Time Series Analysis
Lecture 2

Regression With Time Series, An Introduction to Exploratory Time Series Data Analysis and Time Series Smoothing

datascience@berkeley

Time Series Smoothing Techniques: Introduction and Mathematical Formulation

Introduction to Smoothing Techniques

- Smoothing techniques ("smoothers") are often used to uncover trend and cyclical components of a series.

- The general concept of a smoothing technique that it is formed using a **weighted average of pass values** of a series.

- We will discuss some popular smoothing techniques:
  1. Moving averages
  2. Polynomial and periodic regression smoothers
  3. Spline smoothers
  4. Kernel smoothers

- We will
  - Define the mathematical form of each of these smoothers.
  - Illustrate each of these techniques and their empirical patterns using two examples. The dataset used in one of the examples can be downloaded directly from the Federal Reserve's website.

1. Symmetric Moving Average Smoother

A **symmetric moving average smoother** takes the following formulation:

$$m_t = \sum_{j=-k}^{k} a_j x_{t-j}$$

where $a_j = a_{j-1} \geq 0$ and the sum of the weights equal to one: $\sum_{j=-k}^{k} a_j = 1$

1. Symmetric Moving Average Smoother

Setting $k = 2$ essentially gives a monthly series, if the underlying series is a weekly series, and can help bring out the seasonality pattern, if exists:

$$m_t = \frac{1}{5} \sum_{j=-2}^{2} x_{t-j}$$
$$= \frac{1}{5} \left(x_{t-2} + x_{t-1} + x_t + x_{t+1} + x_{t+2} \right)$$

where $a_j = \frac{1}{5} \forall a_j$

1. Symmetric Moving Average Smoother

Setting $k = 26$ essentially gives an annual series, if the underlying series is a weekly series, and can help identify the long-term trend underlying the series:

$$
\begin{aligned}
m_t &= \frac{1}{53} \sum_{j=-26}^{26} x_{t-j} \\
&= \frac{1}{55} \left(x_{t-26} + x_{t-25} + \cdots + x_t + \cdots + x_{t+25} + x_{t+26} \right)
\end{aligned}
$$

where $a_j = \frac{1}{53} \forall a_j$

2. Regression and Periodic Smoothers

Another class of time series smoothing technique has the following general setup:

$$x_t = f_t + z_t$$

where f_t is some smooth function of time and z_t is a stationary process. One choice of f_t is a polynomial:

$$\boxed{f_t = \sum_{i=0}^{p} \beta_i t^i}$$

For periodic data, **periodic function** is used:

$$f_t = \sum_{i=0}^{p} \alpha_i cos(2\pi\omega_i t)\beta_i sin(2\pi\omega_i t)$$

where $cos(2\pi\omega_0 t) = sin(2\pi\omega_0 t) = 1$, and $\omega_1 \dots \omega_p$ are distinct, specified frequencies.

The polynomial and periodic polynomial functions can be combined as one smoother in a classical linear regression.

3. Spline Smoother

Smoothing splines Extending the polynomial regression as a smoothing technique is to use spline function.

Consider dividing the modeling time horizon into k mutual exclusive and exhaustive intervals:

$$[t_0 = 1, t_1], [t_1 + 1, t_2], \ldots, [t_{k-1} + 1, t_k = n]$$

where $t_0, t_1, \ldots, t_k$ are called knots.

The generalization of the polynomial regression comes from the fact that one fits a regression of the form

$$f_t = \beta_0 + \beta_1 t + \cdots + \beta_p t^p$$

in each of the time intervals defined above. When $p = 3$, it is called *cubic splines*.

3. Spline Smoother

Smoothing splines technique modifies the spline method by incorporating the penalized smoothness component in the objective function such that the minimization problem accounts for the trade-off between the model fit and the degree of smoothness. The objective function is written as

$$\sum_{t=1}^{n}[x_t - f_t]^2 + \lambda \int \left(f_t^{''}\right)^2 dt$$

where f_t is a cubic spline with a knot at each t and λ is the smoothing parameter

4. Kernel Smoother

Kernel Smoothing is a symmetric moving average smoother with a probability density weight function.

$$\hat{f}_t = \sum_{i=1}^{n} w_i(t) x_i$$

where

$$w_i(t) = \frac{K\left(\frac{t-i}{b}\right)}{\sum_{j=1}^{n} K\left(\frac{t-i}{b}\right)}$$

Some example kernel functions are…

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