

T3/2019

CSIRO

Exercises from last time



- → Download and install Isabelle from http://mirror.cse.unsw.edu.au/pub/isabelle/
- → Step through the demo files from the lecture web page
- → Write your own theory file, look at some theorems in the library, try 'find_theorems'
- → How many theorems can help you if you need to prove something containing the term "Suc(Suc x)"?
- → What is the name of the theorem for associativity of addition of natural numbers in the library?

Content

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→ Intro & motivation, getting started	[1]
→ Foundations & Principles	
 Lambda Calculus, natural deduction 	[1,2]
 Higher Order Logic, Isar (part 1) 	[3ª]
Term rewriting	[4]
→ Proof & Specification Techniques	
 Inductively defined sets, rule induction 	[5]
 Datatypes, recursion, induction, Isar (part 2) 	$[6, 7^b]$
 Hoare logic, proofs about programs, invariants 	[8]
 C verification 	[9]
 Practice, questions, exam prep 	[10 ^c]

^aa1 due; ^ba2 due; ^ca3 due

λ -calculus



Alonzo Church

- → lived 1903-1995
- → supervised people like Alan Turing, Stephen Kleene
- → famous for Church-Turing thesis, lambda calculus, first undecidability results
- \rightarrow invented λ calculus in 1930's



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λ -calculus

- → originally meant as foundation of mathematics
- → important applications in theoretical computer science
- → foundation of computability and functional programming



- → turing complete model of computation
- → a simple way of writing down functions



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- → a simple way of writing down functions

Basic intuition:

instead of
$$f(x) = x + 5$$

write $f = \lambda x. x + 5$



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- → a term
- → a nameless function



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Basic intuition:

instead of
$$f(x) = x + 5$$

write $f = \lambda x. x + 5$

$$\lambda x$$
. $x + 5$

- → a term
- → a nameless function
- → that adds 5 to its parameter



For applying arguments to functions

instead of f(a) write f(a)



For applying arguments to functions

instead of
$$f(a)$$
 write $f(a)$

Example: $(\lambda x. x + 5) a$



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 write $f(a)$

Example:
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Evaluating: in
$$(\lambda x. t)$$
 a replace x by a in t (computation!)



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 write $f(a)$

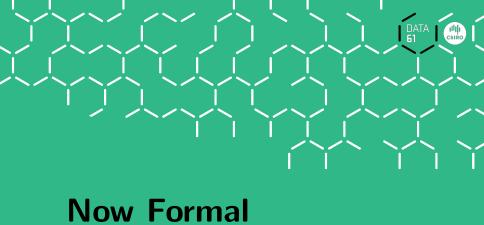
Example:
$$(\lambda x. x + 5) a$$

Evaluating: in
$$(\lambda x. t)$$
 a replace x by a in t

(computation!)

Example:
$$(\lambda x. x + 5) (a + b)$$
 evaluates to $(a + b) + 5$





Syntax



Terms:
$$t ::= v \mid c \mid (t \ t) \mid (\lambda x. \ t)$$

 $v, x \in V, c \in C, V, C \text{ sets of names}$

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Terms:
$$t ::= v \mid c \mid (t \ t) \mid (\lambda x. \ t)$$
 $v, x \in V, \quad c \in C, \quad V, C \text{ sets of names}$

- $\rightarrow V, X$ variables
- → C constants
- \rightarrow $(t \ t)$ application
- \rightarrow $(\lambda x. t)$ abstraction

Conventions



- → leave out parentheses where possible
- ightharpoonup list variables instead of multiple λ

Example: instead of $(\lambda y. (\lambda x. (x y)))$ write $\lambda y. x. x. y$

Conventions



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Rules:

- \rightarrow list variables: $\lambda x. (\lambda y. t) = \lambda x y. t$
- \rightarrow application binds to the left: $x \ y \ z = (x \ y) \ z \neq x \ (y \ z)$
- \rightarrow abstraction binds to the right: $\lambda x. \ x \ y = \lambda x. \ (x \ y) \neq (\lambda x. \ x) \ y$
- → leave out outermost parentheses



$$\lambda x \ y \ z \cdot x \ z \ (y \ z) =$$



$$\lambda x y z. x z (y z) =$$

$$\lambda x \ y \ z. \ (x \ z) \ (y \ z) =$$



$$\lambda x y z. x z (y z) =$$

$$\lambda x y z. (x z) (y z) =$$

$$\lambda x \ y \ z. \ ((x \ z) \ (y \ z)) =$$



$$\lambda x \ y \ z. \ x \ z \ (y \ z) =$$

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$$\lambda x. \ \lambda y. \ \lambda z. \ ((x \ z) \ (y \ z)) =$$



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 $(\lambda x. \ (\lambda y. \ (\lambda z. \ ((x \ z) \ (y \ z)))))$



Intuition: replace parameter by argument

this is called β -reduction

$$(\lambda x \ y. \ f \ (y \ x)) \ 5 \ (\lambda x. \ x) \longrightarrow_{\beta}$$

DATA SIÑO

Intuition: replace parameter by argument

this is called β -reduction

$$(\lambda x \ y. \ f \ (y \ x)) \ 5 \ (\lambda x. \ x) \longrightarrow_{\beta} (\lambda y. \ f \ (y \ 5)) \ (\lambda x. \ x) \longrightarrow_{\beta}$$

DATA SINO

Intuition: replace parameter by argument

this is called β -reduction

$$\begin{array}{cccc} (\lambda x \ y. \ f \ (y \ x)) & 5 & (\lambda x. \ x) \longrightarrow_{\beta} \\ (\lambda y. \ f \ (y \ 5)) & (\lambda x. \ x) \longrightarrow_{\beta} \\ f \ ((\lambda x. \ x) \ 5) \longrightarrow_{\beta} \end{array}$$

DATA CSIRO

Intuition: replace parameter by argument

this is called β -reduction

$$(\lambda x \ y. \ f \ (y \ x)) \ 5 \ (\lambda x. \ x) \longrightarrow_{\beta} (\lambda y. \ f \ (y \ 5)) \ (\lambda x. \ x) \longrightarrow_{\beta} f \ ((\lambda x. \ x) \ 5) \longrightarrow_{\beta} f \ 5$$

Defining Computation



eta reduction:

Defining Computation



eta reduction:

Still to do: define $s[x \leftarrow t]$

Defining Substitution



Easy concept. Small problem: variable capture.

Example: $(\lambda x. \ x \ z)[z \leftarrow x]$

Defining Substitution



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We do **not** want: $(\lambda x. x x)$ as result.

What do we want?

Defining Substitution



Easy concept. Small problem: variable capture.

Example: $(\lambda x. \ x \ z)[z \leftarrow x]$

We do **not** want: $(\lambda x. x x)$ as result.

What do we want?

In $(\lambda y. \ y \ z)$ $[z \leftarrow x] = (\lambda y. \ y \ x)$ there would be no problem.

So, solution is: rename bound variables.

Free Variables



Bound variables: in $(\lambda x. t)$, x is a bound variable.

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Free variables *FV* of a term:

$$FV(x) = \{x\}$$

 $FV(c) = \{\}$
 $FV(s t) = FV(s) \cup FV(t)$
 $FV(\lambda x. t) = FV(t) \setminus \{x\}$

Example: $FV(\lambda x. (\lambda y. (\lambda x. x) y) y x)$

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Term t is called **closed** if $FV(t) = \{\}$

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The substitution example, $(\lambda x. \times z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term "x".



$$x [x \leftarrow t] = t$$

$$y [x \leftarrow t] = y$$

$$c [x \leftarrow t] = c$$

$$(s_1 s_2) [x \leftarrow t] =$$



$$x \begin{bmatrix} x \leftarrow t \end{bmatrix} = t$$

$$y \begin{bmatrix} x \leftarrow t \end{bmatrix} = y$$

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$$(s_1 \ s_2) \begin{bmatrix} x \leftarrow t \end{bmatrix} = (s_1 [x \leftarrow t] \ s_2 [x \leftarrow t])$$

$$(\lambda x. \ s) \begin{bmatrix} x \leftarrow t \end{bmatrix} =$$



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$$(s_1 s_2) [x \leftarrow t] = (s_1[x \leftarrow t] s_2[x \leftarrow t])$$

$$(\lambda x. s) [x \leftarrow t] = (\lambda x. s)$$

$$(\lambda y. s) [x \leftarrow t] =$$



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if $x \neq y$ and $y \notin FV(t)$

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if $x \neq y$
and $z \notin FV(t) \cup FV(s)$

Substitution Example



$$(x (\lambda x. x) (\lambda y. z x))[x \leftarrow y]$$

Substitution Example



$$(x (\lambda x. x) (\lambda y. z x))[x \leftarrow y]$$

$$= (x[x \leftarrow y]) ((\lambda x. x)[x \leftarrow y]) ((\lambda y. z x)[x \leftarrow y])$$

Substitution Example



$$(x (\lambda x. x) (\lambda y. z x))[x \leftarrow y]$$

$$= (x[x \leftarrow y]) ((\lambda x. x)[x \leftarrow y]) ((\lambda y. z x)[x \leftarrow y])$$

$$= y (\lambda x. x) (\lambda y'. z y)$$



Bound names are irrelevant:

 $\lambda x. \ x$ and $\lambda y. \ y$ denote the same function.

α conversion:

 $s =_{\alpha} t$ means s = t up to renaming of bound variables.



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Formally:

$$(\lambda x. \ t) \longrightarrow_{\alpha} (\lambda y. \ t[x \leftarrow y]) \ \text{if} \ y \notin FV(t)$$

$$s \longrightarrow_{\alpha} s' \implies (s \ t) \longrightarrow_{\alpha} (s' \ t)$$

$$t \longrightarrow_{\alpha} t' \implies (s \ t) \longrightarrow_{\alpha} (s \ t')$$

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$$s \longrightarrow_{\alpha} s' \implies (\lambda x. \ s) \longrightarrow_{\alpha} (\lambda x. \ s')$$

$$s =_{\alpha} t \quad \text{iff} \quad s \longrightarrow_{\alpha}^{*} t$$

$$(\longrightarrow_{\alpha}^{*} = \text{transitive, reflexive closure of} \longrightarrow_{\alpha} = \text{multiple steps})$$



Equality in Isabelle is equality modulo α conversion:

if $s =_{\alpha} t$ then s and t are syntactically equal.

$$x (\lambda x y. x y)$$

α Conversion



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$$=_{\alpha} x (\lambda z y. z y)$$

$$\neq_{\alpha} z (\lambda z y. z y)$$

$$\neq_{\alpha}$$
 $z(\lambda z y. z y)$



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$$x (\lambda x y. x y)$$

$$=_{\alpha} x (\lambda y x. y x)$$

$$=_{\alpha} x (\lambda z y. z y)$$

$$=_{\alpha}$$
 $\lambda (\lambda z y. z y)$
 \neq_{α} $z (\lambda z y. z y)$

$$\neq_{\alpha}$$
 $\times (\lambda x \ x. \ x \ x)$



We have defined β reduction: \longrightarrow_{β} Some notation and concepts:

 $\rightarrow \beta$ conversion: $s = \beta t$ iff $\exists n. \ s \longrightarrow_{\beta}^* n \land t \longrightarrow_{\beta}^* n$



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- \rightarrow ($\lambda x. s$) t is called a **redex** (reducible expression)
- → t is reducible iff it contains a redex
- \rightarrow if it is not reducible, t is in **normal form**



$$(\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \longrightarrow_{\beta}$$



$$(\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \longrightarrow_{\beta} (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \longrightarrow_{\beta}$$



No!

$$(\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \longrightarrow_{\beta} (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \longrightarrow_{\beta} \dots$$
$$(\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \longrightarrow_{\beta} \dots$$



No!

$$\begin{array}{l} (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \ \longrightarrow_{\beta} \\ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \ \longrightarrow_{\beta} \\ (\lambda x. \ x \ x) \ (\lambda x. \ x \ x) \ \longrightarrow_{\beta} \dots \end{array}$$

(but:
$$(\lambda x \ y. \ y)$$
 $((\lambda x. \ x \ x) \ (\lambda x. \ x \ x)) \longrightarrow_{\beta} \lambda y. \ y)$



No!

Example:

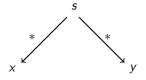
$$(\lambda x. x x) (\lambda x. x x) \longrightarrow_{\beta} (\lambda x. x x) (\lambda x. x x) \longrightarrow_{\beta} (\lambda x. x x) (\lambda x. x x) \longrightarrow_{\beta} \dots$$

$$(\text{but: } (\lambda x y. y) ((\lambda x. x x) (\lambda x. x x)) \longrightarrow_{\beta} \lambda y. y)$$

 λ calculus is not terminating

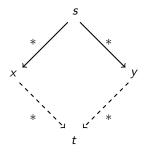


Confluence: $s \longrightarrow_{\beta}^* x \land s \longrightarrow_{\beta}^* y \Longrightarrow \exists t. \ x \longrightarrow_{\beta}^* t \land y \longrightarrow_{\beta}^* t$





Confluence: $s \longrightarrow_{\beta}^* x \land s \longrightarrow_{\beta}^* y \Longrightarrow \exists t. \ x \longrightarrow_{\beta}^* t \land y \longrightarrow_{\beta}^* t$



Order of reduction does not matter for result Normal forms in λ calculus are unique



$$(\lambda x \ y. \ y) ((\lambda x. \ x \ x) \ a)$$

 $(\lambda x \ y. \ y) ((\lambda x. \ x \ x) \ a)$



$$(\lambda x \ y. \ y) \ ((\lambda x. \ x \ x) \ a) \longrightarrow_{\beta} (\lambda x \ y. \ y) \ (a \ a)$$

 $(\lambda x \ y. \ y) \ ((\lambda x. \ x \ x) \ a) \longrightarrow_{\beta} \lambda y. \ y$



$$(\lambda x \ y. \ y) \ ((\lambda x. \ x \ x) \ a) \longrightarrow_{\beta} (\lambda x \ y. \ y) \ (a \ a) \longrightarrow_{\beta} \lambda y. \ y$$

 $(\lambda x \ y. \ y) \ ((\lambda x. \ x \ x) \ a) \longrightarrow_{\beta} \lambda y. \ y$

η Conversion



Another case of trivially equal functions: $t = (\lambda x. \ t \ x)$

η Conversion



Another case of trivially equal functions: $t = (\lambda x. \ t \ x)$

Definition:

$$(\lambda x. \ t \ x) \longrightarrow_{\eta} t \quad \text{if } x \notin FV(t)$$

$$s \longrightarrow_{\eta} s' \Longrightarrow (s \ t) \longrightarrow_{\eta} (s' \ t)$$

$$t \longrightarrow_{\eta} t' \Longrightarrow (s \ t) \longrightarrow_{\eta} (s \ t')$$

$$s \longrightarrow_{\eta} s' \Longrightarrow (\lambda x. \ s) \longrightarrow_{\eta} (\lambda x. \ s')$$

$$s =_{\eta} t \quad \text{iff} \ \exists n. \ s \longrightarrow_{\eta}^{*} n \land t \longrightarrow_{\eta}^{*} n$$

Example:
$$(\lambda x. f x) (\lambda y. g y) \longrightarrow_{\eta}$$

η Conversion



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Example:
$$(\lambda x. f x) (\lambda y. g y) \longrightarrow_{\eta} (\lambda x. f x) g \longrightarrow_{\eta} f g$$

- $\rightarrow \eta$ reduction is confluent and terminating.
- \longrightarrow $\longrightarrow_{\beta\eta}$ is confluent. $\longrightarrow_{\beta\eta}$ means \longrightarrow_{β} and \longrightarrow_{η} steps are both allowed.
- \rightarrow Equality in Isabelle is also modulo η conversion.

In fact ...



Equality in Isabelle is modulo α , β , and η conversion.

We will see later why that is possible.



 λ calculus is very expressive, you can encode:

- → logic, set theory
- → turing machines, functional programs, etc.



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```
true \equiv \lambda x \ y. \ x

false \equiv \lambda x \ y. \ y

if \equiv \lambda z \ x \ y. \ z \ x \ y
```



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- → logic, set theory
- → turing machines, functional programs, etc.

```
\begin{array}{ll} \text{true} & \equiv \lambda x \; y. \; x & \text{if true} \; x \; y \longrightarrow_{\beta}^* x \\ \text{false} & \equiv \lambda x \; y. \; y & \text{if false} \; x \; y \longrightarrow_{\beta}^* y \\ \text{if} & \equiv \lambda z \; x \; y. \; z \; x \; y & \end{array}
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Now, not, and, or, etc is easy:



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```

```
Now, not, and, or, etc is easy:

not \equiv \lambda x. if x false true

and \equiv \lambda x y. if x y false

or \equiv \lambda x y. if x true y
```



Encoding natural numbers (Church Numerals)

```
0 \equiv \lambda f \times x \times 1 = \lambda f \times x \cdot f \times 2 = \lambda f \times x \cdot f \cdot (f \times x) \times 3 = \lambda f \times x \cdot f \cdot (f \cdot (f \times x)) \times \dots
```

Numeral n takes arguments f and x, applies f n-times to x.



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```
0 \equiv \lambda f \ x. \ x
1 \equiv \lambda f \ x. \ f \ x
2 \equiv \lambda f \ x. \ f \ (f \ x)
3 \equiv \lambda f \ x. \ f \ (f \ (f \ x))
...
```

Numeral n takes arguments f and x, applies f n-times to x.

iszero $\equiv \lambda n$. $n (\lambda x$. false) true



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```

Numeral n takes arguments f and x, applies f n-times to x.

```
iszero \equiv \lambda n. \ n \ (\lambda x. \ false) true succ \equiv \lambda n \ f \ x. \ f \ (n \ f \ x)
```



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```
0 \equiv \lambda f \ x. \ x

1 \equiv \lambda f \ x. \ f \ x

2 \equiv \lambda f \ x. \ f \ (f \ x)

3 \equiv \lambda f \ x. \ f \ (f \ (f \ x))

...
```

Numeral n takes arguments f and x, applies f n-times to x.

```
iszero \equiv \lambda n. \ n \ (\lambda x. \ false) true
succ \equiv \lambda n \ f \ x. \ f \ (n \ f \ x)
add \equiv \lambda m \ n. \ \lambda f \ x. \ m \ f \ (n \ f \ x)
```



$$(\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t \longrightarrow_{\beta}$$



$$(\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t \longrightarrow_{\beta} (\lambda f. f ((\lambda x f. f (x x f)) (\lambda x f. f (x x f)) f)) t \longrightarrow_{\beta}$$



```
 \begin{array}{l} (\lambda x \ f. \ f \ (x \ x \ f)) \ (\lambda x \ f. \ f \ (x \ x \ f)) \ t \longrightarrow_{\beta} \\ (\lambda f. \ f \ ((\lambda x \ f. \ f \ (x \ x \ f)) \ (\lambda x \ f. \ f \ (x \ x \ f)) \ f)) \ t \longrightarrow_{\beta} \\ t \ ((\lambda x \ f. \ f \ (x \ x \ f)) \ (\lambda x \ f. \ f \ (x \ x \ f)) \ t) \end{array}
```



$$(\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t \longrightarrow_{\beta} (\lambda f. f ((\lambda x f. f (x x f)) (\lambda x f. f (x x f)) f)) t \longrightarrow_{\beta} t ((\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t)$$

$$\mu = (\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t$$

$$\mu t \longrightarrow_{\beta} t (\mu t) \longrightarrow_{\beta} t (t (\mu t)) \longrightarrow_{\beta} t (t (t (\mu t))) \longrightarrow_{\beta} \dots$$



$$(\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t \longrightarrow_{\beta} (\lambda f. f ((\lambda x f. f (x x f)) (\lambda x f. f (x x f)) f)) t \longrightarrow_{\beta} t ((\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t)$$

$$\mu = (\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t$$

$$\mu t \longrightarrow_{\beta} t (\mu t) \longrightarrow_{\beta} t (t (\mu t)) \longrightarrow_{\beta} t (t (t (\mu t))) \longrightarrow_{\beta} ...$$

 $(\lambda x f. f(x \times f)) (\lambda x f. f(x \times f))$ is Turing's fix point operator



As a mathematical foundation, λ does not work. It is inconsistent.



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- → Russell (1901): Paradox $R \equiv \{X | X \notin X\}$
- → Whitehead & Russell (Principia Mathematica, 1910-1913): Fix the problem
- → Church (1930): λ calculus as logic, true, false, \wedge , ... as λ terms



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$$\{x \mid P \mid x\} \equiv \lambda x. P \mid x \qquad x \in M \equiv M \mid x$$



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with
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 you can write $R \equiv \lambda x. \ \text{not} \ (x \mid x)$



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with
$$\{x \mid P x\} \equiv \lambda x. \ P x$$
 $x \in M \equiv M x$ you can write $R \equiv \lambda x. \ \text{not} \ (x \ x)$ and get $(R \ R) =_{\beta} \ \text{not} \ (R \ R)$

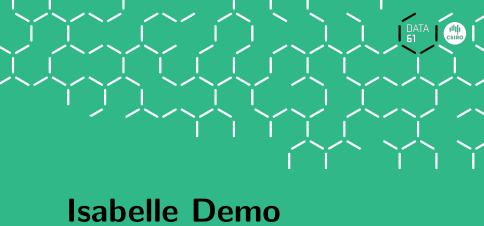


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with
$$\{x \mid P \ x\} \equiv \lambda x. \ P \ x \qquad x \in M \equiv M \ x$$

you can write $R \equiv \lambda x. \ \text{not} \ (x \ x)$
and get $(R \ R) =_{\beta} \ \text{not} \ (R \ R)$
because $(R \ R) = (\lambda x. \ \text{not} \ (x \ x)) \ R \longrightarrow_{\beta} \ \text{not} \ (R \ R)$



We have learned so far...



- $\rightarrow \lambda$ calculus syntax
- → free variables, substitution
- $\rightarrow \beta$ reduction
- $\rightarrow \alpha$ and η conversion
- $\rightarrow \beta$ reduction is confluent
- $\rightarrow \lambda$ calculus is very expressive (turing complete)
- \rightarrow λ calculus is inconsistent