COMP4161: Advanced Topics in Software Verification

$\lambda^{\rightarrow}$

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Last time...

- ➜ λ calculus syntax
- ➜ free variables, substitution
- ➜ β reduction
- ➜ α and η conversion
- ➜ β reduction is confluent
- ➜ λ calculus is expressive (Turing complete)
- ➜ λ calculus is inconsistent (as a logic)

Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction, Isar (part 2) [6, 7[b]]
  - Hoare logic, proofs about programs, invariants [8]
  - C verification [9]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

λ calculus is inconsistent

Can find term R such that $R\ R\ =_\beta\ \texttt{not}(R\ R)$

There are more terms that do not make sense:

$$1\ 2, \quad \texttt{true}\ \texttt{false}, \quad \text{etc.}$$

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Solution: rule out ill-formed terms by using types.
(Church 1940)

Introducing types

Idea: assign a type to each "sensible" λ term.

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  Write: $(\lambda x.\ x) :: \alpha \Rightarrow \alpha$

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- if x has type α then $\lambda x.\ x$ is a function from α to α
  Write: $(\lambda x.\ x) :: \alpha \Rightarrow \alpha$
- for $s\ t$ to be sensible:
  s must be a function
  t must be right type for parameter

  If $s :: \alpha \Rightarrow \beta$ and $t :: \alpha$ then $(s\ t) :: \beta$

That’s about it

Now formally again

Syntax for $\lambda^{\rightarrow}$

Terms: $t \quad ::= \quad v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

Types: $\tau \quad ::= \quad \mathtt{b} \mid \nu \mid \tau \Rightarrow \tau$

$\mathtt{b} \in \{\mathtt{bool}, \mathtt{int}, \ldots\}$ base types

$\nu \in \{\alpha, \beta, \ldots\}$ type variables

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Term t has type τ in context Γ: $\quad \Gamma \vdash t :: \tau$

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A term t is **well typed** or **type correct**
if there are Γ and τ such that $\Gamma \vdash t :: \tau$

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More complex Example

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Formally:

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Type checking and type inference on $\lambda^{\rightarrow}$ are decidable.

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This property is called **subject reduction**

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➜ **$=_{\alpha\beta\eta}$ is decidable**
This is why Isabelle can automatically reduce each term to $\beta\eta$ normal form.

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Each computable function can be encoded as closed, type correct $\lambda^{\rightarrow}$ term using $Y :: (\tau \Rightarrow \tau) \Rightarrow \tau$ with $Y\ t \longrightarrow_\beta t\ (Y\ t)$ as only constant.

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- ➜ Y is called fix point operator
- ➜ used for recursion
- ➜ lose decidability (what does $Y\ (\lambda x.\ x)$ reduce to?)
- ➜ (Isabelle/HOL doesn't have Y; it supports more restricted forms of recursion)

Types and Terms in Isabelle

Types: $\tau \quad ::= \quad \mathtt{b} \mid {}'\nu \mid {}'\nu :: C \mid \tau \Rightarrow \tau \mid (\tau, \ldots, \tau)\ K$

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- **type constructors**: construct a new type out of a parameter type. Example: `int list`
- **type classes**: restrict type variables to a class defined by axioms. Example: $\alpha :: \mathit{order}$
- **schematic variables**: variables that can be instantiated.

Type Classes

➜ similar to Haskell's type classes, but with semantic properties

class order =

assumes order_refl: "$x \leq x$"

assumes order_trans: "$[\![x \leq y; y \leq z]\!] \Longrightarrow x \leq z$"

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  instance nat :: "{order, linorder}" **by** $\ldots$

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Solution:

Isabelle has **free** (x), **bound** (x), and **schematic** (?X) variables.

Only schematic variables can be instantiated.

Free converted into schematic after proof is finished.

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Find substitution σ on variables for terms s, t such that $\sigma(s) = \sigma(t)$

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?X \wedge ?Y & =_{\alpha\beta\eta} & x \wedge x \\
?P\ x & =_{\alpha\beta\eta} & x \wedge x \\
P\ (?f\ x) & =_{\alpha\beta\eta} & ?Y\ x
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Higher Order: schematic variables can be functions.

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- Unification modulo $\alpha\beta\eta$ is undecidable
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- Important fragments (like Higher Order Patterns) are decidable

Higher Order Pattern:

- is a term in β normal form where
- each occurrence of a schematic variable is of the form $?f\ t_1\ \ldots\ t_n$
- and the $t_1\ \ldots\ t_n$ are η-convertible into n distinct bound variables

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- Types and terms in Isabelle

