COMP4161: Advanced Topics in Software Verification

λ

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Exercises from last time

- ➜ Download and install Isabelle from `http://mirror.cse.unsw.edu.au/pub/isabelle/`
- ➜ Step through the demo files from the lecture web page
- ➜ Write your own theory file, look at some theorems in the library, try 'find_theorems'
- ➜ How many theorems can help you if you need to prove something containing the term "Suc(Suc x)"?
- ➜ What is the name of the theorem for associativity of addition of natural numbers in the library?

Content

➜ Intro & motivation, getting started [1]

➜ Foundations & Principles

- Lambda Calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [3[a]]
- Term rewriting [4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [5]
- Datatypes, recursion, induction, Isar (part 2) [6, 7[b]]
- Hoare logic, proofs about programs, invariants [8]
- C verification [9]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

λ-calculus

Alonzo Church

- ➜ lived 1903–1995
- ➜ supervised people like Alan Turing, Stephen Kleene
- ➜ famous for Church-Turing thesis, lambda calculus, first undecidability results
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λ-calculus

- originally meant as foundation of mathematics
- important applications in theoretical computer science
- foundation of computability and functional programming

untyped λ-calculus

- ➜ turing complete model of computation
- ➜ a simple way of writing down functions

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Basic intuition:

instead of	$f(x) = x + 5$
write	$f = \lambda x.\ x + 5$

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instead of	$f(x) = x + 5$
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- a nameless function

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$\lambda x.\ x + 5$

- a term
- a nameless function
- that adds 5 to its parameter

Function Application

For applying arguments to functions

instead of	$f(a)$
write	$f\ a$

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Evaluating: in $(\lambda x.\ t)\ a$ replace x by a in t
(computation!)

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Example: $(\lambda x.\ x + 5)\ a$

Evaluating: (computation!) in $(\lambda x.\ t)\ a$ replace x by a in t

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

That’s it!

Now Formal

Syntax

Terms: $t \quad ::= \quad v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

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$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- v, x variables
- c constants
- $(t\ t)$ application
- $(\lambda x.\ t)$ abstraction

Conventions

- leave out parentheses where possible
- list variables instead of multiple λ

Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

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Rules:

- list variables: $\lambda x.\ (\lambda y.\ t) = \lambda x\ y.\ t$
- application binds to the left: $x\ y\ z = (x\ y)\ z \neq x\ (y\ z)$
- abstraction binds to the right: $\lambda x.\ x\ y = \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$
- leave out outermost parentheses

Getting used to the Syntax

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

Getting used to the Syntax

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$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

Computation

Intuition: replace parameter by argument
this is called β-reduction

Example

$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$

Computation

Intuition: replace parameter by argument
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Example

$$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$$
$$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$$

Computation

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Example

$$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$$
$$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$$
$$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$$

Computation

Intuition: replace parameter by argument
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Example

$$
\begin{array}{l}
(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta \\
(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta \\
f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta \\
f\ 5
\end{array}
$$

Defining Computation

β reduction:

$$
\begin{array}{rcrcrcl}
 & & & & (\lambda x.\ s)\ t & \longrightarrow_\beta & s[x \leftarrow t] \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\
t & \longrightarrow_\beta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\
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Still to do: define $s[x \leftarrow t]$

Defining Substitution

Easy concept. Small problem: variable capture.

Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

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We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

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Easy concept. Small problem: variable capture.
Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

Free Variables

Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

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Free variables FV of a term:

$$\begin{aligned}
FV\ (x) &= \{x\} \\
FV\ (c) &= \{\} \\
FV\ (s\ t) &= FV(s) \cup FV(t) \\
FV\ (\lambda x.\ t) &= FV(t) \setminus \{x\}
\end{aligned}$$

Example: $FV(\quad \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x \quad)$

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Example: $FV(\ \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x\ ) = \{y\}$

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Term t is called **closed** if $FV(t) = \{\}$

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The substitution example, $(\lambda x.\ x\ z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term "x".

Substitution

$$
\begin{aligned}
x\ [x \leftarrow t] &= t \\
y\ [x \leftarrow t] &= y && \text{if } x \neq y \\
c\ [x \leftarrow t] &= c \\
(s_1\ s_2)\ [x \leftarrow t] &=
\end{aligned}
$$

Substitution

$$
\begin{array}{lll}
x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
c\ [x \leftarrow t] & = c & \\
(s_1\ s_2)\ [x \leftarrow t] & = (s_1[x \leftarrow t]\ s_2[x \leftarrow t]) & \\
(\lambda x.\ s)\ [x \leftarrow t] & = &
\end{array}
$$

Substitution

$$
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x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
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 & & \\
(s_1\ s_2)\ [x \leftarrow t] & = (s_1[x \leftarrow t]\ s_2[x \leftarrow t]) & \\
 & & \\
(\lambda x.\ s)\ [x \leftarrow t] & = (\lambda x.\ s) & \\
(\lambda y.\ s)\ [x \leftarrow t] & = &
\end{array}
$$

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 & & \\
(s_1\ s_2)\ [x \leftarrow t] & = (s_1[x \leftarrow t]\ s_2[x \leftarrow t]) & \\
 & & \\
(\lambda x.\ s)\ [x \leftarrow t] & = (\lambda x.\ s) & \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda y.\ s[x \leftarrow t]) & \text{if } x \neq y \text{ and } y \notin FV(t) \\
(\lambda y.\ s)\ [x \leftarrow t] & = &
\end{array}
$$

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(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda z.\ s[y \leftarrow z][x \leftarrow t]) & \text{if } x \neq y \\
 & & \text{and } z \notin FV(t) \cup FV(s)
\end{array}
$$

Substitution Example

$$(x \;\; (\lambda x.\; x) \;\; (\lambda y.\; z\; x))[x \leftarrow y]$$

Substitution Example

$$
\begin{array}{ll}
 & (x\ \ (\lambda x.\ x)\ \ (\lambda y.\ z\ x))[x \leftarrow y] \\
= & (x[x \leftarrow y])\ \ ((\lambda x.\ x)[x \leftarrow y])\ \ ((\lambda y.\ z\ x)[x \leftarrow y])
\end{array}
$$

Substitution Example

$$
\begin{array}{rl}
 & (x\ \ (\lambda x.\ x)\ \ (\lambda y.\ z\ x))[x \leftarrow y] \\
= & (x[x \leftarrow y])\ \ ((\lambda x.\ x)[x \leftarrow y])\ \ ((\lambda y.\ z\ x)[x \leftarrow y]) \\
= & y\ (\lambda x.\ x)\ (\lambda y'.\ z\ y)
\end{array}
$$

α Conversion

Bound names are irrelevant:

$\lambda x.\ x$ and $\lambda y.\ y$ denote the same function.

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$$

$$s =_\alpha t \quad \text{iff} \quad s \longrightarrow_\alpha^* t$$

($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

α Conversion

Equality in Isabelle is equality modulo α conversion:

if $s =_\alpha t$ then s and t are syntactically equal.

Examples:

$x\ (\lambda x\ y.\ x\ y)$

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=_\alpha & x\ (\lambda y\ x.\ y\ x) \\
=_\alpha & x\ (\lambda z\ y.\ z\ y) \\
\neq_\alpha & z\ (\lambda z\ y.\ z\ y)
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$$

Back to β

We have defined β reduction: $\longrightarrow_\beta$
Some notation and concepts:

- **β conversion:** $s =_\beta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$

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- t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- t is reducible iff it contains a redex
- if it is not reducible, t is in **normal form**

Does every λ term have a normal form?

Example:

$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta$$

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(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta
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Does every λ term have a normal form?

No!

Example:

$$\begin{array}{l} (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta \ \ldots \end{array}$$

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(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta \ \ldots
\end{array}
$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \ \longrightarrow_\beta \ \lambda y.\ y$)

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\end{array}
$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \ \longrightarrow_\beta \ \lambda y.\ y$)

λ calculus is not terminating

β reduction is confluent

Confluence: $s \longrightarrow_\beta^* x \wedge s \longrightarrow_\beta^* y \Longrightarrow \exists t.\ x \longrightarrow_\beta^* t \wedge y \longrightarrow_\beta^* t$

$$\begin{tikzcd} & s \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x & & y \end{tikzcd}$$

β reduction is confluent

Confluence: $s \longrightarrow^*_\beta x \wedge s \longrightarrow^*_\beta y \Longrightarrow \exists t.\ x \longrightarrow^*_\beta t \wedge y \longrightarrow^*_\beta t$

$$\begin{tikzcd}
 & s \arrow[dl, "*"'] \arrow[dr, "*"] & \\
x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\
 & t &
\end{tikzcd}$$

Order of reduction does not matter for result
Normal forms in λ calculus are unique

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a)$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a)$

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a)$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta \lambda y.\ y$

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Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$

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η Conversion

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Definition:

$$
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 & & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t & \text{if } x \notin FV(t) \\
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$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta$

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\begin{array}{lllllll}
 & & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t & \text{if } x \notin FV(t) \\
s & \longrightarrow_\eta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s'\ t) \\
t & \longrightarrow_\eta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s\ t') \\
s & \longrightarrow_\eta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\eta & (\lambda x.\ s')
\end{array}
$$

$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta (\lambda x.\ f\ x)\ g \longrightarrow_\eta f\ g$

- ➜ η reduction is confluent and terminating.
- ➜ $\longrightarrow_{\beta\eta}$ is confluent.
  $\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.
- ➜ **Equality in Isabelle is also modulo η conversion.**

In fact ...

Equality in Isabelle is modulo α, β, and η conversion.

We will see later why that is possible.

So, what can you do with λ calculus?

λ calculus is very expressive, you can encode:

- ➜ logic, set theory
- ➜ turing machines, functional programs, etc.

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\texttt{true} &\equiv \lambda x\ y.\ x \\
\texttt{false} &\equiv \lambda x\ y.\ y \\
\texttt{if} &\equiv \lambda z\ x\ y.\ z\ x\ y
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$$\begin{array}{lll} \texttt{true} & \equiv \lambda x\ y.\ x & \qquad \texttt{if true}\ x\ y \longrightarrow_\beta^* x \\ \texttt{false} & \equiv \lambda x\ y.\ y & \qquad \texttt{if false}\ x\ y \longrightarrow_\beta^* y \\ \texttt{if} & \equiv \lambda z\ x\ y.\ z\ x\ y & \end{array}$$

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Now, `not`, `and`, `or`, etc is easy:

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Now, `not`, `and`, `or`, etc is easy:

$$\begin{array}{lll} \texttt{not} & \equiv & \lambda x.\ \texttt{if}\ x\ \texttt{false true} \\ \texttt{and} & \equiv & \lambda x\ y.\ \texttt{if}\ x\ y\ \texttt{false} \\ \texttt{or} & \equiv & \lambda x\ y.\ \texttt{if}\ x\ \texttt{true}\ y \end{array}$$

More Examples

Encoding natural numbers (Church Numerals)

$$
\begin{aligned}
0 &\equiv \lambda f\ x.\ x \\
1 &\equiv \lambda f\ x.\ f\ x \\
2 &\equiv \lambda f\ x.\ f\ (f\ x) \\
3 &\equiv \lambda f\ x.\ f\ (f\ (f\ x)) \\
\ldots
\end{aligned}
$$

Numeral n takes arguments f and x, applies f n-times to x.

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\texttt{add} &\equiv \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)
\end{aligned}
$$

Fix Points

$$(\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ \ t \longrightarrow_\beta$$

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$$
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$$\mu = (\lambda xf.\ f\ (x\ x\ f))\ (\lambda xf.\ f\ (x\ x\ f))$$
$$\mu\ t \longrightarrow_\beta t\ (\mu\ t) \longrightarrow_\beta t\ (t\ (\mu\ t)) \longrightarrow_\beta t\ (t\ (t\ (\mu\ t))) \longrightarrow_\beta \ldots$$

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$$\mu\ t \longrightarrow_\beta t\ (\mu\ t) \longrightarrow_\beta t\ (t\ (\mu\ t)) \longrightarrow_\beta t\ (t\ (t\ (\mu\ t))) \longrightarrow_\beta \dots$$

$(\lambda xf.\ f\ (x\ x\ f))\ (\lambda xf.\ f\ (x\ x\ f))$ is Turing's fix point operator

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  allows arbitrary quantification over predicates
- ➜ **Russell** (1901): Paradox $R \equiv \{X | X \notin X\}$
- ➜ **Whitehead & Russell** (Principia Mathematica, 1910-1913):
  Fix the problem
- ➜ **Church** (1930): λ calculus as logic, `true`, `false`, $\wedge$, ... as λ terms

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Problem:
with $\quad \{x|\ P\ x\} \equiv \lambda x.\ P\ x \qquad x \in M \equiv M\ x$

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Problem:

with $\{x|\ P\ x\} \equiv \lambda x.\ P\ x \qquad x \in M \equiv M\ x$

you can write $R \equiv \lambda x.\ \texttt{not}\ (x\ x)$

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Problem:

with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x$	$x \in M \equiv M\ x$
you can write	$R \equiv \lambda x.\ \texttt{not}\ (x\ x)$	
and get	$(R\ R) =_\beta \texttt{not}\ (R\ R)$	

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Problem:

with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x \qquad x \in M \equiv M\ x$
you can write	$R \equiv \lambda x.\ \texttt{not}\ (x\ x)$
and get	$(R\ R) =_\beta \texttt{not}\ (R\ R)$
because	$(R\ R) = (\lambda x.\ \texttt{not}\ (x\ x))\ R \longrightarrow_\beta \texttt{not}\ (R\ R)$

Isabelle Demo

We have learned so far...

- ➜ λ calculus syntax
- ➜ free variables, substitution
- ➜ β reduction
- ➜ α and η conversion
- ➜ β reduction is confluent
- ➜ λ calculus is very expressive (turing complete)
- ➜ λ calculus is inconsistent