Structural Optimization 8, 42–51 © Springer-Verlag 1994

Bubble method for topology and shape optimization of structures

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Abstract This paper addresses a novel method of topology and shape optimization. The basic idea is the iterative positioning of new holes (so-called "bubbles") into the present structure of the component. This concept is therefore called the "bubble method". The iterative positioning of new bubbles is carried out by means of different methods, among others by solving a variational problem. The insertion of a new bubble leads to a change of the class of topology. For these different classes of topology, hierarchically structured shape optimizations that determine the optimal shape of the current bubble, as well as the other variable boundaries, are carried out.

1 Introduction

The position and arrangement of the structural elements within a component, i.e. the change of its topology, decisively influence the deformation and the stress-field. When developing new structures, it is important to optimally determine the topology. Topology optimization aims at replacing the more intuitive variation of constructions in the design phase by mathematical-mechanical strategies for the sake of greater efficiency.

As there are other types of design variables apart from the topology optimization of structures, a classification according to these types seems sensible in order to discuss the recent developments of topology optimization in the scope of structural optimization (Schmit and Mallet 1963). Many structural improvements can be achieved either by an optimal choice of cross-sections or by a best-possible shaping. Apart from optimally choosing advanced materials, a number of research groups have increasingly dealt with topology optimization. The state-of-the-art in this field of research will be described briefly in the following.

Michell (1904) developed a design theory for the topology of bar structures that are optimal with regard to weight. The bars in these structures are all perpendicular to each other and therefore form an optimal arrangement in the sense of maximum tensile and compressive stresses. The development of finite element procedures and of further efficient structural computation methods in the course of the 1960's has facilitated the structural optimization of more complex problems. In this respect, research into topology optimization is yet in its early stages. Important initial steps in this direction were taken by Prager (1974) and Prager and Rozvany (1977), who solved topology optimization problems by analytical procedures. Kirsch (1990) and Rozvany *et al.* (1989, 1990, 1993) developed optimality criteria for bar structures. These criteria determine the optimal structure from a defined basic structure containing all feasible bar elements.

Proceeding from continua, further research has been carried out during the last years. Atrek (1989) has developed a procedure which divides the topology space into many smaller sub-areas, the thicknesses of which are defined as design variables and are then varied by means of a simple optimality criterion. He defines a decision limit for the element thickness according to which all design variables under this limit are set at "zero" (hole) and those beyond this limit are set at "one" (material). This is called 0-1-check. Based upon works on the homogenization of porous materials (Hashin 1962; Tartar 1973; Bourgat 1973), Bendsøe *et al.* (1988, 1993) have developed the "homogenization method" for topology optimization. In this context they have referred to research activities carried out by Kohn and Strang (1986) and Allaire and Kohn (1993). The homogenization method works without a 0-1-Check since it considers so-called microcells which form the small structural areas of the topology space. These microcells consist of massive material on the one hand, and of a no-material area (hole) on the other, so that a porous material behaviour can be simulated. For the computation, this porous material is then subjected to a homogenization, i.e. it is made indistinct. The basic idea of this type of optimization is a variation of the volumetric efficiency, i.e. of the ratio of volume of massive material to the total volume of the micro-cells in the structural areas. Here, one tries to approach the volumetric efficiency "0" (no material) and "1" (full material). This method takes a further step in the direction of a topology optimization of mechanical structures.

The topology optimization method addressed in this paper uses an iterative positioning and a hierarchically structured shape optimization of new holes, so-called "bubbles". This means that the boundaries of the component are taken as parameters, and that the shape optimization of the new bubbles and of the other variable boundaries of the component is carried out as a parameter optimization (Eschenauer *et al.* 1993).

2 Global and local domain variation

The bubble method combines global with local domain variation. The global domain variation (classical shape optimiza-

*Humboldt scholar, visiting the Institute of Mechanics and Control Engineering, University of Siegen, 1990–1991 tion, see Fig. 1) provides efficient strategies that vary the boundary shape of a component, considering objective and constraint functions. The search for the optimal shape of a component structure is based upon the theory of the "variation with variable domain" (Courant and Hilbert 1968). According to this, the shape optimization problem can be formulated as the integral of the product of a function F_1 and the domain variation δ_s (distance between the old and the new boundary considered in the direction of the normal) along the varied boundary Γ

$$\delta J_1 = \int_\Gamma F_1 \delta s \, \mathrm{d}\Gamma \stackrel{!}{=} 0 . \qquad (1)$$

This problem formulation corresponds to the first variation of a functional J_1.

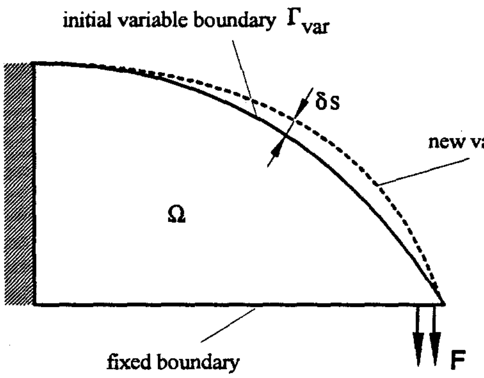


Fig. 1. Classical domain variation

The majority of numerical methods is based upon this first variation according to (1). The function F_1 contains the objective function(s) and the constraints, which are introduced by means of the Lagrangian multipliers. Apart from that, the mechanical basic equations (equilibrium conditions, strain - displacement relations and the constitutive equations) are being considered. In order to explicitly determine function F_1, one can employ existing methods based on the formulation of the problem by means of the Lagrangian function, the consideration of the variation of multi-dimensional problems, and on the fundamental equations of structural mechanics. Global shape optimization does not change the topology of the structure. The optimization results can be found in the topology class of the initial structure.

The above limitation can be overcome by considering the local domain variation, where the goal is to determine the optimal position of a new hole (bubble) within the structure. For this purpose, we insert an infinitesimal bubble $\delta\omega$ (Fig. 2) which changes the stress and deformation state in the structure.

Assuming that

- the macroscopic stress field of the component remains practically unchanged by the infinitesimal bubble (the stress field depends on the outer shape of the component and on the external loads), and that
- a local stress concentration is generated in the vicinity of the bubble so that the microscopic stress field in the domain G depends on the shape of the bubble and on the mean value of the macroscopic stress at the boundary of G (consideration: τ^{ij} = const. at G),

it is sufficient to evaluate the state in area G during the variation δJ_2.

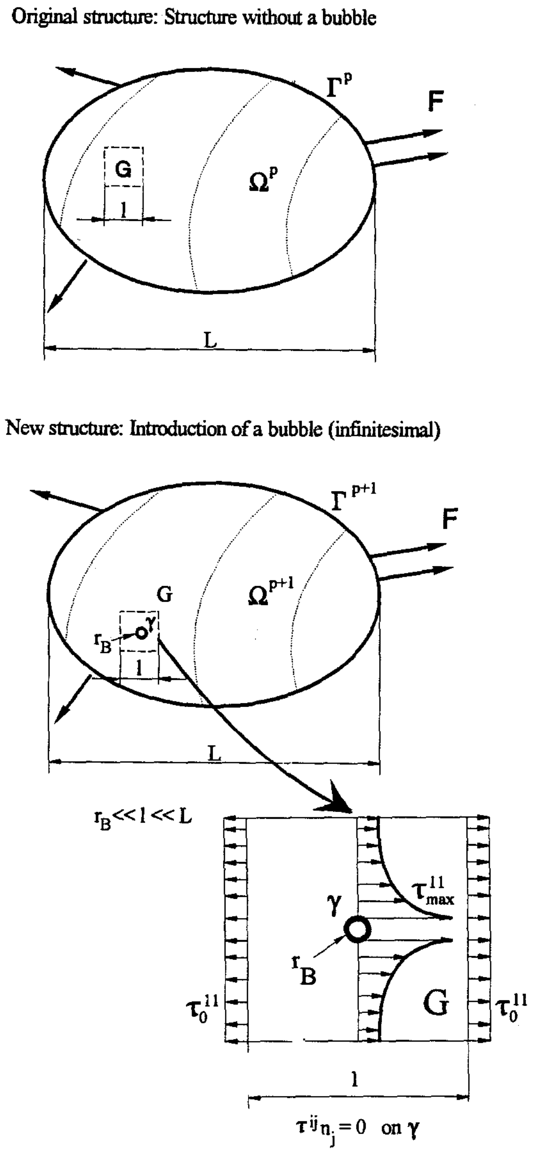


Fig. 2. Local domain variation by inserting a hole (bubble)

The variation can be formulated as an integral of a function F_2 over the volume $\delta\omega$ of the hole

$$\delta J_2 = \int_{\delta\omega} F_2 \, \mathrm{d}(\delta\omega) \stackrel{!}{=} 0 . \qquad (2)$$

The integral can then be used as a positioning criterion for the bubble. The form of F_2 depends on the objective functions and constraints of the optimization problem as well as on the type of load.

3 Structure of the bubble method

3.1 Subdivision into single steps

Optimization via the bubble method is based upon a solution concept comprising an iterative positioning of new bubbles followed by a hierarchically secondary shape optimization of the new bubbles together with all variable boundaries.

The optimization process is divided into the following steps (Fig. 3).

Step 1. For a given topology domain, shape optimization is carried out, considering objective and constraint functions, after which the structure of the component in this topology class cannot be improved any further.

Step 2. By inserting a hole (change of the topology class) we try to achieve improved results. We require the coordinates of the optimal position of the new hole (bubble). The positioning is carried out via a so-called positioning criterion which is to be determined analytically for special objective and constraint functionals (e.g. the global stiffness or the volume) and numerically for general cases.

Step 3. After the positioning, a shape optimization is carried out in order to find the optimal shape of the new bubbles and to determine the influence on the other variable boundaries.

After the shape optimization has been completed, a further bubble is inserted by means of the positioning criterion and is then optimized with respect to its shape. This generates an iterative process, and one obtains a number of possible topologies out of which a suitable variant can be chosen by a criterion still to be developed. The choice, in turn, must follow external demands on the construction (e.g. possibilities of manufacturing).

3.2 First step — direct shape strategy

In order to solve the global domain variation, mathematical programming methods are employed. For this purpose, the problem is first approximated and then solved. Since the approximated solution space approaches the real problem step-wise only, several iterations are required. In order to touch the real problem at all, a suitable definition of the shape optimization problem is essential.

3.2.1 Definition for shape optimization problems

The mathematical formulation of multicriteria shape optimization problems can be written as follows:

$$\mathbf{F}^*[\Gamma_{\mathrm{var}}(\xi^i)] = \min_{\Gamma_{\mathrm{var}}}\{\mathbf{F}[\Gamma_{\mathrm{var}}(\xi^i)] | \Gamma_{\mathrm{var}}(\xi^i) \in G\}\,, \tag{3}$$

with

$$G = \{\Gamma_{\mathrm{var}}(\xi^i) \in I\!R^3 | \mathbf{H}[\Gamma_{\mathrm{var}}(\xi^i)] = 0, \quad \mathbf{G}[\Gamma_{\mathrm{var}}(\xi^i)] \geq 0\}\,,$$

where $\mathbf{F}$ is the vector of objective functionals, $\mathbf{H}$, $\mathbf{G}$ the equality and inequality constraint functionals, and $\Gamma_{\mathrm{var}}(\xi^i)$ the variable boundary of the structure.

The shape of a body can be described by the approach functions $\mathbf{R}(\xi^i, \mathbf{x})$. Thus, the domain variation can be given as follows:

$$\xi^{i*} = \xi^i + \delta[s(\mathbf{R})]\,, \tag{4}$$

with $\mathbf{x}$ the free parameters of the approach functions, ξ^i the coordinates of the boundaries of the structure, ξ^{i*} the coordinates of the new boundaries of the structure, and $\delta[s(\mathbf{R})]$ the variation of the boundaries.

In previous years, suitable approach functions have been implemented into the computer-aided-graphic-design procedures. These tools describe the boundary contours by means

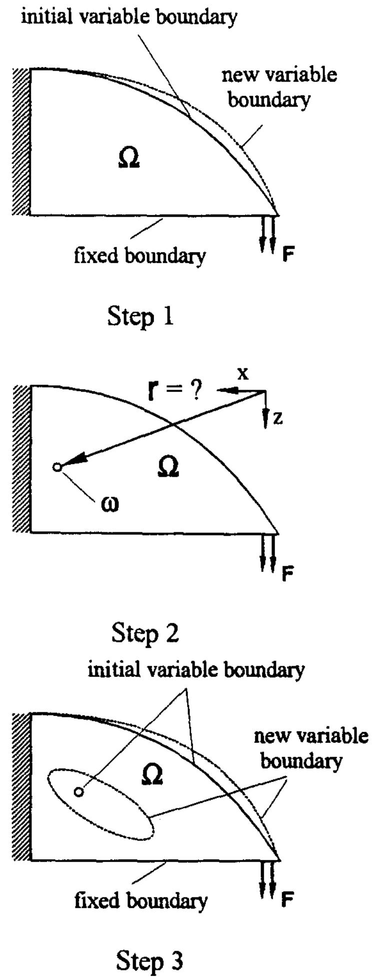


Fig. 3. Steps of the bubble method

of simple approach functions like circle segments, pieces of straight line, surfaces, etc. Apart from that, it is advisable to describe the boundary of the component by suitable parametric approach functions in the form of B-spline or Bezier curves (Eschenauer and Weinert 1992).

In order to provide a suitable way of approaching sharp edges as well, shape optimization contains all those methods which the corresponding literature combines under the term NURBS (non-uniform-rational-B-splines) (Farin 1991).

The general recursive formula for NURBS reads as follows:

$$\mathbf{R}(\xi^i, \mathbf{x}) = \sum_{i=0}^{n} \frac{N_{i,j}(\eta) \cdot w_i}{\sum\limits_{\ell=0}^{n} N_{\ell,j}(\eta) \cdot w_\ell} \mathbf{P}_i\,, \tag{5}$$

with

$$N_{i,j}(\eta) = \frac{(\eta - t_i)N_{i,j-1}(\eta)}{t_{i+j-1} - t_i} + \frac{(t_{i+j} - \eta)N_{i+1,j-1}(\eta)}{t_{i+j} - t_{i+1}}$$

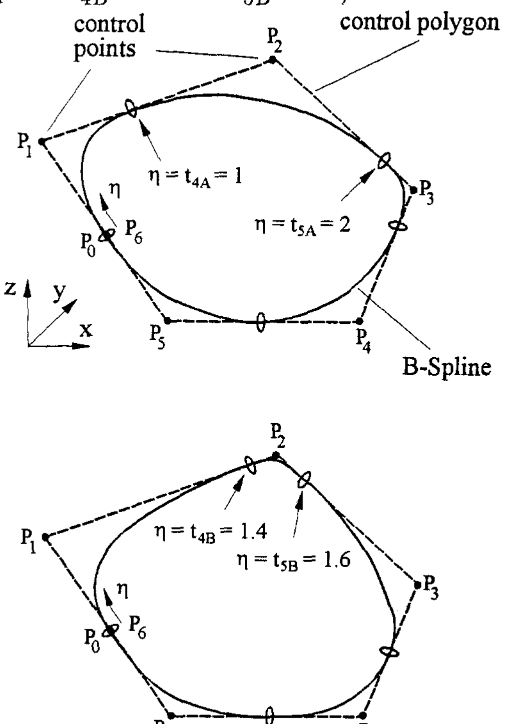
$$N_{i,1}(\eta) = 1 \quad \text{for} \quad t_i \leq \eta \leq t_{i+1}$$

$$N_{i,1}(\eta) = 0 \quad \text{otherwise}\,,$$

where $\mathbf{P}_i$ is the coordinate vector of the control point i, η the variable for the curve description, $N_{i,j}(\eta)$ the basic functions, j the degree of the basic functions, w_i the weightings of the control point i, $\mathbf{t} = (t_0, \ldots, t_m)^T$ the knot vector (0 for $k < j$,

$i-j+1$ for $j \leq k \leq n$, $n-j+2$ for $k > n$), and $n+1$ the number of control points.

Thus, the coordinates of the control points $\mathbf{P}_i$, the weighting of the control points w_i and the knot vector $\mathbf{t}$ can be defined variably. The variation of the knot vector, for example, allows us to generate sharper corners. Start and end values of the knot vector are defined j-fold, while the intermediate values determine the point where the spline touches the control polygon (valid for $j = 3$ only). Figure 4 shows the variation of the knot vector $\mathbf{t}$ in the case of constant coordinates and weighting of the control points of a B-spline curve. By variation of two components of the knot vector ($\mathbf{t}_{4A} = 1$ and $\mathbf{t}_{5A} = 2$) the curve can be forced into the corner P_2 more strongly (the two components of the knot vector moves into the new points $\mathbf{t}_{4B} = 1.4$ and $\mathbf{t}_{5B} = 1.6$).



$\mathbf{t}_A = \{0, 0, 0, 1, 2, 3, 4, 5, 5, 5\}$

$\mathbf{t}_B = \{0, 0, 0, 1.4, 1.6, 3, 4, 5, 5, 5\}$

$\mathbf{w}_A = \mathbf{w}_B = \{1, 1, 1, 1, 1, 1, 1\} = \text{const.}$

$P_A = P_B = \text{const.}$

Fig. 4. Variation of the knot vector $\mathbf{t}$ of a B-spline curve

3.2.2 Solution of shape optimization problems

Shape optimization problems can be solved by means of indirect and direct methods. In indirect procedures, the necessary conditions for the optimal shape are derived using variational principles, and the resulting differential equations are then solved, generally by approximation methods. Here, the direct solution method is applied, and the shape optimization problem is transformed into a parameter optimization problem using the approach functions $\mathbf{R}(\xi^i, \mathbf{x})$.

By means of these approach functions and the design variables $\mathbf{x}$, the design model creates a current design of the component defined by the so-called analysis variables $\mathbf{y}$. In the structural model, the state variables $\mathbf{u}$ are calculated for the current design of the component. The evaluation model determines the constraints $\mathbf{h}$, $\mathbf{g}$ and the objective functions $\mathbf{f}$ from the state variables as well as from the analysis and design variables.

The next step is the sensitivity analysis, which presents a very important task in the optimization loop. When using commercial structural analysis programmes, one requires a sensitivity analysis that is independent of the structural analysis program. The shape optimization in the bubble method is carried out using the finite differences method (Eschenauer *et al.* 1993) on the one hand, and variational sensitivity analysis (Dems 1991; Korycki *et al.* 1993) on the other. The former method requires a high degree of computation time and is relatively inaccurate, but it can be applied to arbitrary optimization functionals. The latter method is suitable for a certain type of optimization functional because it need not carry out a new structural analysis for each design variable.

3.3 Second step — positioning of the bubble

Whenever a new bubble is inserted, the global optimum of its position must be determined, since the hierarchically secondary shape optimizations often find local optima only. Thus, they depend on the initial design and on the shape optimization of the preceding iterations.

When simple optimization functionals are considered, analytical variation expressions that lead to so-called "characteristic functions" may be determined. The optimal position of a bubble within the structure corresponds to the point of minimal value of the above function. Moreover, there are further ways of positioning for arbitrary objective and constraint functions.

3.3.1 Characteristic function for the positioning of the bubbles

For the derivation of the "characteristic function" for problems with global functionals (e.g. complementary energy and volume of the structure), we consider the following optimization problem of the mixed boundary value problem with the given stresses $t^j_{(\Gamma_\tau)} = (\tau^{ij} n_i)_{\Gamma_\tau}$ and the given deformations $v_{i(\Gamma_v)} = (v_i)_{\Gamma_v}$ (Banichuk 1990; Eschenauer and Schnell 1993).

Definition. The objective is to minimize the volume of an elastic body

$$\min_{\Gamma_{\text{var}}} \left(F_0 = \int_{\Omega} \mathrm{d}\Omega \right), \tag{6}$$

while satisfying the following integral relations as constraints

$$G_\nu = \int_{\Omega} g_\nu(\tau^{ij}, \gamma_{ij}, v_i)\, \mathrm{d}\Omega \leq G^0_\nu, \quad \nu = 1, 2, \ldots, N,$$

$$\longrightarrow G_\nu = \int_{\Omega} g_\nu(\tau^{ij}, \gamma_{ij}, v_i)\, \mathrm{d}\Omega = G^0_\nu - \mu^2_\nu, \tag{7}$$

where μ_ν are the slack variables.

The problem can be augmented by adding the known expressions for the equilibrium conditions, the strain-displacement relations and the constitutive equations

$$\tau^{ij}|_j + \rho f^i = 0\,, \tag{8a}$$

$$\gamma_{k\ell} = \frac{1}{2}(v_i|_j + v_j|_i)\,, \tag{8b}$$

$$\tau^{ij} = C^{ijk\ell}\gamma_{k\ell} \quad (C^{ijk\ell} - \text{stiffness matrix})\,. \tag{8c}$$

Thus, together with (6) and (7) we formulate a Lagrangian functional

$$J = \int_\Omega f_L(\tau^{ij}, \gamma_{ij}, v_i)\,\mathrm{d}\Omega + \sum_{\nu=1}^{N} \lambda_\nu(\mu_\nu^2 - G_\nu^0)\,, \tag{9}$$

where

$$f_L = 1 + \sum_{\nu=1}^{N} \lambda_\nu g_\nu(\tau^{ij}, v_i) + \Psi_i(\tau^{ij}|_j + \rho f^i) +$$

$$\chi_{ij}(\tau^{ij} - C^{ijk\ell}\gamma_{k\ell})\,,$$

with Ψ_i and χ_{ij} as adjoint functions (special Lagrangian multipliers of the mechanical problem), and λ_ν are the Lagrangian multipliers for the consideration of the constraints of the optimization problem. Here, the integrand f_L depends on the vector

$$\mathbf{u}_k = (v_1, v_2, v_3, \tau^{11}, \tau^{12}, \ldots, \tau^{33}, \Psi_1, \Psi_2, \Psi_3, \chi_{11}, \chi_{12}, \ldots,$$

$$\chi_{33}, \lambda_1, \ldots, \lambda_N)^T\,.$$

The vector $\mathbf{u}_k$ consists of 18 components for considering the mechanical basic equations and N Lagrangian multipliers. According to Courant and Hilbert (1968), the first variation of (9) for one of the vector components u_k (problem with variable domain) contains an additional term due to domain variation

$$\delta J_k = \int_\Omega \left(\frac{\partial f_L}{\partial u_k} - \frac{\partial}{\partial \xi^i}\frac{\partial f_L}{\partial u_{k|i}}\right)(\delta u_k)_\Gamma\,\mathrm{d}\Omega +$$

$$\int_\Omega \frac{\partial}{\partial \xi^i}\left[\frac{\partial f_L}{\partial u_{k|i}}(\delta u_k)_\Gamma\right]\mathrm{d}\Omega + f_L\delta\xi^i|_{\Omega_{\mathrm{var}}} +$$

$$2\sum_{\nu=1}^{N}(\lambda_\nu\mu_\nu)\delta\mu_\nu = 0\,, \tag{10}$$

where the Lagrangian consideration yields the following variation of $\mathbf{u}_k$:

$$(\delta\mathbf{u}_k)_\Gamma = (\delta\mathbf{u}_k - \mathbf{u}_{k|i}\delta\xi^i)\,. \tag{11}$$

For further computation, the following derivatives with regard to the components of $\mathbf{u}_k$ are calculated by means of (9):

$$\frac{\partial f_L}{\partial v_i} = \sum_{\nu=1}^{N} \lambda_\nu \frac{\partial g_\nu}{\partial v_i}, \quad \frac{\partial f_L}{\partial \gamma_{k\ell}} = -\chi_{ij}C^{ijk\ell}\,,$$

$$\frac{\partial f_L}{\partial \tau^{ij}} = \sum_{\nu=1}^{N} \lambda_\nu \frac{\partial g_\nu}{\partial \tau^{ij}} + \chi_{ij}, \quad \frac{\partial f_L}{\partial \tau^{ij}|_j} = \Psi_i\,. \tag{12a}$$

Furthermore, we formulate for f_L in the state of variation

$$f_L = 1 + \sum_{\nu=1}^{N} \lambda_\nu g_\nu(\tau^{ij}, v_i)\,. \tag{12b}$$

The two middle terms in (10) can be transformed using Green's formula (Courant and Hilbert 1968)

$$\int_\Omega \frac{\partial}{\partial \xi^i}\left[\frac{\partial f_L}{\partial u_{k|i}}(\delta u_k)_\Gamma\right]\mathrm{d}\Omega = \int_\Gamma \left(\frac{\partial f_L}{\partial u_{k|i}} n_i\right)(\delta u_k)_\Gamma\,\mathrm{d}\Gamma\,,$$

$$f_L\delta\xi^i|_{\Omega_{\mathrm{var}}} = \int_{\Gamma_{\mathrm{var}}} (f_L n_i)\delta\xi^i\,\mathrm{d}\Gamma\,. \tag{13}$$

After composing all single parts δJ_k to one and with the symmetry conditions $v_i|_j = v_j|_i$, we obtain

$$\delta J = \int_\Omega \left[\left(\sum_{\nu=1}^{N} \lambda_\nu \frac{\partial g_\nu}{\partial \tau^{ij}} + \chi_{ij}\right) - \frac{\partial}{\partial \xi^i}\Psi_i\right](\delta\tau^{ij} -$$

$$\tau^{ij}|_j\delta\xi^i)\,\mathrm{d}\Omega + \int_\Omega \left[\left(\sum_{\nu=1}^{N} \lambda_\nu \frac{\partial g_\nu}{\partial v_i}\right) - \frac{\partial}{\partial \xi^i}(-\chi_{ij}C^{ijk\ell})\right](\delta v_i -$$

$$\gamma_{ij}\delta\xi^i)\,\mathrm{d}\Omega + \int_\Gamma \Psi_i n_i(\delta\tau^{ij} - \tau^{ij}|_j\delta\xi^i)\,\mathrm{d}\Gamma +$$

$$\int_\Gamma (-\chi_{ij}C^{ijk\ell})n_i(\delta v_i - \gamma_{ij}\delta\xi^i)\,\mathrm{d}\Gamma +$$

$$\int_{\Gamma_{\mathrm{var}}} \left(1 + \sum_{\nu=1}^{N} \lambda_\nu g_\nu\right) n_i\delta\xi^i\,\mathrm{d}\Gamma + 2\sum_{\nu=1}^{N}(\lambda_\nu\mu_\nu)\delta\mu_\nu = 0\,. \tag{14}$$

Variation of (14) leads the following necessary conditions for the total problem:

$$\left(\sum_{\nu=1}^{N} \lambda_\nu \frac{\partial g_\nu}{\partial v_i} + \chi_{ij}|_i C^{ijk\ell}\right)_\Omega = 0\,,$$

$$\left(\sum_{\nu=1}^{N} \lambda_\nu \frac{\partial g_\nu}{\partial \tau^{ij}} + \chi_{ij} - \Psi_i|_j\right)_\Omega = 0\,,$$

$$(\Psi_i n_i)_{\Gamma_v} = 0, \quad (-\chi_{ij}C^{ijk\ell})_{\Gamma_\tau} = 0\,,$$

$$\left(1 + \sum_{\nu=1}^{N} \lambda_\nu g_\nu\right)_{\Gamma_{\mathrm{var}}} = 0, \quad \lambda_\nu\mu_\nu = 0\,,$$

$$(\delta\xi^i)_{\Gamma_{\tau,\mathrm{not-var}}} = 0, \quad (\delta\tau^{ij}n_i)_{\Gamma_{\tau,\mathrm{not-var}}} = 0\,,$$

$$(\delta\xi^i)_{\Gamma_{v,\mathrm{not-var}}} = 0, \quad (v_i)_{\Gamma_{v,\mathrm{not-var}}} = 0\,. \tag{15}$$

Here, we are not interested in developing analytical solutions to the total problem, because structures relevant to practice have become so expensive that an analytical, relatively simple solution of arbitrary problems is no longer possible. In the present case, however, we use the analytical variation for finding the optimal position of the new bubbles. For this purpose, we subdivide the variation of the whole problem δJ into variation over the stress boundary δJ_τ, and variation over the displacement boundary δJ_v

$$\delta J = \delta J_v + \delta J_\tau\,. \tag{16}$$

For variation over the boundary of the new bubbles δJ_γ, part of the variation δJ_τ, we need the special expression of δJ_τ. This special expression is a simple version of (14). The variation only depends on terms of the state at the variable boundary

$$\delta J_T = \int\limits_{\Gamma_{\mathrm{var}}} \left[\Psi_i n_i (\delta\tau^{ij} - \tau^{ij}|_j \delta\xi^i) + \right.$$

$$\left. \left(1 + \sum_{\nu=1}^{N} \lambda_\nu g_\nu \right) n_i \delta\xi^i \right] \mathrm{d}\Gamma . \tag{17}$$

The following boundary condition for an unloaded hole is assumed:

$$t^j_{(\Gamma_\gamma)} = (\tau^{ij} n_i)_{\Gamma_\gamma} = 0 . \tag{18}$$

By means of the scalar $\delta s = n_i \delta\xi^i$ we can formulate for the variation over the boundary of the new bubble δJ_γ:

$$\delta J_\gamma = \int\limits_\gamma \left[1 + \sum_{\nu=1}^{N} \lambda_\nu g_\nu - \frac{\partial}{\partial \xi^i} (\Psi_i \tau^{ij}) \right] \delta s \, \mathrm{d}\Gamma . \tag{19}$$

Now we can introduce various constraints into (19). In the scope of this paper we consider the mean compliance, which in this case corresponds to the complementary energy of the structure

$$g_1 \hat{=} \overline{U}^* = \frac{1}{2} \tau^{ij} \gamma_{ij} = \frac{1}{2} D_{ijk\ell} \tau^{ij} \tau^{k\ell}$$

$$(D_{ijk\ell} - \text{compliance matrix}) . \tag{20}$$

It is obvious that, apart from these global energy functionals, there are also local (pointwise) constraints like stresses or failure criteria in structural optimization which still have to be introduce in future works.

With $\frac{\partial g_1}{\partial v_i} = 0$ and $\frac{\partial g_1}{\partial \tau^{ij}} = D_{ijk\ell} \tau^{ij} = \gamma_{ij}$, the adjoint functions from (15) lead to $\chi_{ij} = 0$ and $\Psi_i = \lambda v_i$. Equation (19) can be re-written as

$$\delta J_\gamma = \int\limits_\gamma \left[1 + \lambda \frac{1}{2} D_{ijk\ell} \tau^{ij} \tau^{k\ell} - \frac{\partial}{\partial \xi^i} (\lambda v_i \tau^{ij}) \right] \delta s \, \mathrm{d}\Gamma =$$

$$\int\limits_\gamma \left(1 + \lambda \frac{1}{2} D_{ijk\ell} \tau^{ij} \tau^{k\ell} - \lambda D_{ijk\ell} \tau^{ij} \tau^{k\ell} \right) \delta s \, \mathrm{d}\Gamma =$$

$$\int\limits_\gamma \left(1 - \lambda \frac{1}{2} D_{ijk\ell} \tau^{ij} \tau^{k\ell} \right) \delta s \, \mathrm{d}\Gamma =$$

$$\int\limits_\gamma \delta s \, \mathrm{d}\Gamma - \lambda \int\limits_\gamma \overline{U}^* \delta s \, \mathrm{d}\Gamma = \delta V - \lambda \delta J_\gamma(\overline{U}^*) , \tag{21}$$

where the first term in (21) is constant. This is due to the fact that the penetration of a bubble takes place with an arbitrary but constant volume δV.

If a circular-shaped bubble in a thin disc (two-dimensional) is considered, we determine the following boundary stresses written in polar coordinates in dependence on the principal stresses (Eschenauer and Schnell 1993):

$$\sigma_{\varphi\varphi} = (\sigma_1 + \sigma_2) - 2(\sigma_1 - \sigma_2) \cos 2\varphi , \quad \sigma_{rr} = \tau_{r\varphi} = 0 . \tag{22}$$

The complementary energy yields

$$\overline{U}^* = \frac{1}{2} D_{\alpha\beta\gamma\delta} \tau^{\alpha\beta} \tau^{\gamma\delta} = \frac{1}{2E} \cdot \left[(\sigma_1 + \sigma_2)^2 + \right.$$

$$\left. 2(\sigma_1 - \sigma_2)^2 - 4(\sigma_1^2 - \sigma_2^2) \cos 2\varphi + 2(\sigma_1 - \sigma_2)^2 \cos 4\varphi \right] . \tag{23}$$

For further consideration, a splitting of the variation expression into

$$\delta J_\gamma(\overline{U}^*) = \Phi(\tau^{\alpha\beta}) \delta V \tag{24}$$

proves useful, where $\delta V = 2\pi r_B t \, \delta s$ is the variation of the volume of the bubbles (t = thickness of the disc) and $\Phi(\tau^{\alpha\beta})$ is the so-called "characteristic" function that depends on the stress state in the component

$$\Phi(\tau^{\alpha\beta}) = \Phi(\sigma_1, \sigma_2) = \frac{1}{2E} \left[(\sigma_1 + \sigma_2) + 2(\sigma_1 - \sigma_2)^2 \right] . \tag{25}$$

In our case, the characteristic function only depends on the principal stresses in two directions of the disc. Note that for $\nu = 1/3$, the first characteristic function assumes the expression of the elastic energy density.

In (24), δV is a virtual, infinitesimally small, but finite value. According to (16), δJ_γ must tend to 0 like the other variational values δJ_T and δJ_v, i.e. the function Φ, which depends on the position coordinates x, y, should possibly attain 0 and in any case becomes very small. This means that the function Φ must attain a minimum from which the position vector $\mathbf{r}$ can be determined.

Thus

$$\delta J_\gamma(\overline{U}^*) \longrightarrow \varepsilon \quad (\varepsilon \text{ is a very small value}) .$$

Now, we have the coordinates of the position to set the bubble in the right position in the structure.

4 Numerical implementation

The bubble method is numerically realized by means of the optimization procedure SAPOP (Structural Analysis Program and Optimization Procedure) (Eschenauer *et al.* 1993). In order to carry out topology optimization using the bubble method, SAPOP had to be augmented by the positioning model (Step 2 in Fig. 3). Figure 5 presents the flow-chart of the bubble method.

Changes of the geometry during the optimization, i.e. the insertion of bubbles into the structure, are thus carried out on the solid model, and the FE-mesh is produced in each iteration by a free mesh-generator. If the positioning criterion in the bubble method decides to form an open hole (bubble) at the boundary of the contour, a notch is created. If, on the other hand, the hole is positioned within the component, the method generates a closed hole (bubble) (see Fig. 6).

The potential of applying the bubble method depends on the work in classical shape optimization. SAPOP offers a large number of different optimization strategies and algorithms which facilitate an effective application of the bubble method. In their application to the bubble method, the Reduced Gradient Procedure (GRG) (Lasdon 1982) and in combination with an SQP procedure (Parkinson and Wilson 1986) and the dual algorithm "Method of Moving Asymptotes (MMA)" (Fleury and Braibant 1986; Svanberg 1987) proved to be highly efficient.

5 Application examples

5.1 *Topology optimization of a cantilever disc*

In order to achieve a simple scheme of positioning and shape optimization of a cantilever disc, certain approach functions are selected, which should have a limited number of design variables (see Fig. 7).

If, with the above standard definition of the approach functions, the positioning criterion for a bubble coincides

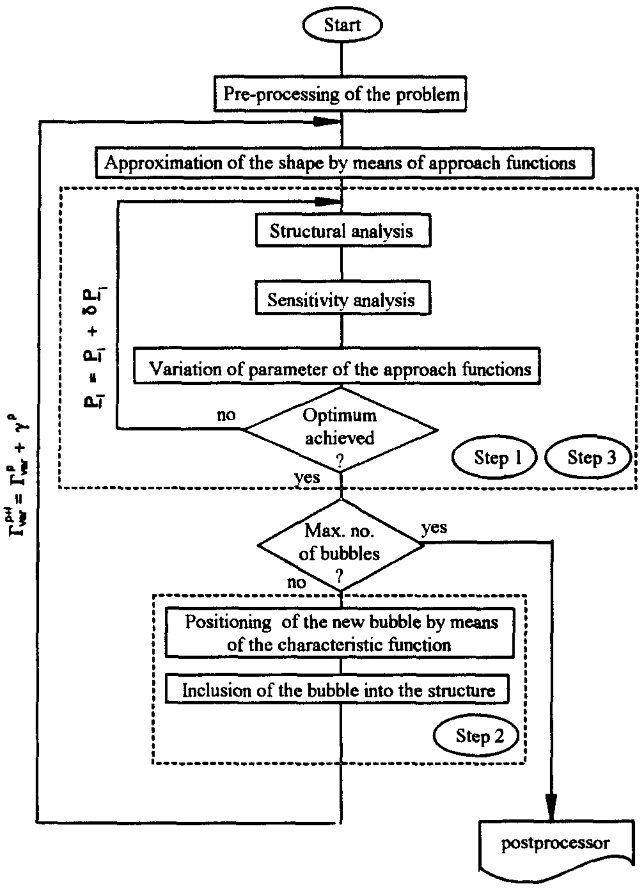


Fig. 5. Flow-chart of the bubble method

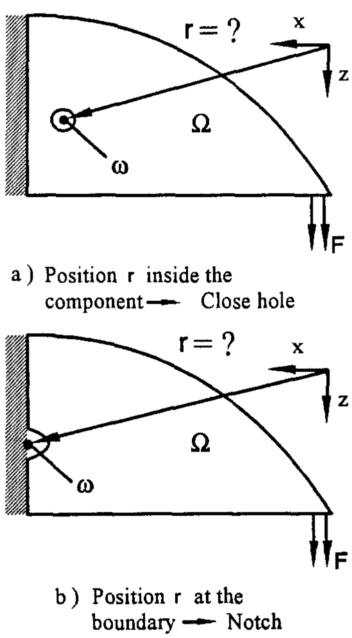


Fig. 6. Positioning of the bubble

with a boundary of the component, further variation possibilities of the approach functions must be made available for this point. For that purpose, the coordinates of the control points that are dependent on the other design variables are transformed into design variables themselves. If necessary, weightings of the control points or the knot vector of the approach function (see Fig. 4) can also be varied.

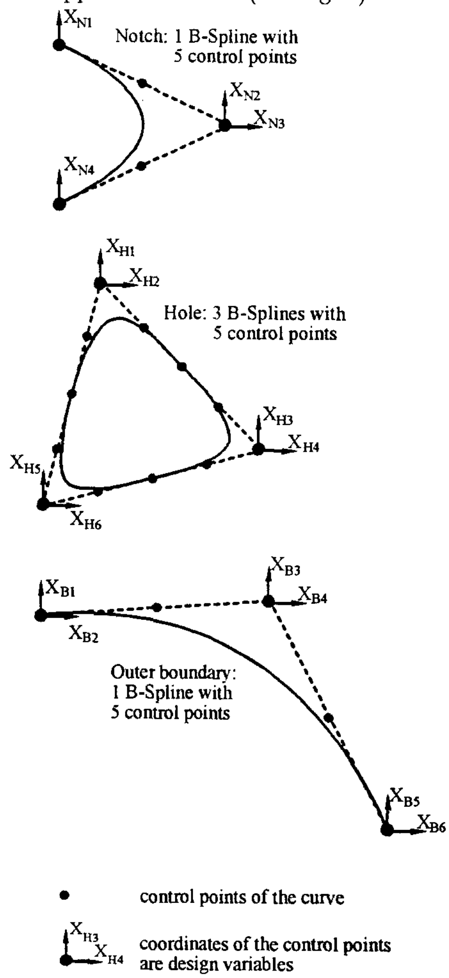


Fig. 7. Standard definition of the approach functions and the design variables of different features ($\mathbf{t} = \{0,0,0,1,2,3,4,5,5,5\}$, $\mathbf{w} = \{1,1,1,1,1,\}$)

A first simple task is to find a best-possible initial design for a prescribed topology domain (Fig. 8). In this case, the problem reads as follows:

$$\min_{\mathbf{x}\in \mathbb{R}^n} \left[U^*(\mathbf{x}) | V(\mathbf{x}) = \frac{1}{2} V_{\text{domain}}, \mathbf{h}(\mathbf{x}) = 0\right], \tag{27}$$

with $U^*(\mathbf{x}) \mathrel{\hat{=}}$ complementary energy, $V(\mathbf{x}) \mathrel{\hat{=}}$ volume of the structure, and $\mathbf{h}(\mathbf{x}) \mathrel{\hat{=}}$ vector of q equality constraints.

The component structure is allowed to develop within the given bounds of the topology domain. The structure is clamped on the left-hand side and is subject to a load on the right-hand side. The optimization problem consists of a minimization of the complementary energy of the structure ($\mathrel{\hat{=}}$ minimization of the mean compliance) while considering a volume constraint and equality conditions. The initial design in the half-filled topology domain is a triangular disc. If a shape optimization is carried out on this structure, a shape as presented in genus 1 is achieved. If a bubble is positioned at the point of minimum of the characteristic function (in the middle of the fixing bound) and a new shape optimization is carried out, we achieve genus 2. By positioning the next bubble at the point of minimum of the characteristic function, genus 3 is created. In the present example, the optimization is terminated at this point.

Further studies of the convergence behaviour are necessary. One possible way of achieving this aim is the application of termination criteria, since they are already used in the hierarchically secondary shape optimization (e.g. improvement of the objective function $\Delta f^{(i)} = |f^{(i)} - f^{(i-1)}| \leq \varepsilon_f, \varepsilon_f$ is a small value of demanded improvement pf the objective function).

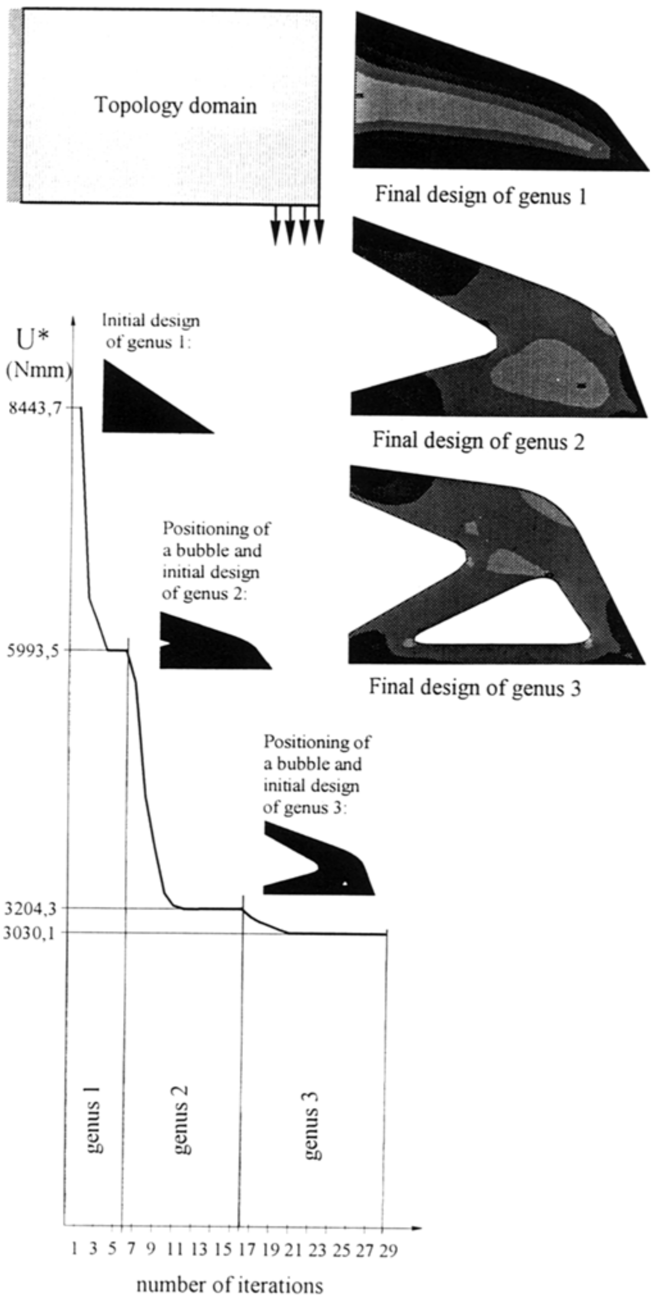


Fig. 8. Topology optimization of a cantilever disc

5.2 Topology optimization of a panel structure

A further application example is the topology optimization of a panel rib (Eschenauer and Schumacher 1993b) (Fig. 9). The first topology domain has a depth of 150 mm (version 1), whereas the second topology domain has a depth of 250 mm (version 2). According to (26), we minimize the complementary energy, considering a volume constraint (volume of the structure = volume of the initial structure). With constant volume, the mean compliance of the structure can be reduced to 39% (starting from 100%).

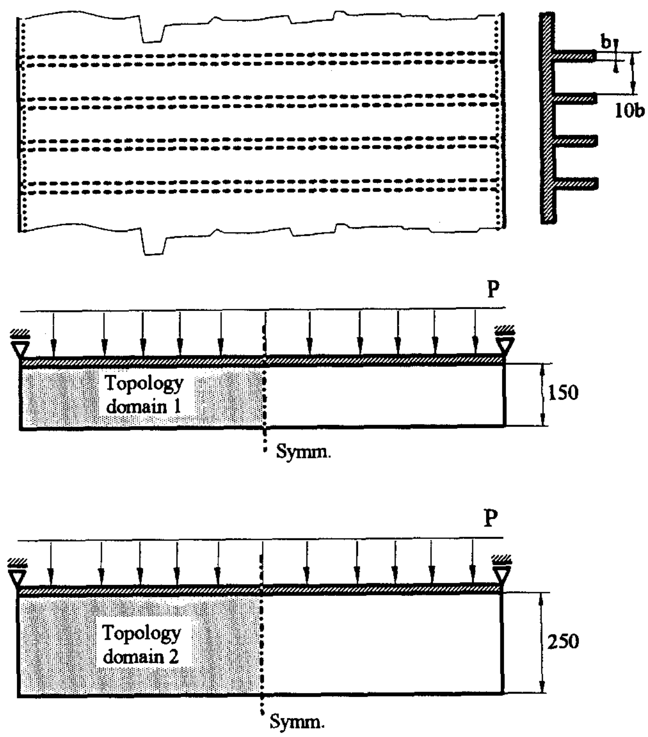


Fig. 9. Topology domains of the panel rib

The initial design of the first problem (version 1) is a retangular disc. If shape optimization is carried out on this structure, the shape shown in Fig. 10 is achieved. If a bubble is positioned at the point of minimum of the characteristic function and a new shape optimization is carried out, we achieve genus 2. By positioning the next bubble at the point of minimum of the characteristic function, genus 3 and 4 are created. The next bubble must be positioned at the boundary of the first bubble. For further calculations, the approach functions of the bubbles must be improved (more control points, variation of the weight of the control points, variation of the knot vector of the approach function).

The topology optimization of version 2 is illustrated in Fig. 11. The optimization results are much better compared to version 1.

6 Conclusions

The research for the bubble method in the topology optimization of component structures is still in its infancy. A number of further ideas call for increased research activities in this field.

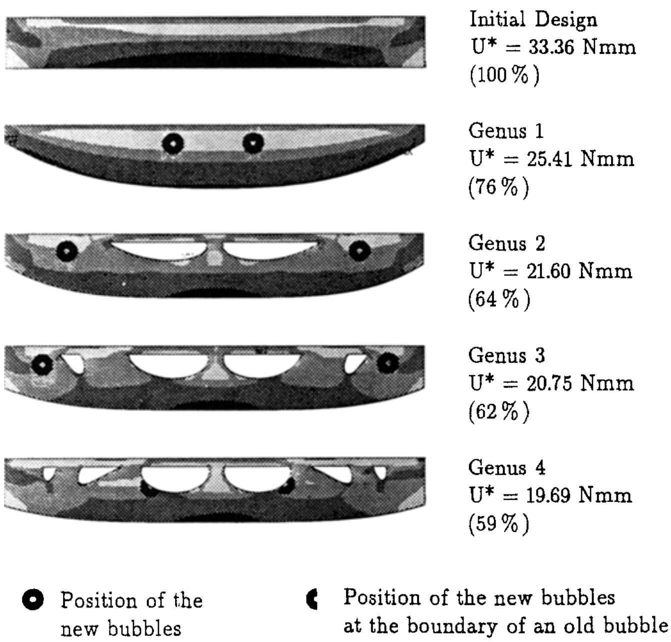


Fig. 10. Topology optimization of the panel rib (version 1)

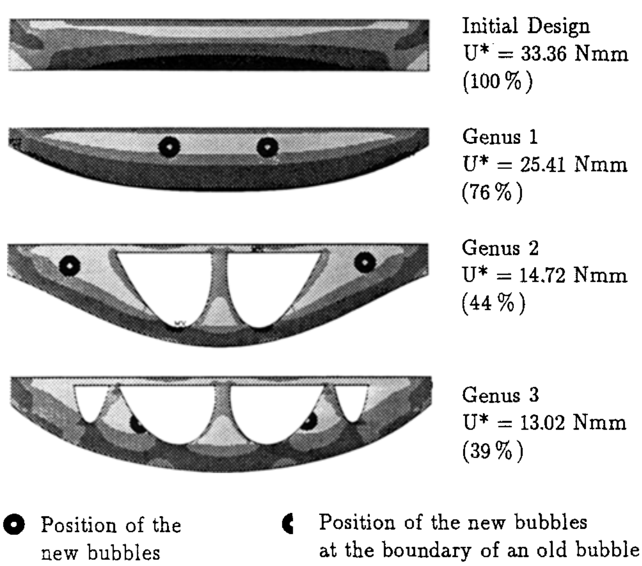


Fig. 11. Topology optimization of the panel rib (version 2)

Here, major emphasis is put on the positioning of the bubble in the structure considering various optimization functionals. In order to minimize local functions such as the equal stress, there are optimality criteria for the positioning of the bubble. Furthermore, methods where the bubble is directly positioned are being investigated. In this type of procedure, a bubble is inserted at an arbitrary initial position in the structure and is then "guided" to the optimal position by means of mathematical programming methods.

Other investigations deal with the positioning of bubbles with a non-circular shape – like an ellipse with a defined orientation, because the optimal position of a bubble depends on its shape.

Further research on the bubble method will aim at its augmentation to plate and shell problems.

Acknowledgement

The authors express their thanks to the German Research Foundation (DFG) for the support of the research project "Topology Optimization of Components using Hole Positioning Criteria" (DFG - No. ES 53/8-1).

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Received Dec. 15, 1993