COMP4161: Advanced Topics in Software Verification

$\lambda^{\rightarrow}$ and HOL

June Andronick, Christine Rizkallah, Miki Tanaka, Johannes Åman Pohjola

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data61.csiro.au

Last time...

- Simply typed lambda calculus: $\lambda^{\rightarrow}$
- Typing rules for $\lambda^{\rightarrow}$, type variables, type contexts
- β-reduction in $\lambda^{\rightarrow}$ satisfies subject reduction
- β-reduction in $\lambda^{\rightarrow}$ always terminates
- Types and terms in Isabelle

Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction, Isar (part 2) [6, 7[b]]
  - Hoare logic, proofs about programs, invariants [8]
  - C verification [9]
  - Practice, questions, exam prep [10[c]]

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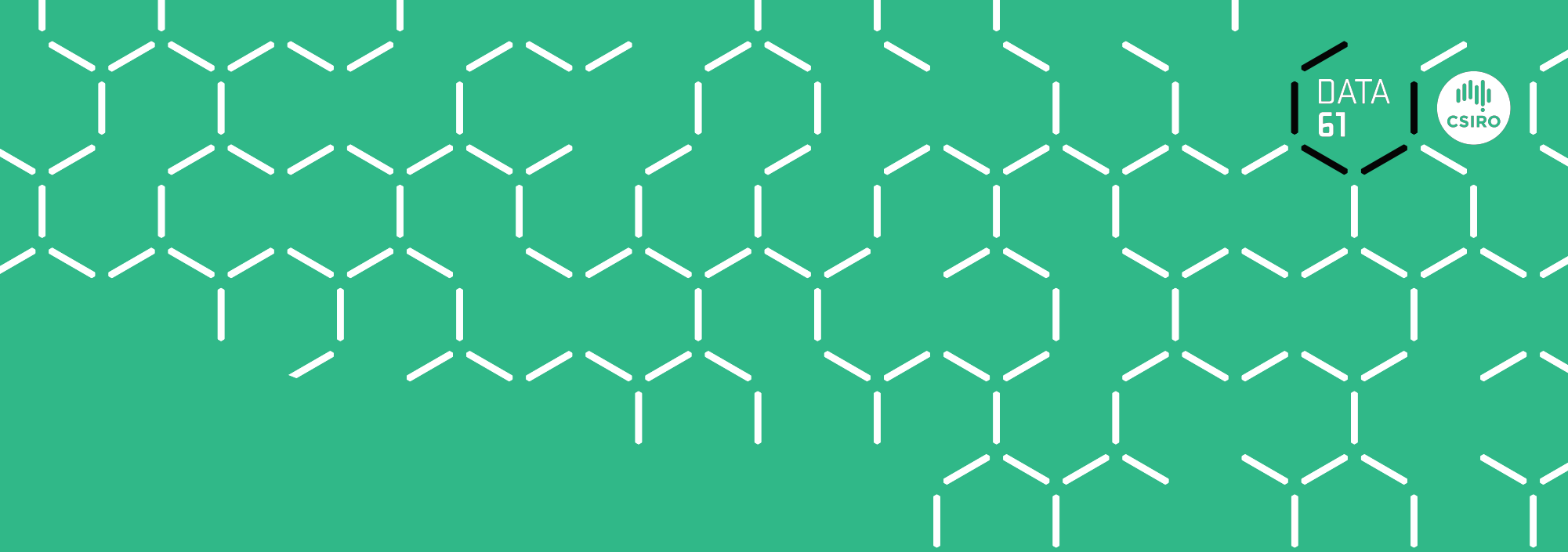
[a] a1 due; [b] a2 due; [c] a3 due



Preview: Proofs in Isabelle

Proofs in Isabelle

General schema:

lemma name: "<goal>"
apply <method>
apply <method>
...
done

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➜ Sequential application of methods until all **subgoals** are solved.

The Proof State

1. $\bigwedge x_1 \ldots x_p . [\![A_1; \ldots; A_n]\!] \Longrightarrow B$
2. $\bigwedge y_1 \ldots y_q . [\![C_1; \ldots; C_m]\!] \Longrightarrow D$

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$x_1 \ldots x_p$	Parameters
$A_1 \ldots A_n$	Local assumptions
B	Actual (sub)goal

Isabelle Theories

Syntax:

`theory` *MyTh*
`imports` $ImpTh_1 \ldots ImpTh_n$
`begin`
(declarations, definitions, theorems, proofs, ...)*
`end`

➜ *MyTh*: name of theory. Must live in file *MyTh*`.thy`
➜ $ImpTh_i$: name of *imported* theories. Import transitive.

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Unless you need something special:

`theory` *MyTh* `imports Main begin ... end`

Natural Deduction Rules

$$\frac{}{A \wedge B}\ \text{conjI} \qquad \frac{A \wedge B}{C}\ \text{conjE}$$

$$\frac{}{A \vee B}\ \frac{}{A \vee B}\ \text{disjI1/2} \qquad \frac{A \vee B}{C}\ \text{disjE}$$

$$\frac{}{A \longrightarrow B}\ \text{impI} \qquad \frac{A \longrightarrow B}{C}\ \text{impE}$$

For each connective ($\wedge$, $\vee$, etc):
introduction and **elimination** rules

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For each connective ($\wedge$, $\vee$, etc):
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Proof by assumption

apply assumption

proves

1. $[\![B_1; \ldots; B_m]\!] \Longrightarrow C$

by unifying C with one of the B_i

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apply assumption

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by unifying C with one of the B_i

There may be more than one matching B_i and multiple unifiers.

Backtracking!

Explicit backtracking command: **back**

Intro rules

Intro rules decompose formulae to the right of $\Longrightarrow$.

apply (rule <intro-rule>)

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Applying rule $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$ to subgoal C:

➜ unify A and C

➜ replace C with n new subgoals $A_1 \ldots A_n$

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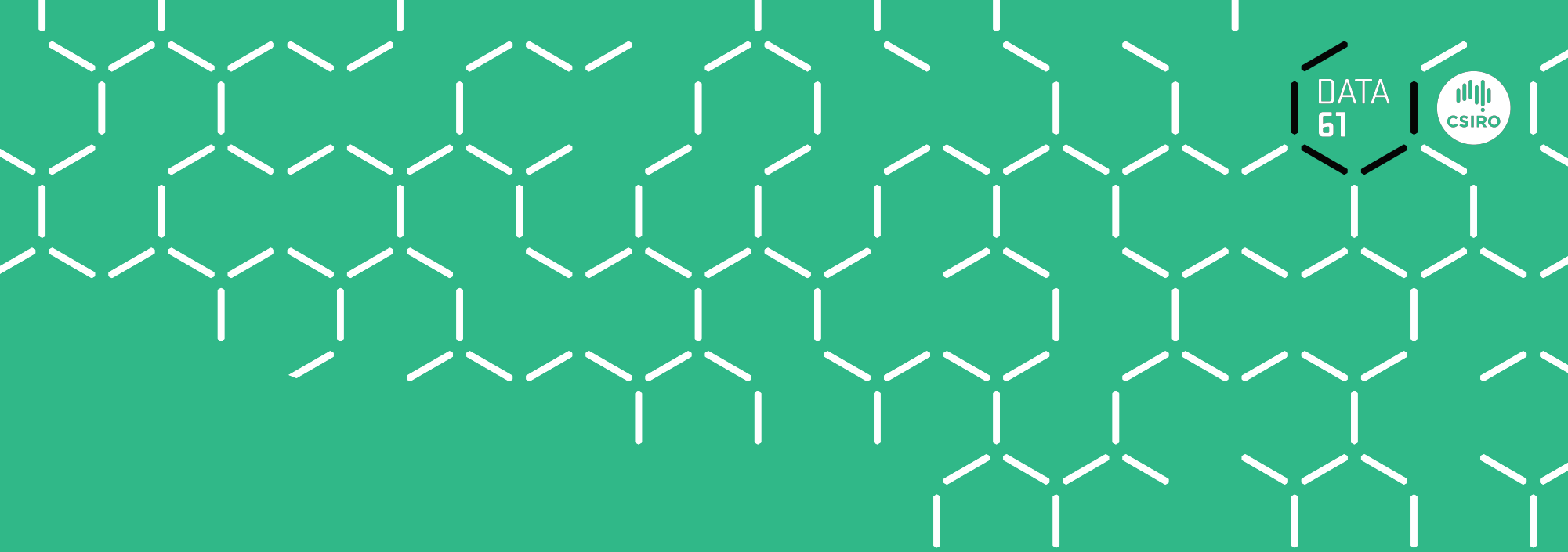
Applying rule $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$ to subgoal C:
Like **rule** but also

➜ unifies first premise of rule with an assumption

➜ eliminates that assumption

Demo



More Proof Rules

Iff, Negation, True and False

$$\frac{}{A = B}\ \text{iffI} \qquad \frac{A = B}{C}\ \text{iffE}$$

$$\frac{A = B}{}\ \text{iffD1} \qquad \frac{A = B}{}\ \text{iffD2}$$

$$\frac{}{\neg A}\ \text{notI} \qquad \frac{\neg A}{P}\ \text{notE}$$

Iff, Negation, True and False

$$\frac{A \Longrightarrow B \quad B \Longrightarrow A}{A = B}\ \text{iffI} \qquad \frac{A = B}{C}\ \text{iffE}$$

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$$\frac{}{\mathit{True}}\ \text{TrueI} \qquad \frac{\mathit{False}}{P}\ \text{FalseE}$$

Equality

$$\frac{}{t = t}\ \text{refl} \qquad \frac{s = t}{t = s}\ \text{sym} \qquad \frac{r = s \quad s = t}{r = t}\ \text{trans}$$

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Rarely needed explicitly — used implicitly by term rewriting

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- ➜ **excluded-middle**, **ccontr** and **classical** not derivable from the other rules.
- ➜ if we include True-or-False, they are derivable

They make the logic "classical", "non-constructive"

Cases

$$\frac{}{P \vee \neg P}\ \text{excluded-middle}$$

is a case distinction on type *bool*

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Isabelle can do case distinctions on arbitrary terms:

apply (case_tac *term*)

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Safe rules preserve provability

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conjI, implI, notI, iffI, refl, ccontr, classical, conjE, disjE

$$\frac{A \quad B}{A \wedge B}\ \text{conjI}$$

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Safe rules preserve provability

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Unsafe rules can turn a provable goal into an unprovable one

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Unsafe rules can turn a provable goal into an unprovable one

disjI1, disjI2, impE, iffD1, iffD2, notE

$$\frac{A}{A \vee B}\ \text{disjI1}$$

Safe and not so safe

Safe rules preserve provability

conjI, impI, notI, iffI, refl, ccontr, classical, conjE, disjE

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Unsafe rules can turn a provable goal into an unprovable one

disjI1, disjI2, impE, iffD1, iffD2, notE

$$\frac{A}{A \vee B}\ \text{disjI1}$$

Apply safe rules before unsafe ones

Demo

What we have learned so far...

- ➜ natural deduction rules for $\wedge$, $\vee$, $\longrightarrow$, $\neg$, iff...
- ➜ proof by assumption, by intro rule, elim rule
- ➜ safe and unsafe rules

- ➜ indent your proofs! (one space per subgoal)
- ➜ prefer implicit backtracking (chaining) or *rule_tac*, instead of *back*
- ➜ *prefer* and *defer*
- ➜ *oops* and *sorry*

Assignment

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- ➜ Assignments and exams are take-home. This does NOT mean you can work in groups. Each submission is personal.
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