① a) $p = 0.5$

$Y = h(x)$: winning

$$E(y) = h(0) \cdot P(0) + h(1)\,p(1) + h(2)\,p(2) + h(3)\,p(3)$$

$$6 = 0 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8} + 4 \cdot \frac{3}{8} + h(3) \cdot \frac{1}{8}$$

$$\boxed{h(3) = 48 - 18 = \$30}$$

b)

$$\boxed{F(y) = \begin{cases} 1/8 & y < 2 \\ 1/2 & 2 \le y < 4 \\ \frac{7}{8} & 4 \le y < 30 \\ 1 & y \ge 30 \end{cases}}$$

(2) (a)

$$F(\ell) = \begin{cases} 0 & \ell \leq 0 \\ \int \frac{\ell}{2} d\ell = \frac{\ell^2}{4} & 0 \leq \ell \leq 2 \\ 1 & \ell < \ell \end{cases}$$

(b)

$$E(\ell) = \int_{-\infty}^{\infty} \ell f(\ell) d\ell$$

$$= \int_0^2 \frac{\ell^2}{2} d\ell$$

$$= \frac{\ell^3}{6} \Big|_0^2 =$$

$$\boxed{E(\ell) = \frac{8}{6} = \frac{4}{3}}$$

3 a

$$g(t) = \$100(1-t)^{1/2}$$

$$E(g(t)) = \int_{-\infty}^{\infty} g(t) \cdot f(t) \cdot dt$$

$$= \int_0^1 100(1-t)^{1/2}$$

$u = 1-t$

$du = -dt$

$$= \int -100\, u^{1/2} du$$

$$= -100\, u^{3/2} \cdot \frac{2}{3}$$

$$= \frac{-200}{3}(1-t)^{3/2} \Big|_0^1$$

$$\boxed{E(g(t)) = \$\,\frac{200}{3}}$$

3 b

i. $X = x \quad \& \; 0 \leq x \leq 100$

$\Leftrightarrow \quad 1 \geq t \geq 0$

$\Leftrightarrow \quad 1 \geq t = 1 - \dfrac{x^2}{100^2} \geq 0$

ii $F(x) = F(T > t) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{100^2} & 0 \leq x \leq 100 \\ 1 & x > 100 \end{cases}$

iii $f(x) = \dfrac{d}{dx} F(x)$

$= \dfrac{2x}{100^2}$

iv $\Rightarrow E(x) = \displaystyle\int_{-\infty}^{\infty} x f(x)\, dx$

$= \displaystyle\int_0^{100} \frac{2x^2}{100^2}\, dx$

$= \dfrac{2}{3} \dfrac{x^3}{100^2} \Big|_0^{100}$

$\boxed{E(x) = \dfrac{200}{3}}$

4 (a) $E(Y) = E(X-t)^2$

$$= E(X^2 - 2Xt + t^2)$$

$$E(Y) = E(X^2) - 2t\,E(X) + t^2$$

$$E(Y) = V(X) + [E(X)]^2 - 2t\,E(X) + t^2$$

$$\boxed{E(Y) = V(X) + (E(X) - t)^2}$$

(b) $\dfrac{d}{dt}\left[E(Y)\right] = 2t - 2E(X) = 0$

$$\boxed{t = E(X)}$$

(c) $\boxed{E(Y) = V(X)}$ $\quad t = E(X)$