Title

Abstract

Abstract stuff

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Introduction

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Deterministic dynamics

2.1 Replicator Dynamics

2.1.1 Two player games with two strategies

Replicator dynamics is the core idea of evolutionary dynamics. Replicator dynamics determines how the frequency of different strategies in a population changes over time. We can analyze the equation used to determine replicator dynamics mathematically. Let's take the frequency of a and b strategies are x_a and x_b and the fitness of each strategy is f_a and f_b. It is a two-player game so we can say, $x_a + x_b = 1$. We can write two differential equations using the above information

$$dx_a/dt = x_a(f_a - \bar{f})(1)$$

$$dx_b/dt = y(f_b - \bar{f})(2)$$

. Now to keep the average fitness of the population constant, we can write an equation,

$$\bar{f} = (x_a f_a + x_b f_b)(3)$$

, also, we can write the previously mentioned equation as $x_b = 1 - x_a$. Now, we can substitute this x_b value into the 3 numbered equation, after substituting, we get the undermentioned equation.

$$\bar{f} = x_a f_a + (1 - x_a) f_b(4)$$

Now let's substitute the (4) numbered equation into the (1) equation

$$dx_a/dt = x_a[f_a - (x_a f_a + (1-x_a)f_b)] = x_a[f_a - x_a f_a - f_b + x_a f_b] = x_a[(1-x_a)f_a - (1-x_a)f_b] = x_a(1-x_a)(f_a - f_b)$$

. This is for a two-player population. If we have n population, we can denote the equation of replicator dynamics as,

$$dx_i/dt = x_i[f_i(x) - \bar{f}]$$

where $i = 1, 2, 3.....(n-1)$

To proceed with the replicator dynamics, we must understand simplex as the next step. Simplex is a tool in evolutionary dynamics that helps us understand and visualize how a system evolved through an evolutionary process. We can denote this for $(n-1)$ strategies. But for simplicity, we took a two-player $(n=2)$ homogenous process in which for a, $x_a = 1$ and $x_b = 0$, or the same goes for b, where, $x_a = 0$ and $x_b = 1$. It will represent a line between points a and b, with the midpoint being $x_a = x_b = 0.5$. For another example, if we homogeneously use $n = 3$, it will represent an equatorial triangle, and the midpoint of the simplex will be $x_a = x_b = x_c = 1/3$. In population genetics, this simplex is known as the de Finetti diagram. Let's get back to the replicator dynamics now,

$$dx_a/dt = x_a(1-x_a)(f_a - f_b)$$

, What does this equation tell us? We can understand that the rate of change of the frequency of a certain type with time depends on the fitness, average fitness of the population, and frequency. From this understanding, we can state that if $f_i(x) - \bar{f} > 0$ then the frequency of this type will increase over time, and if $f_i(x) - \bar{f} < 0$, the frequency of the type will decrease over time. If we make the change of strategy a over time constant, which means $dx_a/dt = 0$ then we can write the replicator dynamics equation as,

$$x_a(1-x_a)(f_a - f_b) = 0$$

Now, if we try to find the certain under which dx_a/dt will be 0. We will have three certain conditions,

$$x = 0, x = 1$$

and

$$f_a = f_b$$

$x = 0$ and $x = 1$ will be the two end points of a single line simplex and the joining point for the frequency graph and the dynamics of this frequency depends on the values of f_a and f_b. If $f_a > f_b$ then the curve will be over the 0 if $f_a < f_b$ then the curve will be under 0, and for the condition $f_a = f_b$, it will make a straight line between the

0 and 1[1].We can calculate the particular frequency where the strategy a becomes abundant from rare or rare from abundant.We can calculate the prticular turning point or saturation point frequency using this $f_a = f_b$ equation.
Let's take this frequency asx^*. as we know $f_a = f_b$ So we can write,

$$xa_0 + (1-x)a_1 = xb_0 + (1-x)b_1$$

$$x(a_0 - a_1) + a_1 = x(b_0 - b_1) + b_1$$

$$x(a_0 - a_1 - b_0 + b_1) = b_1 - a_1$$

$$x = \frac{b_1 - a_1}{a_0 - a_1 - b_0 + b_1}$$

We can write the x as the above mentioned frequency as x^*

$$x^* = \frac{b_1 - a_1}{a_0 - a_1 - b_0 + b_1}$$

For a visual representation of this graph, we can plot this equation for different values of f_a and f_b using Python. We can get different conditions from this equation

and graph. (1) Dominance: Where any of the two strategies will be dominant over another, means, for example, if $f_a > f_b$ that strategy a will be dominant over strategy b. It will be possible if the $a_1 > b_1$ and $a_0 > b_0$ where a_1, a_0, b_1, b_0 are the payoffs of a payoff matrix of a two-player game. (2) Co-existence: It happens when one of the two strategies is rare and has an advantage for that. For example, if strategy a is rare, then $f_a > f_b$ if a becomes abundant, it will lose its advantage, so the inequality will be $f_a < f_b$ and there will be a saturation point while becoming abundant from rare when the equation will $f_a = f_b$. In this scenario, the player should play the rare strategy. (3) Bi-stability: A condition where all a and b will be stable means a strategy will get an advantage if it is abundant. For example, if strategy a is abundant then the inequality will be $f_a > f_b$ and if it becomes rare then the inequality will be $f_a < f_b$ and there will be a saturation point when $f_a = f_b$. For this condition, the player should play the strategy its opponent is playing. (4) Neutrality: Now we have one condition in which both strategies will have the same impact. It does not depend on the change on the strategy. The equation for this strategy will be $f_a = f_b$. We mentioned this neutrality condition as a saturation point before[2].

(This is great! Here, I would analyse the 4 (rather 3) different games with specific entries of the payoffs, calculate the equilibria and plot them. E.g. show the results in figures for the prisoners dilemma, chicken game and the snowdrift game)

2.1.2 Two player games with multiple strategies

In the previous subsection, we analysed two player games with two strategies, now we are going to analyse two player games with multiple strategies.For better understanding of this particular problem, let's take a biological example, strains of *Escherichia coli* are competing for resources available on the media. K strain is the killer strain which produces toxin that will out-compete strain S. So, having toxin or not are the two strategies, we can write this using two strategy payoff matrix but there can be another possibility, where strain R is resistant to the toxin but pay the cost for resistance. SO, the scenario now changed completely, now the payoff matrix will be 3×3. This study was carried out by[3][4] Now, in a population there will be multiple strategies.So, we have to write the replicator dyanmics in $(n-1)$ dimensional simplex.For that we need $n \times n$ payoff matrix.

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & \dots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \end{array}$$

This is a two player game,here $a_{1,2}$ means player A playing strategy 1 agaisnt the strategy 2 of palyer B so we can write the equation using replicator dyanmics,

$$dx_i/dt = x_i[f_i(x) - \bar{f}]$$

We can derive the average fitness of a strategy i from this equation,

$$f_i(x) = a_{i,1}x_1 + a_{i,2}x_2 + ... + a_{i,n}x_n$$

We can write this equation as,

$$f_i(x) = \sum_{j=1}^{n} a_{i,j} x_j$$

and the average fitness of a population is given by

$$\bar{f} = x_1 f_1 + x_2 f_2 + ... + x_n f_n = \sum_{i=1}^{n} x_i f_i$$

(Similar to above .. now I would move to a specific example... analyse the rock paper scissors game. calculate the equilibrium dynamics. How does the stability change according to the magnitudes and asymmetries of the payoffs?)

Probabilistic Dynamics

3.1 Stochastic Process and Finite Population

3.1.1 Two players with two strategies

In the previous subsection, we used an infinite population for the deterministic approach[5]. Now, we will proceed with a finite population. There are mainly two reasons to use a finite population: first, it is realistic, and second, it is a natural way to introduce randomness into the replicator dynamics[6]. Now, we can consider the finite population of a specific game as N.Let's assume there are two strategies a and b. Frequencies for a and b are i and $(N - i)$. We can write the payoff matrix according to this specification,

	a	b
a	(a_1)	(a_0)
b	(b_1)	(b_0)

Now we can denote the average payoff of a and b strategies as π_a and π_b.

$$\pi_a = \frac{i-1}{N-1} a_1 + \frac{N-i}{N-1} a_0$$

$$\pi_b = \frac{i}{N-1} b_1 + \frac{N-i-1}{N-1} b_0$$

Now, a question can arise, why are we using $i-1$ instead of i? We have to understand that instead of an infinitely large population, we are using a real finite population and for that, we have to remove the particular individual who is observing or analysing this situation. So we use $i-1$. In the previous work, we used fitness directly related to payoff. But in the actual population, we introduce a tunable parameter similar to the fitness which controls the game. Nowak et al.[7] introduced this tunable parameter as selection intensity. Because for some situations or equations, we can know the particular relation between the fitness and payoff but for most of the cases, we don't know the particular relation, we can just create a hypothesis.

$$f_i = 1 - w + w\pi_i$$

where w selection intensity and π_i is the payoff. The value of w is bound by 0 and 1. If the value is 0 then $f_i = 1$, we can deduce from this equation that for every value of i, we will face neutrality, and if the value is 1, we will have $f_i = \pi_i$, now it suggests that the game payoff controls the fitness. Now, we can move to the part of the evolutionary dynamics where, for the finite population, there's randomness involved in the population. So, we will go for the stochastic processes. For these stochastic processes, we will focus on the Moran process. It consists of two events: birth and death. For the birth, one subject is chosen randomly, and it produces it's identical copy, and for the death process again, one random subject is chosen from the population and eliminated. It means by introducing randomness with the birth-death process now this theory can change the population with each time step, and for N steps it will control a generation [8] There are three mathematical equations that we can derive from this theory: (1) The number of a individuals increases by 1; (2) the number of a individuals decreases by 1; (3)The number of a individuals remains neutral.

For the (1) scenario, we can write:

$$T_i^+ = \frac{if_a}{if_a + (N-i)f_b}\frac{N-i}{N}$$

For the (2) scenario, we can write:

$$T_i^- = \frac{(N-i)f_a}{if_a + (N-i)f_b}\frac{i}{N}$$

For the (3) scenario, we can write:

$$1 - T_i^+ - T_i^-$$

Now what can we deduce from these equations?
From the (1) equation, we can say that a numbered individuals can increase only when an a individual is chosen for birth and an b individual is chosen for death.
From the (2) equation, we can say that the number of a decreases cause, an a individual is chosen for death and an b individual is chosen for birth.
From the (3) equation, we can deduce that no one is chosen for the birth or death process, so there's a neutrality.
Now, let us jump to the fixation probability.
In genetics, we often ask a question related to the invasion of a gene, how can the gene invade another gene, and how it affect the another gene's population? After understanding the transition probability and moran processes, we can now form a question based on the previous mentioned topic. In population of j numbered a individuals and $(N - j)$ numbered b individuals, what is the probability that the whole population will become of a individuals? This is known as the fixation probability of j numbered a individuals.We can denote this fixation probability as $\rho(j)$. Since, fixation is an absorbing state, so if all of the population once becomes a, it can not be revert. We can try to write a probability balance equation.

$$\rho(j) = T_i^+ \rho_a(j+1) + (1 - T_i^+ - T_i^-)\rho_a(j) + T_i^- \rho_a(j-1)$$

We can derive This discrete stochastic process using the absorbing phenomenon of Markov chain processes. Let's solve this equation.

$$\rho_a(j) - \rho_a(j-1) = \frac{T_i^-}{T_i^+}(\rho_a(j+1) - \rho_a(j))$$

$$T_i^+ + (\rho_a(j) - \rho_a(j-1) = T_i^-(\rho_a(j+1) - \rho_a(j)$$

$$T_i^+ \rho_a(j) - T_i^+ \rho_a(j-1) = T_i^- \rho_a(j+1) - T_i^- \rho_a(j)$$

$$\rho_a(j+1) = \frac{\rho_a(j)(T_i^+ + T_i^-) - T_i^+ \rho_a(j+1)}{T_i^-}$$

Now, let's write the equation.

$Ratio_j = R_j = \frac{T_i^-}{T_i^+} = \frac{(N-j)f_b}{jf_a}$ Let's substitute the value of R_j

$$\rho_a(j+1) = \rho_a(j)(1+R_j) - R_j\rho_a(j-1)$$

Let's solve this equation by putting some real values,

We get this underwritten equation by putting the value of $j = 1$,

$$\rho_a(2) = \rho_a(1)(1+R_1) - R_1\rho_a(0)$$

Let's put $j = 2$.

$$\rho_a(3) = \rho_a(2)(1+R_2) - R_2\rho_a(1)$$

Let's substitute the value of $\rho_a(2)$

$$\rho_a(3) = \rho_a(1)(1+R_1)(1+R_2) - \rho_a(1)$$

$$\rho_a(3) = \rho_a(1)((1+R_1)(1+R_2) - 1)$$

Now we can write a general equation based on this.

$$\rho_a(j) = \left[1 + \sum_{m=1}^{j-1}\prod_{i=1} mR_i\right]\rho_a(1)$$

Since we know that $j = N$, and fixation is guaranteed means $\rho_a(N) = 1$

$$1 = \left[1 + \sum_{m=1}^{N-1}\prod_{i=1}^{m} R_i\right]\rho_a(1)$$

$$\rho_a(1) = \frac{1}{1 + \sum_{m=1}^{N-1}\prod_{i=1}^{m} R_i}$$

We can use this above-mentioned equation when we are proceeding with just 1 player population. So, we can write the final fixation probability equation as,

$$\rho(j) = \frac{1 + \sum_{m=1}^{j-1}\prod_{i=1}^{m} R_i}{1 + \sum_{m=1}^{N-1}\prod_{i=1}^{m} R_i}$$

If we consider a neutral drift, where $f_a = f_b$, then we can write the equation as, $\rho_a = j/N$. If we observe the neutral drift with a single individual, we can write

$\rho_a = 1/N$[9]

We can calculate the fixation probability for strategy b as ρ_b. 1 b individual reaches fixation is equal to the $N-1$ a individuals fails to reach fixation.

$$\begin{aligned}\rho_b &= 1 - \rho_a(N-1) \\ &= 1 - \frac{1+\sum_{m=1}^{N-2}\prod_{i=1}^{m} R_i}{1+\sum_{m=1}^{N-1}\prod_{i=1}^{m} R_i} \\ &= \frac{1}{1+\sum_{m=1}^{N-1}\prod_{i=1}^{m} R_i}(\prod_{i=1}^{N-1} R_i) \\ &= \rho_a(\prod_{i=1}^{N-1} R_i)\end{aligned}$$

(I like the use of R_j as it makes some of the notations quite nice! Next, I would try to simulate the process by starting with a single player of a second strategy and $N-1$ of the first. For the same three games as you would have explored in the two-player two strategy case see what happens as you change the selection intensity. On the graphs then, plot the analytical expression of starting with a single mutant as a line and plot the simulatio results as dots.)

Discussion

Code availability. Appropriate computer code describing the model is available at `REDACTED for review`.

Acknowledgements

REDACTED for review.

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Supplementary material

6.1 Analysis of the simple system