COMP4161: Advanced Topics in Software Verification

HOL

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[a] a1 due; [b] a2 due; [c] a3 due

Defining Higher Order Logic

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➜ **Higher Order Logic:**
  - quantification over everything, including predicates
  - consistency by types
  - formula $=$ term of type bool
  - definition built on $\lambda^{\rightarrow}$ with certain default types and constants

Defining Higher Order Logic

Default types:

Defining Higher Order Logic

Default types:

bool

Defining Higher Order Logic

Default types:

$$\text{bool} \qquad\qquad \_ \Rightarrow \_$$

Defining Higher Order Logic

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$$\longrightarrow \quad :: \quad bool \Rightarrow bool \Rightarrow bool$$

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$$\begin{array}{lll} \longrightarrow & :: & \mathit{bool} \Rightarrow \mathit{bool} \Rightarrow \mathit{bool} \\ = & :: & \alpha \Rightarrow \alpha \Rightarrow \mathit{bool} \end{array}$$

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= & :: & \alpha \Rightarrow \alpha \Rightarrow \mathit{bool} \\
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Higher Order Abstract Syntax

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One approach: $\forall :: \mathit{var} \Rightarrow \mathit{term} \Rightarrow \mathit{bool}$
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$$\lambda$$

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So: Use λ to encode all other binders.

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Example:

$$\mathsf{ALL} :: (\alpha \Rightarrow \mathit{bool}) \Rightarrow \mathit{bool}$$

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ALL $(\lambda x.\ x = 2)$	$\forall x.\ x = 2$

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Isabelle can translate usual binder syntax into HOAS.

Side Track: Syntax Declarations in Isabelle

➜ **mixfix:**
consts drvbl :: $ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ ("_, _ ⊢ _")
Legal syntax now: $\Gamma, \Pi \vdash F$

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  Example: or :: $bool \Rightarrow bool \Rightarrow bool$ (infixr " ∨ " 30)

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- **infixl/infixr**: short form for left/right associative binary operators
  Example: or :: $bool \Rightarrow bool \Rightarrow bool$ (infixr " $\vee$ " 30)
- **binders:** declaration must be of the form
  $c :: (\tau_1 \Rightarrow \tau_2) \Rightarrow \tau_3$ (binder "B" $< p >$)
  $B\ x.\ P\ x$ translated into $c\ P$ (and vice versa)
  Example ALL :: $(\alpha \Rightarrow bool) \Rightarrow bool$ (binder "$\forall$" 10)

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More in Isabelle/Isar Reference Manual (7.2)

Back to HOL

Base: bool, $\Rightarrow$, ind $\quad$ $=$, $\longrightarrow$, ε

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\text{Ex } P & \equiv \\
\text{False} & \equiv \\
\neg P & \equiv \\
P \wedge Q & \equiv \\
P \vee Q & \equiv \\
\text{If } P\ x\ y & \equiv \\
\text{inj } f & \equiv \\
\text{surj } f & \equiv
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\text{Ex } P & \equiv & \forall Q.\ (\forall x.\ P\ x \longrightarrow Q) \longrightarrow Q \\
\text{False} & \equiv & \forall P.\ P \\
\neg P & \equiv & P \longrightarrow \text{False} \\
P \wedge Q & \equiv & \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R \\
P \vee Q & \equiv & \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R \\
\text{If } P\ x\ y & \equiv & \text{SOME } z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y) \\
\text{inj } f & \equiv & \forall x\ y.\ f\ x = f\ y \longrightarrow x = y \\
\text{surj } f & \equiv & \forall y.\ \exists x.\ y = f\ x
\end{array}
$$

The Axioms of HOL

$$\frac{}{t = t}\ \text{refl} \qquad \frac{s = t \quad P\ s}{P\ t}\ \text{subst} \qquad \frac{\bigwedge x.\ f\ x = g\ x}{(\lambda x.\ f\ x) = (\lambda x.\ g\ x)}\ \text{ext}$$

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$$\frac{}{\exists f :: \mathit{ind} \Rightarrow \mathit{ind}.\ \text{inj}\ f \wedge \neg \text{surj}\ f}\ \text{infty}$$

That’s it.

➜ 3 basic constants
➜ 3 basic types
➜ 9 axioms

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With this you can define and derive all the rest.

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Isabelle knows 2 more axioms:

$$\frac{x = y}{x \equiv y}\ \text{eq\_reflection} \qquad \frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Demo:
The Definitions in Isabelle

Deriving Proof Rules

In the following, we will

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Convenient for deriving rules: **named assumptions in lemmas**

lemma [*name* :]
assumes [$name_1$:] "$< prop >_1$"
assumes [$name_2$:] "$< prop >_2$"
$\vdots$
shows "$< prop >$" $< proof >$

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\textbf{lemma}\ [\mathit{name}\ :] \\
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\textbf{assumes}\ [\mathit{name}_2\ :]\ \text{``}<\mathit{prop}>_2\text{''} \\
\vdots \\
\textbf{shows}\ \text{``}<\mathit{prop}>\text{''} \quad <\mathit{proof}>
\end{array}
$$

$$\textbf{proves:}\ \llbracket <\mathit{prop}>_1;\ <\mathit{prop}>_2;\ \ldots \rrbracket \Longrightarrow <\mathit{prop}>$$

True

consts True :: *bool*
True $\equiv (\lambda x :: bool.\ x) = (\lambda x.\ x)$

Intuition:
right hand side is always true

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Proof Rules:

$$\frac{}{\text{True}}\ \text{TrueI}$$

Proof:

$$\dfrac{\overline{(\lambda x :: \mathit{bool}.\ x) = (\lambda x.\ x)}\ \text{refl}}{\text{True}}\ \text{unfold True\_def}$$

Demo

Universal Quantifier

consts ALL :: $(\alpha \Rightarrow \mathit{bool}) \Rightarrow \mathit{bool}$

$$\text{ALL } P \quad \equiv \quad P = (\lambda x.\ \text{True})$$

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Intuition:

- ALL P is Higher Order Abstract Syntax for $\forall x.\ P\ x$.
- P is a function that takes an x and yields a truth value.

Universal Quantifier

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Intuition:

- ➜ ALL P is Higher Order Abstract Syntax for $\forall x.\ P\ x$.
- ➜ P is a function that takes an x and yields a truth value.
- ➜ ALL P should be true iff P yields true for all x, i.e. if it is equivalent to the function $\lambda x.$ True.

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Proof Rules:

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

Proof: Isabelle Demo

False

consts False :: *bool*

$$\text{False} \equiv \forall P. P$$

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Everything can be derived from *False*.

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Proof Rules:

$$\frac{\text{False}}{P}\ \text{FalseE} \qquad \frac{}{\text{True} \neq \text{False}}$$

Proof: Isabelle Demo

Negation

consts Not :: $\mathit{bool} \Rightarrow \mathit{bool}$ ($\neg$ _)

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Proof Rules:

$$\frac{A \Longrightarrow \mathit{False}}{\neg A}\ \text{notI} \qquad \frac{\neg A \quad A}{P}\ \text{notE}$$

Proof: Isabelle Demo

Existential Quantifier

consts $\mathrm{EX} :: (\alpha \Rightarrow \mathit{bool}) \Rightarrow \mathit{bool}$

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- Note that inner $\forall$ binds wide: $(\forall x.\ P\ x \longrightarrow Q)$

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- ➜ Remember lemma from last time: $(\forall x.\ P\ x \longrightarrow Q) = ((\exists x.\ P\ x) \longrightarrow Q)$

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Proof Rules:

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

Proof: Isabelle Demo

Conjunction

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Proof Rules:

$$\frac{A \quad B}{A \wedge B}\ \text{conjI} \qquad \frac{A \wedge B \quad [\![A; B]\!] \Longrightarrow C}{C}\ \text{conjE}$$

Proof: Isabelle Demo

Disjunction

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$P \vee Q \equiv \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$

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- ➜ Try truth table for P, Q, and R

Proof Rules:

$$\frac{A}{A \vee B} \quad \frac{B}{A \vee B}\ \text{disjI1/2} \qquad \frac{A \vee B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C}\ \text{disjE}$$

Proof: Isabelle Demo

If-Then-Else

consts If :: *bool* $\Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha$ (if_ then _ else _)
If $P\ x\ y\ \equiv$ SOME $z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

If-Then-Else

consts If :: $bool \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha$ (if_ then _ else _)
If $P\ x\ y\ \equiv$ SOME $z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

Intuition:

- for $P =$ True, right hand side collapses to SOME $z.\ z = x$

If-Then-Else

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$\text{If } P\ x\ y \ \equiv\ \text{SOME } z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

Intuition:

- for $P = \text{True}$, right hand side collapses to $\text{SOME } z.\ z = x$
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Proof Rules:

$$\frac{}{\text{if True then } s \text{ else } t = s}\ \text{ifTrue} \qquad \frac{}{\text{if False then } s \text{ else } t = t}\ \text{ifFalse}$$

Proof: Isabelle Demo

That was HOL

More on Automation

Last time: safe and unsafe, heuristics: use safe before unsafe

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Application (roughly):

do safe rules first, search/backtrack on unsafe rules only

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Example:

declare attribute globally	**declare** conjI [intro!] allE [elim]
remove attribute gloablly	**declare** allE [rule del]
use locally	**apply** (blast intro: someI)
delete locally	**apply** (blast del: conjI)

Demo: Automation

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➜ More automation

