COMP20007 Design of Algorithms

Hashing

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Lecture 15

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Dictionaries - Recap

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- Average case performance for Search, Insert and Delete: $\Theta(1)$
- Requires a *hash function*: $h(K) \rightarrow i \in [0, m-1]$.
- A hash function should:
    - Be efficient ($\Theta(1)$).
    - Distribute keys evenly (uniformly) along the table.

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- Note that $K \in [0, m - 1]$. In other words *we need to know the maximum number of keys in advance*.
- Sometimes this is possible: postcodes, for example.
- Many times it is not:
  - m is too large (need to preallocate)
  - Unbounded integers (student IDs)
  - Non-integer keys (games)

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- Allow us to set the size m.
- Small m results in lots of collisions, large m takes excessive memory. Best m will vary.

Hashing Strings

- Assume A $\mapsto 0$, B $\mapsto 1$, etc.
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We can think of a string as a long binary number:

M Y K E Y $\mapsto 0110011000010100010011000$ $(= 13379736)$

$$13379736 \bmod 101 = 64$$

So 64 is the position of string M Y K E Y in the hash table.

Hashing Strings

We deliberately chose m to be prime.

$$13379736 = 12 \times 32^4 + 24 \times 32^3 + 10 \times 32^2 + 4 \times 32 + 24$$

With $m = 32$, the hash value of any key is the last character's value!

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Then we have

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where m is a prime number. For example,

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The term between parenthesis can become quite large and result in overflow.

Horner's Rule

Instead of

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Now utilize these properties of modular arithmetic:

$$\begin{aligned} (x + y) \bmod m &= ((x \bmod m) + (y \bmod m)) \bmod m \\ (x \times y) \bmod m &= ((x \bmod m) \times (y \bmod m)) \bmod m \end{aligned}$$

So for each sub-expression it suffices to take values modulo m.

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Practical efficiency will depend on the table **load factor**: $\alpha = n/m$

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- Similar numbers for Insert. Delete virtually impossible.
- Does not require extra memory.
- Worst case $\Theta(n)$ with a bad hash function *and/or* clusters.

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Both Linear Probing and Double Hashing are sometimes referred as *Open Addressing* methods.

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- *Rehashing* allocates a new table (usually around double the size) and move every item from the previous table to the new one.
- *Very expensive operation*, but happens infrequently.

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- Allow $\Theta(1)$ Search, Insert and Delete in the average case.
- Preallocates memory (size m).
- Requires good *hash functions*.
- Requires good collision handling.

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That being said, if hashing is applicable, a well-tuned hash table will typically outperform BSTs.

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Next lecture: what happens if records/data is too large?