Assignment 1

z5141448 Ruofei HUANG

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Definations

Two Dimensional Arrays Expression

To describe variants of two-dimensional arrays we write $(b: k \mapsto j \mapsto x)$ instead of $(b: k \mapsto (b[k]: j \mapsto x))$. We use this new notation to state an instance of the array-assignment axiom we saw already

$$\left\{\phi[^{(b:k\mapsto x)}/_b]\right\}b[k] := x\left\{\phi\right\}$$

for two-dimensional arrays:

$$\left\{\phi[^{(b:k\mapsto j\mapsto x)}/_b]\right\}b[k][j]:=x\left\{\phi\right\}$$

String Length

A string $S \in Letter^*$ which is an array of letters¹. Also, string will be terminate by the null character which is a convention by the C programming language and we will follow this convention in this proof. We write |S| for the number of letters in the string. Formally, we define these two nothion inductively by

$$|S\ell| = |S| + \begin{cases} 1 & \text{if } \ell \neq' \setminus 0' \\ 0 & \text{if } \ell =' \setminus 0' \end{cases}$$

Also, by the convention of C we has this definition for $S \in string$.

$$S[|S|] = ' \setminus 0' \land \forall 0 \le i < |S| (S[i] \ne ' \setminus 0')$$

¹The letter here is a legal charater encode with ASCII, UTF-8 or other charater encoding standard.

Precondiction

In the toy language's programme, the string length must has a way to calculate. Here is the programme to calculate the String Length

```
a \in String
```

Postcondiction

```
a \in String \land aLength = |a|
```

Programme

```
 \begin{aligned} &\{a \in String\} \\ &\{I[^0/_{aLength}]\} \\ &aLength := 0; \\ & \text{while } a[aLength]! =' \setminus 0' \text{ do} \\ &\{I \wedge a[aLength] \neq' \setminus 0'\} \\ &\{I[^{aLength+1}/_{aLength}]\} \\ &aLength := saLength + 1; \\ &\text{od} \\ &\{I \wedge a[aLength] =' \setminus 0'\} \\ &\{a \in String \wedge aLength = |a|\} \end{aligned}
```

Where

```
I = a \in String \land \forall k \in 0..(aLength - 1) (a[k] \neq' \setminus 0')
```

First Implication: $a \in String \Rightarrow I[^0/_{aLength}]$

```
a \in String
\Rightarrow \quad \langle \text{The } \forall k \in 0..(0-1) \, (a[k] \neq' \setminus 0') \text{ is always true.} \rangle
a \in String \land \forall k \in 0..(0-1) \, (a[k] \neq' \setminus 0')
\Leftrightarrow \quad \langle \text{Defination of I.} \rangle
I[^0/_{aLength}]
```

Second Implication: $I \wedge a[aLength] \neq' \backslash 0' \Rightarrow I[^{aLength+1}/_{aLength}]$

$$I \wedge a[aLength] \neq' \setminus 0'$$

$$\Leftrightarrow \quad \langle \text{Defination of I.} \rangle$$

$$a \in String \wedge \forall k \in 0..(aLength - 1) (a[k] \neq' \setminus 0') \wedge a[aLength] =' \setminus 0'$$

$$\Leftrightarrow \quad \langle \text{Merge same condition of } a[i] \neq' \setminus 0' \rangle$$

$$a \in String \wedge \forall k \in 0..(aLength) (a[k] \neq' \setminus 0')$$

$$\Leftrightarrow \quad \langle \text{Substitue back of I.} \rangle$$

$$I[^{aLength+1}/_{aLength}]$$

Third Implication: $I \wedge a[aLength] =' \setminus 0' \Rightarrow a \in String \wedge aLength = |a|$

$$I \wedge a[aLength] =' \setminus 0'$$

 $\Leftrightarrow \quad \langle \text{Definition of I.} \rangle$
 $a \in String \wedge \forall k \in 0..(aLength - 1) (a[k] \neq' \setminus 0') \wedge a[aLength] =' \setminus 0'$
 $\Leftrightarrow \quad \langle \text{Definition of postcondiction and the definition of String.} \rangle$
 $a \in String \wedge aLength = |a|$

String Equals

To describe two string $a, b \ (a, b \in String)$ are equals we write a = b when:

$$a = b \iff |a| = |b| \land \forall j \in 0.. \, |a| \, (a[j] = b[j])$$

Similarly, we write:

$$a \neq b \iff \neg(a = b)$$

Comparing String

After defining what is string equal, we need a pieces programme in toy language to do the dirty work which could compare two strings. Also we need a flag variable to store the truth value of a=b

Precondiction

$$a, b \in String$$

Postcondiction

$$flag = (a = b)$$

Programme

```
\{a,b\in String\}
\left\{I[^{	ext{TRUE}}/_{flag}][^{0}/_{i}]
ight\}
i := 0;
flag := 1;
\{I\}
if |a| = |b| then
      \{I \wedge |a| = |b|\}
       \{J\}
       while i \leq |a| do
              \{J \wedge i \leq |a|\}
              {K}
              if a[i] = b[i] then
                     \{K \wedge a[i] = b[i]\}
                     skip
              else
                      \{K \wedge a[i] \neq b[i]\}
                     \left\{J[^{\mathrm{false}}/_{flag}][^{i+1}/_{i}]\right\}
                     flag := 0;
                     \left\{J[^{i+1}/_i]\right\}
              fi
              \left\{J[^{i+1}/_i]\right\}
              i := i + 1;
              \{J\}
       od
       \{J \wedge i > |a|\}
       \{I\}
else
       {I \wedge |a| \neq |b|}
       \{I[^{\mathrm{FALSE}}/_{flag}]\}
       flag := 0;
```

```
\begin{split} &\{I\} \\ \mathbf{fi} \\ &\{I\} \\ &\{flag = (a = b)\} \end{split}
```

Where TRUE = 1, FALSE = 0, and

$$\begin{split} I &= a, b \in String \land flag = (|a| = |b| \land 0 \leq i \leq (|a| + 1) \land \forall k \in 0..(i - 1) \ (a[i] = b[i])) \\ J &= a, b \in String \land flag = (0 \leq i \leq (|a| + 1) \land \forall k \in 0..(i - 1) \ (a[i] = b[i])) \\ K &= \left(\begin{array}{c} a[i] \neq b[i] \Rightarrow I^{\text{\tiny FALSE}}/flag][^{i+1}/i] \\ a[i] &= b[i] \Rightarrow I^{[i+1}/i] \end{array} \right) \end{split}$$

First Implication: $a,b \in String \Rightarrow I[^{\mathrm{true}}/_{flag}][^{0}/_{i}]$

$$a,b \in String$$
 \Rightarrow $\langle \text{The append junction is alway true.} \rangle$
$$a,b \in String \land |a| = |b| \land 0 \leq i \leq 0 \land \forall k \in 0..(i-1) \ (a[i] = b[i])$$
 \Rightarrow $\langle flag = \text{True and true} = \text{True} \rangle$
$$a,b \in String \land flag = (|a| = |b| \land 0 \leq i \leq 0 \land \forall k \in 0..(i-1) \ (a[i] = b[i]))$$
 \Leftrightarrow $\langle \text{Defination of I.} \rangle$
$$I[^{\text{True}}/_{flag}]$$

Second Implication: $I \wedge |a| = |b| \Rightarrow J \wedge [^0/_i]$

$$I \wedge |a| = |b|$$
 $\Rightarrow \quad \langle \text{Definiation of I. } i = 0 \text{ when right junction is always true.} \rangle$
 $a, b \in String \wedge flag = (|a| = |b| \wedge 0 \leq 0 \leq (|a| + 1))$
 $\wedge \forall k \in 0..(i - 1) (a[i] = b[i]))$
 $\Rightarrow \quad \langle \text{Definition of J.} \rangle$
 $J \wedge [0/i]$

Third Implication: $J \wedge i \leq |a| \Rightarrow K$

$$J \wedge i \leq |a|$$

$$\Leftrightarrow \quad \langle \text{Definition of J} \rangle$$

$$a,b \in String \wedge flag = (0 \leq i \leq (|a|+1) \wedge \forall k \in 0..(i-1) (a[i]=b[i])) \wedge i \leq |a|$$

$$\Rightarrow \quad \langle \text{By definition of two string is equal, two case to consider.} \rangle$$

$$\begin{pmatrix} a[i] \neq b[i] \Rightarrow a,b \in String \wedge flag = \text{FALSE} \\ a[i] = b[i] \Rightarrow a,b \in String \wedge flag = (|a|=|b| \\ \wedge 0 \leq i+1 \leq (|a|+2) \wedge \forall k \in 0..i (a[i+1]=b[i+1])) \wedge i \leq |a| \end{pmatrix}$$

$$\Rightarrow \quad \langle \text{By definition of two string is equal, two case to consider.} \rangle$$

$$\begin{pmatrix} a[i] \neq b[i] \Rightarrow a,b \in String \wedge flag = \text{FALSE} \\ a[i] = b[i] \Rightarrow a,b \in String \wedge flag = (|a|=|b| \\ \wedge 0 \leq i+1 \leq (|a|+1) \wedge \forall k \in 0..i (a[i+1]=b[i+1])) \end{pmatrix}$$

$$\Leftrightarrow \quad \langle \text{Definition of } I \rangle$$

$$\begin{pmatrix} a[i] \neq b[i] \Rightarrow I[^{\text{FALSE}}/_{flag}][^{i+1}/_{i}] \\ a[i] = b[i] \Rightarrow I[^{i+1}/_{i}] \end{pmatrix}$$

$$\Leftrightarrow \quad \langle \text{Defination of } K \rangle$$

$$K$$

Forth Implication: $I \wedge |a| \neq |b| \Rightarrow I[false]_{flag}$

$$I \wedge |a| \neq |b|$$
 $\Rightarrow \quad \langle \text{Definition of two string is not equal.} \rangle$
 $I \wedge |a| \neq |b| \wedge a \neq b$
 $\Rightarrow \quad \langle flag = \text{False and false} = \text{false} \rangle$
 $a, b \in String \wedge flag = (a = b) \wedge a \neq b$
 $\Leftrightarrow \quad \langle \text{Definition of I.} \rangle$
 $I[\text{False}/flag]$

Fifth Implication: $I \Rightarrow flag = (a = b)$

$$I$$
 \Leftrightarrow \quad \text{Defination of the purpose of } flag\rangle \quad flag = $(a = b)$

String Assign

To assign a string to another string array, we will denote as

$$a := b$$

instead of a long programme of our toy language:

```
 \begin{aligned} & \{a,b \in String\} \\ & \{I[^0/i]\} \\ & i := 0; \\ & \{I\} \\ & \text{while } i \leq |b| \text{ do } \\ & \{I \wedge i \leq |b|\} \\ & \{I[^{i+1}/_i][^{a:i \mapsto b[i]}/_a]\} \\ & a[i] := b[i]; \\ & \{I[^{i+1}/_i]\} \\ & i := i+1; \\ & \{I\} \\ & \text{od}; \\ & \{I \wedge i > |b|\} \\ & \{a,b \in String \wedge a = b\} \end{aligned}
```

when our invariant is

$$I = a, b \in String \land 0 < i < (|b| + 1) \land \forall k \in 0..(i - 1) (a[k] = b[k])$$

Here are the proofs of the implications:

First Implication for String assign: $a, b \in String \Rightarrow I^{[0]}_{i}$

```
a,b \in String
\Rightarrow \quad \langle \text{using } |b| \in \mathbb{N} \text{ and realising that the last conjunct is vacuously true} \rangle
a,b \in String \land 0 \leq 0 \leq (|b|+1) \land \forall k \in 0..(0-1) (a[k]=b[k])
\Leftrightarrow \quad \langle \text{definition of I and substitution} \rangle
I[^0/_i]
```

Second Implication $I \wedge i \leq |b| \Rightarrow I[^{i+1}/_i][^{a:i \mapsto b[i]}/_a]$

We first look at the LHS:

```
\begin{split} I \wedge i &\leq |b| \\ \Leftrightarrow & \langle \text{Substitue I} \; \rangle \\ a, b &\in String \wedge 0 \leq i \leq (|b|+1) \wedge \forall k \in 0..(i-1) \, (a[k]=b[k]) \wedge i \leq |b| \\ \Leftrightarrow & \langle \text{Conjunct } i \leq (|b|+1) \text{ and } i \leq |b| \rangle \\ a, b &\in String \wedge 0 \leq i \leq |b| \wedge \forall k \in 0..(i-1) \, (a[k]=b[k]) \end{split}
```

We then expand RHS:

$$\begin{split} I[^{i+1}/_i][^{a:i\mapsto b[i]}/_a] \\ \Leftrightarrow & \langle \text{substitute } i=i+1 \text{ and } a[i]=b[i] \text{ by definition} \rangle \\ a,b \in String \land 0 \leq i+1 \leq (|b|+1) \land \forall k \in 0..((i+1)-1) \, (a[k]=b[k]) \land a[i] := b[i] \end{split}$$

We then have a clear imply

$$a, b \in String \land 0 \le i \le |b| \land \forall k \in 0..(i-1) (a[k] = b[k])$$

 $\Rightarrow \quad \langle i \le |b| \Rightarrow i+1 \le |b|+1 \text{ and } a[i] := b[i] \rangle$
 $a, b \in String \land 0 \le i+1 \le (|b|+1) \land \forall k \in 0..((i+1)-1) (a[k] = b[k]) \land a[i] := b[i]$

Third Implication $I \wedge i > |b| \Rightarrow a, b \in String \wedge a = b$

$$\begin{split} I \wedge i > |b| \\ \Leftrightarrow & \langle \text{substitution of I} \rangle \\ a, b \in String \wedge 0 \leq i \leq (|b|+1) \wedge \forall k \in 0...(i-1) \, (a[k] := b[k]) \wedge i > |b| \\ \Leftrightarrow & \langle i > |b| \text{and } i \leq (|b|+1) \text{ with some calculation} \rangle \\ a, b \in String \wedge \forall k \in 0... |b| \, (a[k] := b[k]) \\ \Rightarrow & \langle \text{Definition of two string equal} \rangle \\ a, b \in String \wedge a = b \end{split}$$

1 Task 1

Since we have define some manipulation of String, we can see a string as a whole. So the input is an array of String. Also, the ouput is store in b which is an empty array (type is String* too). Hence we can define our precondiction as:

$$a, b \in String^* \land |a| = n$$

As the post condition as:

$$a, b \in String \land \forall i < n (a[i] = b[m(a, i)])$$

Where m is a mapping function define recursively as follow:

$$m(a,i) = \begin{cases} -1 & \text{if } i < 0\\ 0 & \text{if } i = 0\\ m(a,i-1) & \text{if } a[i] = a[i-1]\\ m(a,i-1) + 1 & \text{if } a[i] \neq a[i-1] \end{cases}$$

2 Task 2

We propose the following proof outline to demonstrate the correctness of our code (in black).

```
\{a, b \in String^* \land |a| = n\}
                                                                                                          (1)
\{J\}
                                                                                                          (2)
if |a| > 0 then
                                                                                                          (3)
      {J \wedge |a| > 0}
                                                                                                          (4)
      \{I[^1/_j][^1/_i][^{b:0\mapsto a[0]}/_b]\}
                                                                                                          (5)
      b[0] := a[0];
                                                                                                          (6)
      \{I[^1/_j][^1/_i]\}
                                                                                                          (7)
     i = 1; j = 1;
                                                                                                          (8)
      \{I\}
                                                                                                          (9)
else
                                                                                                        (10)
      {J \wedge |a| \leq 0}
                                                                                                        (11)
      \{I[^0/_i][^0/_j]\}
                                                                                                        (12)
     i := 0; j := 0;
                                                                                                        (13)
      \{I\}
                                                                                                        (14)
fi
                                                                                                        (15)
\{I\}
                                                                                                        (16)
while i < |a| do
                                                                                                        (17)
      \{I \wedge i < |a|\}
                                                                                                        (18)
      \{K\}
                                                                                                        (19)
      if a[i] \neq a[i-1] then
                                                                                                        (20)
            \{K \land a[i] \neq a[i-1]\}
                                                                                                        (21)
            \{I[^{i+1}/_i][^{j+1}/_j][^{b:j\mapsto a[i]}/_b]\}
                                                                                                        (22)
            b[j] := a[i];
                                                                                                        (23)
            \{I[^{i+1}/_i][^{j+1}/_j]\}
                                                                                                        (24)
            j := j + 1;
                                                                                                        (25)
            \{I^{[i+1]}_i\}
                                                                                                        (26)
      else
                                                                                                        (27)
            skip
                                                                                                        (28)
      fi
                                                                                                        (29)
      \left\{I[^{i+1}/_i]\right\}
                                                                                                        (30)
      i := i + 1
                                                                                                        (31)
```

$$\{I\} \tag{32}$$

$$od (33)$$

$$\{I \land i \ge |a|\} \tag{34}$$

$$\{a, b \in String \land \forall i < n \ (a[i] = b[m(a, i)])\}$$

$$(35)$$

Here are the invariants of this programme 2 :

$$\begin{split} I &= a, b \in String^* \land |a| = n \land 0 \le i \le |a| \land \forall k \in 0..(i-1) \ (a[k] = b[m(a,k)]) \\ \land j &= |b| = m(a,i-1) + 1 \\ J &= \begin{pmatrix} |a| > 0 \Rightarrow I[^1/_j][^1/_i][^{b:0 \mapsto a[0]}/_b] \\ |a| \le 0 \Rightarrow I[^0/_j][^0/_i] \end{pmatrix} \\ K &= \begin{pmatrix} a[i] \neq a[i-1] \Rightarrow I[^{i+1}/_i][^{j+1}/_j][^{b:j \mapsto a[i]}/_b] \\ a[i] &= a[i-1] \Rightarrow I[^{i+1}/_i] \end{pmatrix} \end{split}$$

2.1 First Implication: $a, b \in String^* \land |a| = n \Rightarrow J$

$$a,b \in String^* \wedge |a| = n$$

$$\Rightarrow \quad \langle \text{The right append junction is always true.} \, \rangle$$

$$\begin{pmatrix} |a| > 0 \Rightarrow a,b \in String^* \wedge |a| = n \wedge 0 \leq 1 \leq |a| \wedge \\ \forall k \in 0..(1-1) \, (a[k] = b[m(a,k)]) \wedge j = |b| = m(a,i-1) + 1 \\ |a| = 0 \Rightarrow a,b \in String^* \wedge |a| = n \wedge 0 \leq 0 \leq |a| \wedge \\ \forall k \in 0..(0-1) \, (a[k] = b[m(a,k)]) \wedge j = |b| = m(a,i-1) + 1 \end{pmatrix}$$

$$\Rightarrow \quad \langle \text{Substitution of I } \rangle$$

$$\begin{pmatrix} |a| > 0 \Rightarrow I[^1/_j][^1/_i][^{b:0 \mapsto a[0]}/_b] \\ |a| \leq 0 \Rightarrow I[^0/_j][^0/_i] \end{pmatrix}$$

$$\Leftrightarrow \quad \langle \text{Definition of J} \rangle$$

²The invariant is following the case study in week 8, might not be true for the stuff we study for now. But I have to use this tool otherwise I couldn't countinue this proof

2.2 Second Implication: $I \wedge i < |a| \Rightarrow K$

```
I \wedge i < |a|
\Leftrightarrow \(\rightarrow\) Definition of I\)
        a, b \in String^* \land |a| = n \land 0 \le i \le |a| \land \forall k \in 0..(i-1) (a[k] = b[m(a,k)]) \land i < |a|
                    \langle m(a,i) \text{ need to consider two situation of } a[i] \ (a[i] = a[i-1] \text{ and } a[i] \neq a[i-1]) \rangle
           \begin{pmatrix} a[i] \neq a[i-1] \Rightarrow a, b \in String^* \land |a| = n \land \\ 0 \leq i+1 \leq |a| \land \forall k \in 0..(i-1) (a[k] = b[m(a,k)]) \\ \land a[i] = b[j] \\ a[i] = a[i-1] \Rightarrow a, b \in String^* \land |a| = n \land \\ 0 \leq i+1 \leq |a| \land \forall k \in 0..(i-1) (a[k] = b[m(a,k)]) \\ \land a[i] = b[j-1] 
                      (By the recursively definition of m(a, i))
          \begin{cases} a[i] \neq a[i-1] \Rightarrow a, b \in String^* \land |a| = n \land \\ 0 \leq i+1 \leq |a| \land \forall k \in 0..(i-1) \ (a[k] = b[m(a,k)]) \\ \land a[i] = b[m(a,i)] \\ a[i] = a[i-1] \Rightarrow a, b \in String^* \land |a| = n \land \\ 0 \leq i+1 \leq |a| \land \forall k \in 0..(i-1) \ (a[k] = b[m(a,k)]) \\ \land a[i] = b[m(a,i)] \end{cases}
                    (Merge two cases of k (k = i and k \in 0...(i-1)).)
           \left( \begin{array}{l} a[i] \neq a[i-1] \Rightarrow a,b \in String^* \wedge |a| = n \wedge \\ 0 \leq i+1 \leq |a| \wedge \forall k \in 0..(i) \left( a[k] = b[m(a,k)] \right) \\ a[i] = a[i-1] \Rightarrow a,b \in String^* \wedge |a| = n \wedge \\ 0 \leq i+1 \leq |a| \wedge \forall k \in 0..(i) \left( a[k] = b[m(a,k)] \right) \end{array} \right) 
         (Definition of K
\Leftrightarrow
        K
```

The meaning of "By the recursively definition of m(a,i)" is when a[i] = a[i-1], a[i] = b[m(a,i)] = b[m(a,i-1)], which m(a,i-1) = j-1 since j = |b|. In another case, $a[i] \neq a[i-1]$, b[j] = a[i] (where |b| = j+1)

2.3 Third Implication:

```
\begin{split} I \wedge i &\geq |a| \Rightarrow a, b \in String \wedge \forall i < n \ (a[i] = b[m(a,i)]) \\ &I \wedge i \geq |a| \\ \Leftrightarrow & \langle \text{Definations of I} \rangle \\ &a, b \in String^* \wedge |a| = n \wedge 0 \leq i \leq |a| \wedge \forall k \in 0...(i-1) \ (a[k] = b[m(a,k)]) \wedge i \geq |a| \\ \Leftrightarrow & \langle i \leq |a| \wedge i \geq |a| \Leftrightarrow i = |a| \rangle \\ &a, b \in String^* \wedge |a| = n \wedge \forall k \in 0...(|a|-1) \ (a[k] = b[m(a,k)]) \\ \Rightarrow & \langle |a| = n \ \text{and} \ i \in \mathbb{N} \rangle \\ &a, b \in String^* \wedge \forall i \in 0...(n-1) \ (a[i] = b[m(a,i)]) \\ \Leftrightarrow & \langle \text{Defination of post condition.} \rangle \\ &a, b \in String \wedge \forall i < n \ (a[i] = b[m(a,i)]) \end{split}
```

3 Task 3

```
int length_str(char *a){
 1
 2
         int i = 0;
 3
         while(a[i]!='\setminus 0'){
 4
             i = i+1;
 5
         }
 6
         return i;
    }
 7
 8
    void copy_str(char *a, char *b){
 9
10
         // copy a particular string from a to b
         int i = 0;
11
12
         \mathbf{while}(i \le length\_str(a))
             b[i] = a[i];
13
14
             i++;
         }
15
    }
16
17
    int compare_str(char *a, char *b){
18
19
         int flag = 1;
20
         if(length\_str(a) == length\_str(b))
21
         {
22
             int i = 0:
             while (i \le length\_str(a))
23
                 if (a[i]!=b[i]){
24
```

```
flag = 0;
25
                    }
26
27
                    else{}
28
                         // skip
29
30
                    i++;
               }
31
32
          }
33
          \stackrel{\cdot}{\mathbf{else}}\{
34
               flag = 0;
35
          return flag;
36
37
     }
38
     unsigned int uniq(unsigned int n, char *a[], char *b[]){
39
          int i,j;
40
          if(n > 0){
41
               copy\_str(a[0],b[0]);
42
43
               i = j = 1;
          }
44
45
          \mathbf{else} \{
               i = j = 0;
46
47
48
          \mathbf{while}(i < n){
               if (!compare\_str(a[i], a[i-1])){
49
                    copy\_str(a[i],b[j]);
50
                    j++;
51
               }
52
               \stackrel{,}{\mathbf{else}}\{
53
                    // skip
54
55
56
               i++;
          }
57
58
          return j;
    }
59
```

4 Task 4