Time Series Analysis

Lecture 5

Vector Autoregressive (VAR) Models

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Multivariate Time Series Models: An Introduction to Vector Autoregressive Models

Vector AR Models: Mathematical Formulation

- The good news of studying multivariate time series models in the form of vector autoregressive models is that the univariate autoregressive models can be easily generalized to their multivariate counterpart, although the mathematical notation is heavier.

- Consider a simple vector autoregressive process of order 1 (VAR(1)):

$$x_t = \phi_{11} x_{t-1} + \phi_{12} y_{t-1} + \omega_{x,t}$$
$$y_t = \boxed{\phi_{21}} x_{t-1} + \phi_{22} y_{t-1} + \omega_{y,t}$$

where $\{w_{x,t}\}$ and $\{w_{y,t}\}$ are bivariate white noise and ϕ_{ij} are the model parameters.

VAR Models in Matrix Form

Exercise: Show that if the white noise processes have mean 0 and the VAR(1) process is stationary, then both $\{x_t\}$ and $\{y_t\}$ have mean 0.

As in AR(p) models, to incorporate a mean, simply define $\{x_t\}$ and $\{y_t\}$ as deviations from their means.

To derive properties of the VAR(p) process, it is easier to use matrix notations:

$$\mathbf{Z_t} = \Psi \mathbf{Z_{t-1}} + \mathbf{w_t}$$

$$\phi \qquad \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix}$$

$$\underset{2\times 1}{\mathbf{Z_t}} = \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \quad \mathbf{w_t} = \begin{pmatrix} w_{x,t} \\ w_{y,t} \end{pmatrix}$$

VAR Models: Stationarity Condition

Expressed in backward shift operator:

$$(\mathbf{I} - \Psi(B))\,\mathbf{Z_t} = \theta(B)\mathbf{Z_t} = \mathbf{w_t}$$

where θ is a matrix polynomial of order 1 and $\mathbf{I}$ is a 2 x 2 identity matrix.

The mechanics and concepts of VAR(1) process can be easily extended to the general VAR(p) process of m time series, in which case θ becomes a matrix polynomial of order p and is a m x m matrix of parameters, $\mathbf{I}$ becomes a $(m \text{ x } m)$ identity matrix, and $\mathbf{Z_t}$ is a m x 1 matrix of time series variables, and $\mathbf{w_t}$ is a multivariate white noise.

Stationarity of the VAR(p) model is similarly defined as that of the AR(p) model: **The roots of the characteristic equation all lie outside of the unity circle.**

In the case of VAR(p) model, the characteristic equation is given by the determinant of the ***theta*** matrix defined above.

Finding the Roots of the Characteristic Polynomial

Back to the VAR(1) model, the determinant is

$$\begin{vmatrix} 1-\theta_{11}x & -\theta_{12} \\ -\theta_{21}x & 1-\theta_{22}x \end{vmatrix} = (1-\theta_{11}x)\,(1-\theta_{11}x) - \theta_{12}\theta_{21}x^2$$

- However, although this polynomial can be easily solved using papers and pencils, we can use ***R*** functions ***polyroot*** and **Mod**.
- The function ***polyroot***, coming with the base package, is used to find zeros of a real or complex polynomial of the form $p(x) = z_1 + z_2 x + \ldots + z_n x^{n-1}$. For our purpose, it suffices to know how to use this function to find roots in ***R***.
- Interested readers can refer to the documentation https://stat.ethz.ch/R-manual/R-devel/library/base/html/polyroot.html and the reference therein.

Vector AR Models: Introduction

The function **Mod** is one of the basic functions that supports complex arithmetic in **R**. Detailed documentation can be found here `https://stat.ethz.ch/R-manual/R-patched/library/base/html/complex.html` or within **R** by typing help(Mod). We will have to use this function because roots of characteristic polynomials can be complex roots. Since our interests lie only in the absolute value of the roots, we will apply the **Mod** function to roots of the characteristic polynomials and examine if the absolute value of the roots are bigger than 1.

As an example, if the VAR(1) model has the parameter matrix

$$\Psi = \begin{array}{c} \begin{array}{cc} \phi_{11} & \phi_{12} \end{array} \\ \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \end{pmatrix} \\ \begin{array}{cc} \phi_{21} & \phi_{22} \end{array} \end{array}$$

and the corresponding characteristic polynomial is given by

$$\theta = \begin{vmatrix} 1-0.4x & -0.3x \\ -0.2x & 1-0.1x \end{vmatrix} = (1-0.5x-0.02x^2)$$

The absolute value of the roots of this equation can be found using ***R***: **Mod(polyroot(c($1, -0.5, -0.02$))).**

```
> Mod(polyroot(c(1, -0.5, -0.02)))
[1] 1.9 26.9
```

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