Lab 2: Probability Theory

W203: Statistics for Data Science

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1. Meanwhile, at the Unfair Coin Factory...

You are given a bucket that contains 100 coins. 99 of these are fair coins, but one of them is a trick coin that always comes up heads. You select one coin from this bucket at random. Let T be the event that you select the trick coin. This means that $P(T) = 0.01$.

a. To see if the coin you have is the trick coin, you flip it k times. Let H_k be the event that the coin comes up heads all k times. If you see this occur, what is the conditional probability that you have the trick coin? In other words, what is $P(T|H_k)$.

Let $P(NT)$ be the probability of getting a fair coin from the 100 coins. Then:

$$P(T|H_k) = \frac{P(H_k|T)P(T)}{P(H_k|NT) * P(NT) + P(H_k|T) * P(T)} = \frac{1 * 0.01}{0.5^k * 0.99 + 1 * 0.01}$$

b. How many heads in a row would you need to observe in order for the conditional probability that you have the trick coin to be higher than 99%?

$$P(T|H_k) > 0.99 => k \geq 13.25 => k \geq 14$$

2. Wise Investments

You invest in two startup companies focused on data science. Thanks to your growing expertise in this area, each company will reach unicorn status (valued at $1 billion) with probability 3/4, independent of the other company. Let random variable X be the total number of companies that reach unicorn status. X can take on the values 0, 1, and 2. Note: X is what we call a binomial random variable with parameters $n = 2$ and $p = 3/4$.

a. Give a complete expression for the probability mass function of X.

$$f(x) = \begin{cases} P(X=0) = \frac{1}{16} \\ P(X=1) = \frac{6}{16} \\ P(X=2) = \frac{9}{16} \end{cases}$$

b. Give a complete expression for the cumulative probability function of X.

$$F(x) = \begin{cases} 0 & X < 0 \\ \frac{1}{16} & 0 \leq X < 1 \\ \frac{7}{16} & 1 \leq X < 2 \\ 1 & X \geq 2 \end{cases}$$

c. Compute $E(X)$.

$$E(X) = \sum x \cdot f(x) = 1.5$$

d. Compute $var(X)$.

$$V(X) = E(X^2) - E(X)^2 = \frac{21}{8} - 1.5^2 = 0.375$$

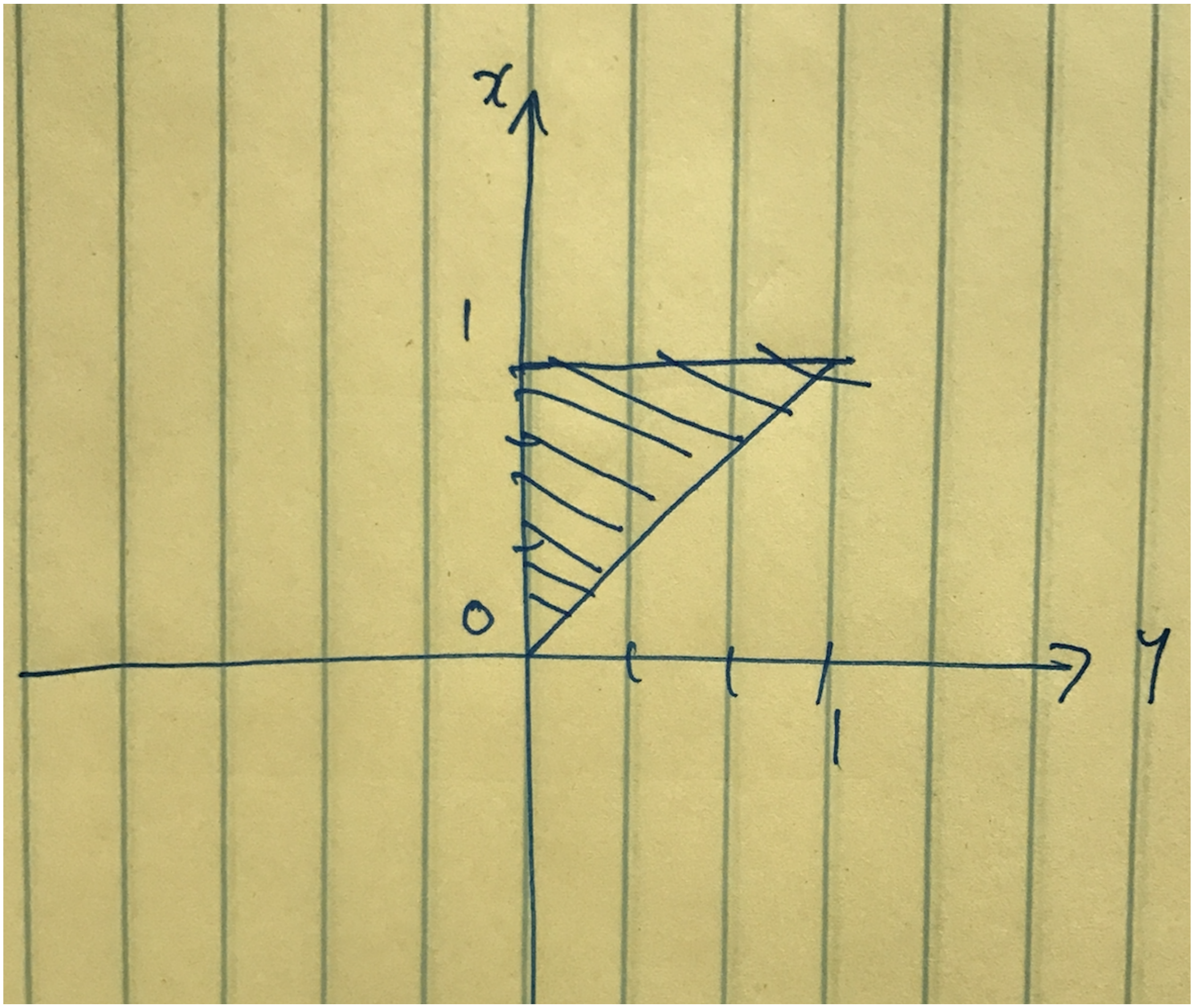
3. Relating Min and Max

Continuous random variables X and Y have a joint distribution with probability density function,

$$f(x, y) = \begin{cases} 2, & 0 < y < x < 1 \\ 0, & otherwise. \end{cases}$$

You may wonder where you would find such a distribution. In fact, if A_1 and A_2 are independent random variables uniformly distributed on $[0, 1]$, and you define $X = max(A_1, A_2)$, $Y = min(A_1, A_2)$, then X and Y will have exactly the joint distribution defined above.

a. Draw a graph of the region for which X and Y have positive probability density.



b. Derive the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy = \int_0^x 2dy = 2x$$

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

c. Derive the unconditional expectation of X.

$$E(X) = \int x \cdot f_X(x) = \int_0^1 x \cdot 2x dx = \frac{2}{3}x^3|_0^1 = \frac{2}{3}$$

d. Derive the conditional probability density function of Y, conditional on X, $f_{Y|X}(y|x

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{x}0 \leq x \leq 1$$

e. Derive the conditional expectation of Y, conditional on X, $E(Y|X)$.

$$E(Y|X) = \int_0^x y \cdot f_{Y|X}(y|x) = \frac{x}{2}$$

f. Derive $E(XY)$. Hint: if you take an expectation conditional on X, X is just a constant inside the expectation. This means that $E(XY|X) = XE(Y|X)$.

$$E(XY) = E(E(XY|X)) = E(XE(Y|X)) = E(X \cdot \frac{X}{2}) = \frac{1}{2}E(X^2) = \frac{x^4}{4}|_0^1 = \frac{1}{4}$$

g. Using the previous parts, derive $cov(X, Y)$

$$E(Y) = E(E(Y|X)) = E(\frac{X}{2}) = \frac{x^3}{3}|_0^1 = \frac{1}{3}$$

$$cov(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{36}$$

4. Circles, Random Samples, and the Central Limit Theorem

Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be independent random samples from a uniform distribution on $[-1, 1]$. Let D_i be a random variable that indicates if (X_i, Y_i) falls within the unit circle centered at the origin. We can define D_i as follows:

$$D_i = \begin{cases} 1, & X_i^2 + Y_i^2 < 1 \\ 0, & otherwise \end{cases}$$

Each D_i is a Bernoulli variable. Furthermore, all D_i are independent and identically distributed.

a. Compute the expectation of each indicator variable, $E(D_i)$. Hint: your answer should involve a Greek letter.

$$E(D_i) = \int_{-\infty}^{\infty} D_i \cdot P(D_i) = 1 \cdot P(1) = \frac{Area(X_i^2 + Y_i^2 = 1)}{Area(SampleSpace)} = \frac{\pi}{4} = 0.7853982$$

b. Compute the standard deviation of each D_i.

$$\sigma_{D_i} = \sqrt{E(D_i^2) - E(D_i)^2} = \sqrt{\frac{4\pi - \pi^2}{16}}$$

c. Let $\bar{D}$ be the sample average of the D_i. Compute the standard error of $\bar{D}$. This should be a function of sample size n.

$$se = \frac{\sigma_{D_i}}{\sqrt{n}}$$

d. Now let n=100. Using the Central Limit Theorem, compute the probability that $\bar{D}$ is larger than 3/4. Make sure you explain how the Central Limit Theorem helps you get your answer.

$$P(\bar{D} \geq \frac{3}{4}) = P(Z \geq \frac{\frac{3}{4} - E(D_i)}{se}) = 1 - P(Z \leq -0.86) = 0.806$$

The sample size is 100, which is larger than 30 and we can use CLT that $\bar{D}$ follows a normal distribution. We can compute the Z-value and estimate the coressponding probability i.e. area under the curve.

```
#code
sigma = sqrt(4*pi - pi^2)/4
se = sigma/10
Z = (3/4 - pi/4)/se
print(Z)
```

```
## [1] -0.8622219
```

```
probability = 1 - pnorm(Z)
print(probability)
```

```
## [1] 0.8057173
```

e. Now let $n = 100$. Use R to simulate a draw for $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$. Calculate the resulting values for $D_1, D_2, ...D_n$. Create a plot to visualize your draws, with X on one axis and Y on the other. We suggest using a command like the following to assign a different color to each point, based on whether it falls inside the unit circle or outside it. Note that we pass $d + 1$ instead of d into the color argument because 0 corresponds to the color white.

```
x = runif(100,min=-1,max = 1)
y = runif(100,min=-1,max = 1)
d = x^2 + y^2 <1
plot(x,y, col=d+1, asp=1)
```

f. What value do you get for the sample average, $\bar{D}$? How does it compare to your answer for part a?

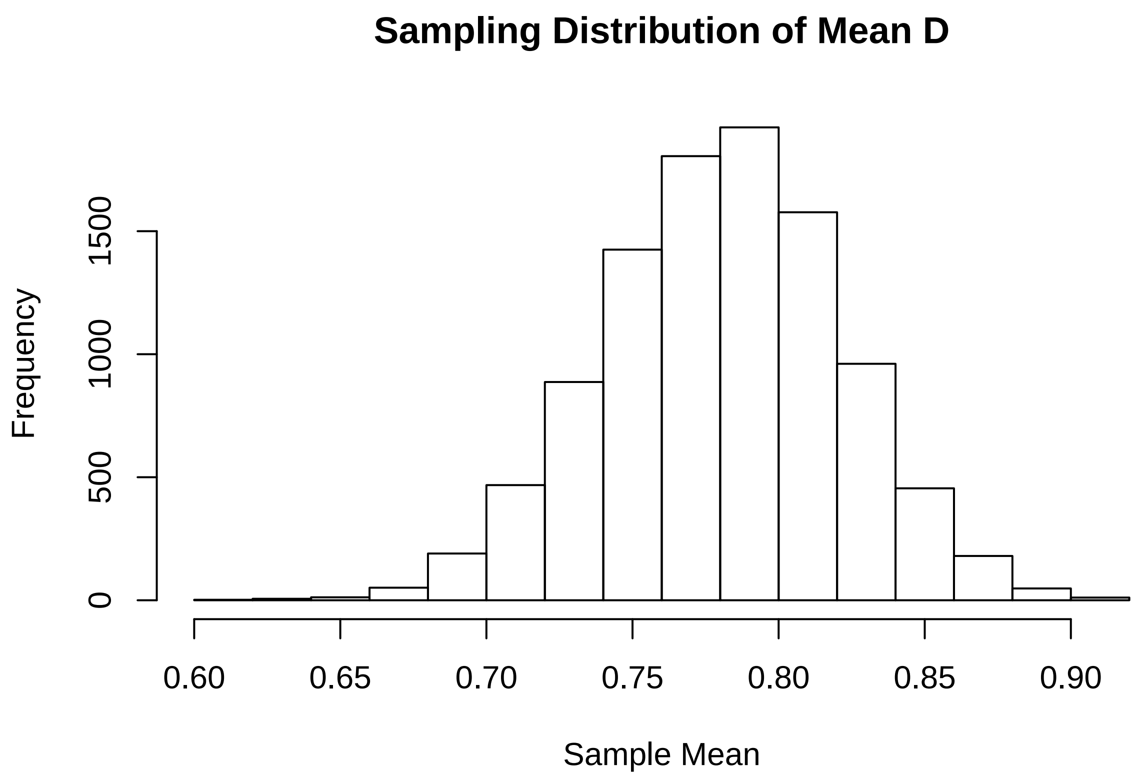
```
x = runif(100,min=-1,max = 1)
y = runif(100,min=-1,max = 1)
d = x^2 + y^2 <1
mean(d)
```

```
## [1] 0.75
```

The sample average fluctuates around the result of part a.

g. Now use R to replicate the previous experiment 10,000 times, generating a sample average of the D_i each time. Plot a histogram of the sample averages.

```
getaverage = function(size){
x = runif(size, min = -1, max = 1)
y = runif(size, min = -1, max = 1)
d = x^2 + y^2 < 1
return(mean(d))
}
experiment = replicate(10000,getaverage(100))
hist(experiment, main = "Sampling Distribution of Mean D", xlab = "Sample Mean")
```



h. Compute the standard deviation of your sample averages to see if it's close to the value you expect from part c.

```
std_deviation_expr = sd(experiment)
print(std_deviation_expr)
```

```
## [1] 0.04097296
```

```
std_deviation_calculated = 1/4*sqrt(4*pi - pi^2)/10
print(std_deviation_calculated)
```

```
## [1] 0.04105458
```

The two values are almost identical.

i. Compute the fraction of your sample averages that are larger that 3/4 to see if it's close to the value you expect from part d.

```
sum(experiment > 0.75)/10000
```

```
## [1] 0.7742
```

This value is close to our calculated value of 0.806 in part d.