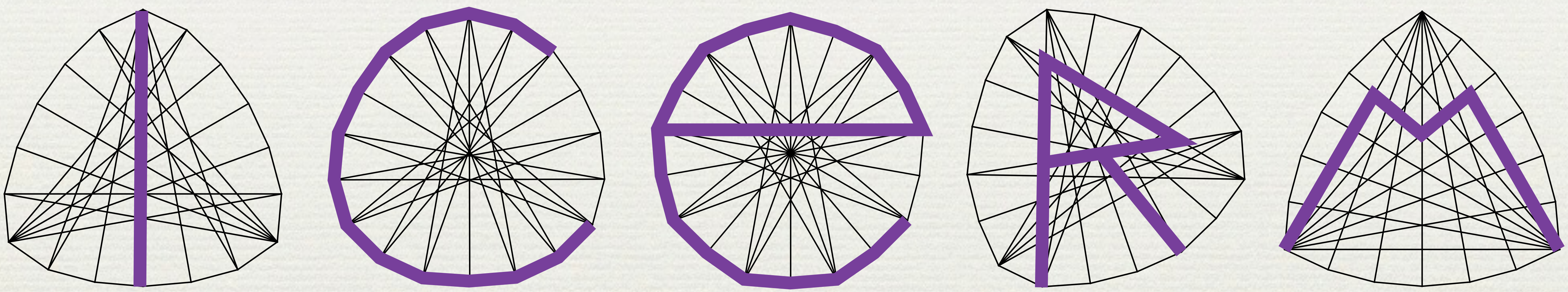

ICERM

I ❤ Reinhardt Polygons

Michael Mossinghoff
Davidson College



Introduction to Topics
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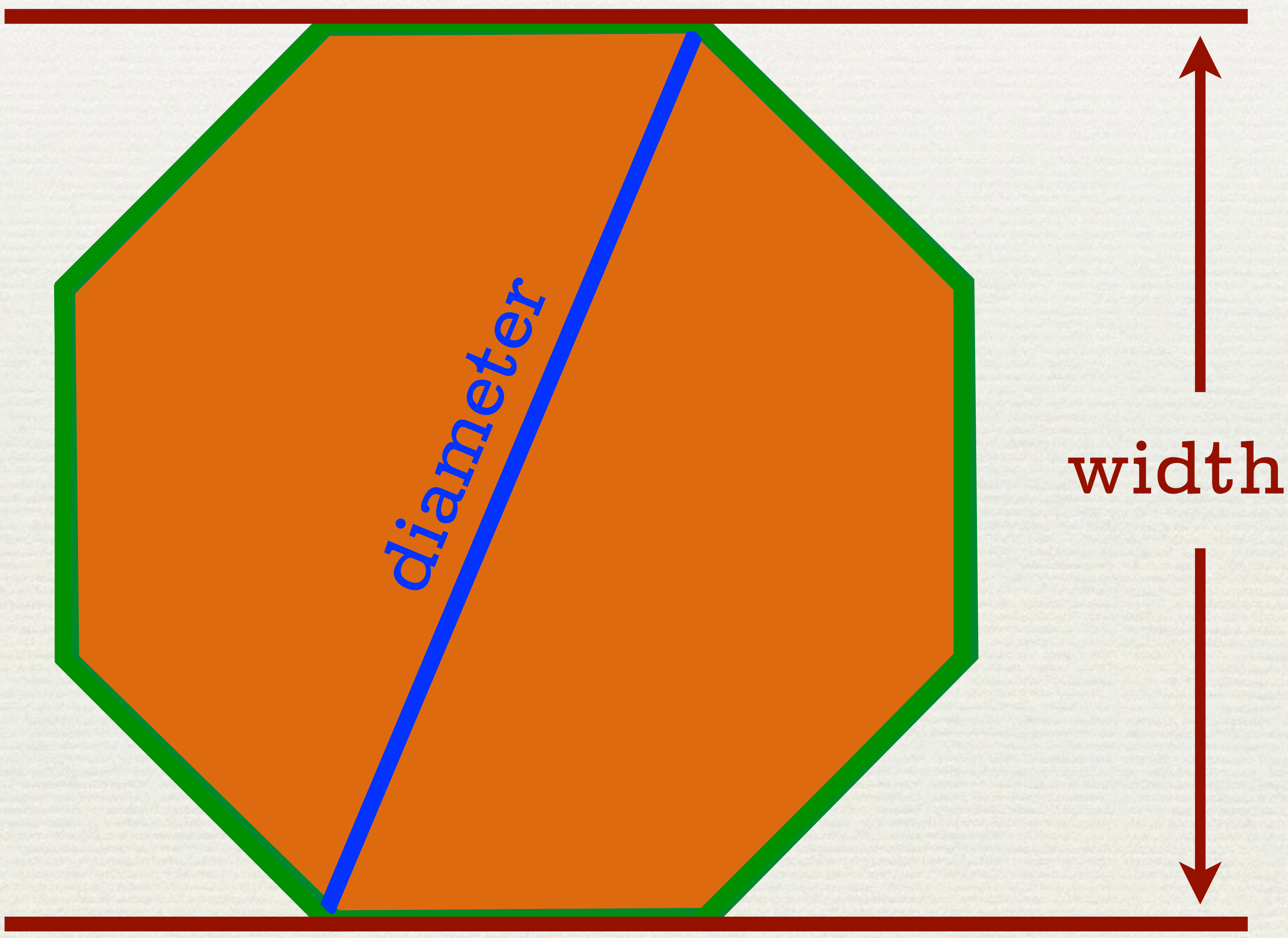
Brown University

Quantities of Interest

- For a convex polygon P, several quantities:
- Area, A.
- Perimeter, L.
- Diameter, d.
- Width, w.



Some Problems on Polygons

- P a convex polygon in the plane.
- Fix number of sides, n.
- Fix one of area, perimeter, diameter, and width, and optimize another.
- Produces six nontrivial problems.
- Isoperimetric problem: regular n-gon is the unique solution.

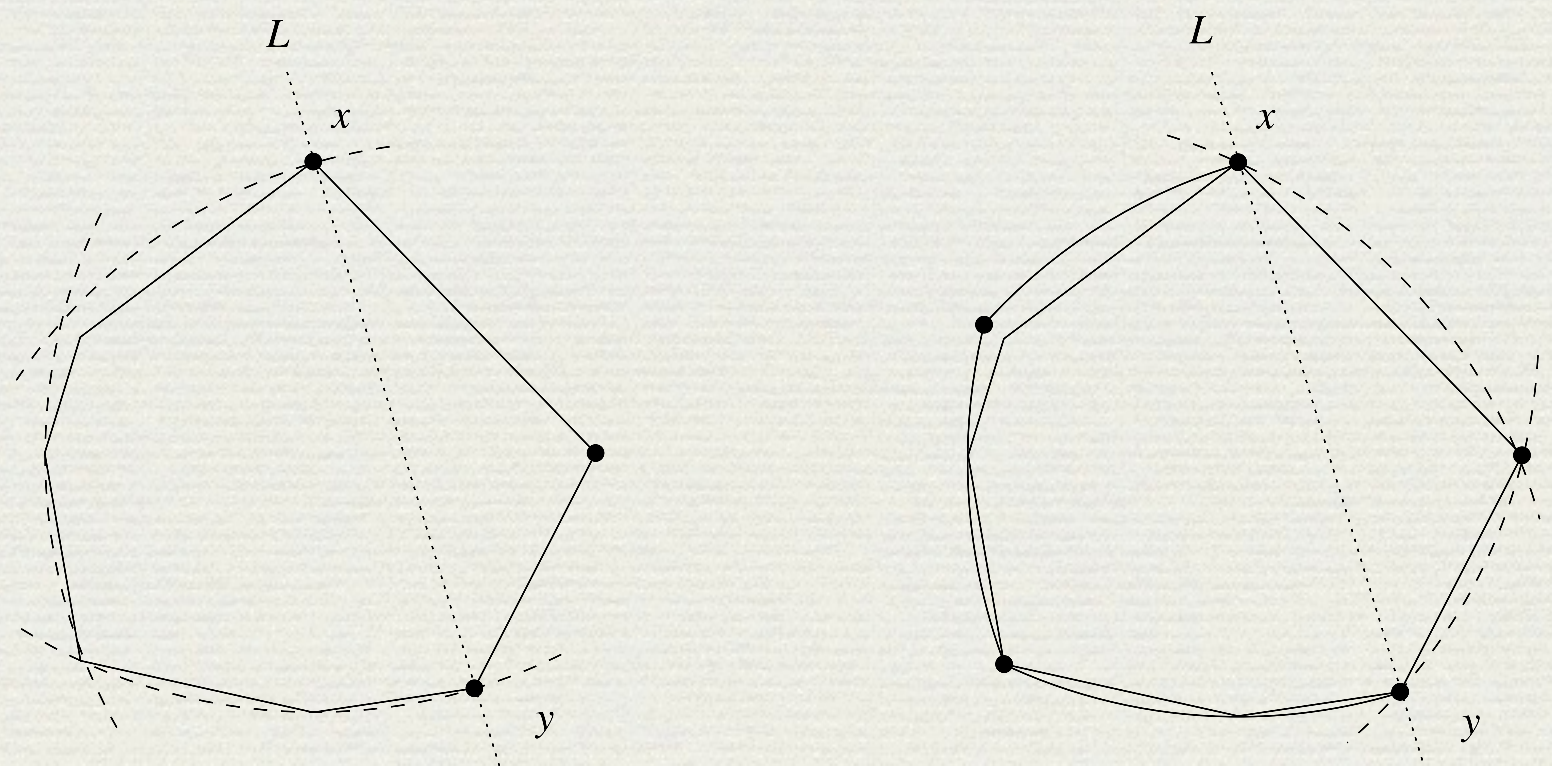
Three Extremal Problems

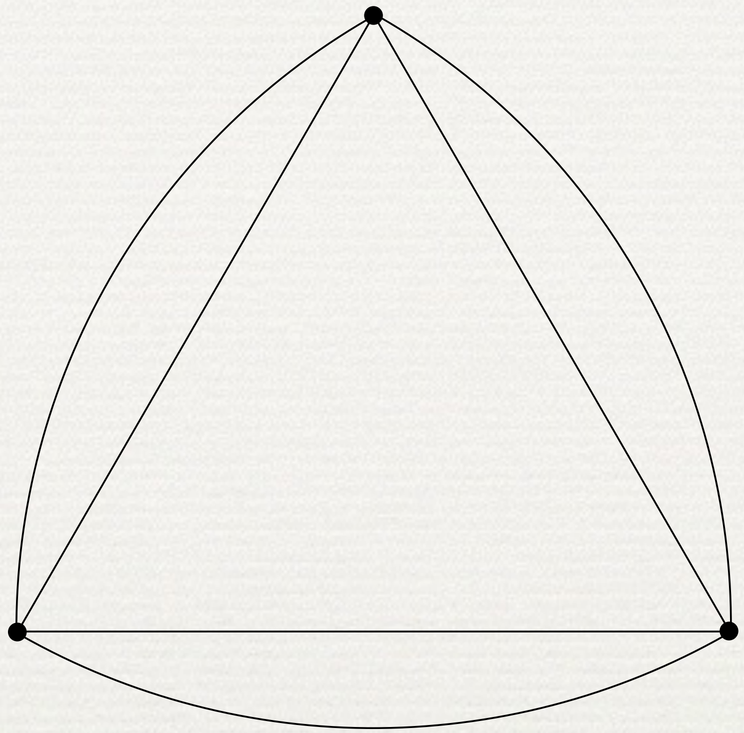
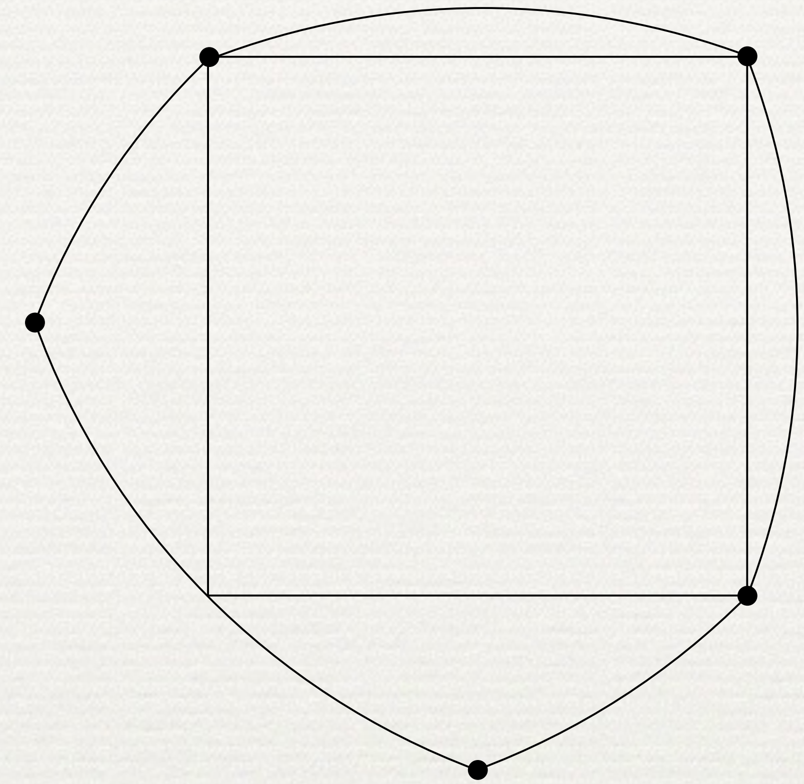
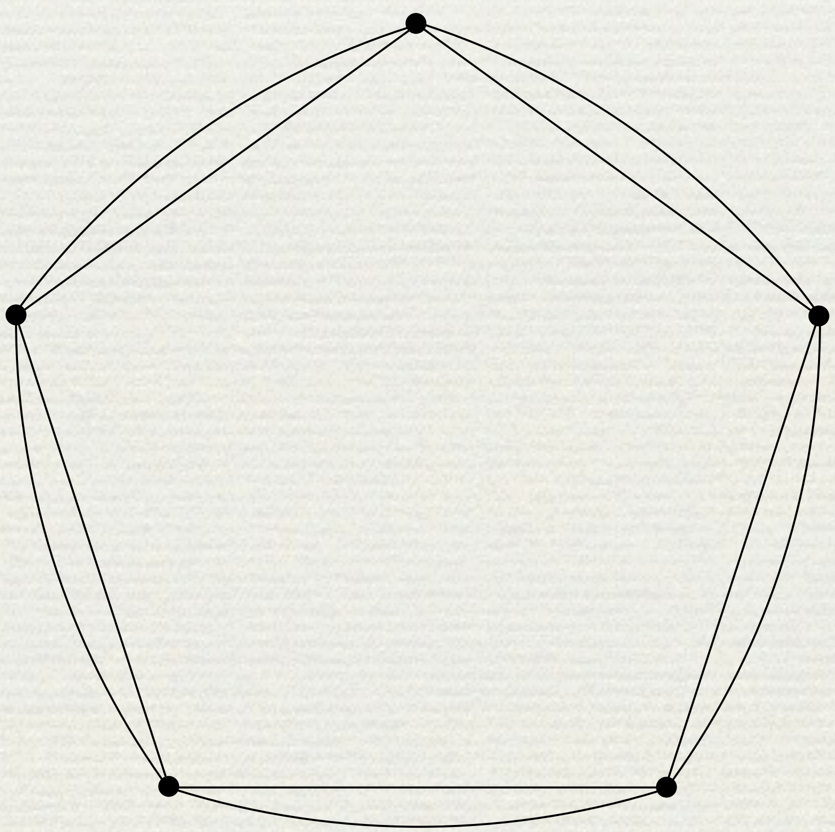
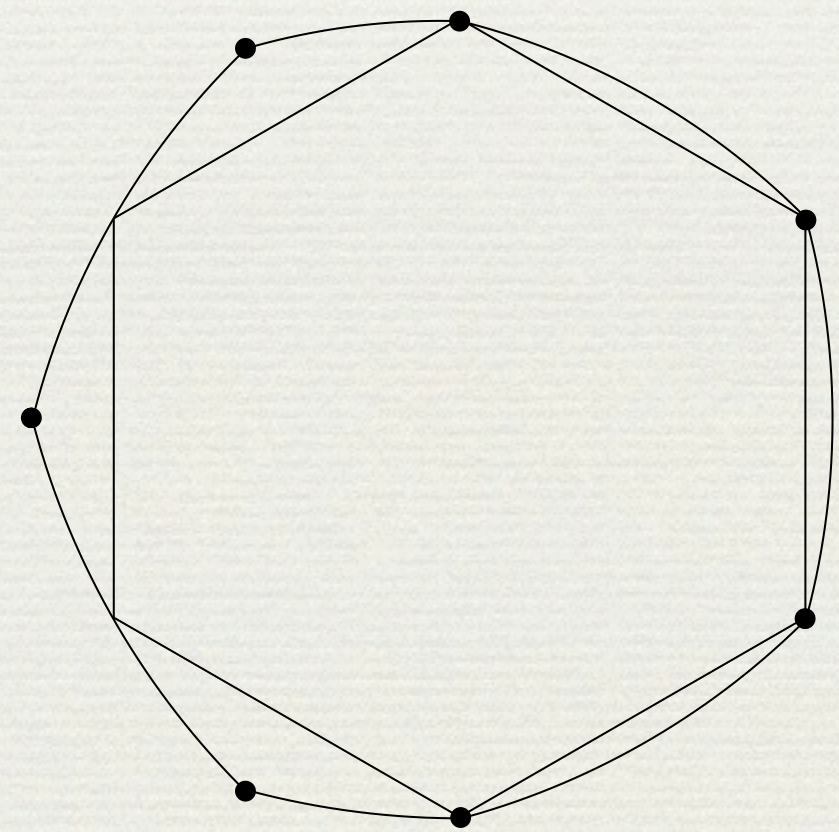
- *Fix diameter, maximize perimeter.*
  - Reinhardt (1922), Vincze (1950), Larman & Tamvakis (1984), Datta (1997).

- *Fix diameter, maximize width.*
  - Bezdek & Fodor (2000).
- *Fix perimeter, maximize width.*
  - Audet, Hansen, & Messine (2009).
- When $n \neq 2^m$, precisely the same polygons are optimal in all three problems: *Reinhardt polygons*.

Reuleaux Polygons

- Convex planar region bounded by a finite number of circular arcs of the same radius.
- Constant width.
- Perimeter $= \pi d$.
- If P has diameter d, then there exists a Reuleaux polygon with diameter d containing P.



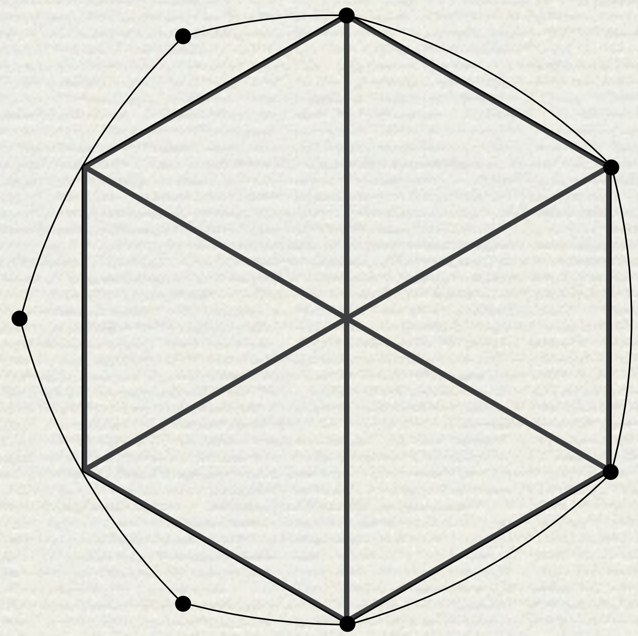





- Reuleaux polygons have an odd number of vertices.

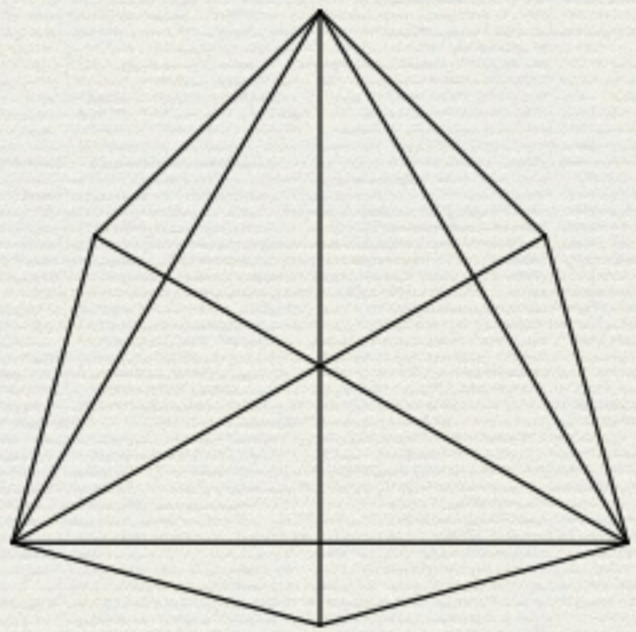
Spotting Reuleaux Polygons

Reinhardt Polygons

- Equilateral.
- If all vertices of P at maximal distance are connected, then a cycle occurs (star polygon).
    - I.e., P may be inscribed in a Reuleaux polygon R with the property that every vertex of R is a vertex of P.

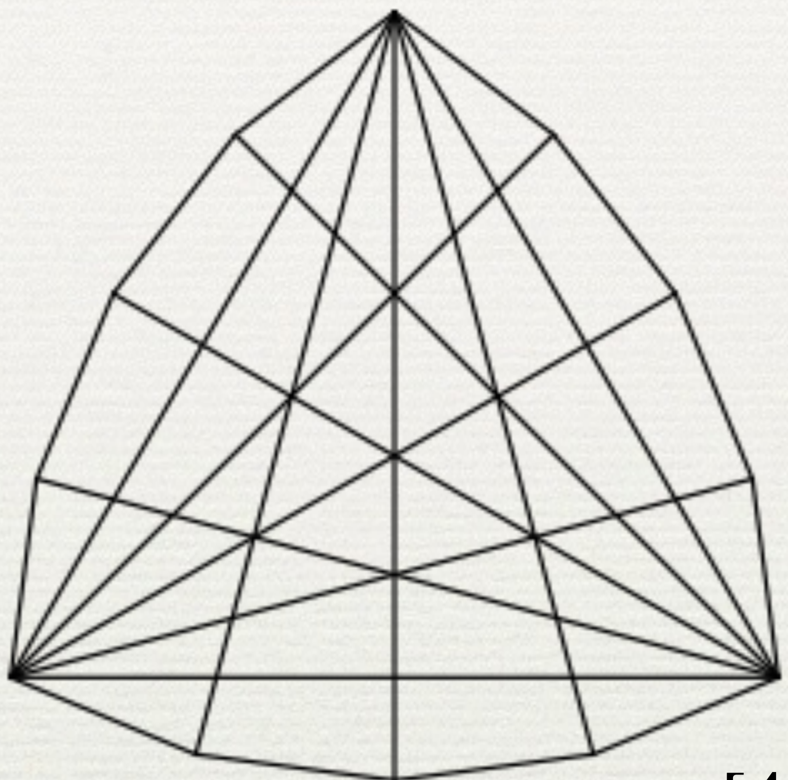


No!

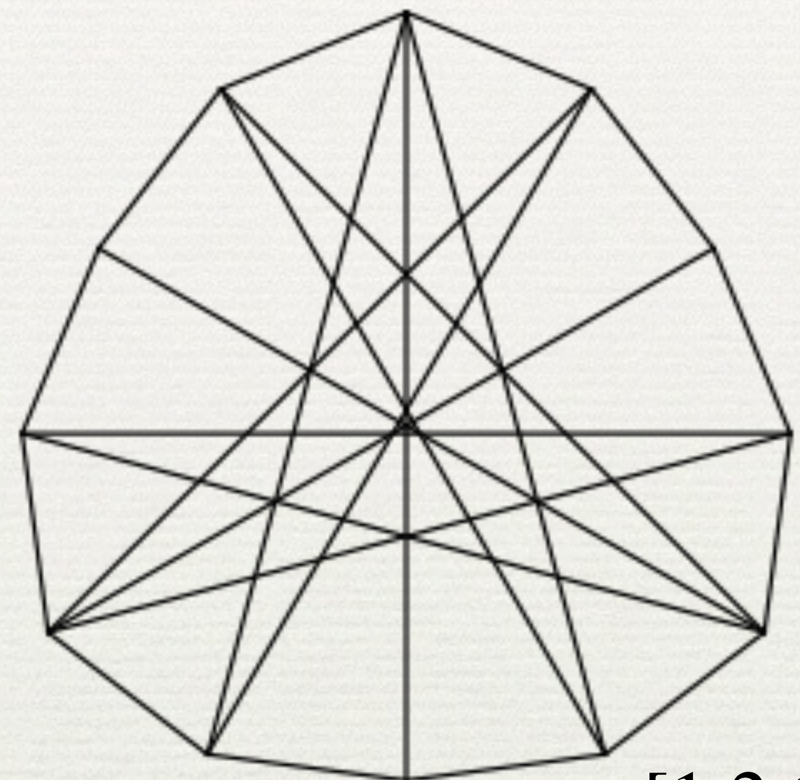


Yes!

- $n = 12$: Two Reinhardt polygons.



[4,4,4]



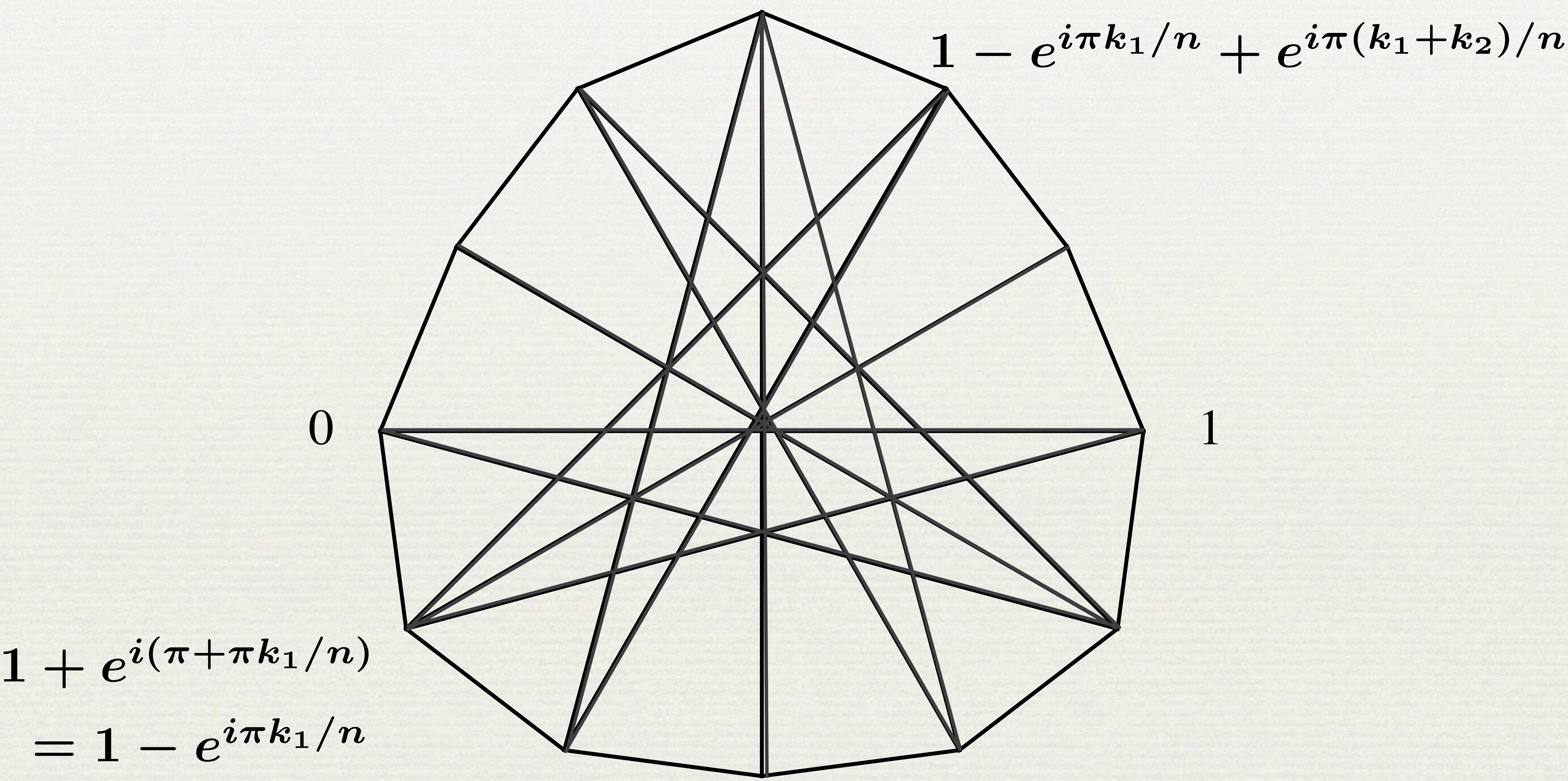
[1,2,1,1,2,1,1,2,1]

- Each interior angle of the star polygon is an integer multiple of π/n.

- How many Reinhardt polygons are there for fixed n?

- Dihedral equivalence classes.

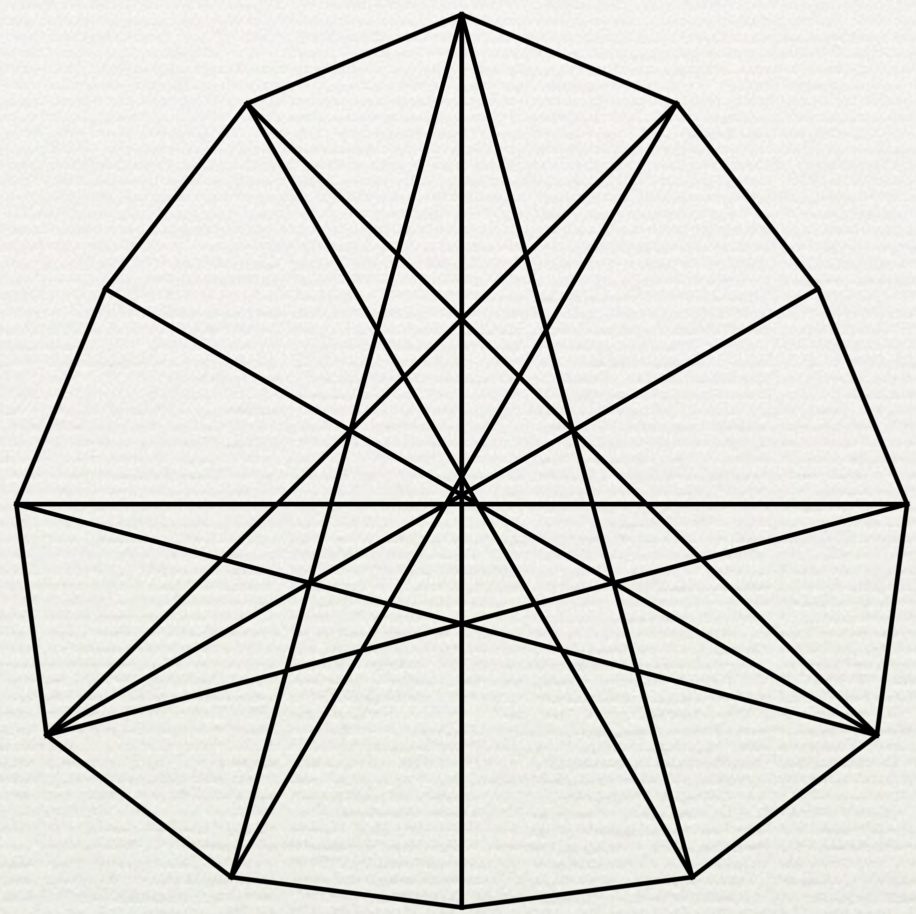
Example: Construct P for $[1, 2, 1, 1, 2, 1, 1, 2, 1]$.



Requires:

$$\mathbf{1} - e^{i\pi k_1/n} + e^{i\pi(k_1+k_2)/n} - \cdots + e^{i\pi(k_1+\cdots+k_{r-1})/n} = \mathbf{0}.$$

Example: Construct P for $[1, 2, 1, 1, 2, 1, 1, 2, 1]$.



$$1 - e^{i\pi/12} + e^{3i\pi/12} - e^{4i\pi/12} + e^{5i\pi/12}$$
$$- e^{7i\pi/12} + e^{8i\pi/12} - e^{9i\pi/12} + e^{11i\pi/12} = 0.$$

$$1 - z + z^3 - z^4 + z^5 - z^7 + z^8 - z^9 + z^{11}$$
$$= (z^3 - z + 1)\Phi_{24}(z).$$

Cyclotomic Polynomials

$$\Phi_n(z) = \frac{z^n - 1}{\prod_{\substack{d|n \\ d \neq n}} \Phi_d(z)}.$$

$$\Phi_1(z) = z - 1,$$

$$\Phi_2(z) = \frac{z^2 - 1}{z - 1} = z + 1,$$

$$\Phi_3(z) = \frac{z^3 - 1}{z - 1} = z^2 + z + 1,$$

$$\Phi_4(z) = \frac{z^4 - 1}{(z - 1)(z + 1)} = z^2 + 1,$$

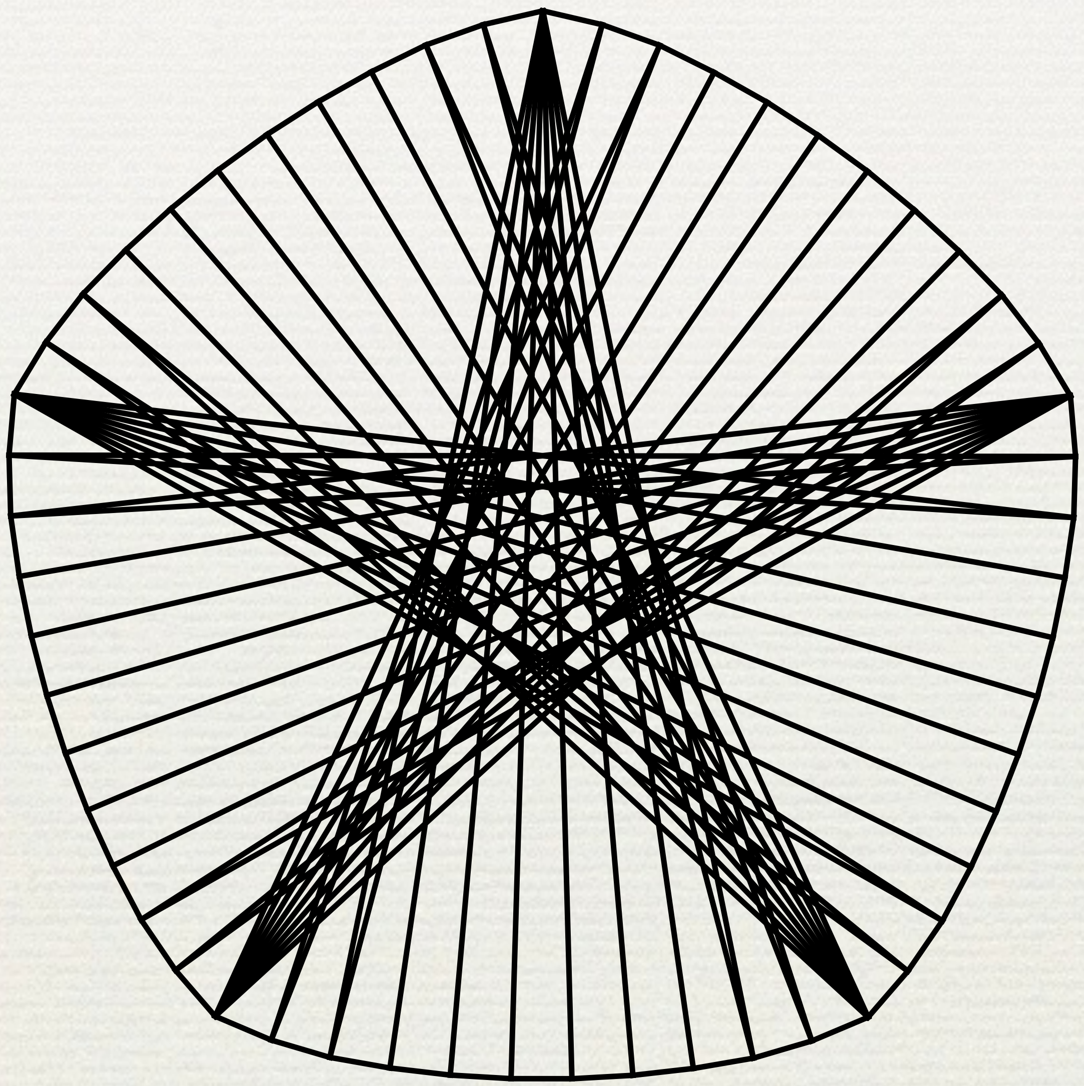
$$\Phi_p(z) = 1 + z + \cdots + z^{p-1}.$$

For $n > 1$ odd: $\Phi_{2n}(z) = \Phi_n(-z)$.

Equivalent Polynomial Problem

- A Reinhardt polygon corresponds to a polynomial $F(z)$ satisfying:
  - $\deg(F) < n$.
  - $F(0) = 1$.
  - Nonzero coefficients of F alternate ± 1.
  - Odd number of terms.
  - $F(e^{i\pi/n}) = 0$, i.e., $\Phi_{2n}(z) \mid F(z)$.

Example: $n = 55$

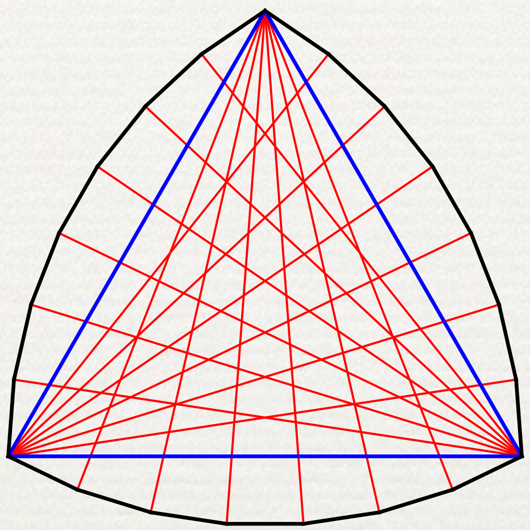


$$z^{54} - z^{53} + z^{52} - z^{51} +$$
$$z^{44} - z^{43} + z^{42} - z^{41} +$$
$$z^{40} - z^{33} + z^{32} - z^{31} +$$
$$z^{30} - z^{29} + z^{22} - z^{21} +$$
$$z^{20} - z^{19} + z^{18} - z^{11} +$$
$$z^{10} - z^{9} + z^{8} - z^{7} + 1$$

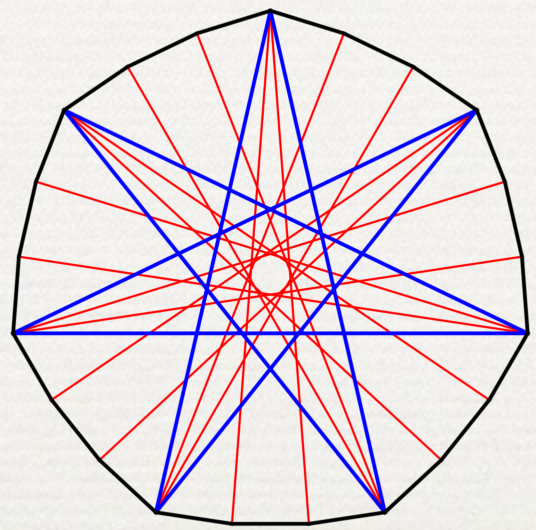
$$[7,1,1,1,1,7,1,1,1,1,7,1,1,1,1,7,1,1,1,1,7,1,1,1,1]$$

$$= [(7,1,1,1,1)^5].$$

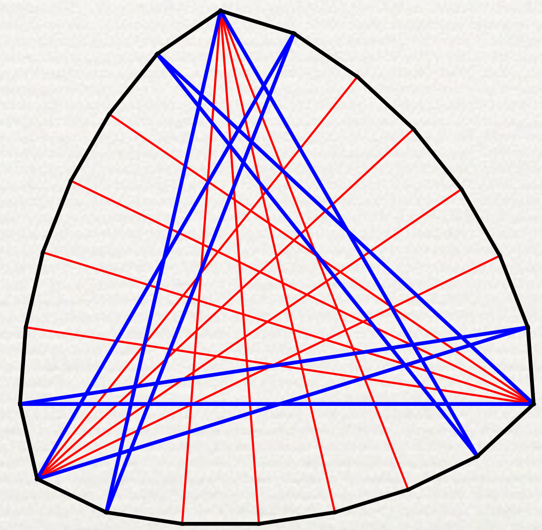
$n = 21$: Reinhardt Henicosagons



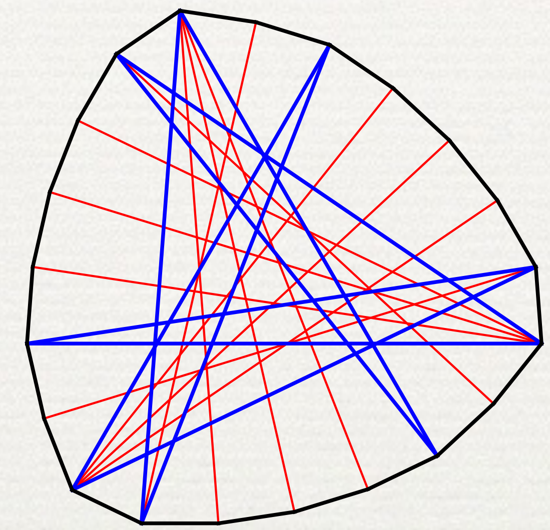
$[(7)^3]$



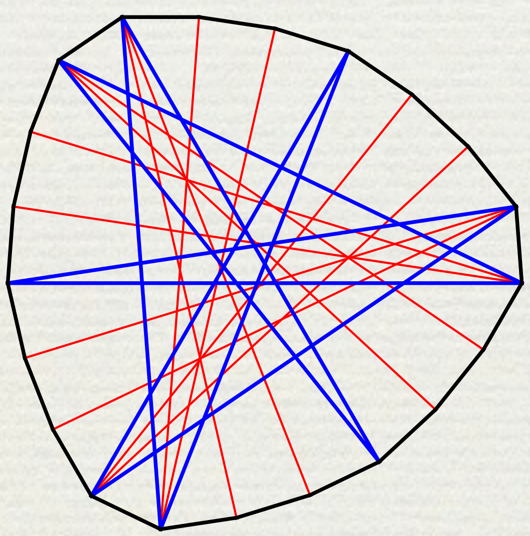
$[(3)^7]$



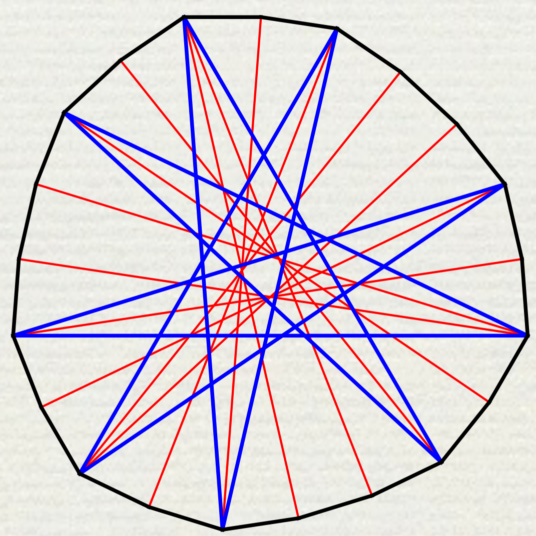
$[(5,1,1)^3]$



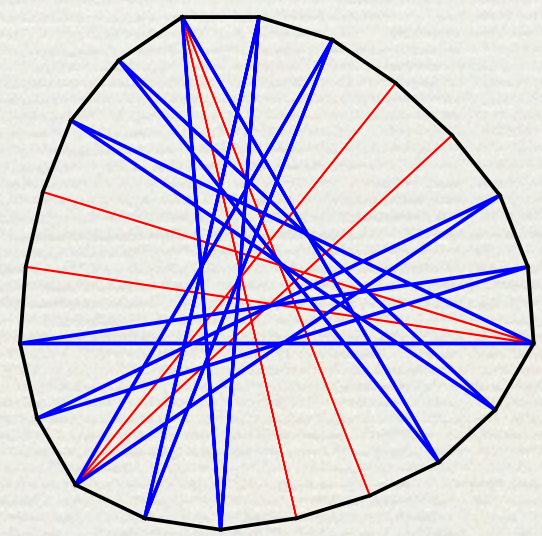
$[(4,2,1)^3]$



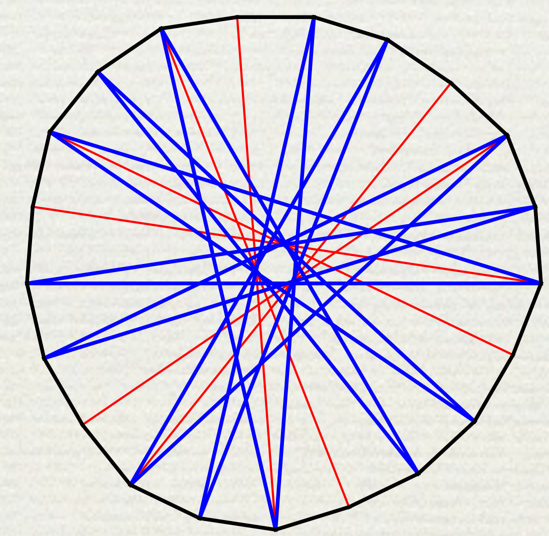
$[(3,3,1)^3]$



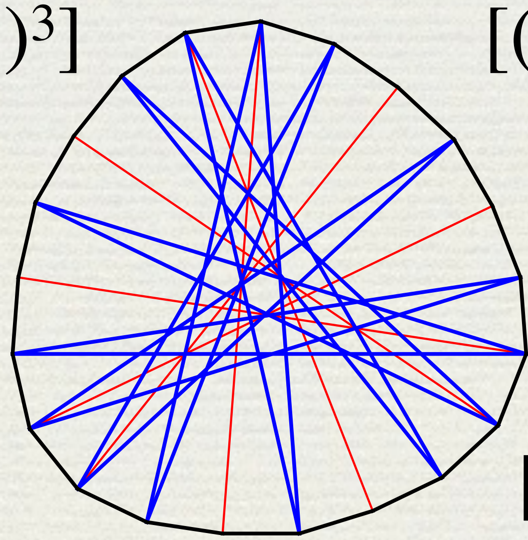
$[(3,2,2)^3]$



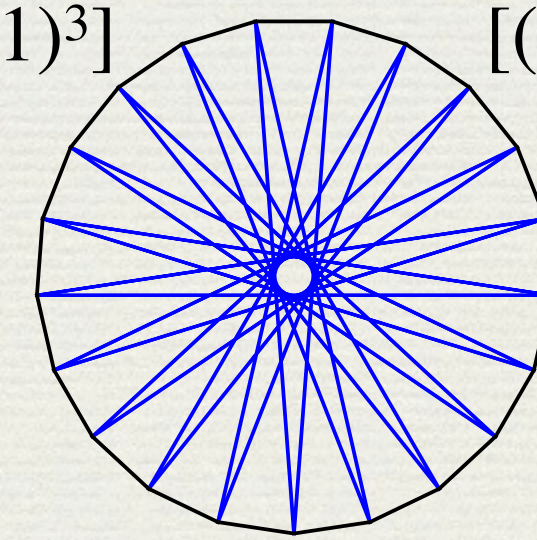
$[(3,1,1,1,1)^3]$



$[(2,2,1,1,1)^3]$



$[(2,1,2,1,1)^3]$



$[(1)^{21}]$

Compositions

- *Composition* of n = <u>sequence</u> of positive integers whose sum is n.

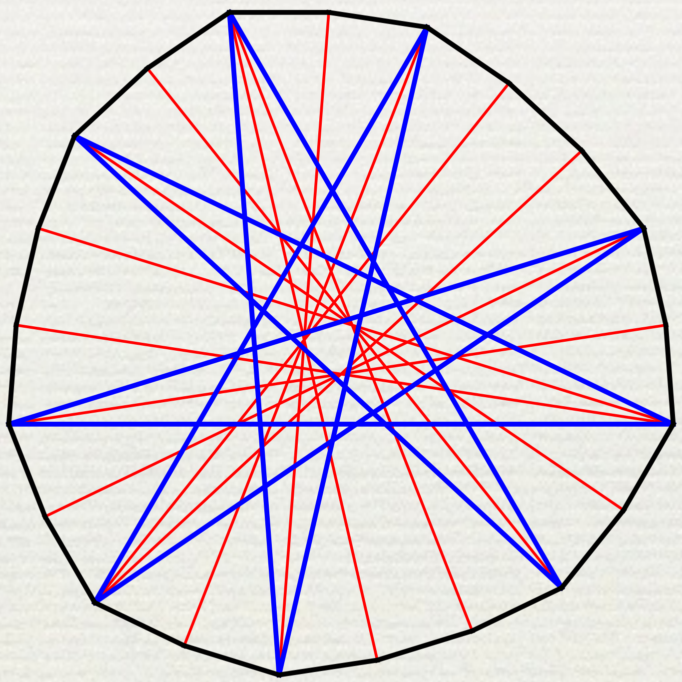
- Number of compositions of n is 2^{n-1}.

- *Partition* of n = equivalence class of compositions under action by the symmetric group.

- *Dihedral composition*: equivalence class of compositions under action by the dihedral group.

Dihedral

Reinhardt Polygon → Composition of n into an odd number of parts



→ $[(3,2,2)^3]$

- Not every dihedral composition with an odd number of parts produces a Reinhardt polygon.

- **Theorem:** Every *periodic* such composition does.

- Let $E_0(n)$ = number of *periodic* Reinhardt n-gons.
- So $E_0(n)$ = number of periodic dihedral compositions of n into an odd number of parts.

Theorem: (M., 2011) Let $n \neq 2^m$. Then

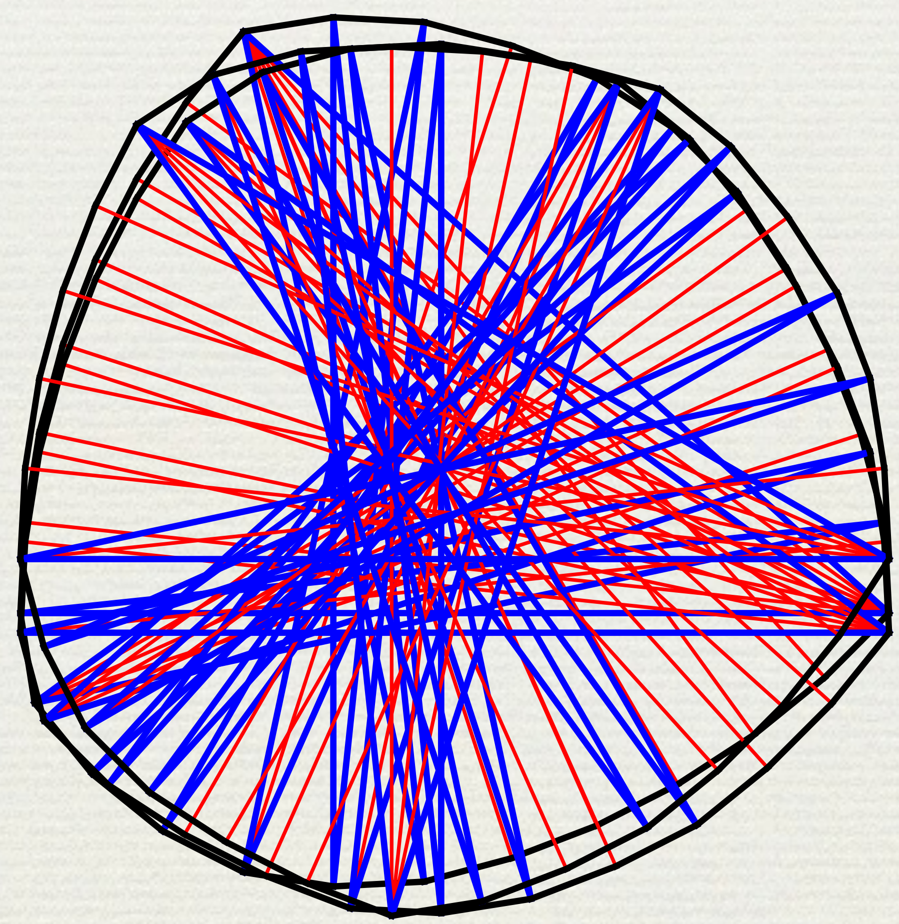
$$E_0(n) = \sum_{\substack{d \mid n \\ d>1}} \mu(2d) D(n/d),$$

where

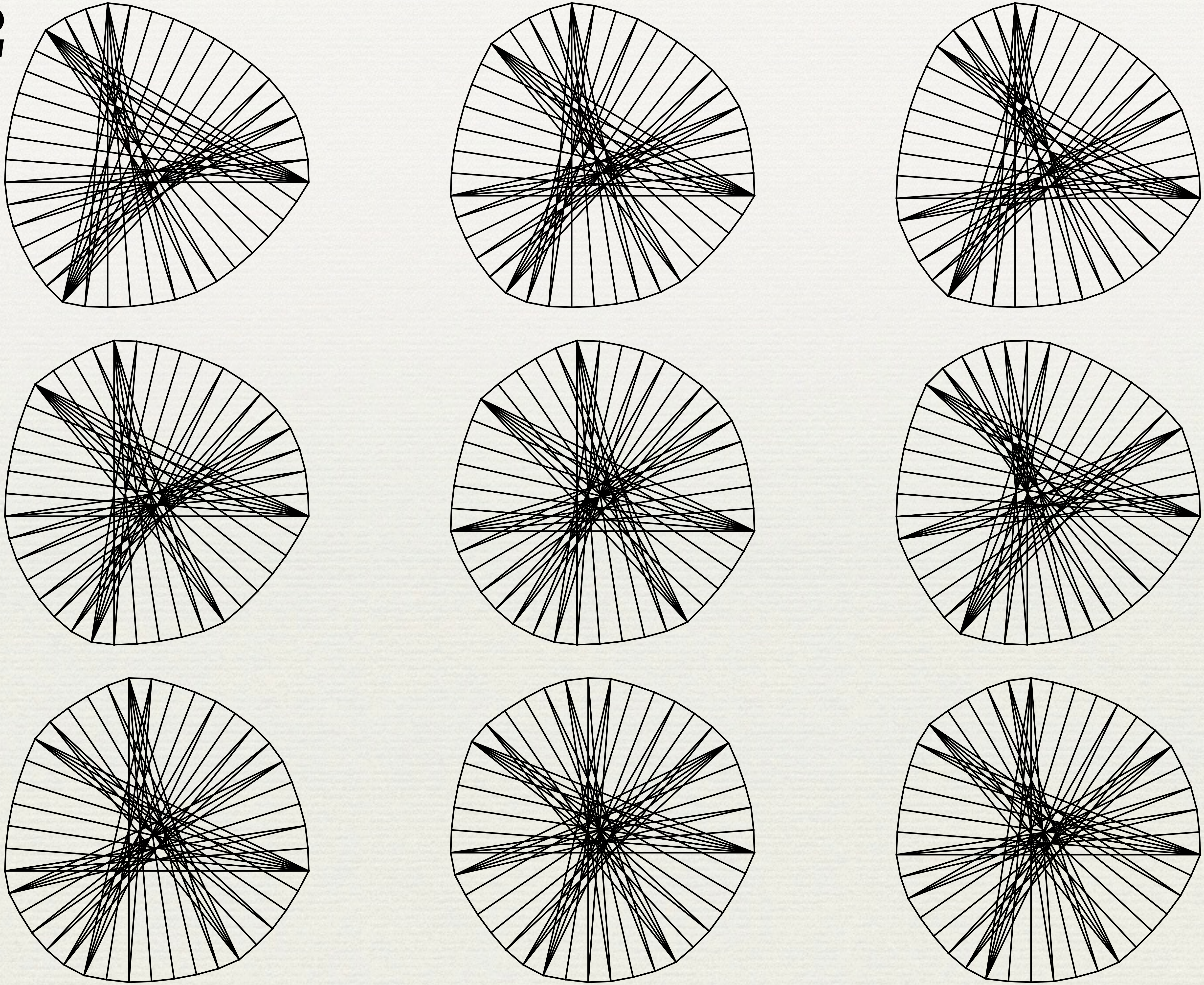
$$D(m) = 2^{\lfloor (m-3)/2 \rfloor} + \frac{1}{4m} \sum_{\substack{d \mid m \\ 2 \nmid d}} 2^{m/d} \varphi(d).$$

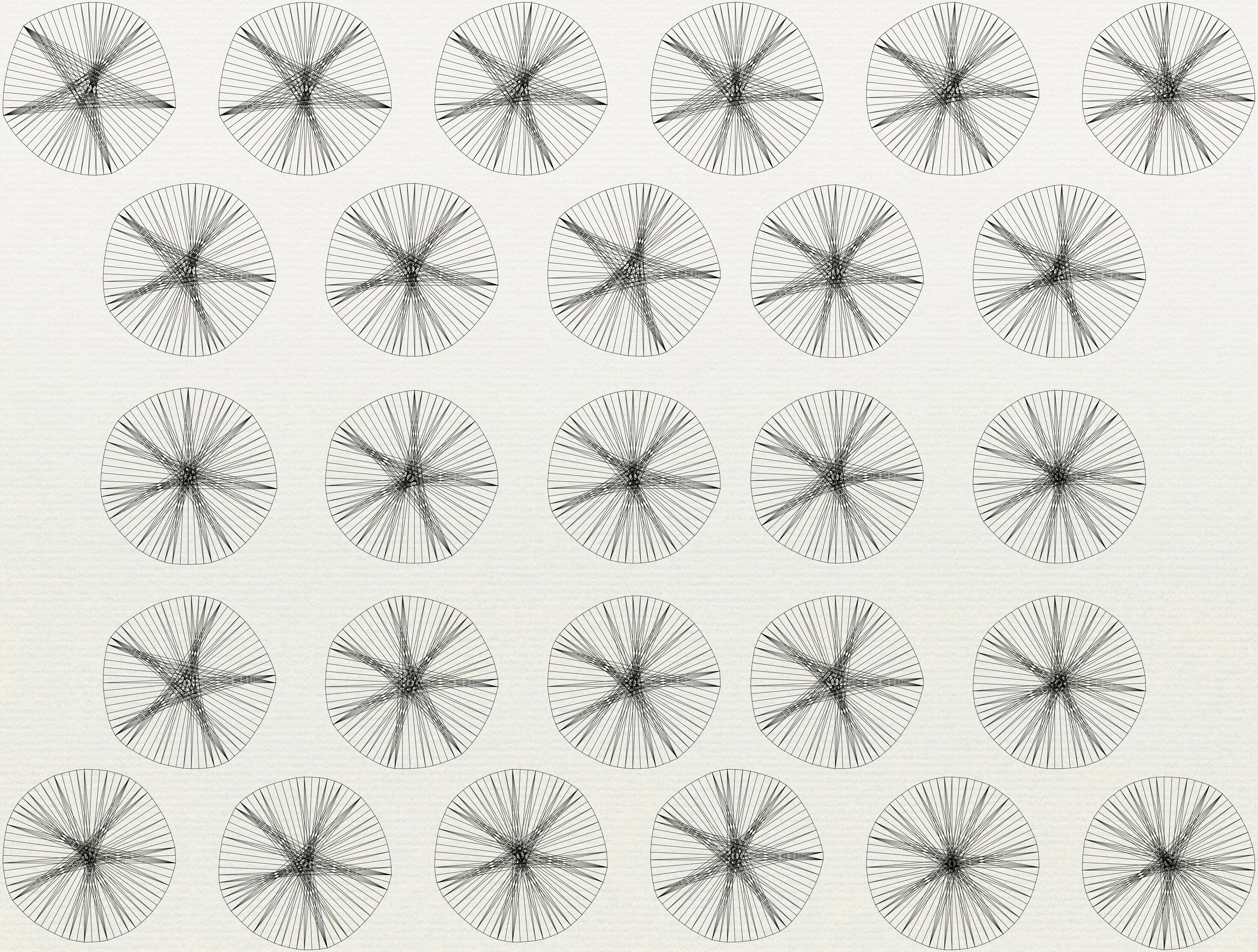
- E.g., $E_0(21) = D(7) + D(3) - 1 = 9 + 2 - 1 = 10.$

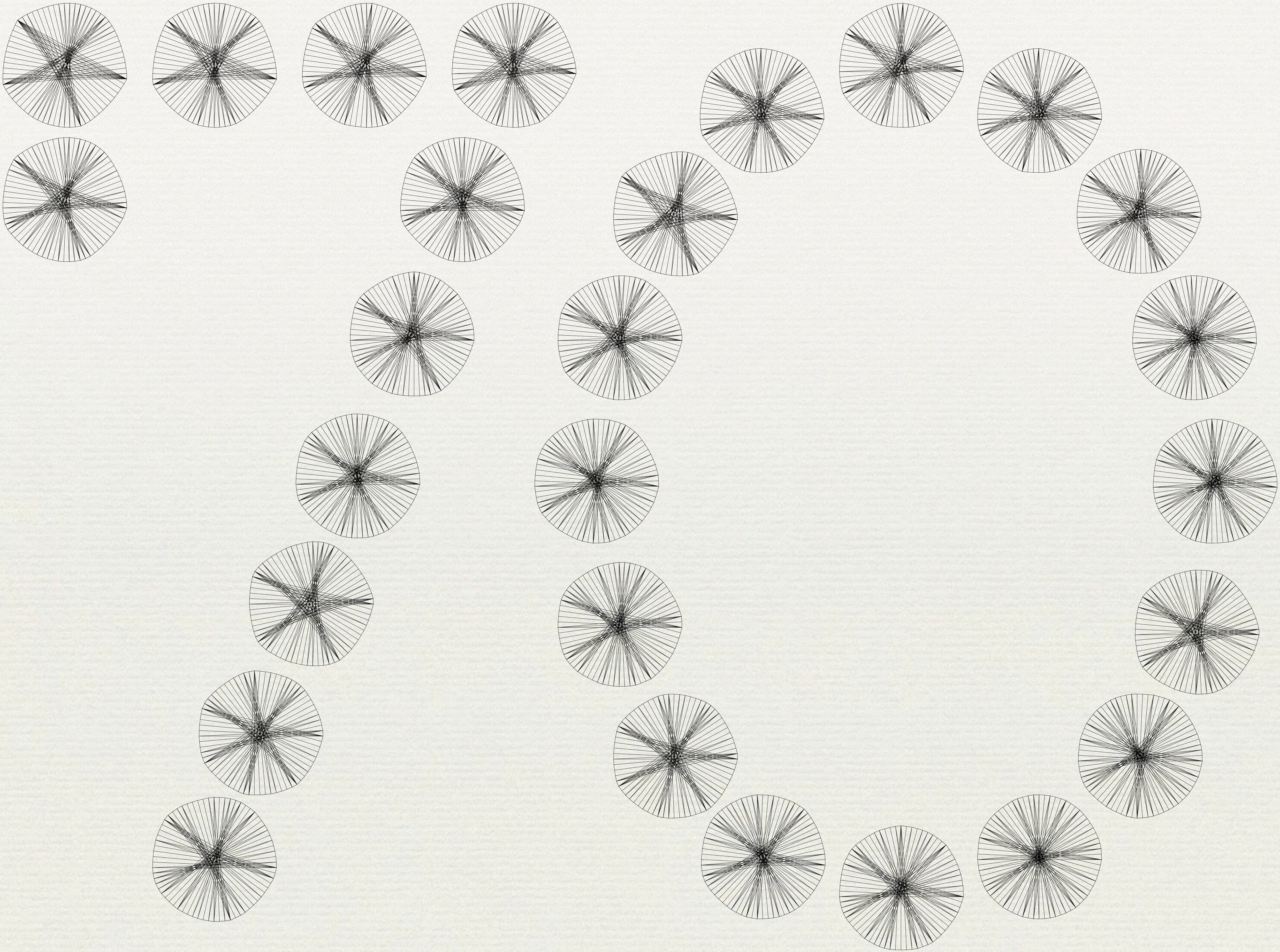
- Are all Reinhardt polygons periodic?
- Let $E_1(n)$ = number of *sporadic* Reinhardt polygons.
- $E_1(n) = 0$ for $n < 30$.

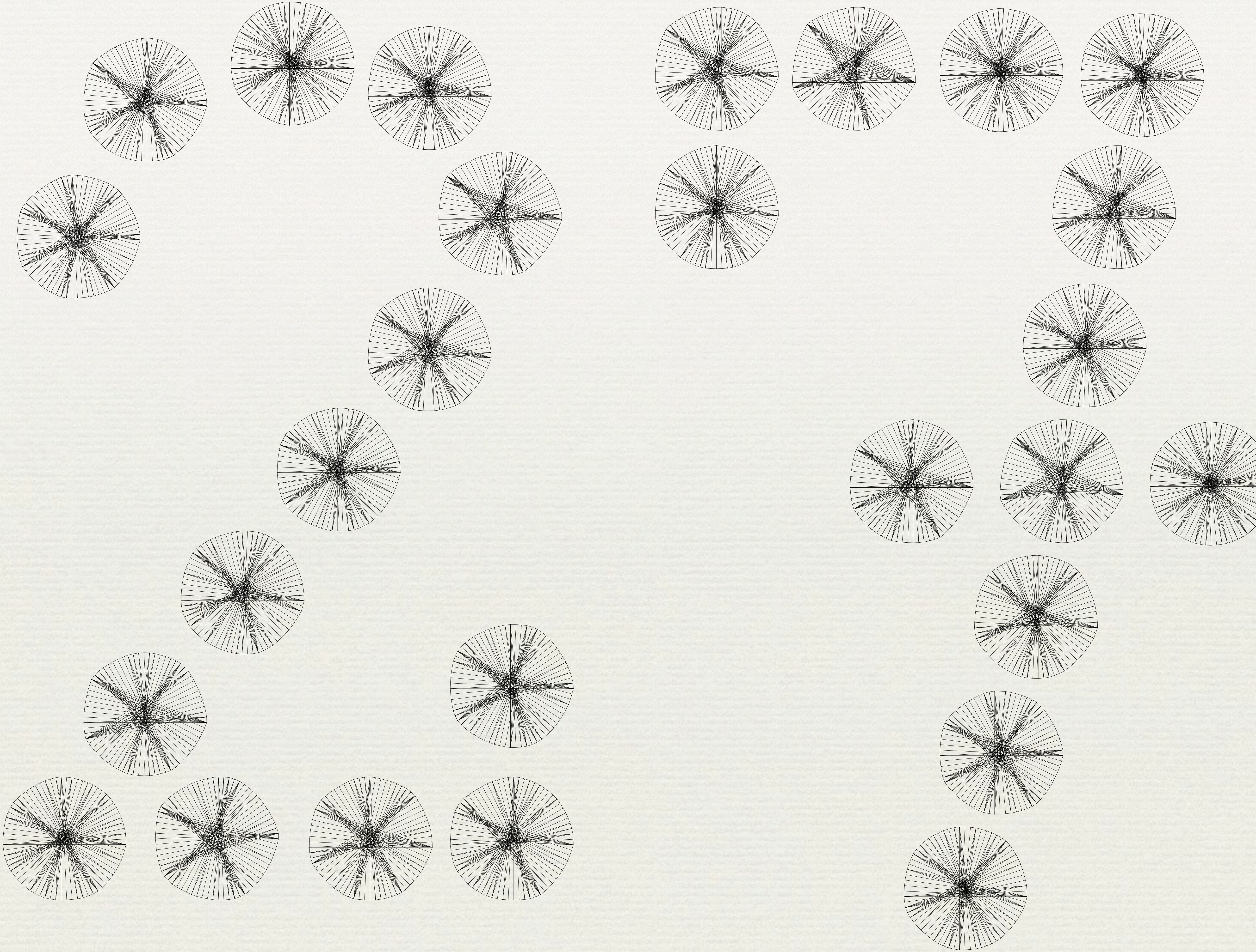


[5, 4, 1, 2, 1, 1, 4, 3, 2, 3, 2, 1, 4, 1, 2]









All $n < 100$ with $E_1(n) > 0$

n	Factorization	E	E
30	$2 \cdot 3 \cdot 5$	38	3
42	$2 \cdot 3 \cdot 7$	329	9
45	3	633	144
60	2	13,464	4392
63	3	25,503	1308
66	$2 \cdot 3 \cdot 11$	48,179	93
70	$2 \cdot 5 \cdot 7$	358	27
75	3	338,202	153,660
78	$2 \cdot 3 \cdot 13$	647,330	315
84	2	2,400,942	161,028
90	2	8,959,826	5,385,768
99	3	65,108,083	192,324

More n with $E_1(n) > 0$

n	Factorization	E	E
102	$2\cdot 3\cdot 17$	126,355,340	3855
110	$2\cdot 5\cdot 11$	48,208	279
114	$2\cdot 3\cdot 19$	1,808,538,359	13,797
117	3	3,524,338,001	2,587,284
130	$2\cdot 5\cdot 13$	647,359	945
140	2	2,414,204	633,528
154	$2\cdot 7\cdot 11$	48,499	837
170	$2\cdot 5\cdot 17$	126,355,369	11,565
182	$2\cdot 7\cdot 13$	647,650	2835
190	$2\cdot 5\cdot 19$	1,808,538,388	41,391
238	$2\cdot 7\cdot 17$	126,355,660	34,695
286	$2\cdot 11\cdot 13$	695,500	29,295

Results (Hare & M.; 2011, 2013)

Theorem: If n has exactly one odd prime divisor, then $E_1(n) = 0$.

Proof: Suppose $n = 2^a p^{b+1}$ and $F(z)$ is a Reinhardt polynomial for n.

$$F(z) = \Phi_{2n}(z) f(z),\ \deg(F) < n,$$

$$\deg(\Phi_{2n}) = \varphi(2n) = n - n/p,$$

$$\deg(f) < n/p,$$

$$\Phi_{2n}(z) = 1 - z^{n/p} + z^{2n/p} - \cdots + z^{(p-1)n/p},$$

$$f(z) = 1 - z^{a_1} + z^{a_2} - \cdots + z^{a_t}.$$

Results (Hare & M.; 2011, 2013)

Theorem: There is exactly one Reinhardt n-gon precisely when $n = p$ or $2p$, for p an odd prime.

Theorem: Let p and q be distinct odd primes. Then $E_1(pq) = 0$.

Theorem: Let p and q be distinct odd primes, and let $r \geq 2$. Then $E_1(pqr) > 0$.

Question: Is $E_1(n)$ ever larger than $E_0(n)$?

Key Fact

- de Bruijn (1953): If n has distinct prime divisors p_1, ..., p_r, then the ideal $(\Phi_n(z))$ is generated by $\{\Phi_{p_i}(z^{n/p_i}) : 1 \leq i \leq r\}$.

- It follows that if $F(z)$ is a Reinhardt polynomial for n, with odd prime divisors p_1, ..., p_r, then there exist polynomials $f_1(z)$, ..., $f_r(z)$ so that
$$F(z) = f_0(z)(z^n + 1) + \sum_{i=1}^{r} f_i(z)\Phi_{p_i}(-z^{n/p_i}).$$

- Periodic case: each $f_i(z) = 0$ except one with $i>0$.

Constructing Sporadic Polygons

- Let $n = pqr$, p and q distinct odd primes, $r \geq 2$.
- Construct nontrivial $f_1(z)$ and $f_2(z)$ so:
    - $F(z) = f_1(z)\Phi_q(-z^{pr}) + f_2(z)\Phi_p(-z^{qr})$.
    - $F(0) = 1$, $\deg(F) < n$, leading coefficient 1, and nonzero coefficients alternate ± 1.
- Then $F(z)$ corresponds to a Reinhardt polygon.
- Verify it is sporadic.

- Take $f_1(z) = 1 - z$.
- Take $f_2(z) =$ a polynomial with coefficient sequence: $0\ A_1\ B_1\ A_2\ B_2 \cdots A_t\ B_t\ C$, where
  - $t = (q-1)/2$,
  - Each A_i and B_i has length r, C has length $r-1$, each one a sequence over $\{-1, 0, +1\}$.
  - Nonzero entries in each A_i and C alternate ± 1, beginning and ending with $+1$.
  - Nonzero entries in each B_i alternate ∓ 1, beginning and ending with -1.

$n = 30$: $p = 3$, $q = 5$, $r = 2$

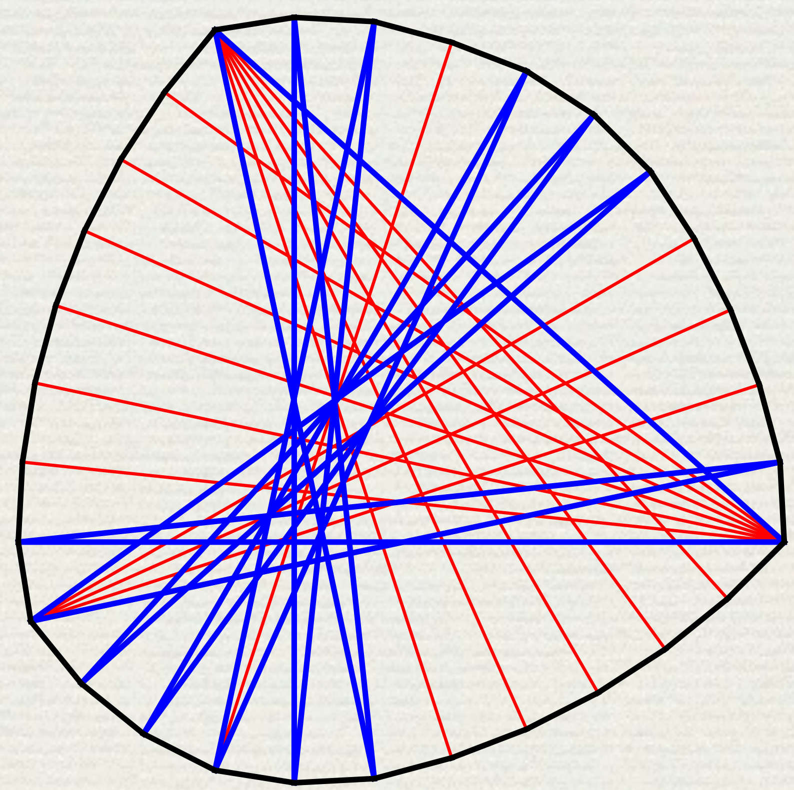
$A_1 = 0+$, $B_1 = 0-$, $A_2 = +0$, $B_2 = 0-$, $C = +$.

f_1 +−0000 −+0000 +−0000 −+0000 +−0000

f_2 00+0−+00−+ 00−0+−00+− 00+0−+00−+

F +−+0−+−+−+000−+−000000+00000−+

[7, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 4, 1, 1]



$n = 30$: $p = 3$, $q = 5$, $r = 2$

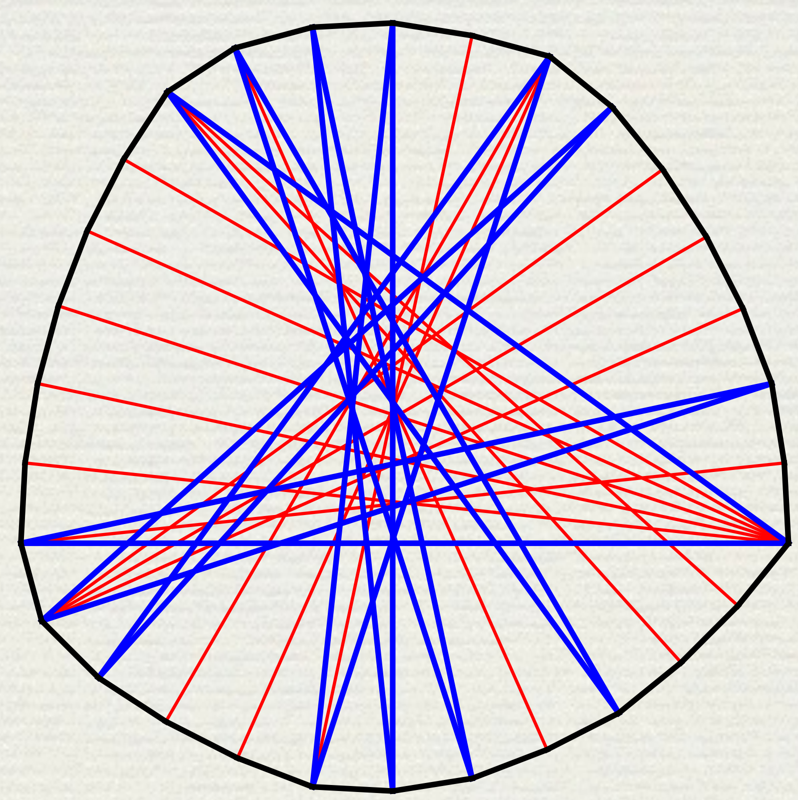
$A_1 =$ 0+, $B_1 =$ 0−, $A_2 =$ ⊕⊕, $B_2 =$ 0−, $C =$ +.

f_1 +−0000−+0000+−0000−+0000+−0000

f_2 00+0−⊕⊕0−+00−0+0−0+−00+0−⊕⊕0−+

F +−+0−⊕⊖+−+000−+⊖⊖00000+00⊖⊕0−+

[6, 3, 1, 2, 1, 1, 1, 1, 2, 3, 1, 1, 4, 1, 2]

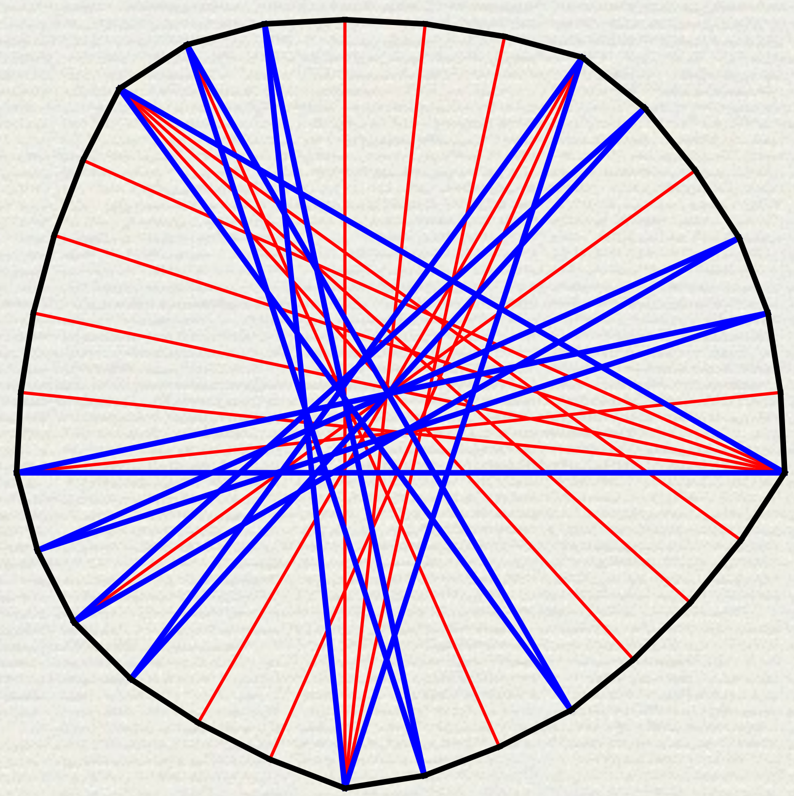


$n = 30$: $p = 3$, $q = 5$, $r = 2$

$A_1 =$ 0+, $B_1 =$ 0−, $A_2 =$ 0+, $B_2 =$ θθ, $C =$ +.

f_1	+−0000−+0000+−0000−+0000+−0000
f_2	00+0−0+θθ+00−0+0−+0−00+0−0+θθ+
F	+−+0−00⦶θ+000−+0−⦶θ000+00−+θθ+

[5, 4, 1, 2, 1, 1, 4, 3, 1, 1, 2, 1, 1, 1, 2]



Sporadic Polygons

- Construction produces a sporadic polygon, unless $A_1 = \cdots = A_t = C0 = -B_1 = \cdots = -B_t$.
- Sporadic polygons constructed: $2^{q(r-1)-1} - 2^{r-2}$.
- Even more: $2^p - 2$ choices for $f_1(z)$.

Number Constructed, $\hat{E}_1(n)$

n	Factorization	E	$\hat{E}$
30	$2 \cdot 3 \cdot 5$	3	3
42	$2 \cdot 3 \cdot 7$	9	9
45	3	144	144
60	2	4392	3492
63	3	1308	1308
66	$2 \cdot 3 \cdot 11$	93	93
70	$2 \cdot 5 \cdot 7$	27	27
75	3	153,660	107,400
78	$2 \cdot 3 \cdot 13$	315	315
84	2	161,028	150,444
90	2	5,385,768	3,371,568
99	3	192,324	192,324

Number Constructed, $\hat{E}_1(n)$

n	Factorization	E	$\hat{E}$
102	$2 \cdot 3 \cdot 17$	3855	3855
110	$2 \cdot 5 \cdot 11$	279	279
114	$2 \cdot 3 \cdot 19$	13,797	13,797
117	3	2,587,284	2,587,284
130	$2 \cdot 5 \cdot 13$	945	945
140	2	633,528	478,548
154	$2 \cdot 7 \cdot 11$	837	837
170	$2 \cdot 5 \cdot 17$	11,565	11,565
182	$2 \cdot 7 \cdot 13$	2835	2835
190	$2 \cdot 5 \cdot 19$	41,391	41,391
238	$2 \cdot 7 \cdot 17$	34,695	34,695
286	$2 \cdot 11 \cdot 13$	29,295	29,295

Number of Sporadic Polygons

- If n has smallest odd prime divisor p then

$$E_0(n) \sim \frac{p}{4n} \cdot 2^{n/p}.$$

- Let $E(n) = E_0(n) + E_1(n)$.

Theorem (Hare & M., 2013): If $p < q$ are odd primes, $\epsilon > 0$, and r is sufficiently large with no prime divisor less than p, then

$$\frac{E_1(pqr)}{E(pqr)} > \frac{2^p - 2}{p2^q + 2^p - 2} - \epsilon.$$

- $n = 15r : > 5.8\%$ sporadic.

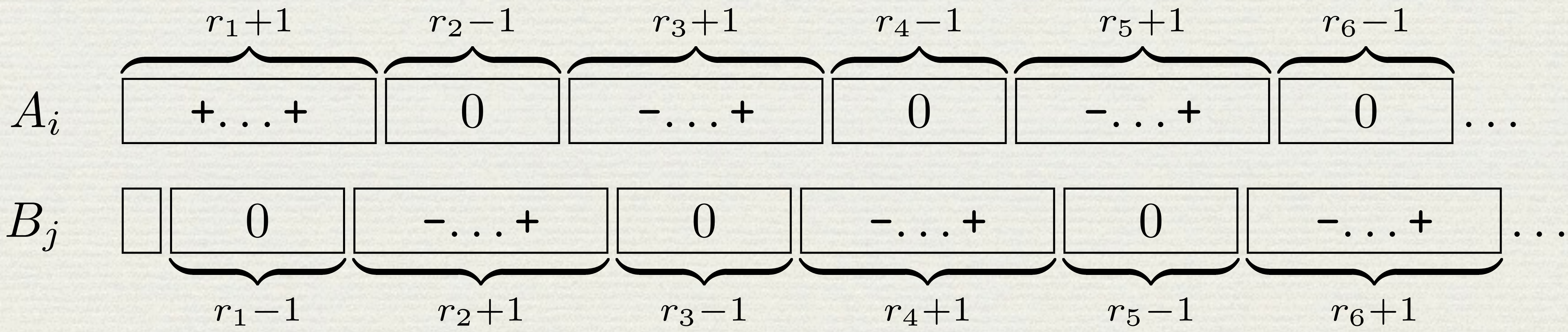
More Recent Work

- Hare & M., 2014.
- Generalized construction for $n = pqr$, p, q distinct odd primes, $r \geq 2$.

- Form $f_1(z)$ from A_1, ..., A_p; $f_2(z)$ from B_1, ..., B_q; each size r.

- Choose a composition of r into an even number of parts, $(r_1, r_2, ..., r_{2m})$.

- Use the composition to guide selections of the blocks.



More Recent Work

- Results from Hare & M., 2014:
  - As $r \to \infty$, $\dfrac{E_1(n)}{E_0(n)} > \dfrac{r(2^{p-1})}{p2^{q-1}}(1 + o(1))$.
  - $E_1(n) > E_0(n)$ for almost all n.
  - First occurs at $n = 105$.
  - $E_1(2pq) = \dfrac{2^{p-1} - 1}{p} \cdot \dfrac{2^{q-1} - 1}{q}$.

Number Constructed, $\ddot{E}_1(n)$

n	Factors	E	$\hat{E}$	$\ddot{E}$
60	2	4392	3492	4392
75	3	153,660	107,400	153,660
84	2	161,028	150,444	161,028
90	2	5,385,768	3,371,568	5,385,768
140	2	633,528	478,548	633,528
105	$3 \cdot 5 \cdot 7$	?	126,714,582	211,752,810

- $E_0(105) = 245{,}518{,}324$, $E_1(105) \geq 249{,}597{,}286$.
- Some polygons for $n = 105$ need three terms for their construction.

Problems

- Can the construction methods for sporadic Reinhardt polygons be generalized to use three distinct odd prime divisors?

- E.g., say $n = lpqr$, l, p, q distinct odd primes, $r \geq 1$.

- Construct nontrivial $f_1(z)$, $f_2(z)$, $f_3(z)$ so

  $$F(z) = f_1(z)\Phi_q(-z^{lpr}) + f_2(z)\Phi_p(-z^{lqr}) + f_3(z)\Phi_l(-z^{pqr}).$$

Problems

- Arbitrary number of odd prime divisors?
- Can one find new lower bounds on $E_1(n)$ for some n?
- Are there more nice formulas for $E_1(n)$ in other cases?

Warm-Ups

- Determine all Reinhardt polynomials for $n = 15$ (say) by searching for suitable multiples of $\Phi_{2n}(z)$.

- Construct some polynomials corresponding to sporadic Reinhardt polygons with $n = 42$ sides.

Possible Avenues

- Generalize one of the constructions to three-term expressions.

- Test if a new construction produces additional polynomials at $n = 105$.

- Find representations of missing 105-gons as three-term sums.

- Look for patterns that might indicate a method of construction.

- New bound for $E_1(105)$? For $E_1(n)$?

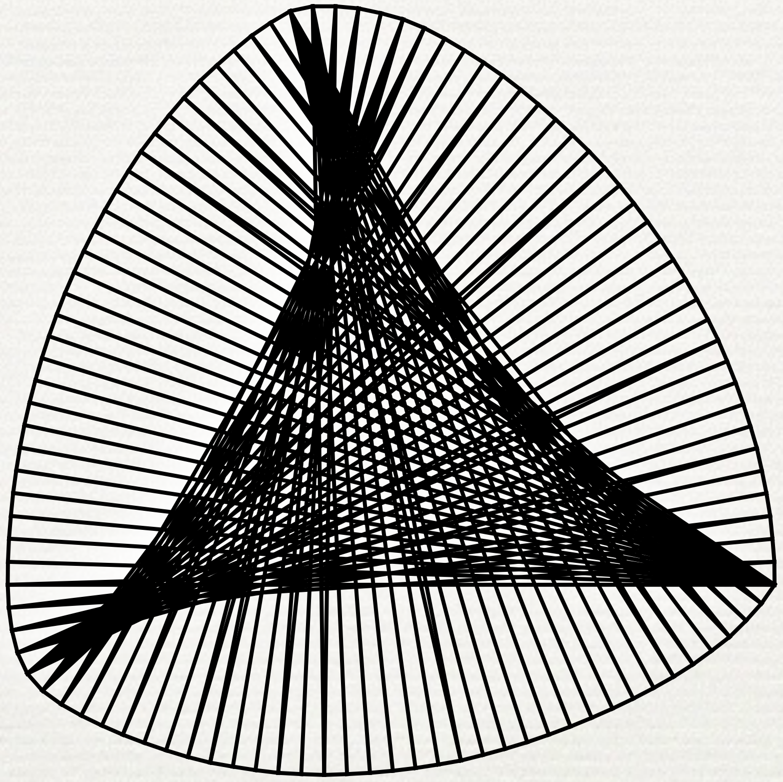
Resources

- M., *A \$1 Problem*, Amer. Math. Monthly **113** (2006), no. 5, 385-402. (Expository.)

- M., *Enumerating isodiametric and isoperimetric polygons*, J. Combin. Theory Ser. A **118** (2011), no. 6, 1801-1815. (Periodic case.)

- K. Hare & M., *Sporadic Reinhardt polygons*, Discrete Comput. Geom. **49** (2013), no. 3, 540-557. (Sporadic construction.)

- K. Hare & M., *Sporadic Reinhardt polygons, II*, arXiv: 1405.5233, 2014. (More general sporadic construction.)

U ♥ 

Good Luck!