# Understanding Backpropagation Algorithm

Learn the nuts and bolts of a neural network's most important ingredient

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"A man is running on a highway" — photo by Andrea Leopardi on Unsplash

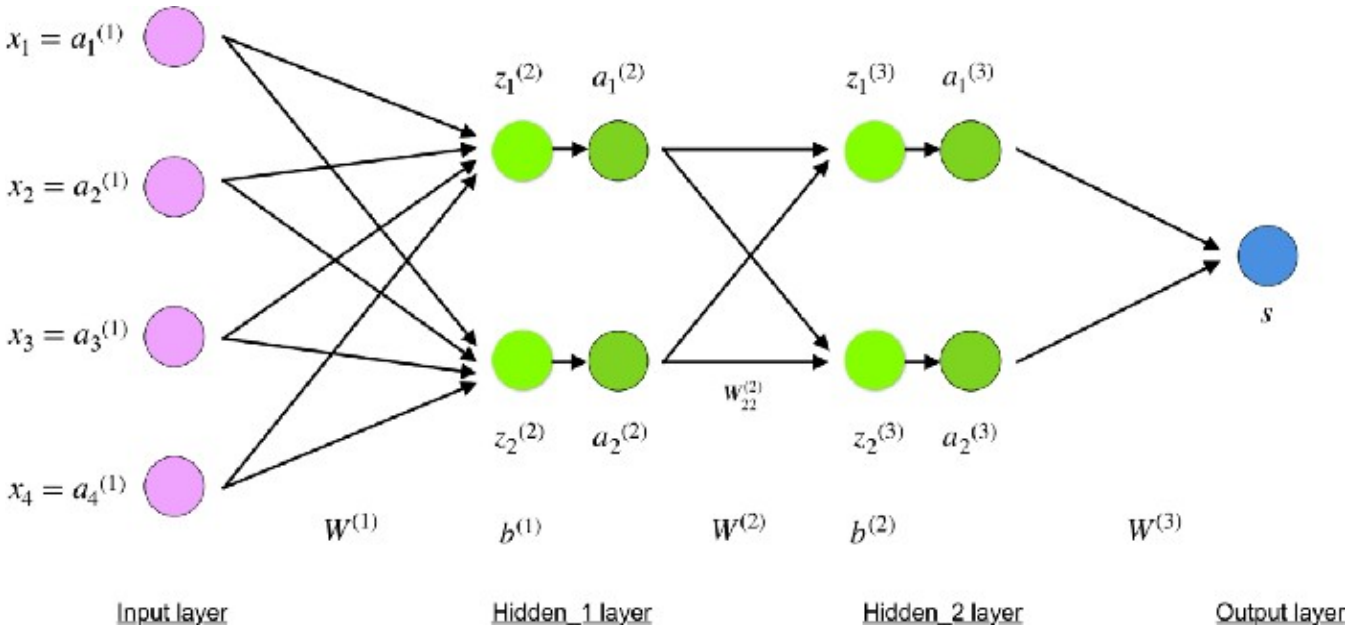
**Backpropagation algorithm** is probably the most fundamental building block in a neural network. It was first introduced in 1960s and almost 30 years later (1989)

**The algorithm is used to effectively train a neural network through a method called chain rule.** In simple terms, after each forward pass through a network, backpropagation performs a backward pass while adjusting the model's parameters (weights and biases).

In this article, I would like to go over the mathematical process of training and optimizing a simple 4-layer neural network. *I believe this would help the reader understand how backpropagation works as well as realize its importance.*

## Define the neural network model

The 4-layer neural network consists of 4 neurons for the **input layer**, 4 neurons for the **hidden layers** and 1 neuron for the **output layer**.




Simple 4-layer neural network illustration

### Input layer

The neurons, colored in **purple**, represent the input data. These can be as simple as scalars or more complex like vectors or multidimensional matrices.

$$x_i = a_i^{(1)}, i \in 1,2,3,4$$

Equation for input x_i

## Hidden layers

The final values at the hidden neurons, colored in **green**, are computed using $z^{\wedge}l$ — weighted inputs in layer $l$, and $a^{\wedge}l$— activations in layer $l$. For layer 2 and 3 the equations are:

- $l = 2$

$$z^{(2)} = W^{(1)}x + b^{(1)}$$

$$a^{(2)} = f(z^{(2)})$$

Equations for $z^2$ and $a^2$

- $l = 3$

$$z^{(3)} = W^{(2)}a^{(2)} + b^{(2)}$$

$$a^{(3)} = f(z^{(3)})$$

Equations for $z^3$ and $a^3$

$W^2$ and $W^3$ are the weights in layer 2 and 3 while $b^2$ and $b^3$ are the biases in those layers.

Activations $a^2$ and $a^3$ are computed using an activation function $f$. Typically, this **function $f$ is non-linear** (e.g. sigmoid, ReLU, tanh) and allows the network to learn complex patterns in data. We won't go over the details of how activation functions work, but, if interested, I strongly recommend reading this great article.

Looking carefully, you can see that all of $x$, $z^2$, $a^2$, $z^3$, $a^3$, $W^1$, $W^2$, $b^1$ and $b^2$ are missing their subscripts presented in the 4-layer network illustration above. **The reason is that we have combined all parameter values in matrices, grouped by layers.** This is the standard way of working with neural networks and one should be comfortable with the calculations. However, I will go over the equations to clear out any confusion.

- $W^1$ is a weight matrix of shape *(n, m)* where *n* is the number of output neurons (neurons in the next layer) and *m* is the number of input neurons (neurons in the previous layer). For us, $n = 2$ and $m = 4$.

$$W^{(1)} = \begin{bmatrix} W_{11}^{(1)} & W_{12}^{(1)} & W_{13}^{(1)} & W_{14}^{(1)} \\ W_{21}^{(1)} & W_{22}^{(1)} & W_{23}^{(1)} & W_{24}^{(1)} \end{bmatrix}$$

Equation for $W^1$

**NB: The first number in any weight's subscript matches the index of the neuron in the next layer** (in our case this is the *Hidden_2 layer*) **and the second number matches the index of the neuron in previous layer** (in our case this is the *Input layer*).

- *x* is the input vector of shape *(m, 1)* where *m* is the number of input neurons. For us, $m = 4$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Equation for x

- $b^1$ is a bias vector of shape *(n , 1)* where *n* is the number of neurons in the current layer. For us, $n = 2$.

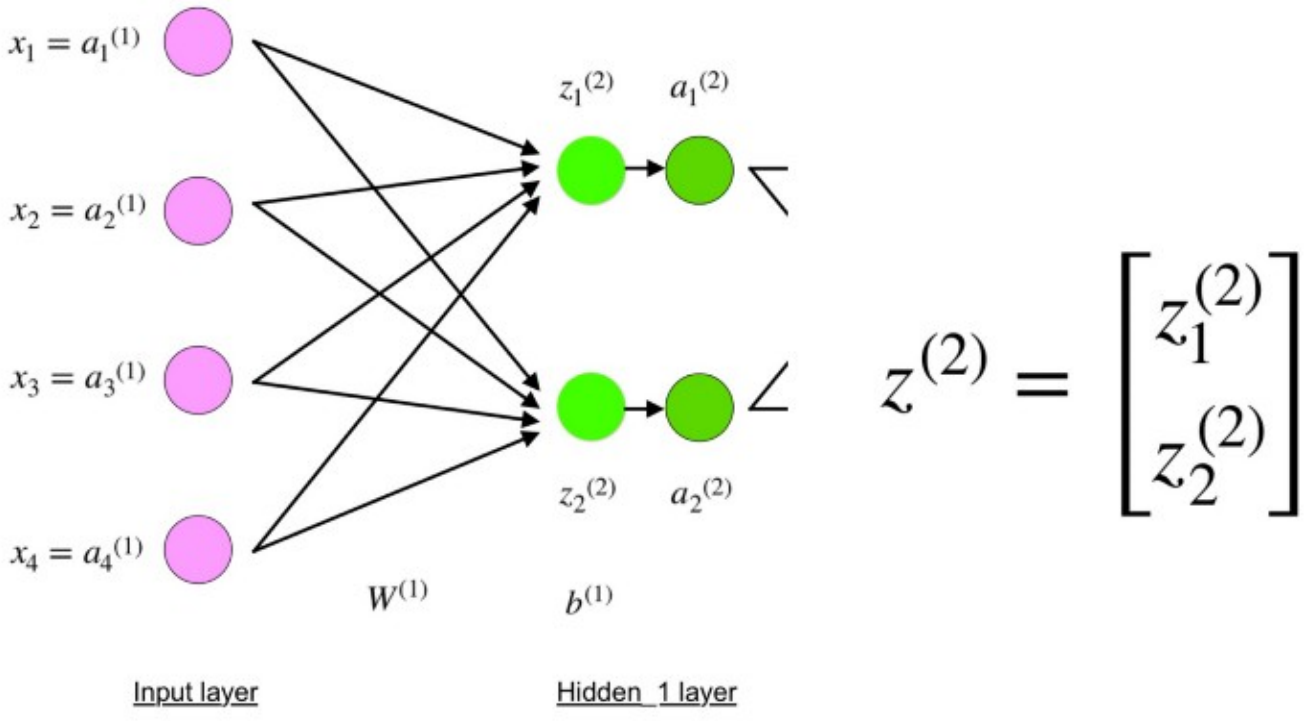
$$b^{(1)} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \end{bmatrix}$$

Following the equation for $z^2$, we can use the above definitions of $W^1$, $x$ and $b^1$ to derive "*Equation for* $z^2$":

$$z^{(2)} = \begin{bmatrix} W_{11}^{(1)}x_1 + W_{12}^{(1)}x_2 + W_{13}^{(1)}x_3 + W_{14}^{(1)}x_4 \\ W_{21}^{(1)}x_1 + W_{22}^{(1)}x_2 + W_{23}^{(1)}x_3 + W_{24}^{(1)}x_4 \end{bmatrix} + \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \end{bmatrix}$$

Equation for z²

Now carefully observe the neural network illustration from above.




Input and Hidden_1 layers

You will see that $z^2$ can be expressed using $(z\_1)^2$ and $(z\_2)^2$ where $(z\_1)^2$ and $(z\_2)^2$ are the sums of the multiplication between every input $x\_i$ with the corresponding weight $(W\_ij)^1$.

This leads to the same "*Equation for* $z^2$" and proofs that the matrix representations for $z^2$, $a^2$, $z^3$ and $a^3$ are correct.

## Output layer

evaluated as follows:

$$s = W^{(3)}a^{(3)}$$

Equation for output s

Again, we are using the matrix representation to simplify the equation. One can use the above techniques to understand the underlying logic. **Please leave any comments below if you find yourself lost in the equations — I would love to help!**

## Forward propagation and evaluation

The equations above form network's forward propagation. Here is a short overview:

$$x = a^{(1)} \qquad Input\ layer$$

$$z^{(2)} = W^{(1)}x + b^{(1)} \qquad neuron\ value\ at\ Hidden_1\ layer$$

$$a^{(2)} = f(z^{(2)}) \qquad activation\ value\ at\ Hidden_1\ layer$$

$$z^{(3)} = W^{(2)}a^{(2)} + b^{(2)} \qquad neuron\ value\ at\ Hidden_2\ layer$$

$$a^{(3)} = f(z^{(3)}) \qquad activation\ value\ at\ Hidden_2\ layer$$

$$s = W^{(3)}a^{(3)} \qquad Output\ layer$$

Overview of forward propagation equations colored by layer

The final step in a forward pass is to evaluate the **predicted output *s*** against an **expected output *y***.

The output *y* is part of the training dataset *(x, y)* where *x* is the input (as we saw in the previous section).

Evaluation between *s* and *y* happens through a **cost function**. This can be as simple as MSE (mean squared error) or more complex like cross-entropy.

$$C = cost(s, y)$$

Equation for cost function C

were *cost* can be equal to MSE, cross-entropy or any other cost function.

Based on *C*'s value, the model "knows" how much to adjust its parameters in order to get closer to the expected output *y*. This happens using the backpropagation algorithm.

## Backpropagation and computing gradients

According to the paper from 1989, backpropagation:

> *repeatedly adjusts the weights of the connections in the network so as to minimize a measure of the difference between the actual output vector of the net and the desired output vector.*

and

> *the ability to create useful new features distinguishes back-propagation from earlier, simpler methods…*

In other words, **backpropagation aims to minimize the cost function by adjusting network's weights and biases.** The level of adjustment is determined by the gradients of the cost function with respect to those parameters.

One question may arise — **why computing gradients**?

To answer this, we first need to revisit some calculus terminology:

- *Gradient of a function C(x_1, x_2, …, x_m) in point x is a vector of the partial derivatives of C in x.*

$$\frac{\partial C}{\partial x} = [\frac{\partial C}{\partial x_1}, \frac{\partial C}{\partial x_2}, \ldots, \frac{\partial C}{\partial x_m}]$$

Equation for derivative of C in x

*words, the derivative tells us the direction C is going.*

- *The gradient shows how much the parameter x needs to change (in positive or negative direction) to minimize C.*

Compute those gradients happens using a technique called chain rule.

For a single weight *(w_jk)^l,* the gradient is:

$$\frac{\partial C}{\partial w_{jk}^l} = \frac{\partial C}{\partial z_j^l}\frac{\partial z_j^l}{\partial w_{jk}^l} \qquad chain\ rule$$

$$z_j^l = \sum_{k=1}^{m} w_{jk}^l a_k^{l-1} + b_j^l \qquad by\ definition$$

$$m\ -\ number\ of\ neurons\ in\ l-1\ layer$$

$$\frac{\partial z_j^l}{\partial w_{jk}^l} = a_k^{l-1} \qquad by\ differentiation\ (calculating\ derivative)$$

$$\frac{\partial C}{\partial w_{jk}^l} = \frac{\partial C}{\partial z_j^l}a_k^{l-1} \qquad final\ value$$

Equations for derivative of C in a single weight (w_jk)^l

Similar set of equations can be applied to *(b_j)^l*:

$$\frac{\partial C}{\partial b_j^l} = \frac{\partial C}{\partial z_j^l}\frac{\partial z_j^l}{\partial b_j^l} \qquad chain\ rule$$

$$\frac{\partial z_j^l}{\partial b_j^l} = 1 \qquad by\ differentiation\ (calculating\ derivative)$$

$$\frac{\partial C}{\partial} \quad \frac{\partial C}{}$$

Equations for derivative of C in a single bias (b_j)^l

The common part in both equations is often called *"local gradient"* and is expressed as follows:

$$\delta_j^l = \frac{\partial C}{\partial z_j^l} \qquad local\ gradient$$

Equation for local gradient

The *"local gradient"* can easily be determined using the chain rule. I won't go over the process now but if you have any questions, please comment below.

. . .

The gradients allow us to optimize the model's parameters:

$$while\ (termination\ condition\ not\ met)$$

$$w := w - \epsilon \frac{\partial C}{\partial w}$$

$$b := b - \epsilon \frac{\partial C}{\partial b}$$

$$end$$

Algorithm for optimizing weights and biases (also called "Gradient descent")

- Initial values of $w$ and $b$ are randomly chosen.
- Epsilon ($e$) is the learning rate. It determines the gradient's influence.

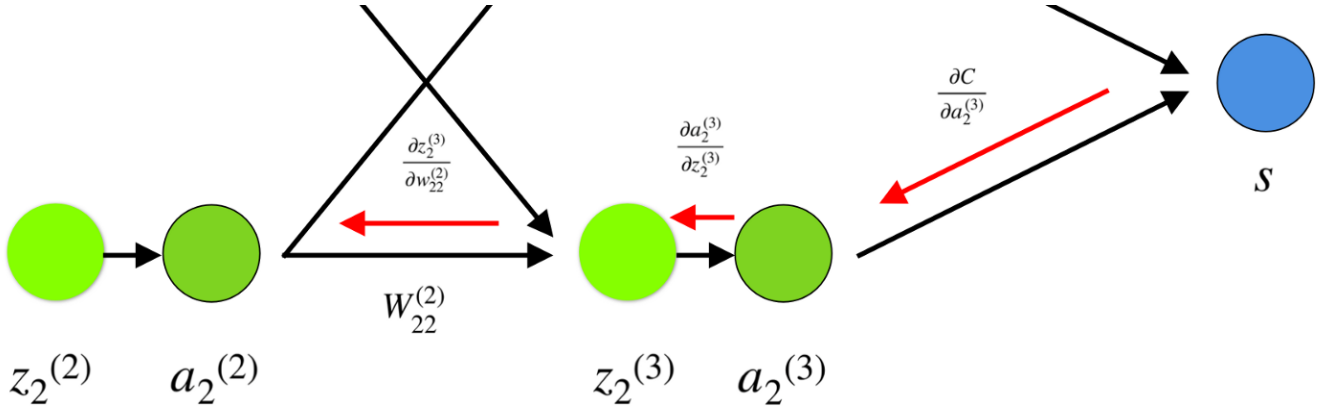
biases.

- Termination condition is met once the cost function is minimized.

. . .

I would like to dedicate the final part of this section to a simple example in which we will calculate the gradient of $C$ with respect to a single weight $(w\_22)^2$.

Let's zoom in on the bottom part of the above neural network:




Visual representation of backpropagation in a neural network

Weight $(w\_22)^2$ connects $(a\_2)^2$ and $(z\_2)^2$, so computing the gradient requires applying the chain rule through $(z\_2)^3$ and $(a\_2)^3$:

$$\frac{\partial C}{\partial w_{22}^{(2)}} = \frac{\partial C}{\partial z_2^{(3)}} \cdot \frac{\partial z_2^{(3)}}{\partial w_{22}^{(2)}} = \frac{\partial C}{\partial a_2^{(3)}} \cdot \frac{\partial a_2^{(3)}}{\partial z_2^{(3)}} \cdot a_2^{(2)} = \frac{\partial C}{\partial a_2^{(3)}} \cdot f'(z_2^{(3)}) \cdot a_2^{(2)}$$

Equation for derivative of C in $(w\_22)^2$

Calculating the final value of derivative of $C$ in $(a\_2)^3$ requires knowledge of the function $C$. Since $C$ is dependent on $(a\_2)^3$, calculating the derivative should be fairly straightforward.

I hope this example manages to throw some light on the mathematics behind computing gradients. To further enhance your skills, I strongly recommend watching

## Final remarks

In this article, I went through a detailed explanation of how backpropagation works under the hood using mathematical techniques like computing gradients, chain rule etc. *Knowing the nuts and bolts of this algorithm will fortify your neural networks knowledge and make you feel comfortable to take on more complex models. Enjoy your deep learning journey!*

## Thank you for the reading. Hope you enjoyed the article 😃 and I wish you a great day!