SGDs

Game1

In[•]:=
```
d1 = 12;
r1 = 3; σ = 1;
```

In[•]:=
```
sgd1fitness[strat_, x1_, x2_, x3_] :=
  {1/(d1 (-1 + x3)^2 x3) (r1 x3 (-1 + x1 + x3) (-1 + x3^d1) +
      d1 (-1 + x3) (-x3 (-1 + r1 x1 + x3) - (-1 + x3) x3^d1 (-1 + r1 - σ))),
   σ x3^(d1-1) + r1 x1/(1 - x3) (1 - (1 - x3^d1)/(d1 (1 - x3))), σ}[[strat]];
```

In[•]:=
```
sgd1avgfit[x1_, x2_, x3_] := {x1, x2, x3}.sgd1fitness[All, x1, x2, x3];
```

In[•]:=
```
sgd1avgfit[x1, x2, x3]
```

Out[•]=
$$x3 + \frac{x1\left(3\,x3\,(-1+x1+x3)\left(-1+x3^{12}\right)+12\,(-1+x3)\left(-(-1+x3)\,x3^{12}-x3\,(-1+3\,x1+x3)\right)\right)}{12\,(-1+x3)^2\,x3} + x2\left(x3^{11}+\frac{3\,x1\left(1-\frac{1-x3^{12}}{12\,(1-x3)}\right)}{1-x3}\right)$$

In[•]:=
```
sgd1fitness[1, 0.1, 0.1, 0.8]
```

Out[•]=
```
0.996151
```

In[•]:=
```
Solve[(sgd1fitness[1, x, y, 1 - x - y] - sgd1fitness[2, x, y, 1 - x - y]) == 0 &&
   (sgd1fitness[1, x, y, 1 - x - y] - sgd1fitness[3, x, y, 1 - x - y]) == 0, {x, y}] // N
```

Out[•]=
```
{{x → 0., y → 0.}, {x → 0.098223, y → 0.098223}, {x → 0.966219, y → 0.966219},
 {x → 0.179562 - 0.338877 i, y → 0.179562 - 0.338877 i},
 {x → 0.179562 + 0.338877 i, y → 0.179562 + 0.338877 i},
 {x → 0.423321 - 0.459915 i, y → 0.423321 - 0.459915 i},
 {x → 0.423321 + 0.459915 i, y → 0.423321 + 0.459915 i},
 {x → 0.688403 - 0.42647 i, y → 0.688403 - 0.42647 i},
 {x → 0.688403 + 0.42647 i, y → 0.688403 + 0.42647 i},
 {x → 0.890778 - 0.254274 i, y → 0.890778 - 0.254274 i},
 {x → 0.890778 + 0.254274 i, y → 0.890778 + 0.254274 i}}
```

In[•]:=
```
tlim = 500;
```

```
sgd1sol[init_] := NDSolve[{
    x1'[t] == x1[t]
      (sgd1fitness[1, x1[t], x2[t], x3[t]] - sgd1avgfit[x1[t], x2[t], x3[t]]),
    x2'[t] == x2[t] (sgd1fitness[2, x1[t], x2[t], x3[t]] -
        sgd1avgfit[x1[t], x2[t], x3[t]]),
    x3'[t] == x3[t] (sgd1fitness[3, x1[t], x2[t], x3[t]] -
        sgd1avgfit[x1[t], x2[t], x3[t]]),
    x1[0] == init[[1]], x2[0] == init[[2]], x3[0] == 1 - init[[1]] - init[[2]]},
   {x1, x2, x3}, {t, 0, tlim}]
```

```
TA = {{-1/2, -1}, {Sqrt[3]/2, 0}};
new[r_] := TA.{r[[1]], r[[2]]} + {1, 0};
list1 = Map[new[#] &,
   Flatten[Table[Evaluate[{x1[t], x2[t], x3[t]} /. sgd1sol[{0.4, 0.15}]],
     {t, 0, tlim, 0.05}], 1]];
list2 = Map[new[#] &, Flatten[Table[Evaluate[
      {x1[t], x2[t], x3[t]} /. sgd1sol[{0.5, 0.3}]], {t, 0, tlim, 0.05}], 1]];
list3 = Map[new[#] &, Flatten[Table[Evaluate[{x1[t], x2[t], x3[t]} /.
       sgd1sol[{0.5, 0.2}]], {t, 0, tlim, 0.05}], 1]];
(*list2=Map[new[#]&,Flatten[Table[Evaluate[z/.sgd1sol[init_]],
     {t,0,tlim,0.05}],1]];*)
```

```
raninits = Table[temp = RandomReal[1, 3];
   temp/Total[temp][[1 ;; 2]], {i, 1, 50}];
```

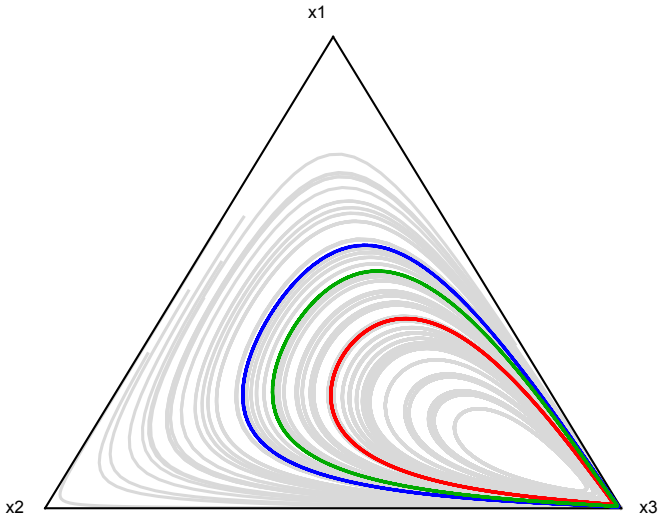
```
rantimeseries =
  Table[Flatten[Table[Evaluate[{x1[t], x2[t], x3[t]} /. sgd1sol[raninits[[i]]],
     {t, 0, tlim, 0.05}], 1], {i, 1, Length[raninits]}];
```

```
ranlist =
  Table[Map[new[#] &, rantimeseries[[i]][[ ;;  ;; 5]]], {i, 1, Length[raninits], 1}];
```

```
randoms = ListPlot[ranlist, Joined → True,
   PlotRange → {{-0.1, 1.1}, {-0.05, Sqrt[3]/2 + 0.1}}, PlotStyle → LightGray,
   Axes → None, AspectRatio → 0.8, Epilog → {Text[x1, new[{1.05, 0, 0}]],
     Text[x2, new[{0, 1.05, 0}]], Text[x3, new[{0, -0.05, 1}]]}];
```

In[•]:=
```
Show[randoms, ListPlot[{{{0, 0}, {1/2, Sqrt[3]/2}}, {{1, 0}, {1/2, Sqrt[3]/2}}, {{0, 0}, {1, 0}}, list1[[;; ;; 5]], list2[[;; ;; 5]], list3[[;; ;; 5]]},
 Joined → True, PlotRange → {{-0.1, 1.1}, {-0.05, Sqrt[3]/2 + 0.1}},
 PlotStyle → {{Black, Thickness[0.003]}, {Black, Thickness[0.003]}, {Black, Thickness[0.003]}, Red, {Blue, Dashed}, {Darker[Green], DotDashed}},
 Axes → None, AspectRatio → 0.8, Epilog → {Text[x1, new[{1.05, 0, 0}]], Text[x2, new[{0, 1.05, 0}]], Text[x3, new[{0, -0.05, 1}]]}]]
```

Out[•]=



Game2

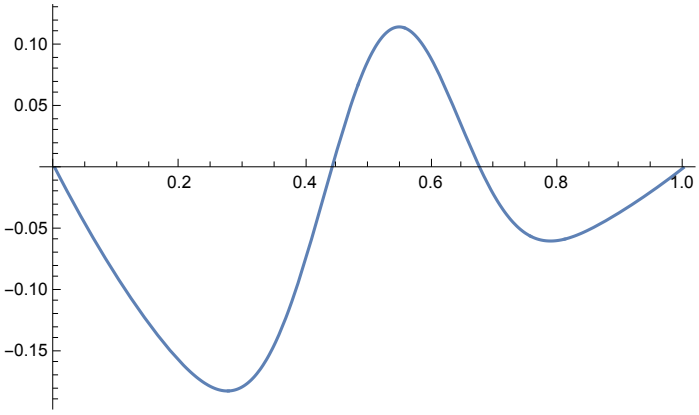
In[•]:=
```
d = 20;
m = 10;
r = 12;
c = 1;
defpay[k_] := k r c / d If[k ≤ m, 0, 1];
cooppay[k_] := k r c / d If[k ≤ m, 0, 1] - c;
```

In[•]:=
```
sgd2fitness[strat_, x1_, x2_] := {Sum[(Binomial[d - 1, k] x1^k x2^(d-1-k) cooppay[k + 1]), {k, 0, d-1}], Sum[(Binomial[d - 1, k] x1^k x2^(d-1-k) defpay[k]), {k, 0, d-1}]}[[strat]];
```

In[•]:=
```
sgd2avgfit[x1_, x2_] := {x1, x2}.sgd2fitness[All, x1, x2];
```

In[•]:=
```
Plot[x (1 - x) (sgd2fitness[1, x, 1 - x] - sgd2fitness[2, x, 1 - x]) // N, {x, 0, 1}]
```

Out[•]=



In[•]:=
```
x (1 - x) (sgd2fitness[1, x, 1 - x] - sgd2fitness[2, x, 1 - x])
```

Out[•]=

$$(1-x)\,x\left(-(1-x)^{19}-19\,(1-x)^{18}\,x-171\,(1-x)^{17}\,x^2-969\,(1-x)^{16}\,x^3-\right.$$
$$3876\,(1-x)^{15}\,x^4-11\,628\,(1-x)^{14}\,x^5-27\,132\,(1-x)^{13}\,x^6-50\,388\,(1-x)^{12}\,x^7-$$
$$75\,582\,(1-x)^{11}\,x^8-92\,378\,(1-x)^{10}\,x^9+\frac{2\,586\,584}{5}\,(1-x)^9\,x^{10}-$$
$$\frac{151\,164}{5}\,(1-x)^8\,x^{11}-\frac{100\,776}{5}\,(1-x)^7\,x^{12}-\frac{54\,264}{5}\,(1-x)^6\,x^{13}-\frac{23\,256}{5}\,(1-x)^5\,x^{14}-$$
$$\left.\frac{7752}{5}\,(1-x)^4\,x^{15}-\frac{1938}{5}\,(1-x)^3\,x^{16}-\frac{342}{5}\,(1-x)^2\,x^{17}-\frac{38}{5}\,(1-x)\,x^{18}-\frac{2\,x^{19}}{5}\right)$$

In[•]:=
```
(sgd2fitness[1, x, 1 - x] - sgd2fitness[2, x, 1 - x]) // N
```

Out[•]=

$$-1.\,(1.-1.\,x)^{19}-19.\,(1.-1.\,x)^{18}\,x-171.\,(1.-1.\,x)^{17}\,x^2-969.\,(1.-1.\,x)^{16}\,x^3-$$
$$3876.\,(1.-1.\,x)^{15}\,x^4-11\,628.\,(1.-1.\,x)^{14}\,x^5-27\,132.\,(1.-1.\,x)^{13}\,x^6-$$
$$50\,388.\,(1.-1.\,x)^{12}\,x^7-75\,582.\,(1.-1.\,x)^{11}\,x^8-92\,378.\,(1.-1.\,x)^{10}\,x^9+$$
$$517\,317.\,(1.-1.\,x)^9\,x^{10}-30\,232.8\,(1.-1.\,x)^8\,x^{11}-20\,155.2\,(1.-1.\,x)^7\,x^{12}-$$
$$10\,852.8\,(1.-1.\,x)^6\,x^{13}-4651.2\,(1.-1.\,x)^5\,x^{14}-1550.4\,(1.-1.\,x)^4\,x^{15}-$$
$$387.6\,(1.-1.\,x)^3\,x^{16}-68.4\,(1.-1.\,x)^2\,x^{17}-7.6\,(1.-1.\,x)\,x^{18}-0.4\,x^{19}$$

In[•]:=
```
Solve[x (1 - x) (sgd2fitness[1, x, 1 - x] - sgd2fitness[2, x, 1 - x]) == 0 // Simplify,
  x, Reals] // N
```

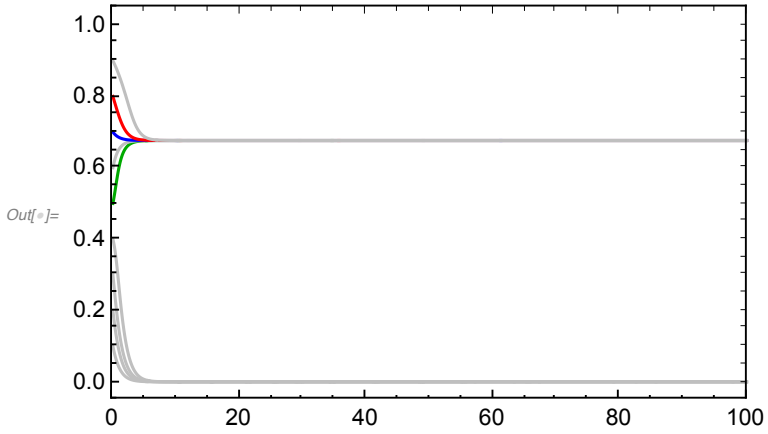
Out[•]=
```
{{x → 0.}, {x → 1.}, {x → -0.220814}, {x → 0.4407}, {x → 0.676432}}
```

In[•]:=
```
tlim = 100;
```

In[•]:=
```
sgd2sol[init_] := NDSolve[{
   x1'[t] == x1[t] (sgd2fitness[1, x1[t], x2[t]] - sgd2avgfit[x1[t], x2[t]]),
   x2'[t] == x2[t] (sgd2fitness[2, x1[t], x2[t]] - sgd2avgfit[x1[t], x2[t]]),
   x1[0] == init, x2[0] == 1 - init}, {x1, x2}, {t, 0, tlim}]
```

In[•]:=
```
sgd2pl = Plot[{
    Evaluate[{x1[iter]} /. sgd2sol[0.1]],
    Evaluate[{x1[iter]} /. sgd2sol[0.2]],
    Evaluate[{x1[iter]} /. sgd2sol[0.3]],
    Evaluate[{x1[iter]} /. sgd2sol[0.4]],
    Evaluate[{x1[iter]} /. sgd2sol[0.5]],
    Evaluate[{x1[iter]} /. sgd2sol[0.6]],
    Evaluate[{x1[iter]} /. sgd2sol[0.7]],
    Evaluate[{x1[iter]} /. sgd2sol[0.8]],
    Evaluate[{x1[iter]} /. sgd2sol[0.9]]}, {iter, 0, 100},
  PlotRange → {{-0.1, 100.1}, {-0.05, 1.05}}, Frame → True,
  PlotStyle → {Lighter[Gray, 0.5], Lighter[Gray, 0.5], Lighter[Gray, 0.5],
    Lighter[Gray, 0.5], Darker[Green], Lighter[Gray, 0.5], Blue,
    Red, Lighter[Gray, 0.5]}, Frame → True, FrameTicksStyle →
   Directive[Black, 12], FrameStyle → Directive[Thickness[0.003]]]
```

Out[•]=



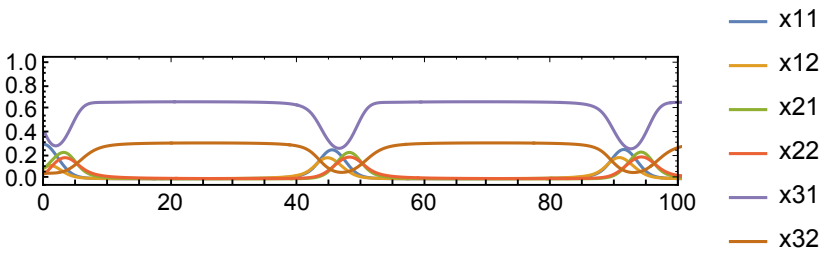
MGD

This is a function for

```
In[•]:= payoffmgd[x11_, x12_, x21_, x22_, x31_, x32_] :=
  {(sgd1fitness[1, x11 + x12, x21 + x22, x31 + x32] -
      sgd1avgfit[x11 + x12, x21 + x22, x31 + x32]) + (sgd2fitness[1, x11 + x21 + x31,
       x12 + x22 + x32] - sgd2avgfit[x11 + x21 + x31, x12 + x22 + x32]),
   (sgd1fitness[1, x11 + x12, x21 + x22, x31 + x32] - sgd1avgfit[x11 + x12,
       x21 + x22, x31 + x32]) + (sgd2fitness[2, x11 + x21 + x31, x12 + x22 + x32] -
      sgd2avgfit[x11 + x21 + x31, x12 + x22 + x32]),
   (sgd1fitness[2, x11 + x12, x21 + x22, x31 + x32] - sgd1avgfit[x11 + x12,
       x21 + x22, x31 + x32]) + (sgd2fitness[1, x11 + x21 + x31, x12 + x22 + x32] -
      sgd2avgfit[x11 + x21 + x31, x12 + x22 + x32]),
   (sgd1fitness[2, x11 + x12, x21 + x22, x31 + x32] - sgd1avgfit[x11 + x12,
       x21 + x22, x31 + x32]) + (sgd2fitness[2, x11 + x21 + x31, x12 + x22 + x32] -
      sgd2avgfit[x11 + x21 + x31, x12 + x22 + x32]),
   (sgd1fitness[3, x11 + x12, x21 + x22, x31 + x32] - sgd1avgfit[x11 + x12,
       x21 + x22, x31 + x32]) + (sgd2fitness[1, x11 + x21 + x31, x12 + x22 + x32] -
      sgd2avgfit[x11 + x21 + x31, x12 + x22 + x32]),
   (sgd1fitness[3, x11 + x12, x21 + x22, x31 + x32] - sgd1avgfit[x11 + x12,
       x21 + x22, x31 + x32]) + (sgd2fitness[2, x11 + x21 + x31, x12 + x22 + x32] -
      sgd2avgfit[x11 + x21 + x31, x12 + x22 + x32])};

In[•]:= mgdfitness[strat_, x11_, x12_, x21_, x22_, x31_, x32_] :=
  {payoffmgd[x11, x12, x21, x22, x31, x32]}[[1, strat]];

In[•]:= mgdavgfit[x11_, x12_, x21_, x22_, x31_, x32_] :=
  {x11, x12, x21, x22, x31, x32}.payoffmgd[x11, x12, x21, x22, x31, x32];

In[•]:= tlim = 500;
mgdsol[init_] := NDSolve[{
   x11'[t] == x11[t] (mgdfitness[1, x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]] -
       mgdavgfit[x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]]),
   x12'[t] == x12[t] (mgdfitness[2, x11[t], x12[t], x21[t], x22[t], x31[t],
        x32[t]] - mgdavgfit[x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]]),
   x21'[t] == x21[t] (mgdfitness[3, x11[t], x12[t], x21[t], x22[t], x31[t],
        x32[t]] - mgdavgfit[x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]]),
   x22'[t] == x22[t] (mgdfitness[4, x11[t], x12[t], x21[t], x22[t], x31[t],
        x32[t]] - mgdavgfit[x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]]),
   x31'[t] == x31[t] (mgdfitness[5, x11[t], x12[t], x21[t], x22[t], x31[t],
        x32[t]] - mgdavgfit[x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]]),
   x32'[t] == x32[t] (mgdfitness[6, x11[t], x12[t], x21[t], x22[t], x31[t],
        x32[t]] - mgdavgfit[x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]]),
   x11[0] == init[[1]], x12[0] == init[[2]], x21[0] == init[[3]], x22[0] == init[[4]],
   x31[0] == init[[5]], x32[0] == 1 - init[[1]] - init[[2]] - init[[3]] - init[[4]] - init[[5]]},
  {x11, x12, x21, x22, x31, x32}, {t, 0, tlim}]

In[•]:= mgdinits = {{0.3, 0.1, 0.1, 0.05, 0.4, 0.05},
   {0.4, 0.1, 0.2, 0.1, 0.1, 0.1}, {0.2, 0.3, 0.1, 0.1, 0.2, 0.1}};
```

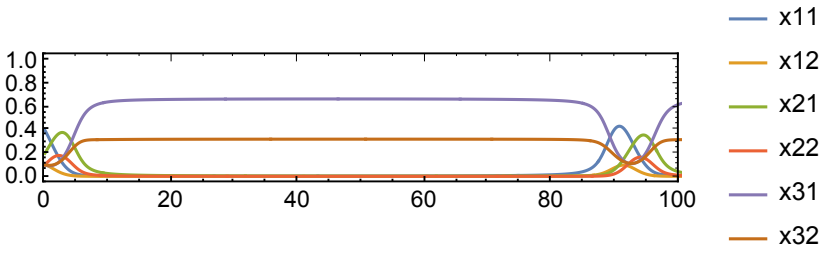
In[•]:=
```
Plot[
 Evaluate[
  {x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]} /. mgdsol[mgdinits[[1]]]],
{t, 0, tlim}, PlotRange → {{-0.1, 100.1}, {-0.05, 1.05}},
Frame → True, PlotStyle → Automatic
(*{Red,Lighter[Red,0.6],Red,Lighter[Red,0.6],Red,Lighter[Red,0.6]}*),
Frame → True, FrameTicksStyle → Directive[Black, 12],
FrameStyle → Directive[Thickness[0.003]],
PlotLegends → {"x11", "x12", "x21", "x22", "x31", "x32"}, AspectRatio → 0.2]
```

Out[•]=



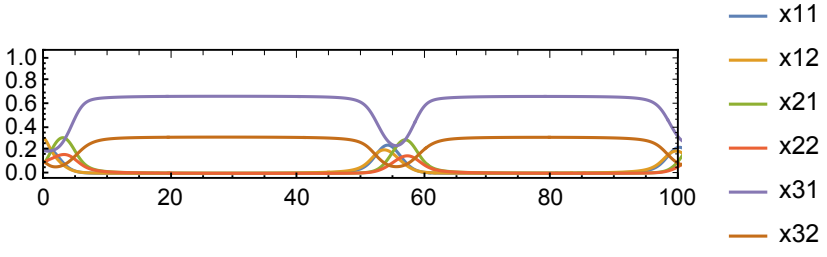
In[•]:=
```
Plot[
 Evaluate[
  {x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]} /. mgdsol[mgdinits[[2]]]],
{t, 0, 200}, PlotRange → {{-0.1, 100.1}, {-0.05, 1.05}}, Frame → True,
Frame → True, FrameTicksStyle → Directive[Black, 12],
FrameStyle → Directive[Thickness[0.003]], PlotStyle → Automatic,
PlotLegends → {"x11", "x12", "x21", "x22", "x31", "x32"}, AspectRatio → 0.2]
```

Out[•]=



In[•]:=
```
Plot[
 Evaluate[
  {x11[t], x12[t], x21[t], x22[t], x31[t], x32[t]} /. mgdsol[mgdinits[[3]]]],
{t, 0, tlim}, PlotRange → {{-0.1, 100.1}, {-0.05, 1.05}}, Frame → True,
Frame → True, FrameTicksStyle → Directive[Black, 12],
FrameStyle → Directive[Thickness[0.003]], PlotStyle → Automatic,
PlotLegends → {"x11", "x12", "x21", "x22", "x31", "x32"}, AspectRatio → 0.2]
```

Out[•]=



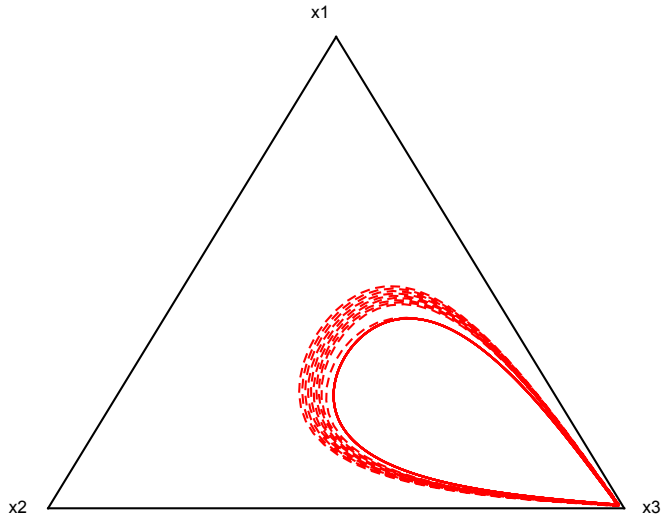
In[•]:=
```
recoverlist1 = Map[new[#] &,
   Flatten[Table[Evaluate[{x11[t] + x12[t], x21[t] + x22[t], x31[t] + x32[t]} /.
       mgdsol[{0.3, 0.1, 0.1, 0.05, 0.4, 0.05}]], {t, 0, tlim, 0.05}], 1]];
recoverlist2 = Map[new[#] &, Flatten[Table[Evaluate[
      {x11[t] + x12[t], x21[t] + x22[t], x31[t] + x32[t]} /.
       mgdsol[{0.4, 0.1, 0.2, 0.1, 0.1, 0.1}]], {t, 0, tlim, 0.05}], 1]];
recoverlist3 = Map[new[#] &, Flatten[Table[Evaluate[
      {x11[t] + x12[t], x21[t] + x22[t], x31[t] + x32[t]} /.
       mgdsol[{0.2, 0.3, 0.1, 0.1, 0.2, 0.1}]], {t, 0, tlim, 0.05}], 1]];
```

In[•]:=
```
recoverlist1[[ ;;  ;; 5]] // Length
```

Out[•]=
```
2001
```

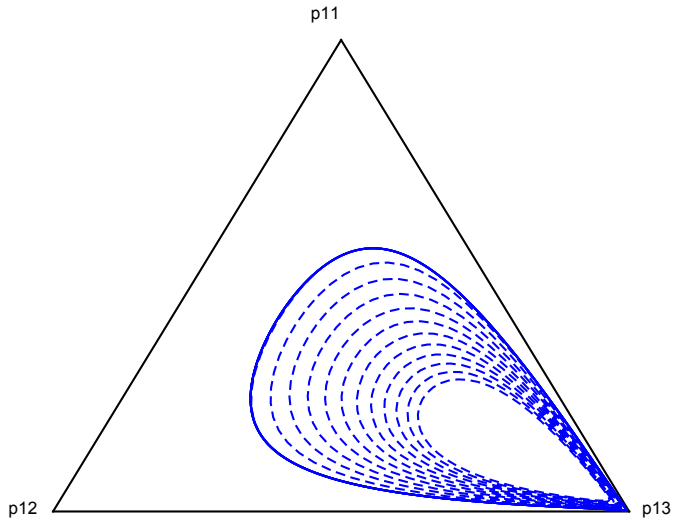
In[•]:=
```
ListPlot[{{{{0, 0}, {1/2, Sqrt[3]/2}}}, {{1, 0}, {1/2, Sqrt[3]/2}}},
  {{0, 0}, {1, 0}}, list1, recoverlist1[[ ;;  ;; 5]]},
 Joined → True, PlotRange → {{-0.1, 1.1}, {-0.05, Sqrt[3]/2 + 0.1}},
 PlotStyle → {{Black, Thickness[0.003]}, {Black, Thickness[0.003]}, {Black,
    Thickness[0.003]}, {Red, Thickness[0.003]}, {Red, Thickness[0.003], Dashed}},
 Axes → None, AspectRatio → 0.8, Epilog → {Text[x1, new[{1.05, 0, 0}]],
   Text[x2, new[{0, 1.05, 0}]], Text[x3, new[{0, -0.05, 1}]]}]
```

Out[•]=



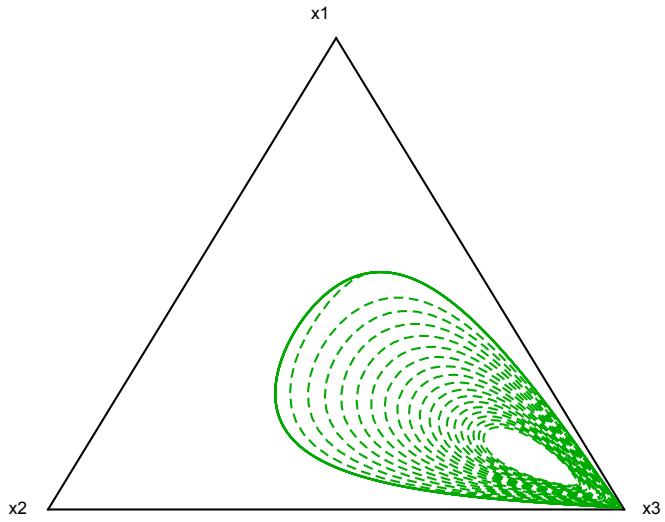
In[•]:=
```
ListPlot[{{{{0, 0}, {1/2, Sqrt[3]/2}}}, {{1, 0}, {1/2, Sqrt[3]/2}}},
  {{0, 0}, {1, 0}}, list2, recoverlist2[[;; ;; 5]]},
 Joined → True, PlotRange → {{-0.1, 1.1}, {-0.05, Sqrt[3]/2 + 0.1}},
 PlotStyle → {{Black, Thickness[0.003]},
   {Black, Thickness[0.003]}, {Black, Thickness[0.003]},
   {Blue, Thickness[0.003]}, {Blue, Thickness[0.003], Dashed}},
 Axes → None, AspectRatio → 0.8, Epilog → {Text[p11, new[{1.05, 0, 0}]],
   Text[p12, new[{0, 1.05, 0}]], Text[p13, new[{0, -0.05, 1}]]}]
```

Out[•]=



In[•]:=
```
ListPlot[{{{{0, 0}, {1/2, Sqrt[3]/2}}}, {{1, 0}, {1/2, Sqrt[3]/2}}},
  {{0, 0}, {1, 0}}, list3, recoverlist3[[ ;;  ;; 5]]},
 Joined → True, PlotRange → {{-0.1, 1.1}, {-0.05, Sqrt[3]/2 + 0.1}},
 PlotStyle → {{Black, Thickness[0.003]}, {Black, Thickness[0.003]},
   {Black, Thickness[0.003]}, {Darker[Green], Thickness[0.003]},
   {Darker[Green], Thickness[0.003], Dashed}}, Axes → None,
 AspectRatio → 0.8, Epilog → {Text[x1, new[{1.05, 0, 0}]],
   Text[x2, new[{0, 1.05, 0}]], Text[x3, new[{0, -0.05, 1}]]}]
```

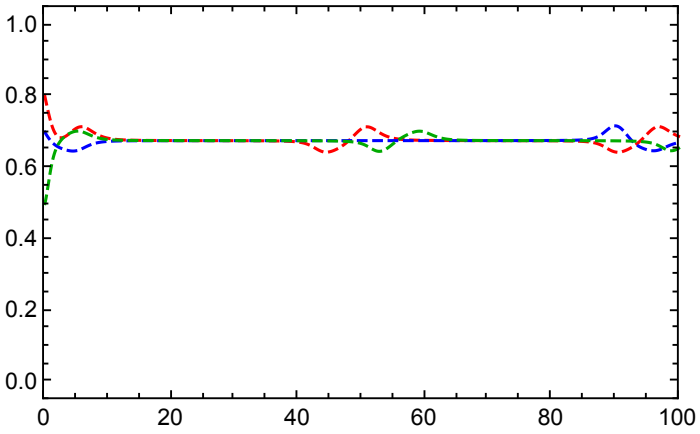
Out[•]=




```
recover1sgd2 =
  Table[{iter, Evaluate[{x11[iter] + x21[iter] + x31[iter], + x12[iter] +
          x22[iter] + x32[iter]} /. mgdsol[
         {0.3, 0.1, 0.1, 0.05, 0.4, 0.05}]]} // Flatten, {iter, 0, 100, 0.05}];
recover2sgd2 = Table[{iter, Evaluate[{x11[iter] + x21[iter] + x31[iter],
         + x12[iter] + x22[iter] + x32[iter]} /.
       mgdsol[{0.4, 0.1, 0.2, 0.1, 0.1, 0.1}]]} // Flatten, {iter, 0, 100, 0.05}];
recover3sgd2 = Table[{iter, Evaluate[{x11[iter] + x21[iter] + x31[iter],
         + x12[iter] + x22[iter] + x32[iter]} /.
       mgdsol[{0.2, 0.3, 0.1, 0.1, 0.2, 0.1}]]} // Flatten, {iter, 0, 100, 0.05}];
```

In[•]:=
```
recoveredsgd2pl = ListPlot[{recover1sgd2[[All, {1, 2}]], recover2sgd2[[All, {1, 2}]],
    recover3sgd2[[All, {1, 2}]]}, Joined → True, DataRange → {0, 100},
  PlotStyle → {{Red, Dashed}, {Blue, Dashed}, {Darker[Green], Dashed}},
  PlotRange → {{-0.1, 100.1}, {-0.05, 1.05}}, Frame → True,
  Frame → True, FrameTicksStyle → Directive[Black, 12],
  FrameStyle → Directive[Thickness[0.003]]]
```

Out[•]=



In[•]:=
```
Show[recoveredsgd2pl, sgd2pl, Frame → True,
 FrameTicksStyle → Directive[Black, 12],
 FrameStyle → Directive[Thickness[0.003]]]
```

Out[•]=