := y_D /. Solve[$dx_{\tau_C 0}$[x, $y_{C,\tau_C 0}$, y_D] == 0, y_D][[1]]

Then, we calculate x

In[711]:= $x_{2\,\tau_{C0}}$ =
 x /.
  Solve[$dy_{D,\tau_C 0}$[x, $y_{C,\tau_C 0}$, $y_{D,\tau_{C0}}$[x]] == 0 && τ_C > 0 && τ_D > 0 && 0 < c < -1 + b &&
    $\tau_D \neq \tau_C$ && 0 < x < 1 && b < 2 + $\sqrt{5}$, x][[1]]

Out[711]=

$$\frac{2 b - c + c \tau_D - 2 b c \tau_D + 2 c^2 \tau_D}{c^2 \tau_D} - \sqrt{\frac{4 b^2 - 4 b c + c^2 + 4 b c \tau_D - 8 b^2 c \tau_D - 2 c^2 \tau_D + 8 b c^2 \tau_D + c^2 \tau_D^2}{c^4 \tau_D^2}} \quad \text{if condition}$$

In[679]:=

```
Simplify[Normal[x2 τc0], τC > 0 && τD > 0 && 0 < c < -1 + b && τD ≠ τC && b < 2 + √5]
```

Out[679]=

$$-\frac{-2\,b+c-c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D}$$

and y_D takes the form:

In[697]:=

```
Simplify[yD,τc0[Normal[x2 τc0]]]
```

Out[697]=

$$\frac{1}{2\,c^3\,\tau_D}\left(2\,b-c-c\,\tau_D\left(-1+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)\left(-2\,b+c+c\,\tau_D\left(-1+2\,b-c+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)$$

If it exists the internal stationary state takes the following form:

$$\tilde{e}_2=\left(-\frac{-2\,b+c-c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D},\right.$$
$$\left.0,\ \frac{\left(2\,b-c-c\,\tau_D\left(-1+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)\left(-2\,b+c+c\,\tau_D\left(-1+2\,b-c+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)}{2\,c^3\,\tau_D}\right)$$

Now we show, that in the limit of $\tau_C \to 0$ the internal stationary state e_2 goes to $\tilde{e}_2$.

In[684]:=

```
Simplify[Limit[x2, τC → 0] , τC > 0 && τD > 0 && 0 < c < -1 + b && τD ≠ τC && b < 2 + √5]
```

Out[684]=

$$-\frac{-2\,b+c-c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D}$$

In[712]:=

$$\mathrm{Reduce}\left[-\frac{-2\,b+c-c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D} == -\frac{-2\,b+c-c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D}\right]$$

Out[712]= True

In[698]:=

```
Limit[x2 τc0, τD → 0, Direction → "FromAbove"]
```

Out[698]=

$$\frac{2\,(b-c)}{2\,b-c}\quad \text{if}\quad 1<b<2+\sqrt{5}\ \&\&\ 0<c<-1+b\ \&\&\ \tau_C>0$$

In[699]:=

```
Limit[yC,τc0, τD → 0]
```

Out[699]= 0

In[700]:=
```
Limit[y_{D,τc0}[x], τ_D → 0]
```

Out[700]= 0

In the limit of no delays the stationary state approaches the stationary state of the traditional Snowdrift game

No cooperator delay (τ_C=0), $b >\ = 2 + \sqrt{5}$

First, we determine the values of the possible internal stationary state by solving the system of ODEs.

First, we notice that the cooperator kindergarten is empty, hence we have:

In[701]:=
```
y_{C,τc 0} = 0;
```

Next, we obtain the value of y_D

In[702]:=
```
y_{D,τc0}[x_] := y_D /. Solve[dx_{τc 0}[x, y_{C,τc 0}, y_D] == 0, y_D][[1]]
```

Then, we calculate x

In[703]:=
```
x_{2 τc0} =
  x /.
   Solve[dy_{D,τc 0}[x, y_{C,τc 0}, y_{D,τc0}[x]] == 0 && τ_C > 0 && τ_D > 0 && 0 < c < -1 + b &&
      τ_D ≠ τ_C && 0 < x < 1 && b ≥ 2 + √5 , x][[1]]
x_{3 τc0} =
  x /.
   Solve[dy_{D,τc 0}[x, y_{C,τc 0}, y_{D,τc0}[x]] == 0 && τ_C > 0 && τ_D > 0 && 0 < c < -1 + b &&
      τ_D ≠ τ_C && 0 < x < 1 && b ≥ 2 + √5 , x][[2]]
```

Out[703]= $\frac{2b - c + c\,\tau_D - 2bc\,\tau_D + 2c^2\,\tau_D}{c^2\,\tau_D} - \sqrt{\frac{4b^2 - 4bc + c^2 + 4bc\,\tau_D - 8b^2c\,\tau_D - 2c^2\,\tau_D + 8bc^2\,\tau_D + c^2\,\tau_D^2}{c^4\,\tau_D^2}}$ if condition

Out[704]= $\frac{2b - c + c\,\tau_D - 2bc\,\tau_D + 2c^2\,\tau_D}{c^2\,\tau_D} + \sqrt{\frac{4b^2 - 4bc + c^2 + 4bc\,\tau_D - 8b^2c\,\tau_D - 2c^2\,\tau_D + 8bc^2\,\tau_D + c^2\,\tau_D^2}{c^4\,\tau_D^2}}$ if condition

In[706]:=
```
Simplify[Normal[x_{2 τc0}], τ_C > 0 && τ_D > 0 && 0 < c < -1 + b && τ_D ≠ τ_C && b ≥ 2 + √5]
Simplify[Normal[x_{3 τc0}], τ_C > 0 && τ_D > 0 && 0 < c < -1 + b && τ_D ≠ τ_C && b ≥ 2 + √5]
```

Out[706]= $-\frac{-2b + c - c\left(1 - 2b + 2c\right)\tau_D + \sqrt{\left(-2b + c\right)^2 - 2c\left(-2b + 4b^2 + c - 4bc\right)\tau_D + c^2\,\tau_D^2}}{c^2\,\tau_D}$

Out[707]= $\frac{2b - c + c\left(1 - 2b + 2c\right)\tau_D + \sqrt{\left(-2b + c\right)^2 - 2c\left(-2b + 4b^2 + c - 4bc\right)\tau_D + c^2\,\tau_D^2}}{c^2\,\tau_D}$

and y_D takes the form:

In[708]:=
```
Simplify[y_{D,τC0}[Normal[x_{2 τC0}]]]
Simplify[y_{D,τC0}[Normal[x_{3 τC0}]]]
```

Out[708]=
$$\frac{1}{2\,c^3\,\tau_D}\left(2\,b-c-c\,\tau_D\left(-1+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,\left(-2\,b+4\,b^2+c-4\,b\,c\right)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)\left(-2\,b+c+c\,\tau_D\left(-1+2\,b-c+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,\left(-2\,b+4\,b^2+c-4\,b\,c\right)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)$$

Out[709]=
$$-\frac{1}{2\,c^3\,\tau_D}\left(2\,b-c+c\,\tau_D\left(1+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,\left(-2\,b+4\,b^2+c-4\,b\,c\right)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)\left(2\,b-c+c\,\tau_D\left(1-2\,b+c+c\,\sqrt{\frac{(-2\,b+c)^2-2\,c\,\left(-2\,b+4\,b^2+c-4\,b\,c\right)\,\tau_D+c^2\,\tau_D^2}{c^4\,\tau_D^2}}\right)\right)$$

If they exist the internal stationary states take the following forms:

$$\tilde{e}_2=\left(\frac{-2b+c-c(1-2b+2c)\tau_D+\sqrt{(-2b+c)^2-2c(-2b+4b^2+c-4bc)\tau_D+c^2\tau_D^2}}{c^2\tau_D},\right.$$
$$\left.0,\ \frac{\left(2b-c-c\tau_D\left(-1+c\sqrt{\frac{(-2b+c)^2-2c(-2b+4b^2+c-4bc)\tau_D+c^2\tau_D^2}{c^4\tau_D^2}}\right)\right)\left(-2b+c+c\tau_D\left(-1+2b-c+c\sqrt{\frac{(-2b+c)^2-2c(-2b+4b^2+c-4bc)\tau_D+c^2\tau_D^2}{c^4\tau_D^2}}\right)\right)}{2c^3\tau_D}\right)$$

$$\tilde{e}_3=\left(\frac{2b-c+c(1-2b+2c)\tau_D+\sqrt{(-2b+c)^2-2c(-2b+4b^2+c-4bc)\tau_D+c^2\tau_D^2}}{c^2\tau_D},\right.$$
$$\left.0,\ -\frac{\left(2b-c+c\tau_D\left(1+c\sqrt{\frac{(-2b+c)^2-2c(-2b+4b^2+c-4bc)\tau_D+c^2\tau_D^2}{c^4\tau_D^2}}\right)\right)\left(2b-c+c\tau_D\left(1-2b+c+c\sqrt{\frac{(-2b+c)^2-2c(-2b+4b^2+c-4bc)\tau_D+c^2\tau_D^2}{c^4\tau_D^2}}\right)\right)}{2c^3\tau_D}\right)$$

Now we show, that in the limit of $\tau_C \to 0$ the internal stationary state e_2 goes to $\tilde{e}_2$.

In[718]:=
```
Simplify[Limit[x2, τC → 0] , τC > 0 && τD > 0 && 0 < c < -1 + b && τD ≠ τC && b ≥ 2 + √5]
```

Out[718]=
$$-\frac{-2\,b+c-c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,\left(-2\,b+4\,b^2+c-4\,b\,c\right)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D}$$

In[713]:=
```
Reduce[-(-2 b + c - c (1 - 2 b + 2 c) τD + √((-2 b + c)^2 - 2 c (-2 b + 4 b^2 + c - 4 b c) τD + c^2 τD^2))/(c^2 τD) ==
  Simplify[Normal[x_{2 τC0}], τC > 0 && τD > 0 && 0 < c < -1 + b && τD ≠ τC && b ≥ 2 + √5]]
```

Out[713]= True

In[714]:=
```
Limit[x2 τC0, τD → 0, Direction → "FromAbove"]
```

Out[714]=
$$\frac{2\,(b-c)}{2\,b-c} \quad \text{if} \quad 1<b<2+\sqrt{5}\;\&\&\;0<c<-1+b\;\&\&\;\tau_C>0$$

In[715]:=
```
Limit[yC,τC0, τD → 0]
```

Out[715]= 0

In[716]:=
```
Limit[yD,τC0[x], τD → 0]
```

Out[716]= 0

In the limit of no delays the stationary state approaches the stationary state of the traditional Snowdrift game

Similarly, we analyse the behaviour of e_3

In[719]:=
```
Simplify[Limit[x3, τC → 0] , τC > 0 && τD > 0 && 0 < c < -1 + b && τD ≠ τC && b ≥ 2 + √5]
```

Out[719]=
$$\frac{2\,b-c+c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D}$$

In[720]:=
$$\text{Reduce}\Big[\frac{2\,b-c+c\,(1-2\,b+2\,c)\,\tau_D+\sqrt{(-2\,b+c)^2-2\,c\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_D+c^2\,\tau_D^2}}{c^2\,\tau_D} ==$$
$$\text{Simplify}[\text{Normal}[x_{3\,\tau_{C0}}],\ \tau_C>0\;\&\&\;\tau_D>0\;\&\&\;0<c<-1+b\;\&\&\;\tau_D\neq\tau_C\;\&\&\;b\geq 2+\sqrt{5}]\Big]$$

Out[720]= True

In[721]:=
```
Limit[x3 τC0, τD → 0, Direction → "FromAbove"]
```

Out[721]= Undefined if condition

In the limit of no delays e_3 does not exists.

No defector delay (τ_D=0)

In the case of no defector delay we only consider e_2

First, we determine the values of the possible internal stationary state by solving the system of ODEs.

First, we notice that the defector kindergarten is empty, hence we have:

In[722]:=
```
yD,τD0 = 0;
```

Next, we obtain the value of y_C

In[732]:=
```
yC,τD0[x_] := yC /. Solve[dxτD0[x, yC, yD,τD0] == 0, yC]〚1〛
```

Then, we calculate x

In[751]:=
```
xτD0 =
  x /. Solve[dyC,τD0[x, yC,τD0[x], yD,τD0] == 0 && τC > 0 && 0 < c < -1 + b && τD ≠ τC && 0 < x < 1,
     x]〚1〛
```

Out[751]=
$$\frac{-2\,b+c+2\,b\,\tau_C}{4\,b^2\,\tau_C}+\frac{1}{4}\sqrt{\frac{4\,b^2-4\,b\,c+c^2-8\,b^2\,\tau_C+16\,b^3\,\tau_C+4\,b\,c\,\tau_C-16\,b^2\,c\,\tau_C+4\,b^2\,\tau_C^2}{b^4\,\tau_C^2}}$$
$$\text{if}\quad b>1\;\&\&\;0<c<-1+b\;\&\&\;\tau_C>0$$

In[737]:=
```
Simplify[x_{τD0}, τC > 0 && τD > 0 && 0 < c < -1 + b && τD ≠ τC]
```

Out[737]=
$$\frac{-2\,b + c + 2\,b\,\tau_C + \sqrt{(-2\,b + c)^2 + 4\,b\,(-2\,b + 4\,b^2 + c - 4\,b\,c)\,\tau_C + 4\,b^2\,\tau_C^2}}{4\,b^2\,\tau_C}$$

and y_D takes the form:

In[736]:=
```
Simplify[y_{C,τD0}[Normal[x_{τD0}]]]
```

Out[736]=
$$\frac{\left(-2\,b + c + b\,\tau_C\left(2 + b\,\sqrt{\frac{(-2\,b+c)^2+4\,b\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_C+4\,b^2\,\tau_C^2}{b^4\,\tau_C^2}}\right)\right)^2}{16\,b^3\,\tau_C}$$

If it exists the internal stationary state takes the following form:

$$\overline{e}_2 = \left(\frac{-2\,b+c+2\,b\,\tau_C+\sqrt{(-2\,b+c)^2+4\,b\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_C+4\,b^2\,\tau_C^2}}{4\,b^2\,\tau_C},\ \frac{\left(-2\,b+c+b\,\tau_C\left(2+b\,\sqrt{\frac{(-2\,b+c)^2+4\,b\,(-2\,b+4\,b^2+c-4\,b\,c)\,\tau_C+4\,b^2\,\tau_C^2}{b^4\,\tau_C^2}}\right)\right)^2}{16\,b^3\,\tau_C},\ 0\right)$$

Now we investigate the limit of $\tau_D \to 0$ of the internal stationary state e_2.

In[740]:=
```
Reduce[Simplify[Limit[Normal[x_2], τD → 0], τC > 0 && 0 < c < -1 + b] ==
  Simplify[x_{τD0}, τC > 0 && τD > 0 && 0 < c < -1 + b && τD ≠ τC]]
```

Out[740]=
```
True
```

Lastly, we check the limit of no delays:

In[752]:=
```
Simplify[Limit[x_{τD0}, τC → 0, Direction → "FromAbove"], c > 0 && b ≥ 1 + c ]
```

Out[752]=
$$\frac{2\,(b - c)}{2\,b - c} \quad \text{if} \quad b > 1 + c$$

In the limit of both delays going to 0 the stationary state approaches the stationary state of the Snowdrift game with no delays.

Examples

Next, we present two examples of the Snowdrift game and show the analysis of the Kindergarten model in that game.

Example 1

We will analyse the following game, corresponding to the gray region in the above figure, characterised by at most one internal stationary state:

In[753]:=
```
b = 5;
c = 2;
matrix
```

Out[755]//MatrixForm=
$$\begin{pmatrix} 4 & 3 \\ 5 & 0 \end{pmatrix}$$

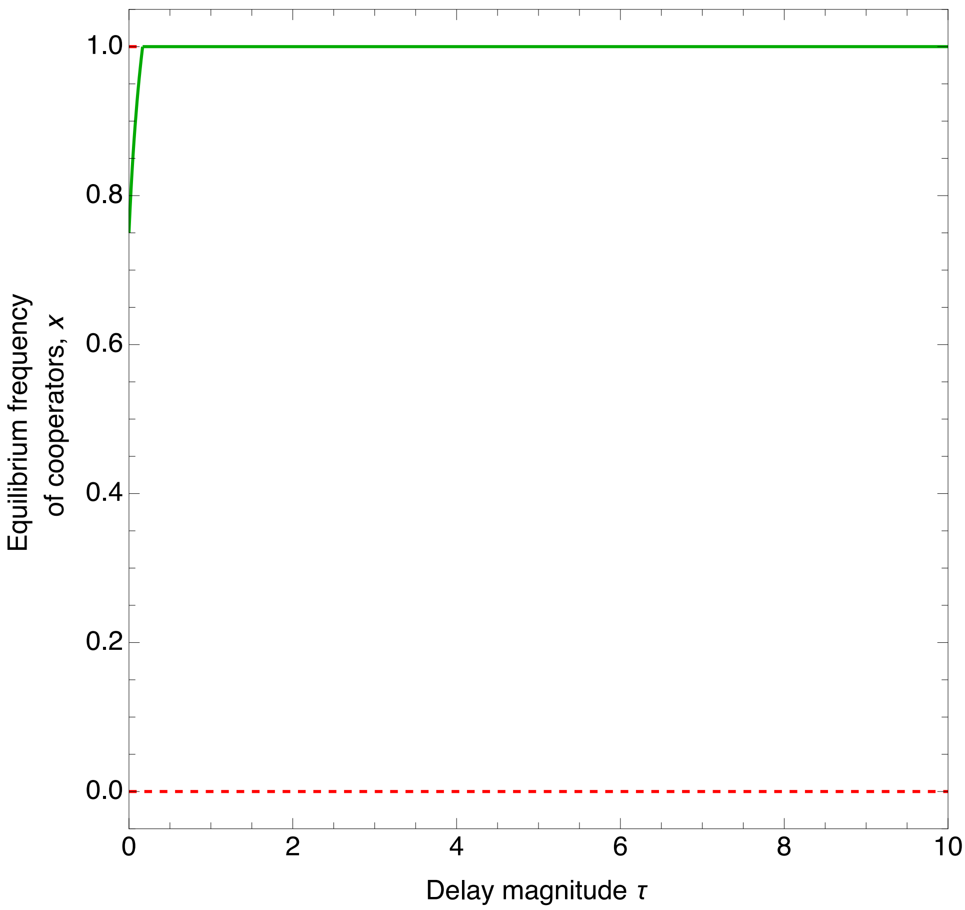
We will plot the value of x_2 for different values of delays

In[790]:=
```
Plotx[tc_, td_] := Plot[If[1 > x2 > 0 /. {τC → tc, τD → td}, {x2 /. {τC → tc, τD → td}}],
  {τ, 0, 10}, PlotRange → {{0, 10}, {-0.05, 1.05}}, PlotStyle → {{Darker[Green]}},
  Ticks → {Automatic, {0, 0.2, 0.4, 0.8, 1}}, Frame → True,
  FrameLabel → {"Delay magnitude τ",
    "Equilibrium frequency \n of cooperators, x"},
  LabelStyle → {14, Black, FontFamily → "Helvetica"}, AspectRatio → 1,
  ImageSize → 500]
Plot0[tc_, td_] := Plot[{0}, {τ, 0, 10}, PlotStyle → {Red, Dashed}]
Plot1st[tc_, td_] := Plot[If[td ≥ n /. {τC → tc, τD → td}, 1], {τ, 0, 10},
  PlotStyle → Darker[Green]]
Plot1unst[tc_, td_] := Plot[If[td < n /. {τC → tc, τD → td}, 1], {τ, 0, 10},
  PlotStyle → {Red, Dashed}]
```

$\tau_C = \tau, \tau_D = 2\tau$

In[794]:=
```
panela = Show[Plotx[τ, 2 τ], Plot0[τ, 2 τ], Plot1st[τ, 2 τ], Plot1unst[τ, 2 τ]]
```

Out[794]=



$\tau_C = \tau, \tau_D = 2$

In[801]:=
```
panelb = Show[Plotx[τ, 2], Plot0[τ, 2], Plot1st[τ, 2], Plot1unst[τ, 2]]
```

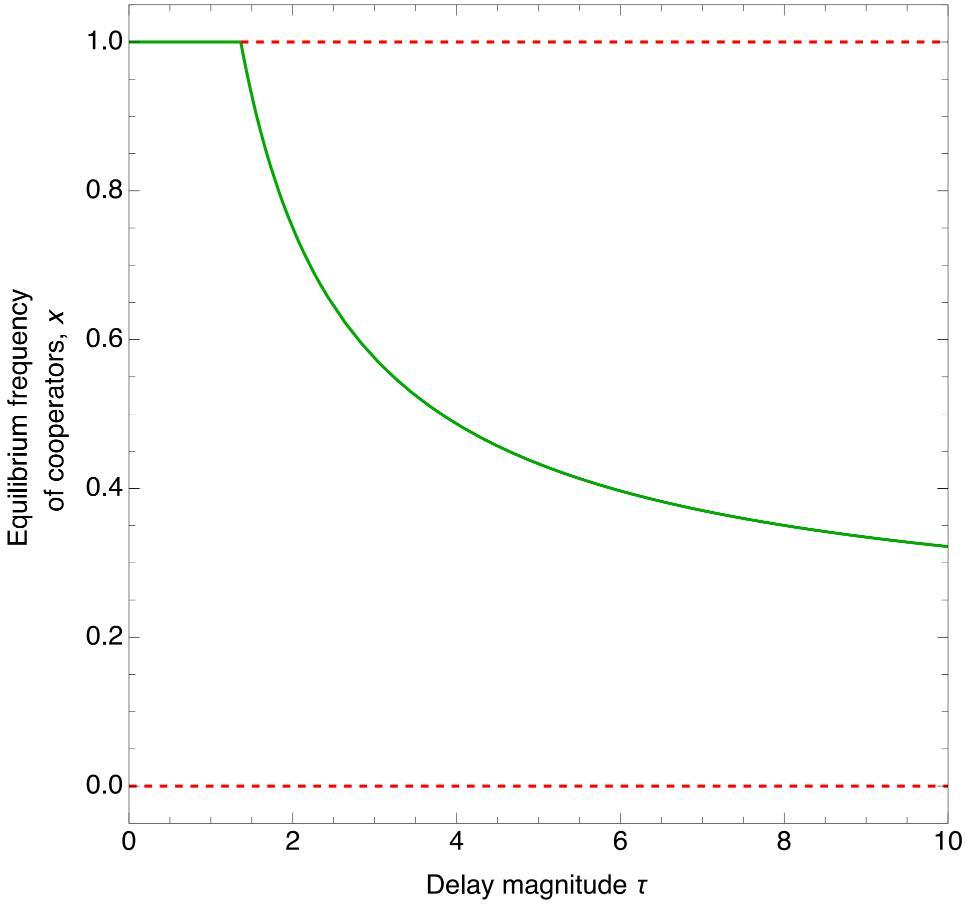
Greater : Invalid comparison with −1.50041 − 2.29273 i attempted.

Greater : Invalid comparison with −1.50041 − 2.29273 i attempted.

Greater : Invalid comparison with −1.50041 − 2.29273 i attempted.

General : Further output of Greater::nord will be suppressed during this calculation.

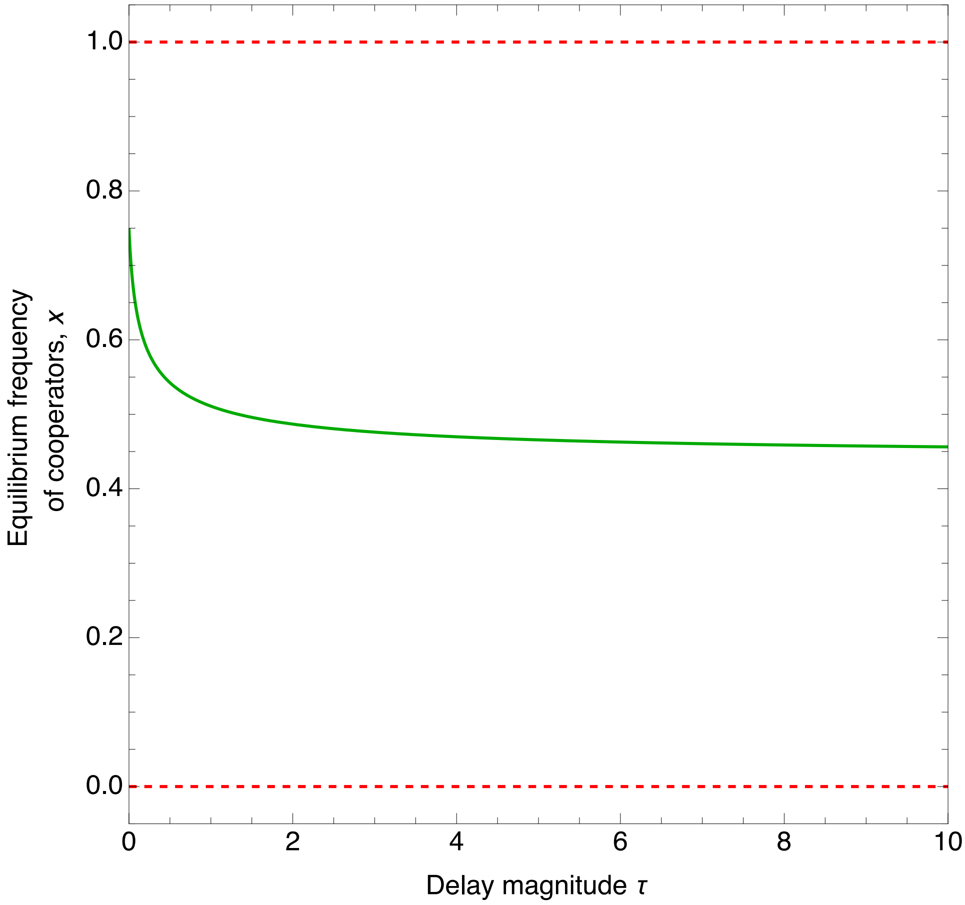
Out[801]=



$\tau_C = 2\ \tau, \tau_D = \tau$

In[799]:=
```
panelc = Show[Plotx[2 τ, τ], Plot0[2 τ, τ], Plot1st[2 τ, τ], Plot1unst[2 τ, τ]]
```

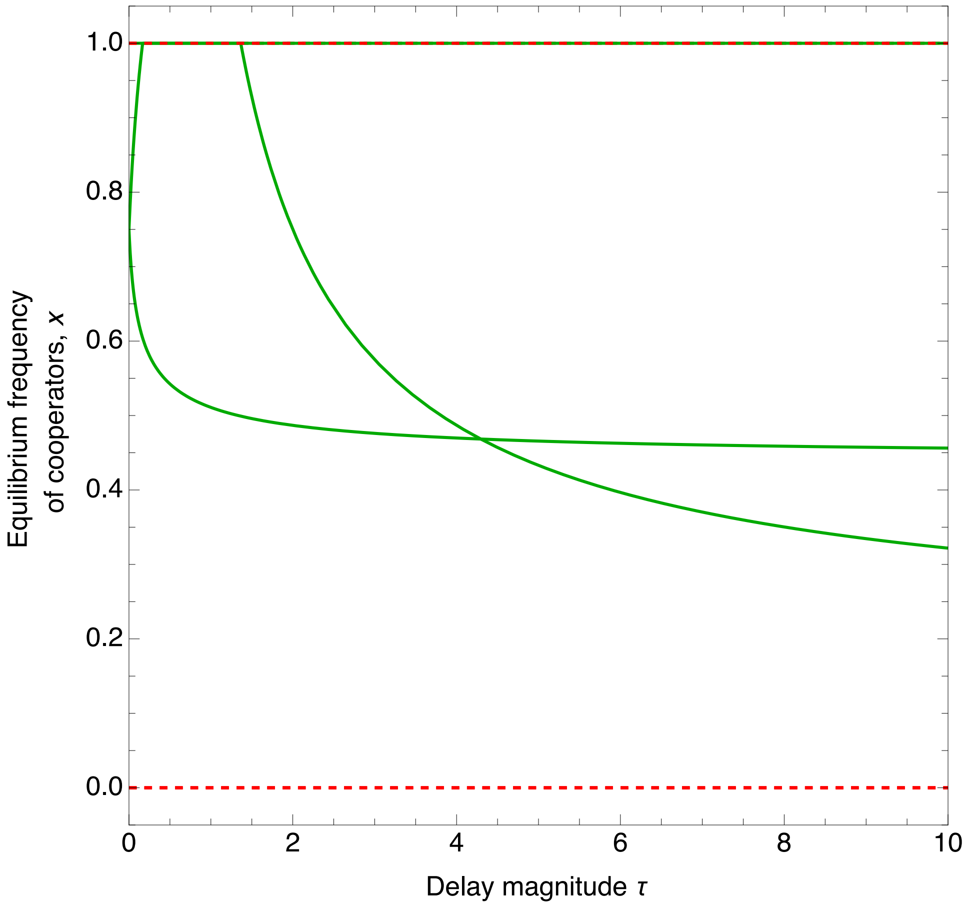
Out[799]=



Figure

In[802]:=
```
Show[panelb, panela, panelc]
```

Out[802]=



Stability

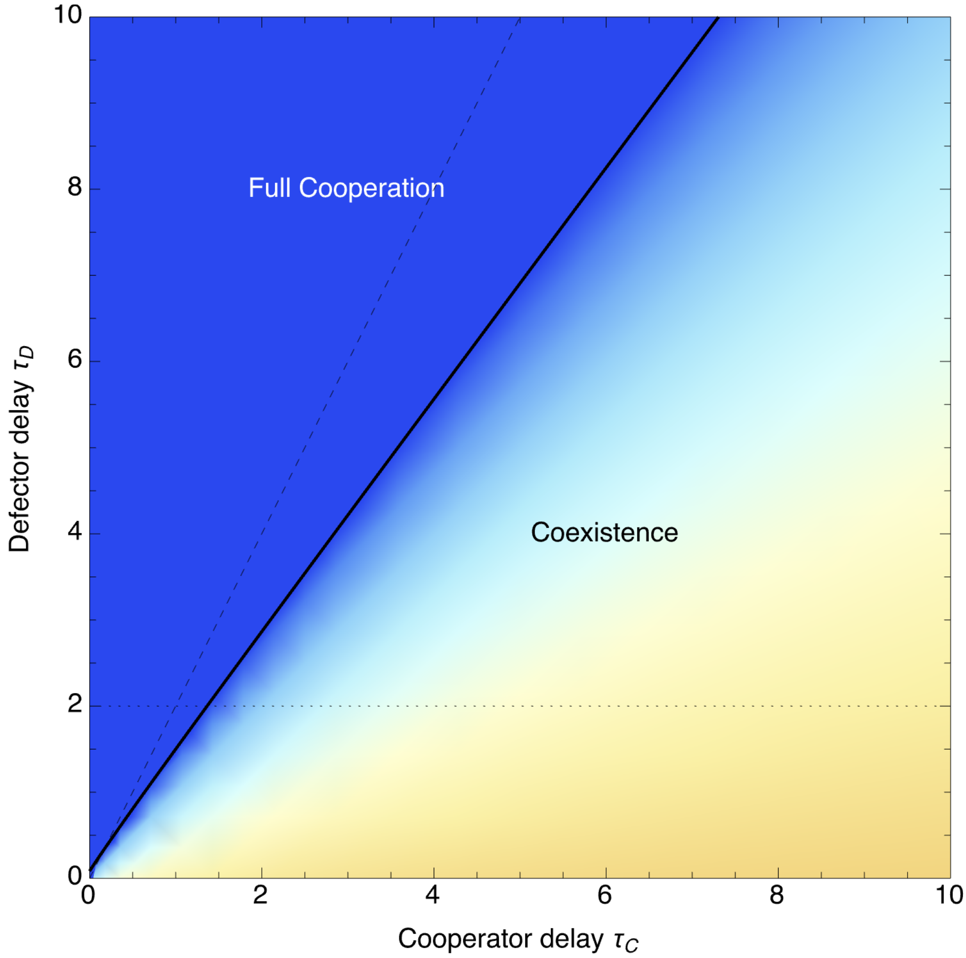
Lastly, we plot the value of x in the internal stationary state depending on the values of delays.

In[803]:=
```
boundary = Solve[ τD == n && τC > 0 && τD ≠ τC && 0 < x < 1, τD, Reals][[1]];
bound = Plot[τD /. Normal[boundary], {τC, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → Black];
```

In[817]:=
```
rainbow = DensityPlot[x₂, {τC, 0, 10}, {τD, 0, 10},
   PlotRange → {{0, 10}, {0, 10}, {0, 1} },
   ColorFunction → (ColorData["LightTemperatureMap", (1 - #)] &),
   FrameLabel → {"Cooperator delay τC", "Defector delay τD"}, Frame → True,
   LabelStyle → {14, Black, FontFamily → "Helvetica"},
   ColorFunctionScaling → False, ColorFunctionScaling → False];
allcoop = DensityPlot[1, {τC, 0, 10}, {τD, 0, 10},
   PlotRange → {{0, 10}, {0, 10}, {0, 1} },
   ColorFunction → (ColorData["LightTemperatureMap", (1 - #)] &),
   ColorFunctionScaling → False,
   FrameLabel → {"Cooperator delay τC", "Defector delay τD"}, Frame → True,
   LabelStyle → {14, Black, FontFamily → "Helvetica"}, ImageSize → Medium];
text =
  Graphics[{Text[Style["Coexistence", 14, Black, FontFamily → "Helvetica"],
     {6, 4} ],
    Text[Style["Full Cooperation", 14, White, FontFamily → "Helvetica"],
     {3, 8}]}];
otherpanelsa = Plot[2 tc,  {tc, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → {Black, Dashed}];
otherpanelsb = Plot[2,  {tc, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → {Black, Dotted}];
otherpanelsc = Plot[1 / 2 tc,  {tc, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
   PlotStyle → {Black, DotDashed}];
```

In[823]:=
```
paneld = Show[allcoop, rainbow, bound, text, otherpanelsa, otherpanelsb,
  ImageSize → 500]
```

Out[823]=



Example 2

We will analyse the following game, corresponding to the purple region in the above figure, characterised by at most two internal stationary states:

In[824]:=
```
b = 5;
c = 3.9;
matrix
```

Out[826]//MatrixForm=

$$\begin{pmatrix} 3.05 & 1.1 \\ 5 & 0 \end{pmatrix}$$

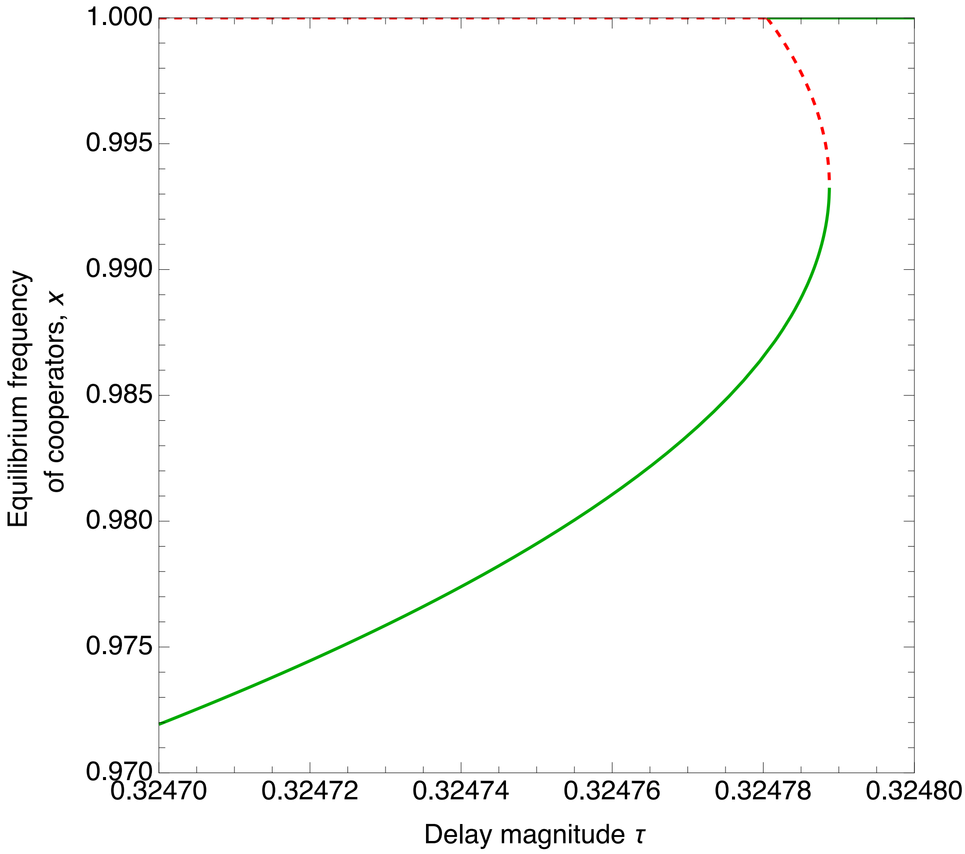
We will plot the value of x_2 and x_3 for different values of delays

In[857]:=
```
Plotx2[tc_, td_] := Plot[ {x2 /. {τC → tc, τD → td}}, {τ, 0.3247, 0.3248},
  PlotRange → {{0.3247, 0.3248}, { 0.97, 1}}, PlotStyle → {{Darker[Green]}},
  Ticks → {Automatic, {0, 0.2, 0.4, 0.8, 1}}, Frame → True,
  FrameLabel → {"Delay magnitude τ",
    "Equilibrium frequency \n of cooperators, x" },
  LabelStyle → {14, Black, FontFamily → "Helvetica"}, AspectRatio → 1,
  ImageSize → 500]
Plotx3[tc_, td_] := Plot[ {x3 /. {τC → tc, τD → td}}, {τ,  0.3247, 0.3248},
  PlotRange → {{0.3247, 0.3248}, { 0.97, 1}}, PlotStyle → {{Red, Dashed}},
  Ticks → {Automatic, {0, 0.2, 0.4, 0.8, 1}}, Frame → True,
  FrameLabel → {"Delay magnitude τ",
    "Equilibrium frequency \n of cooperators, x" },
  LabelStyle → {14, Black, FontFamily → "Helvetica"}, AspectRatio → 1,
  ImageSize → 500]
Plot1st[tc_, td_] := Plot[If[td ≥ n /. {τC → tc, τD → td}, 1], {τ, 0.3247, 0.3248},
  PlotStyle → Darker[Green]]
Plot1unst[tc_, td_] := Plot[If[td < n /. {τC → tc, τD → td}, 1],
  {τ, 0.3247, 0.3248}, PlotStyle → {Red, Dashed}]
```

τ_C = **0.005**, τ_D = τ

In[861]:=
```
panelb = Show[Plotx2[0.005, τ], Plotx3[0.005, τ], Plot1st[0.005, τ],
  Plot1unst[0.005, τ]]
```

Out[861]=



We can observe a small interval of delay values when the two internal stationary states coexist.

Prisoner's Dilemma game

The Prisoner's Dilemma game is characterised by the following payoff matrix:

```
Clear[b, c]
R = b;
S = 0;
T = b + c;
P = c;
matrixSH = {{R, S}, {T, P}} // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} b & 0 \\ b+c & c \end{pmatrix}$$

where $b > c > 0$

In this game, the system characterizing the Kindergarten model becomes:

In[∘]:=
```
FullSimplify[dx[x, y_C, y_D]]
FullSimplify[dy_C[x, y_C, y_D]]
FullSimplify[dy_D[x, y_C, y_D]]
```

Out[∘]= $-\frac{(-1+x)\, y_C}{\tau_C} - \frac{x\, y_D}{\tau_D}$

Out[∘]= $b\, x^2 + y_C \left(1 - \frac{1+y_C}{\tau_C} - \frac{y_D}{\tau_D}\right)$

Out[∘]= $-\left((-1+x)\,(c+b\,x)\right) + y_D \left(1 - \frac{y_C}{\tau_C} - \frac{1+y_D}{\tau_D}\right)$

Homogenous stationary states

First, we analyse the full defection stationary state. We know, that the fraction of cooperators is equal to 0. Then, we determine the relative sizes of kindergartens in the stationary states

In[∘]:=
```
x_0 = 0;
y_C,0 = y_C /. Solve[dx[x_0, y_C, y_D] == 0 && dy_C[x_0, y_C, y_D] == 0, y_C][[1]]
```

Out[∘]= 0

With no cooperators present in the population, the cooperator kindergarten is also empty.

In[∘]:=
```
y_D,0 = y_D /. Solve[dy_D[x_0, y_C,0, y_D] == 0 && τ_C > 0 && τ_D > 0 && b > c > 0 && y_D > 0, y_D]
```

Out[∘]= $\left\{ \frac{1}{2}(-1+\tau_D) + \frac{1}{2}\sqrt{1 - 2\,\tau_D + 4\,c\,\tau_D + \tau_D^2} \text{ if } \tau_C > 0 \,\&\&\, \tau_D > 0 \,\&\&\, c > 0 \,\&\&\, b > c \right\}$

Full defection stationary state is of the following form: $e_0 = \{0, 0, \frac{1}{2}(-1+\tau_D) + \frac{1}{2}\sqrt{1 - 2\,\tau_D + 4\,c\,\tau_D + \tau_D^2}\}$

Next, we check for which parameter values the population grows exponentially and does not go extinct in e_0

In[∘]:=
```
Reduce[Normal[y_C,0/τ_C + y_D,0/τ_D] > 1 && τ_C > 0 && τ_D > 0 && b > c > 0]
```

Out[∘]= $\tau_C > 0 \,\&\&\, \tau_D > 0 \,\&\&\, c > 1 \,\&\&\, b > c$

In the full defection stationary state the population grows when $c > 1$.

Next, we investigate the full cooperation ($x_1 = 1$) stationary state:

In[ ]:=
```
x1 = 1;
yD,1 = yD /. Solve[dx[x1, yC, yD] == 0 && dyD[x1, yC, yD] == 0, yD][[1]]
```
Out[ ]= 0

With no defectors present in the population, the defector kindergarten is also empty.

In[ ]:=
```
yC,1 = yC /. Solve[dyC[x1, yC, yD,1] == 0 && τC > 0 && τD > 0 && b > c > 1 && yC > 0, yC]
```
Out[ ]= $\left\{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\tau_C+4b\tau_C+\tau_C^2}\ \text{if}\ \tau_D>0\ \&\&\ \tau_C>0\ \&\&\ b>1\ \&\&\ 1<c<b\right\}$

Full cooperation equilibrium is of the following form: $e_1=\{1, \frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\tau_C+4b\tau_C+\tau_C^2}, 0\}$

The population grow exponentially in this stationary state if:

In[ ]:=
$$\text{Reduce}\left[\text{Normal}\left[\frac{y_{C,1}}{\tau_C}+\frac{y_{D,1}}{\tau_D}\right]>1\ \&\&\ \tau_C>0\ \&\&\ \tau_D>0\ \&\&\ b>c>1\right]$$

Out[ ]= $\tau_D>0\ \&\&\ \tau_C>0\ \&\&\ b>1\ \&\&\ 1<c<b$

In the full cooperation stationary state the population grows if $b>1$.

Heterogenous stationary states

Next, we investigate the existence of internal stationary states. First, we find the value of y_C depending on x and y_D

In[ ]:=
```
Simplify[ Solve[dx[x, yC, yD] == 0, yC]]
```
Out[ ]= $\left\{\left\{y_C\to\frac{x\,y_D\,\tau_C}{\tau_D-x\,\tau_D}\right\}\right\}$

In[ ]:=
$$y_{C,2}[x\_,\ yd\_] := \frac{x\ yd\ \tau_C}{\tau_D-x\ \tau_D};$$

Next, we determine the value of y_D depending on x

In[ ]:=
```
Simplify[
 Solve[dyD[x, yC,2[x, yD], yD] == 0 && τC > 0 && τD > 0 && b > c > 1 && 0 < x < 1 && yD > 0,
  yD]]
```
Out[ ]= $\left\{\left\{y_D\to -\frac{1}{2}(-1+x)\left(-1+\tau_D+\sqrt{1+(-2+4c+4bx)\tau_D+\tau_D^2}\right)\ \text{if}\ \tau_C>0\ \&\&\ \tau_D>0\ \&\&\ c>1\ \&\&\ b>c\ \&\&\ 0<x<1\right\}\right\}$

In[ ]:=
$$y_{D,2}[x\_] := -\frac{1}{2}(-1+x)\left(-1+\tau_D+\sqrt{1+(-2+4c+4bx)\tau_D+\tau_D^2}\right);$$

Then, y_C becomes:

In[ ]:=
```
yC,2,x[x_] := yC,2[x, yD,2[x]]
Simplify[yC,2,x[x]]
```
Out[ ]=
$$\frac{x\,\tau_C\left(-1+\tau_D+\sqrt{1+(-2+4c+4bx)\tau_D+\tau_D^2}\right)}{2\,\tau_D}$$

Finally, we find the possible values of x

In[ ]:=
```
solutions =
 Solve[FullSimplify[dy_C[x, y_{C,2,x}[x], y_{D,2}[x]]] == 0 && τ_C > 0 && τ_D > 0 &&
   b > c > 1 && 0 < x < 1, x, Reals]
```

$$\left\{\left\{x \to \frac{1}{2}\sqrt{\frac{4\,c - \tau_C + \tau_D}{b^2\,(-\tau_C + \tau_D)}} + \frac{-\tau_C + 2\,c\,\tau_C + \tau_D}{2\,b\,(-\tau_C + \tau_D)} \text{ if } \left(0 < \tau_C < \frac{\tau_D}{2}\,\&\&\, b > c\,\&\&\, c > \frac{-\tau_C + 2\,\tau_C^2 + \tau_D - 3\,\tau_C\,\tau_D + \tau_D^2}{4\,\tau_C^2 - 4\,\tau_C\,\tau_D + \tau_D^2}\,\&\&\,\tau_D > 0\right) \,||\, \left(0 < \tau_C < \frac{\tau_D}{2}\,\&\&\, 1 < c < \frac{-\tau_C + 2\,\tau_C^2 + \tau_D - 3\,\tau_C\,\tau_D + \tau_D^2}{4\,\tau_C^2 - 4\,\tau_C\,\tau_D + \tau_D^2}\,\&\&\, b > \frac{1}{2}\sqrt{\frac{4\,c - \tau_C + \tau_D}{-\tau_C + \tau_D}} + \frac{-\tau_C + 2\,c\,\tau_C + \tau_D}{2\,(-\tau_C + \tau_D)}\,\&\&\,\tau_D > 0\right) \,||\, \left(\frac{\tau_D}{2} < \tau_C < \tau_D\,\&\&\, b > \frac{1}{2}\sqrt{\frac{4\,c - \tau_C + \tau_D}{-\tau_C + \tau_D}} + \frac{-\tau_C + 2\,c\,\tau_C + \tau_D}{2\,(-\tau_C + \tau_D)}\,\&\&\, c > 1\,\&\&\,\tau_D > 0\right)\right\}\right\}$$

In[ ]:=
$$x_2 = \frac{1}{2}\sqrt{\frac{4\,c - \tau_C + \tau_D}{b^2\,(-\tau_C + \tau_D)}} + \frac{-\tau_C + 2\,c\,\tau_C + \tau_D}{2\,b\,(-\tau_C + \tau_D)};$$

We observe only one possible value of internal stationary state. Notably, the internal stationary state only exists in the interesting interval when $\tau_C < \tau_D$.

The possible internal equilibrium takes the following form: $e_2 = \left\{x_2,\ y_D,\ \frac{x_2\,y_D\,\tau_C}{\tau_D - x_2\,\tau_D}\right\}$ where $x_2 = \frac{1}{2}\sqrt{\frac{4\,c-\tau_C+\tau_D}{b^2\,(-\tau_C+\tau_D)}} + \frac{-\tau_C+2\,c\,\tau_C+\tau_D}{2\,b\,(-\tau_C+\tau_D)}$ and $y_D = -\frac{1}{2}\,(-1 + x)\left(-1 + \tau_D + \sqrt{1 + (-2 + 4\,c + 4\,b\,x)\,\tau_D + \tau_D^2}\right)$.

Lastly, we check when does the population grow in this stationary state and is not threatened by extinction

In[ ]:=
$$\text{Reduce}\left[\text{Normal}\left[\frac{y_{C,2,x}[x]}{\tau_C} + \frac{y_{D,2}[x]}{\tau_D}\right] \geq 1\,\&\&\,\tau_C > 0\,\&\&\,\tau_D > 0\,\&\&\, b > c > 1\,\&\&\, 0 < x < 1\right]$$

Out[ ]=
$$\tau_C > 0\,\&\&\,\tau_D > 0\,\&\&\, c > 1\,\&\&\, b > c\,\&\&\, 0 < x < 1$$

In the internal stationary state the population never goes extinct.

Stability analysis

Full defection stationary state

We perform the stability analysis of full defection stationary state e_0.

First, we determine the eigenvalues of the system of ODEs:

In[∘]:=
```
system = {dx[x, yC, yD], dyC[x, yC, yD] , dyD[x, yC, yD]};
J = D[system, {{x, yC, yD}}] ;
J // MatrixForm
Jstar = J /. {x → Normal[x0], yD → Normal[yD,0]〚1〛, yC → Normal[yC,0]};
Jstar // MatrixForm
eigens = Eigenvalues[Jstar];
eigens // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{y_C}{\tau_C}-\frac{y_D}{\tau_D} & \frac{1-x}{\tau_C} & -\frac{x}{\tau_D} \\ 2\,b\,x & -\frac{2\,y_C}{\tau_C}+\frac{-1+\tau_C}{\tau_C}-\frac{y_D}{\tau_D} & -\frac{y_C}{\tau_D} \\ b\,(1-x)-c\,(1-x)-(b+c)\,x & -\frac{y_D}{\tau_C} & -\frac{y_C}{\tau_C}-\frac{2\,y_D}{\tau_D}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,c\,\tau_D+\tau_D^2}}{\tau_D} & \frac{1}{\tau_C} & 0 \\ 0 & \frac{-1+\tau_C}{\tau_C}-\frac{\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,c\,\tau_D+\tau_D^2}}{\tau_D} & 0 \\ b-c & -\frac{\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,c\,\tau_D+\tau_D^2}}{\tau_C} & \frac{-1+\tau_D}{\tau_D}-\frac{2\left(\frac{1}{2}(-1+\tau_D)+\frac{1}{2}\sqrt{1-2\,\tau_D+4\,c\,\tau_D+\tau_D^2}\right)}{\tau_D} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{1-2\,\tau_D+4\,c\,\tau_D+\tau_D^2}}{\tau_D} \\ -\frac{-1+\tau_D+\sqrt{1-2\,\tau_D+4\,c\,\tau_D+\tau_D^2}}{2\,\tau_D} \\ \frac{\tau_C-2\,\tau_D+\tau_C\,\tau_D-\tau_C\sqrt{1-2\,\tau_D+4\,c\,\tau_D+\tau_D^2}}{2\,\tau_C\,\tau_D} \end{pmatrix}$$

Next, we determine when the stationary state is stable

In[∘]:=
```
Reduce[eigens〚1〛 < 0 && eigens〚2〛 < 0 && eigens〚3〛 < 0 && τC > 0 && τD > 0 &&
  τC ≠ τD && b > c > 1]
```

Out[∘]= $\tau_C > 0$ && $((0 < \tau_D < \tau_C$ && $c > 1$ && $b > c)$ || $(\tau_D > \tau_C$ && $c > 1$ && $b > c))$

Full defection stationary state is always stable.

Full cooperation stationary state

We perform the stability analysis of full cooperation stationary state e_1 .

First, we determine the eigenvalues of the system of ODEs:

In[ ]:=
```
system = {dx[x, yC, yD], dyC[x, yC, yD] , dyD[x, yC, yD]};
J = D[system, {{x, yC, yD}}] ;
J // MatrixForm
Jstar = J /. {x → Normal[x1], yD → Normal[yD,1], yC → Normal[yC,1][[1]]};
Jstar // MatrixForm
eigens = Eigenvalues[Jstar];
eigens // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{y_C}{\tau_C}-\frac{y_D}{\tau_D} & \frac{1-x}{\tau_C} & -\frac{x}{\tau_D} \\ 2\,b\,x & -\frac{2\,y_C}{\tau_C}+\frac{-1+\tau_C}{\tau_C}-\frac{y_D}{\tau_D} & -\frac{y_C}{\tau_D} \\ b\,(1-x)-c\,(1-x)-(b+c)\,x & -\frac{y_D}{\tau_C} & -\frac{y_C}{\tau_C}-\frac{2\,y_D}{\tau_D}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2}}{\tau_C} & 0 & -\frac{1}{\tau_D} \\ 2\,b & \frac{-1+\tau_C}{\tau_C}-\frac{2\left(\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2}\right)}{\tau_C} & -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2}}{\tau_D} \\ -b-c & 0 & -\frac{\frac{1}{2}(-1+\tau_C)+\frac{1}{2}\sqrt{1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2}}{\tau_C}+\frac{-1+\tau_D}{\tau_D} \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2}}{\tau_C} \\ \frac{-\tau_C+\tau_D-\sqrt{1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2}\,\tau_D-\sqrt{\tau_C^2-2\,\tau_C^2\,\tau_D+4\,b\,\tau_C^2\,\tau_D+4\,c\,\tau_C^2\,\tau_D-\tau_D^2+2\,\tau_C\,\tau_D^2-4\,b\,\tau_C\,\tau_D^2+\left(1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2\right)\tau_D^2}}{2\,\tau_C\,\tau_D} \\ \frac{-\tau_C+\tau_D-\sqrt{1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2}\,\tau_D+\sqrt{\tau_C^2-2\,\tau_C^2\,\tau_D+4\,b\,\tau_C^2\,\tau_D+4\,c\,\tau_C^2\,\tau_D-\tau_D^2+2\,\tau_C\,\tau_D^2-4\,b\,\tau_C\,\tau_D^2+\left(1-2\,\tau_C+4\,b\,\tau_C+\tau_C^2\right)\tau_D^2}}{2\,\tau_C\,\tau_D} \end{pmatrix}$$

Next, we determine when the stationary state is stable

In[ ]:=
```
Reduce[eigens[[1]] < 0 && eigens[[2]] < 0 && eigens[[3]] < 0 && τC > 0 && τD > 0 &&
   τC ≠ τD && b > c > 1, τC]
```

Out[ ]=

$$b>1\ \&\&\ 1<c<b\ \&\&\ \tau_D>\frac{c}{-b+b^2}\ \&\&$$
$$0<\tau_C<\frac{-c-2\,b\,\tau_D+2\,b^2\,\tau_D-c\,\tau_D+2\,b\,c\,\tau_D}{2\left(-b+b^2-c+2\,b\,c+c^2\right)}-\frac{1}{2}\sqrt{\frac{c^2-2\,c^2\,\tau_D+4\,b\,c^2\,\tau_D+4\,c^3\,\tau_D+c^2\,\tau_D^2}{\left(-b+b^2-c+2\,b\,c+c^2\right)^2}}$$

In[ ]:=

$$r=\mathrm{FullSimplify}\left[\frac{-c-2\,b\,\tau_D+2\,b^2\,\tau_D-c\,\tau_D+2\,b\,c\,\tau_D}{2\left(-b+b^2-c+2\,b\,c+c^2\right)}-\frac{1}{2}\sqrt{\frac{c^2-2\,c^2\,\tau_D+4\,b\,c^2\,\tau_D+4\,c^3\,\tau_D+c^2\,\tau_D^2}{\left(-b+b^2-c+2\,b\,c+c^2\right)^2}},\ b>c>1\right]$$

Out[ ]=

$$\frac{\left(-c+2\,b\,(-1+b+c)\right)\tau_D-c\left(1+\sqrt{1+\tau_D\,(-2+4\,b+4\,c+\tau_D)}\right)}{2\,(-1+b+c)\,(b+c)}$$

Full cooperation stationary state is stable when $\tau_D > \frac{c}{-b+b^2}$ and $\tau_C < r$, where $r = \frac{\left(-c+2\,b\,(-1+b+c)\right)\tau_D-c\left(1+\sqrt{1+\tau_D\,(-2+4\,b+4\,c+\tau_D)}\right)}{2\,(-1+b+c)\,(b+c)}$

The effect of delays on the internal stationary state

Now, we analyse the effect of each of the delays on the value of x in the internal stationary state e_2.

First, we check the effect of cooperator delay:

In[ ]:= `Reduce[D[x2, τC] > 0 && τC > 0 && τD > 0 && b > c > 1 && τC < τD]`

Out[ ]= $\tau_D > 0 \,\&\&\, 0 < \tau_C < \tau_D \,\&\&\, c > 1 \,\&\&\, b > c$

The value of x_2 always increases with τ_C

Next, we check when $x = 1$, that is, when the two stationary states (e_1 and e_2) collide.

In[ ]:= `Reduce[x2 == 1 && τC > 0 && τD > 0 && τC ≠ τD && b > c > 1 && τC < τD, τC]`

Out[ ]= $b > 1 \,\&\&\, 1 < c < b \,\&\&\, \tau_D > \frac{c}{-b+b^2} \,\&\&$
$$\tau_C = \frac{-c - 2\,b\,\tau_D + 2\,b^2\,\tau_D - c\,\tau_D + 2\,b\,c\,\tau_D}{2\left(-b + b^2 - c + 2\,b\,c + c^2\right)} - \frac{1}{2}\sqrt{\frac{c^2 - 2\,c^2\,\tau_D + 4\,b\,c^2\,\tau_D + 4\,c^3\,\tau_D + c^2\,\tau_D^2}{\left(-b + b^2 - c + 2\,b\,c + c^2\right)^2}}$$

In[ ]:= $\text{Reduce}\left[\frac{-c - 2\,b\,\tau_D + 2\,b^2\,\tau_D - c\,\tau_D + 2\,b\,c\,\tau_D}{2\left(-b + b^2 - c + 2\,b\,c + c^2\right)} - \frac{1}{2}\sqrt{\frac{c^2 - 2\,c^2\,\tau_D + 4\,b\,c^2\,\tau_D + 4\,c^3\,\tau_D + c^2\,\tau_D^2}{\left(-b + b^2 - c + 2\,b\,c + c^2\right)^2}} == \right.$
$\left.\text{Normal}[r] \,\&\&\, b > c > 1 \,\&\&\, \tau_D > 0,\ \text{Reals}\right]$

Out[ ]= $b > 1 \,\&\&\, 1 < c < b \,\&\&\, \tau_D > 0$

At the point $\tau_C = r$ the internal stationary state reaches the full cooperation and disappears. At the same time, the full cooperation stationary state changes stability and becomes unstable.
Next, we investigate the effects of defector delay:

In[ ]:= `Reduce[D[x2, τD] > 0 && τC > 0 && τD > 0 && b > c > 1]`

Out[ ]= `False`

In[ ]:= `Reduce[D[x2, τD] < 0 && τC > 0 && τD > 0 && b > c > 1 && τC < τD]`

Out[ ]= $\tau_C > 0 \,\&\&\, \tau_D > \tau_C \,\&\&\, c > 1 \,\&\&\, b > c$

The value of x_2 always decreases with τ_2 .

Now, we check what happens in the limit of $\tau_D \to\infty$

In[ ]:= `Simplify[Limit[x2, τD → Infinity], b > c > 1 && τD > 0]`

Out[ ]= $\frac{1}{b}$

With the value of defector delay growing the value of x in e_2 approaches $1/b$. This ensures that the population always exists in the internal stationary state and confirms that full defector stationary state is always stable (never collides with e_2).

One delay present

No cooperator delay (τ_C=0)

First, we determine the values of the possible internal stationary state by solving the system of ODEs.

First, we notice that the cooperator kindergarten is empty, hence we have:

In[ ]:= $y_{C,\tau_C 0} = 0;$

Next, we obtain the value of y_D

In[ ]:= $y_{D,\tau_{C0}}[x\_] := y_D \text{ /. Solve}\left[dx_{\tau_C 0}\left[x,\ y_{C,\tau_C 0},\ y_D\right] == 0,\ y_D\right][[1]]$

Then, we calculate x

In[∘]:=
```
x_{τC0} =
  x /.
   Solve[dy_{D,τC 0}[x, y_{C,τC 0}, y_{D,τC0}[x]] == 0 && τC > 0 && τD > 0 && b > c > 1 && τD ≠ τC && 0 < x < 1,
     x][[1]]
```

Out[∘]= $\frac{1}{2\,b}+\frac{1}{2}\sqrt{\frac{4\,c+\tau_D}{b^2\,\tau_D}}$ if $c>1\,\&\&\,b>c\,\&\&\,\tau_D>\frac{c}{-b+b^2}\,\&\&\,\tau_C>0$

and y_D takes the form:

In[∘]:=
```
Simplify[y_{D,τC0}[Normal[x_{τC0}]]]
```

Out[∘]= $$-\frac{\tau_D\left(1+b\sqrt{\frac{4\,c+\tau_D}{b^2\,\tau_D}}\right)\left(1+b\left(-2+\sqrt{\frac{4\,c+\tau_D}{b^2\,\tau_D}}\right)\right)}{4\,b}$$

If it exists the internal stationary state takes the following form:

$$\tilde{e}_2=\left(\frac{1}{2b}+\frac{1}{2}\sqrt{\frac{4c+\tau_D}{b^2\tau_D}},\ 0,\ -\frac{\tau_D\left(1+b\sqrt{\frac{4c+\tau_D}{b^2\tau_D}}\right)\left(1+b\left(-2+\sqrt{\frac{4c+\tau_D}{b^2\tau_D}}\right)\right)}{4b}\right)$$

Now we show, that in the limit of $\tau_C\to 0$ the internal stationary state e_2 goes to $\tilde{e}_2$.

In[∘]:=
```
Reduce[Limit[x_2, τC → 0] == Normal[x_{τC0}] && τC > 0 && τD > 0 && b > c > 1 && τD ≠ τC]
```

Out[∘]= $b>1\,\&\&\,1<c<b\,\&\&\,\tau_C>0\,\&\&\,(0<\tau_D<\tau_C\,||\,\tau_D>\tau_C)$

In[∘]:=
```
Reduce[Limit[y_{C,2,x}[x_2], τC → 0] == y_{C,τC 0} && τC > 0 && τD > 0 && b > c > 1 && τD ≠ τC]
```

Out[∘]= $b>1\,\&\&\,1<c<b\,\&\&\,\tau_C>0\,\&\&\,(0<\tau_D<\tau_C\,||\,\tau_D>\tau_C)$

In[∘]:=
```
Reduce[Limit[y_{D,2}[x_2], τC → 0] == y_{D,τC0}[Normal[x_{τC0}]] && τC > 0 && τD > 0 &&
   b > c > 1 && τD ≠ τC]
```

Out[∘]= $\tau_D>0\,\&\&\,c>1\,\&\&\,b>c\,\&\&\,(0<\tau_C<\tau_D\,||\,\tau_C>\tau_D)$

In the limit of $\tau_D\to 0$ $\tilde{e}_2$ does not exist. This can be explained by the fact that with no delays present in the Prisoner's Dilemma there is no internal stationary state.

In[∘]:=
```
Limit[x_{τC0}, τD → 0, Direction → "FromAbove"]
```

Out[∘]= Undefined if $c>1\,\&\&\,b>c\,\&\&\,\tau_C>0$

In[∘]:=
```
Limit[y_{C,τC 0}, τD → 0]
```

Out[∘]= 0

In[∘]:=
```
Limit[y_{D,τC0}[x], τD → 0]
```

Out[∘]= 0

No defector delay (τ_D=0)

First, we determine the values of the possible internal stationary state by solving the system of ODEs.

First, we notice that the defector kindergarten is empty, hence we have:

In[ ]:=
```
y_{D,τ_D 0} = 0;
```

Next, we obtain the value of y_C

In[ ]:=
```
y_{C,τ_D0}[x_] := y_C /. Solve[dx_{τ_D 0}[x, y_C, y_{D,τ_D 0}] == 0, y_C][[1]]
```

Then, we calculate x

In[ ]:=
```
x_{τ_D0} =
  x /.
   Solve[dy_{C,τ_D 0}[x, y_{C,τ_D0}[x], y_{D,τ_D 0}] == 0 && τ_C > 0 && τ_D > 0 && b > c > 1 && τ_D ≠ τ_C && 0 < x < 1,
     x][[1]]
```

Part : Part 1 of {} does not exist.

ReplaceAll : {{}[[1]]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

Solve : When parameter values satisfy the condition b > 1 && 1 < c < b && τ_C > 0 && τ_D > 0, the solution set contains a full –dimensional component; use Reduce for complete solution information.

Part : Part 1 of {} does not exist.

ReplaceAll : {{}[[1]]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing.

Out[ ]=
```
x /. {}[[1]]
```

We see that with no defector delay, no internal stationary state can exist in the Prisoner's Dilemma game.

Now we investigate the limit of $\tau_D \to 0$ of the internal stationary state e_2.

In[ ]:=
```
Simplify[Limit[Normal[x_2], τ_D → 0], τ_C > 0 && b > c > 1]
```

Out[ ]=

$$\frac{1 - 2\,c + \sqrt{1 - \frac{4\,c}{\tau_C}}}{2\,b}$$

In[ ]:=

$$\text{Reduce}\left[\frac{1 - 2\,c + \sqrt{1 - \frac{4\,c}{\tau_C}}}{2\,b} > 0\ \&\&\ \tau_C > 0\ \&\&\ b > c > 1\right]$$

Out[ ]=
```
False
```

We see that in the limit of no defector delay in value of internal stationary state is always negative, which means that it does not exist in the interesting interval of (0, 1)

Example

Next, we present an example of the Prisoner's Dilemma game and show the analysis of the Kindergarten model in that game.

We will analyse the following game:

In[ ]:=
```
b = 3;
c = 2;
matrix
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 3 & 0 \\ 5 & 2 \end{pmatrix}$$

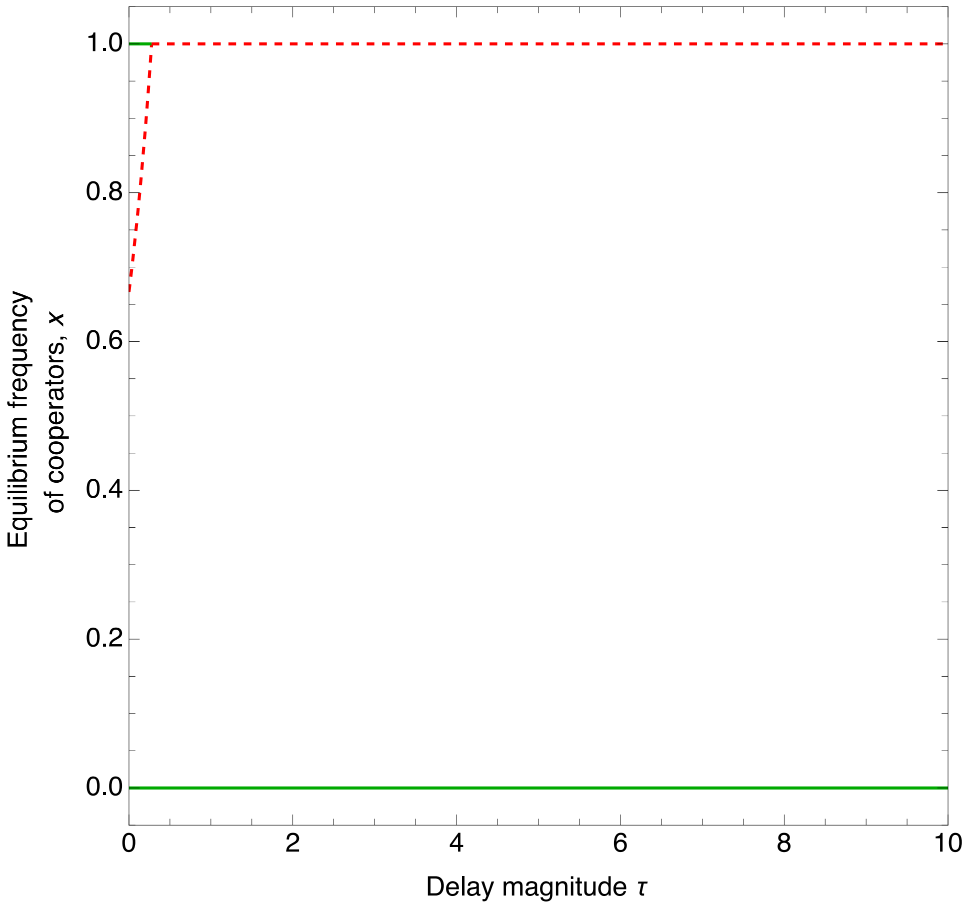
We will plot the value of x_2 for different values of delays

In[•]:=
```
Plotx[tc_, td_, linestyle_] :=
 Plot[If[1 > x2 > 0 /. {τC → tc, τD → td}, {x2 /. {τC → tc, τD → td}}], {τ, 0, 10},
  PlotRange → {{0, 10}, {-0.05, 1.05}}, PlotStyle → {{Red, linestyle}},
  Ticks → {Automatic, {0, 0.2, 0.4, 0.8, 1}}, Frame → True,
  FrameLabel → {"Delay magnitude τ",
    "Equilibrium frequency \n of cooperators, x"},
  LabelStyle → {14, Black, FontFamily → "Helvetica"}, AspectRatio → 1,
  ImageSize → 500]
Plot0[tc_, td_] := Plot[{0}, {τ, 0, 10}, PlotStyle → Darker[Green]]
Plot1st[tc_, td_] := Plot[If[tc < r /. {τC → tc, τD → td}, 1], {τ, 0, 10},
  PlotStyle → Darker[Green]]
Plot1unst[tc_, td_, linestyle_] :=
 Plot[If[tc ≥ r /. {τC → tc, τD → td}, 1], {τ, 0, 10}, PlotStyle → {Red, linestyle}]
```

$\tau_C = \tau, \tau_D = 1$

In[•]:=
```
panela = Show[Plotx[τ, 1, Dashed], Plot0[τ, 1], Plot1st[τ, 1],
  Plot1unst[τ, 1, Dashed]]
```

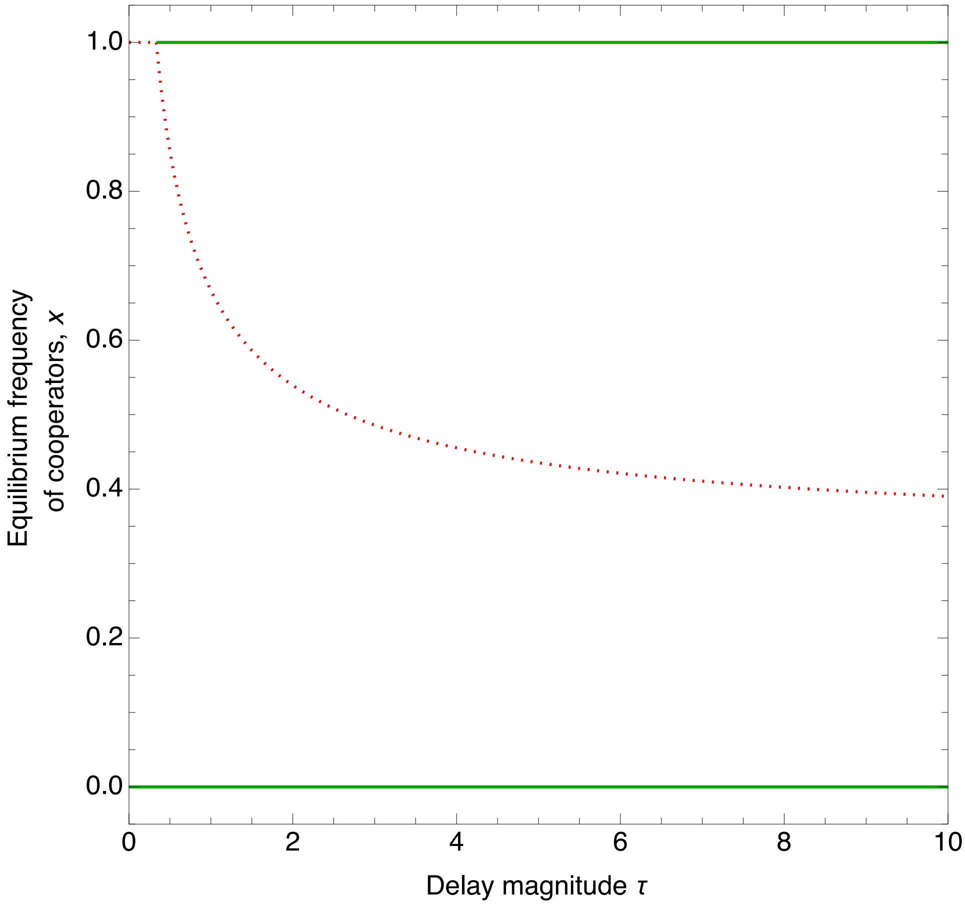
Out[•]=



$\tau_C = 0, \tau_D = \tau$

In[ ]:=
```
panelb = Show[Plotx[0, τ, Dotted], Plot0[0, τ], Plot1st[0, τ],
  Plot1unst[0, τ, Dotted]]
```

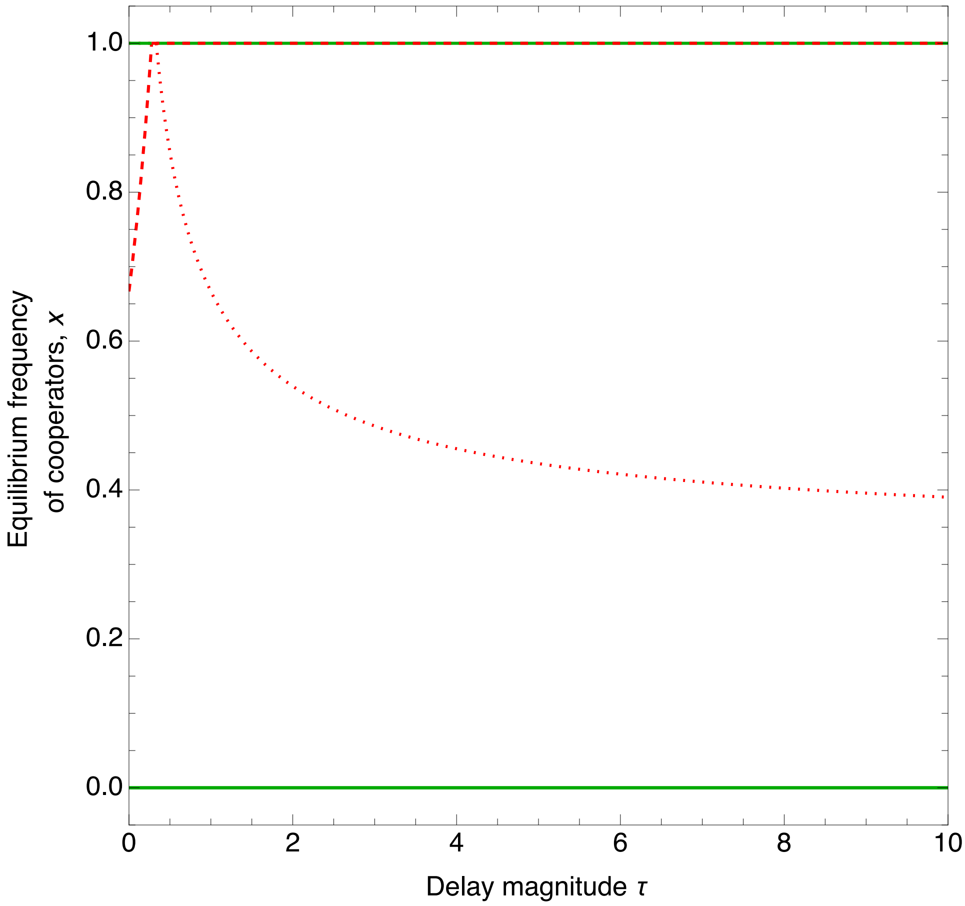
Out[ ]=



Figure

In[◦]:=
```
Show[panelb, panela]
```

Out[◦]=



Stability

Lastly, we plot the value of x in the internal stationary state depending on the values of delays.

In[◦]:=
```
boundary = Solve[τC == r && τC > 0 && τD ≠ τC && 0 < x < 1, τD, Reals]〚1〛;
bound = Plot[τD /. Normal[boundary], {τC, 0, 10}, PlotRange → {{0, 10}, {0, 10}},
    PlotStyle → Black];
```

In[ ]:=
```
rainbow = DensityPlot[x2, {τC, 0, 10}, {τD, 0, 10},
   PlotRange → {{0, 10}, {0, 10}, {0, 1} },
   ColorFunction → (ColorData["LightTemperatureMap", (1 - #)] &),
   FrameLabel → {"Cooperator delay τC", "Defector delay τD"}, Frame → True,
   LabelStyle → {14, Black, FontFamily → "Helvetica"},
   ColorFunctionScaling → False, ColorFunctionScaling → False];
alldefect = DensityPlot[0, {tc, 0, 10}, {td, 0, 10},
   PlotRange → {{0, 10}, {0, 10}, {0, 1} },
   ColorFunction → (ColorData["LightTemperatureMap", (1 - #)] &),
   PlotLegends → Placed[BarLegend[Automatic, LegendLayout → "Column",
      LegendLabel → "value of internal equilibrium"], Right],
   FrameLabel → {"Cooperator delay τC", "Defector delay τD"}, Frame → True,
   LabelStyle → {14, Black, FontFamily → "Helvetica"}, ImageSize → Medium];
text =
  Graphics[{Text[Style["Bistability", 14, FontFamily → "Helvetica"], {2, 8} ],
    Text[Style["Full Defection", 14, White, FontFamily → "Helvetica"], {6, 5}]}];
otherpanelsa = Plot[1, {tc, 0, 10}, PlotRange → {{0, 10}, {10, 10}},
   PlotStyle → {Black, Dashed}];
otherpanelsb =
  Graphics[{Black, Dotted, Thickness[0.02], Line[{{0, 0}, {0, 10}}]}];
```

Power : Infinite expression $\frac{1}{0.}$ encountered.

Power : Infinite expression $\frac{1}{0.}$ encountered.

Infinity : Indeterminate expression ComplexInfinity + ComplexInfinity encountered.

Power : Infinite expression $\frac{1}{0.}$ encountered.

General : Further output of Power::infy will be suppressed during this calculation.

In[ ]:=
```
paneld = Show[alldefect, rainbow, bound, text, otherpanelsa, otherpanelsb,
  ImageSize → 500]
```

Out[ ]=