Contemporary Mathematics

# An Isodiametric Problem for Equilateral Polygons

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Abstract. The maximal perimeter of an equilateral convex polygon with unit diameter and $n = 2^m$ edges is not known when $m \geq 4$. Using experimental methods, we construct improved polygons for $m \geq 4$, and prove that the perimeters we obtain cannot be improved for large $n$ by more than $c/n^4$, for a particular constant $c$.

## 1. Introduction

There are two classical isodiametric problems for curves in the plane. First, determine the maximal area enclosed by a closed planar curve with unit diameter. Second, determine the maximal perimeter of a closed, convex planar curve with unit diameter. It is well known that the unique solution in both problems is attained by a circle. The area problem was solved by Bieberbach in 1915 [8], and the perimeter question was answered by Rosenthal and Szász in 1916 [17].

We can ask the same questions for polygons with a fixed number of sides. First, determine the maximal area $A(P)$ enclosed by a polygon $P$ with $n$ sides and unit diameter. Second, determine the maximal perimeter $L(P)$ of a convex polygon $P$ with $n$ sides and unit diameter. These problems were first investigated by Reinhardt in 1922 [16]. In that article, Reinhardt established several results on these problems, which we briefly summarize here.

- If $n$ is odd, then the regular $n$-gon with unit diameter attains both the maximal area and the maximal perimeter.
- The regular $n$-gon is the unique solution in the area problem when $n$ is odd, but it is the unique solution in the perimeter problem when $n$ is odd only if $n$ is prime.
- If $n$ is even and $n \geq 6$, then the regular $n$-gon with unit diameter does not attain the maximal area, nor the maximal perimeter.
- If $n$ has an odd divisor, then every polygon with maximal perimeter is equilateral.

Proofs of these statements, along with some additional history and background on these problems, may be found in [14].

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Reinhardt also established an upper bound on the perimeter of a convex $n$-gon with unit diameter, which we denote by $\overline{L}_n$. Its value is

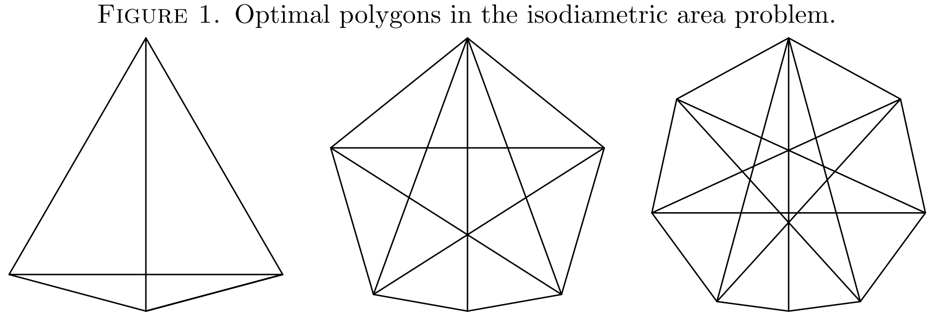
$$\overline{L}_n := 2n\sin(\pi/2n). \tag{1.1}$$

By combining this bound with the well-known isoperimetric inequality for polygons (see for instance [14]), we also obtain an upper bound on the area of an $n$-gon with unit diameter, which we denote by $\overline{A}_n$:

$$\overline{A}_n := \frac{n}{2}\cos\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{2n}\right).$$

Much more is now known in these isodiametric problems for polygons, and we briefly survey some of this progress before we turn to the main topic of this paper, which is a third isodiametric problem for polygons. For the isodiametric area problem, we describe some results for the interesting case when $n$ is even. For $n = 4$, it is easy to show that there are infinitely many quadrilaterals with unit diameter and maximal area 1/2, including the square. For $n = 6$, in 1961 Bieri [9] constructed the hexagon with maximal area and unit diameter, in response to a question of Lenz that appeared in the unsolved problems section of the Swiss journal, *Elemente der Mathematik* [13]. His proof however assumed the existence of an axis of symmetry. In 1975, Graham independently constructed the same hexagon [12], and proved it is optimal, without any assumption of symmetry. More recently, in 2002, Audet, Hansen, Messine, and Xiong [2] combined Graham's strategy with methods of global optimization to determine the optimal octagon in the area problem. The optimal hexagon and octagon are shown in Figure 1, along with one of the optimal quadrilaterals.

FIGURE 1. Optimal polygons in the isodiametric area problem.



(a) $n = 4$. (b) $n = 6$. (c) $n = 8$.

We define the *skeleton* of a polygon as the collection of its vertices, together with all the line segments of maximal length that join two of its vertices. The illustrations in Figure 1, and in all the subsequent diagrams in this article, display the skeleton lying within each polygon.

In 2006, the author [15] used an experimental strategy to construct improved polygons in the area problem for all even integers $n \geq 10$. This method had two principal phases. First, for each even integer $n$ with $10 \leq n \leq 20$, a polygon $Q_n$ with area larger than that of the corresponding regular polygon was constructed by using heuristic optimization methods. These constructions assumed an axis of symmetry and a conjecture of Graham's from [12] that prescribed the pairs of vertices in the polygon that were maximally distant. (Graham's conjecture was recently proved by Foster and Szabo [11].) Constructing an $n$-gon here required optimizing a complicated expression in $n/2-2$ variables. These improved polygons $Q_n$ are displayed in Figure 2. In the second phase, the data from these examples suggested a method for constructing improved polygons in the general case by optimizing an expression in just five parameters. It was shown that for each even integer $n \geq 6$ there exists a polygon $S_n$ with unit diameter for which

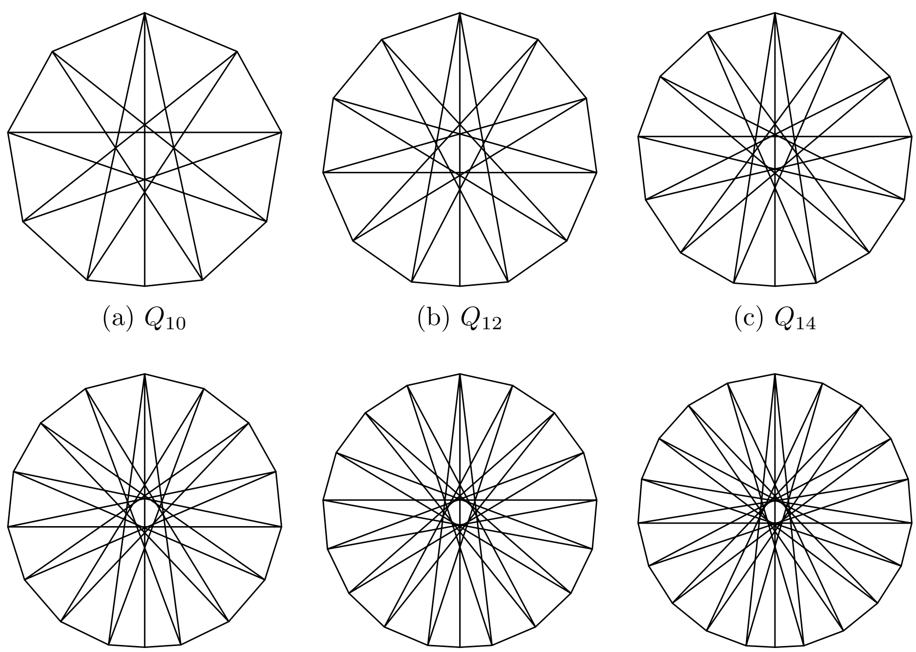
$$\overline{A}_n - A(S_n) < \frac{2\pi^3}{17n^3} + O\left(\frac{1}{n^4}\right).$$

By contrast, the regular $n$-gon with unit diameter, denoted by $P_n$, satisfies

$$\overline{A}_n - A(P_n) = \frac{\pi^3}{16n^2} + O\left(\frac{1}{n^4}\right)$$

when $n$ is even.

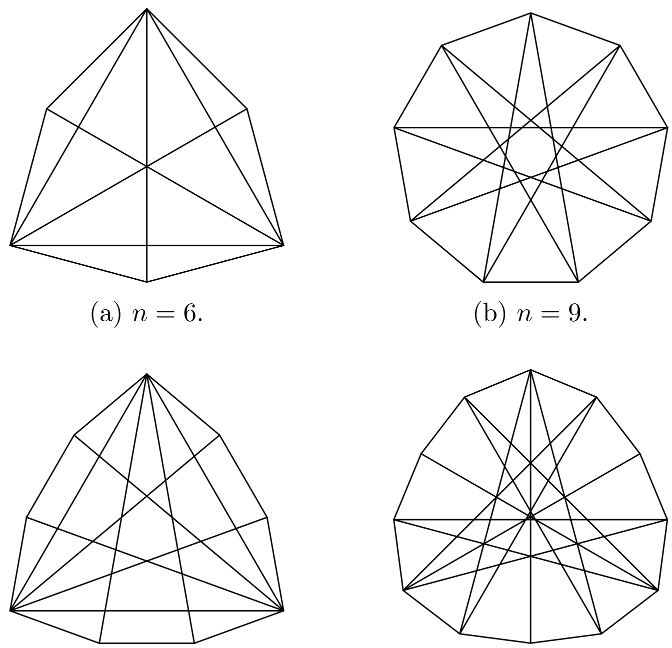
FIGURE 2. Improved polygons $Q_n$ in the area problem for $n = 10$ through $n = 20$.



(a) $Q_{10}$ (b) $Q_{12}$ (c) $Q_{14}$ (d) $Q_{16}$ (e) $Q_{18}$ (f) $Q_{20}$

In the isodiametric perimeter problem, an optimal $n$-gon may be constructed in the following way if $n$ has an odd divisor $m \geq 3$: Simply subdivide each edge of a regular $m$-gon with unit diameter into $n/m$ edges of equal length, placing the new vertices at unit distance from the vertex that lies opposite the original edge. For example, Figure 3(a) shows the optimal hexagon constructed by subdividing the edges of an equilateral triangle (so $m = 3$); this is in fact the unique convex hexagon with unit diameter and maximal perimeter. Figures 3(b) and 3(c) display the two nine-sided polygons (enneagons) with optimal perimeter. In one we choose $m = 9$ in the construction, and in the other, $m = 3$. This method however does not construct all optimal $n$-gons. For example, Figure 3(d) illustrates a convex dodecagon with unit diameter and maximal perimeter which cannot be constructed by this method. (There is one other optimal dodecagon, constructed by subdividing the edges of an equilateral triangle similar to Figure 3(c).) Reinhardt analyzed the number of optimal $n$-gons in the perimeter problem in [16], and studied how this number depends on the factorization of $n$. A more extensive analysis was undertaken by Datta in 1997 [10].

FIGURE 3. Some polygons with maximal perimeter.



(a) $n = 6$. (b) $n = 9$. (c) $n = 9$. (d) $n = 12$.

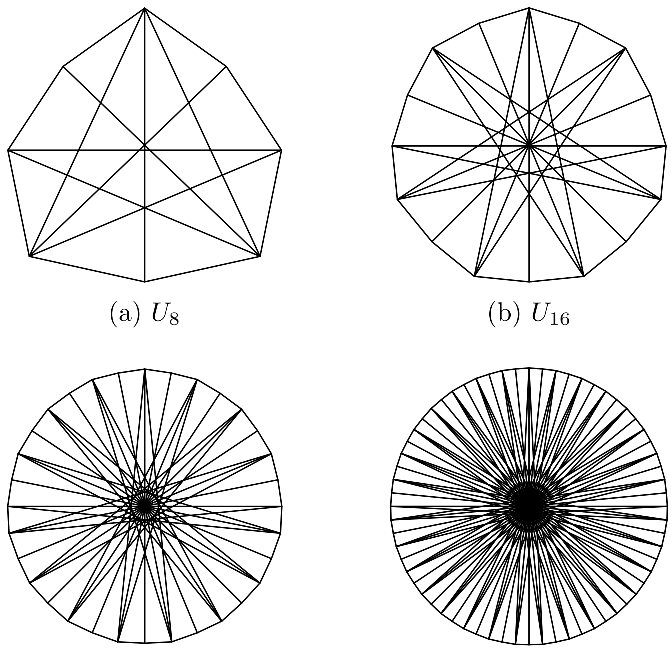
Reinhardt noted that the upper bound $\overline{L}_n$ on the perimeter is in fact attained when $n$ has an odd divisor, and he also showed that no polygon realizes $\overline{L}_n$ when $n$ is a power of 2. The optimal quadrilateral was apparently first found by Tamvakis [18] (proofs also appear in [10, 14]); it is shown in Figure 1(a) and is obtained by subdividing just one edge of an equilateral triangle.

In [15], the author employed an experimental strategy to study the perimeter problem in the case when $n$ is a power of 2. As in the area problem, improved polygons $U_n$ were first constructed for $n = 8$, 16, and 32. These are displayed in Figure 4(a–c). Figure 4(d) also exhibits the polygon $U_{64}$, which was not reported in [15]. Its perimeter is $L(U_{64}) = 3.1412772498200286\ldots$. Each of these constructions required heuristic optimization of an expression in $n/4 - 1$ variables. The data obtained from these polygons then suggested a method for constructing an improved $2^m$-gon in the general case, and this polygon was calculated by optimizing an expression with just four free parameters. It was shown that in general when $n = 2^m$ there exists a convex polygon $V_n$ with unit diameter whose perimeter satisfies

$$\overline{L}_n - L(V_n) = \frac{\pi^5}{16n^5} + O\left(\frac{1}{n^6}\right).$$

By contrast, the regular $n$-gon with unit diameter has $\overline{L}_n - L(P_n) \sim \pi^3/8n^2$, and the best prior known family of polygons $T_n$ (see [14]) satisfies $\overline{L}_n - L(T_n) \sim \pi^3/4n^4$.

FIGURE 4. Improved polygons $U_n$ in the perimeter problem for $n = 8$, 16, 32, and 64.



(a) $U_8$ (b) $U_{16}$

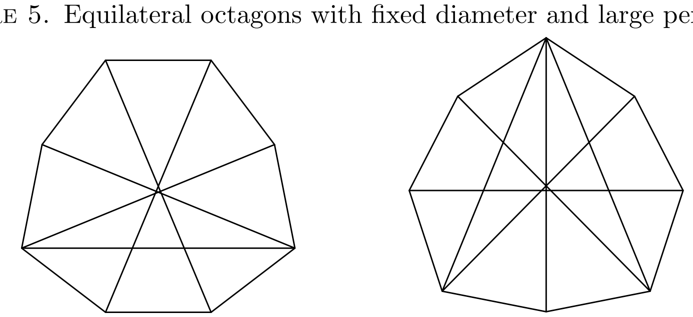
(c) $U_{32}$ (d) $U_{64}$

Independently of [15], Audet, Hansen, and Messine [5] established the optimal octagon in the perimeter problem. This octagon is exactly the one illustrated in Figure 4(a).

In 1950, Vincze [19] studied a similar problem regarding perimeters of polygons with a fixed number of sides, motivated by a question of Erdős. Instead of fixing the diameter of an $n$-gon and maximizing the perimeter, he fixed the perimeter and wished to minimize the diameter, and, more importantly, added a further constraint: Each edge of the polygon was required to have unit length. Vincze independently discovered several of Reinhardt's results, including for instance the fact that the upper bound $\overline{L}_n$ is attained by an equilateral $n$-gon when $n$ has a nontrivial odd divisor. He also described an octagon with unit length sides having diameter $2.588687\ldots$; it is illustrated in Figure 5(a). This value is smaller than that exhibited by the regular octagon with perimeter 8, which has diameter $2.613125\ldots$. Vincze in fact attributed the construction of this improved octagon to his wife, which explains the subtitle of the 2004 article of Audet, Hansen, Messine, and Perron [1]. In this latter article, the authors proved that Vincze's octagon is not the best possible, and they determined the optimal octagon, which is displayed in Figure 5(b). Assuming unit length sides, its diameter is $2.584305\ldots$.

In 1951, Bateman and Erdős [6] considered the related problem of determining the minimal diameter $D(n)$ of a set of $n$ points, none of which lies within unit distance from any of the others. For $n \leq 5$, the optimal configuration is realized by the vertices of a regular polygon with $n$ sides and unit edge length, so $D(2) = D(3) = 1$, $D(4) = \sqrt{2}$, and $D(5) = (1 + \sqrt{5})/2$. For $n = 6$, it is obtained by the vertices of a regular pentagon of unit circumradius together with its circumcenter, so $D(6) = 2\sin(2\pi/5) = 1.902113\ldots$. Bateman and Erdős determined that the optimal configuration for $n = 7$ is attained only by the vertices and circumcenter of a regular hexagon of edge length 1, so $D(7) = 2$. More recently, in 1999 Bezdek and Fodor [7] proved that the optimal configuration for $n = 8$ points occurs only if their convex hull is a regular heptagon of edge length 1, so $D(8) = \frac{1}{2}\csc(\pi/14) = 2.246979\ldots$. For $n = 9$, Audet, Hansen, Messine, and Perron remarked in [1] that placing an additional point in the interior of their octagon of Figure 5(b) establishes that $D(9) \leq 2.584305\ldots$.

FIGURE 5. Equilateral octagons with fixed diameter and large perimeter.



(a) Vincze's octagon. (b) Optimal octagon, $A_8$.

None of the polygons $U_n$ or $V_n$ from Figure 4 and [15] is equilateral. In this article, we investigate the isodiametric problem concerning the perimeter of equilateral polygons in the general open case when $n = 2^m$ with $m \geq 4$. We take the point of view that the diameter is fixed, and we wish to determine a convex, equilateral polygon with $2^m$ sides and especially large perimeter. Our outlook is experimental, and follows the method used in [14] to investigate the first two isodiametric problems for polygons. In Section 2, we describe some experiments in attempting to construct improved polygons for the cases $n = 16$ and $n = 32$. Then, armed with successful examples for these cases, in Section 3 we describe a method for constructing a convex, equilateral $2^m$-gon with unit diameter and large perimeter for arbitrary $m \geq 4$, and establish that the polygons we construct have perimeter quantifiably larger than that of the regular polygon with unit diameter and the same number of sides. We prove the following theorem in Section 3.

THEOREM 1.1. *Suppose $n = 2^m$ with $m \geq 4$. Let $\overline{L}_n$ denote the upper bound on the perimeter of a convex $n$-gon with unit diameter given by* (1.1)*, and let $P_n$ denote the regular $n$-gon with unit diameter. Then there exists a convex, equilateral* *polygon $W_n$ with $n$ sides and unit diameter whose perimeter $L(W_n)$ satisfies*

$$L(W_n) - L(P_n) = \frac{\pi^3}{8n^2} + O\left(\frac{1}{n^4}\right)$$

*and*

$$\overline{L}_n - L(W_n) = \frac{3\pi^4}{n^4} - \frac{9\pi^5}{n^5} + O\left(\frac{1}{n^6}\right).$$

Section 4 adds some additional remarks regarding our constructions, including an application to the problem of Bateman and Erdős, and an indication that further improvements may be possible. Section 5 includes some detailed data on the polygons that we construct.

Additional information on the three isodiametric problems for octagons may be found in [3], and a survey treating a number of extremal problems on polygons, including the three isodiametric problems we discuss here, appears in [4].

## 2. Improved polygons for $\mathbf{n = 16}$ and $\mathbf{n = 32}$

We describe some experiments that lead us to some improved polygons in the isodiametric perimeter problem for equilateral $n$-gons for the cases $n = 16$ and $n = 32$.

**2.1. The case $\mathbf{n = 16}$.** We might begin by studying the optimal octagon in the equilateral perimeter problem, shown in Figure 5(b). Its skeleton has two connected components: the horizontal crossbar is one, and the other is formed by the remaining five segments shown in the interior of the octagon. We might attempt to construct a hexadecagon with an analogous skeleton. First, place a vertex $v_0$ at $(0, -1/2)$ in the plane, and another vertex $v_1$ at $(0, 1/2)$. Then, for $1 \le k \le 6$, we may compute the location of the vertex $v_{k+1}$ by rotating the segment joining $v_{k-1}$ and $v_k$ about the point $v_k$ counterclockwise by an angle $\theta$. Here, $\theta$ is a parameter that will be selected later. Writing $v_k = (x_k, y_k)$ and letting $\mathcal{R}_\theta$ denote the standard linear transformation on the plane that encodes counterclockwise rotation by $\theta$, we see that

$$\begin{bmatrix} x_{k+1} - x_k \\ y_{k+1} - y_k \end{bmatrix} = -\mathcal{R}_\theta \begin{bmatrix} x_k - x_{k-1} \\ y_k - y_{k-1} \end{bmatrix}.$$

It follows that

$$x_{k+1} = x_k - (-1)^k \sin(k\theta),$$

and

$$y_{k+1} = y_k + (-1)^k \cos(k\theta).$$

We then obtain six more vertices by symmetry, adding the points $w_k = (-x_k, y_k)$ for $2 \le k \le 7$. The final two vertices form the horizontal crossbar in the skeleton, and we denote these by $v_8$ and $w_8$. Their coordinates are $v_8 = (-1/2, y_8)$ and $w_8 = (1/2, y_8)$, and $y_8$ is chosen so that the distance $d(v_7, v_8)$ from $v_7$ to $v_8$ matches the distance $d(v_8, w_6)$ from $v_8$ to $w_6$. We compute

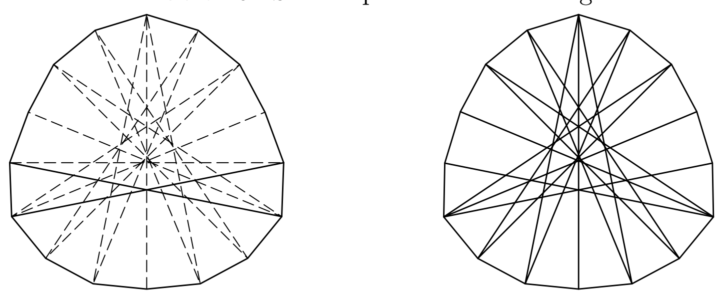
$$y_8 = \frac{\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1}{\cos 3\theta + \sin 3\theta}.$$

Last, we determine $\theta$ by the constraint $d(v_0, v_2) = d(v_7, v_8)$, and this produces $\theta = 0.1967414682\ldots$, which determines our hexadecagon.

We must still check that the resulting polygon has unit diameter, but unfortunately this hexadecagon does not. Two pairs of vertices violate this requirement:

$d(v_6, v_8) = d(w_6, w_8) = 1.01217\ldots$. The hexadecagon constructed is illustrated in Figure 6(a), where the dashed lines in the interior show the pairs of vertices which are unit distance apart, and the solid lines inside the polygon mark the longer segments of the skeleton.

FIGURE 6. Some equilateral hexadecagons.



(a) First attempt. (b) Successful construction, $W_{16}$.

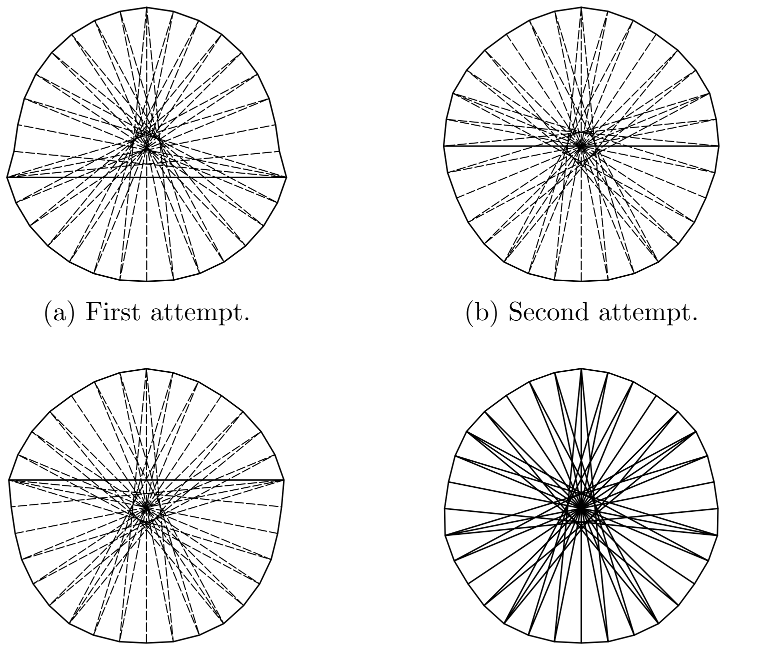
Our initial construction however suggests a second line of attack. We can drop our original demand that the horizontal crossbar connecting $v_8$ and $w_8$ have unit length, and instead require that the two solid lines in the interior of Figure 6(a) have this property. The amended construction is then similar to the first one, but now $v_8$ is obtained by rotating the segment connecting $v_6$ and $v_7$ about $v_6$ by an angle of $\theta$. (Again, $w_8 = (-x_8, y_8)$.) Choosing $\theta$ again so that $d(v_0, v_2) = d(v_6, w_8)$ produces a slightly smaller angle, $\theta = 0.1962338627\ldots$, and this time the hexadecagon constructed is convex and has unit diameter. It is displayed in Figure 6(b). Its perimeter is $3.1347065475\ldots$; the regular hexadecagon with unit diameter has perimeter $16\sin(\pi/16) = 3.1214451522\ldots$.

**2.2. The case $\mathbf{n = 32}$.** We may now attempt to construct an improved equilateral polygon with 32 sides (a triacontakaidigon), using the new hexadecagon of Figure 6(b) as a guide. As a first attempt, we might try constructing a polygon whose skeleton is much like that of Figure 6(b): connected, symmetric, and with exactly three vertices of degree 3 in the skeleton. We number the vertices as before, so $v_0 = (0, -1/2)$, $v_1 = (0, 1/2)$, and several subsequent $v_k$ are obtained by rotating the line connecting $v_{k-1}$ and $v_{k-2}$ about $v_{k-1}$ counterclockwise through an angle $\theta$. In our first construction, we compute the vertices $v_k$ in this way for $2 \le k \le 14$, then make $v_{14}$ a vertex of degree 3 in the skeleton, rotating the segment connecting $v_{13}$ and $v_{14}$ twice to obtain $v_{15}$ and $v_{16}$. Again, the vertices $w_k$ for $2 \le k \le 16$ are obtained by symmetry. This way, the three vertices of degree 3 occur just where they did in Figure 6(b): at $v_1$, and at the end of the two long paths in the skeleton which begin at $v_1$.

Figure 7(a) exhibits the polygon constructed with this strategy, once $\theta$ is chosen so that the resulting polygon is equilateral ($\theta = 0.09838064077\ldots$). However, this polygon is not convex, and does not have diameter 1. The dashed lines in the diagram illustrate pairs of vertices which are unit distance apart, and the solid horizontal line in the skeleton has length $1.02110\ldots$.

As a second attempt, we try a configuration where each of five vertices of the polygon have unit distance from three other vertices. Figure 7(b) shows the location of the vertices of degree 3 in this construction. Here, we set $\theta = 0.09819823267\ldots$, and we find that the resulting equilateral polygon is convex, but again does not have unit diameter. The horizontal line here has length $1.00503\ldots$. A third construction, illustrated in Figure 7(c), tests a slight variation in the placement of the vertices of degree 3. It suffers the same fate. With $\theta = 0.09819113099\ldots$, we obtain a convex, equilateral polygon with 32 sides, but its diameter again exceeds 1. Here, the solid horizontal line in Figure 7(c) has length $1.00382\ldots$.

Figure 7. Some equilateral triacontakaidigons.



(a) First attempt. (b) Second attempt.

(c) Third attempt. (d) Successful construction, $W_{32}$.

A fourth construction posits seven vertices of degree 3 in the skeleton, arranged as shown in Figure 7(d). Here, we select $\theta = 0.09816859586\ldots$, and this time we find a convex triacontakaidigon with unit diameter. Its perimeter is $3.1401338090\ldots$, which exceeds that of the regular polygon with 32 sides and unit diameter, $L(P_{32}) = 32\sin(\pi/32) = 3.1365484905\ldots$.

## 3. Proof of Theorem 1.1

Our constructions for $n = 16$ and $n = 32$ shown in Figures 6(b) and 7(d) indicate a method for constructing an improved polygon in the general case when $n = 2^m$ and $m \geq 4$. It seems clear that we should plan for a skeleton having $\frac{n}{4} - 1$ vertices of order 3, arranged in a particular way. Again, we take $v_0 = (0, -1/2)$ and $v_1 = (0, 1/2)$, and number the subsequent vertices $v_k$ (and $w_k$) in the same way as before. Certainly the topmost vertex $v_1$ should have order 3 in the general construction, and proceeding clockwise around the perimeter, we see that the $\frac{n}{8} - 1$ vertices of degree 3 on the right side of the polygon should be rather evenly spaced around the boundary. We place the first such vertex fifth from the top after $v_1$, and place the subsequent vertices of degree 3 at every fourth vertex thereafter as we trace along the boundary from top to bottom. See Figure 8 for the case $n = 64$. This way, except for a slight break in the pattern near the top and bottom of the polygon, the vertices of degree 3 around the boundary alternate perfectly with the vertices which connect to a vertex of degree 3 on the opposite side of the polygon. Using this alternating pattern, we can determine the coordinates of the vertex $v_{n/2}$ with the following calculations. We find that

$$\begin{aligned} x_{n/2} &= \sin\theta - \sin 2\theta + \sin 3\theta \\ &\quad + \sum_{k=1}^{\frac{n}{8}-1} (-1)^k \big(\sin 4k\theta - \sin((4k+1)\theta) + \sin((4k+3)\theta)\big), \end{aligned}$$

and

$$\begin{aligned} y_{n/2} &= \tfrac{1}{2} - \cos\theta + \cos 2\theta - \cos 3\theta \\ &\quad - \sum_{k=1}^{\frac{n}{8}-1} (-1)^k \big(\cos 4k\theta - \cos((4k+1)\theta) + \cos((4k+3)\theta)\big). \end{aligned}$$

Let

$$\begin{aligned} S(m,\theta) &:= \sum_{k=0}^{m-1} (-1)^k \sin k\theta \\ &= \operatorname{Im}\left( \sum_{k=0}^{m-1} \left(-e^{i\theta}\right)^k \right) \\ &= -\frac{(-1)^m \sin((m-\frac{1}{2})\theta) + \sin(\frac{\theta}{2})}{2\cos(\frac{\theta}{2})}, \end{aligned}$$

and let

$$\begin{aligned} C(m,\theta) &:= \sum_{k=0}^{m-1} (-1)^k \cos k\theta \\ &= \operatorname{Re}\left( \sum_{k=0}^{m-1} \left(-e^{i\theta}\right)^k \right) \\ &= \frac{1}{2} - (-1)^m \frac{\cos((m-\frac{1}{2})\theta)}{2\cos(\frac{\theta}{2})}. \end{aligned}$$

Then a short calculation reveals that

$$\begin{aligned} x_{n/2} &= 2\sin\theta - \sin 2\theta + S(n/8, 4\theta)(1 - \cos\theta + \cos 3\theta) \\ &\quad + C(n/8, 4\theta)(\sin 3\theta - \sin\theta), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} y_{n/2} &= \tfrac{3}{2} - 2\cos\theta + \cos 2\theta - C(n/8, 4\theta)(1 - \cos\theta + \cos 3\theta) \\ &\quad + S(n/8, 4\theta)(\sin 3\theta - \sin\theta). \end{aligned} \tag{3.2}$$

The vertex $v_{n/2-2}$ has degree 3 in the skeleton and has unit distance from $v_{n/2}$. We can describe its coordinates easily from our expressions for $x_{n/2}$ and $y_{n/2}$. They are

$$
\begin{aligned}
x_{n/2-2} &= x_{n/2} + \sin((n/2-1)\theta), \\
y_{n/2-2} &= y_{n/2} - \cos((n/2-1)\theta).
\end{aligned} \tag{3.3}
$$

Also, the two vertices nearest $v_{n/2}$ in the polygon are $v_{n/2-1}$ and $w_{n/2-2}$. Certainly $w_{n/2-2} = (-x_{n/2-2}, y_{n/2-2})$, and we see that

$$
\begin{aligned}
x_{n/2-1} &= x_{n/2-2} - \sin((n/2-2)\theta), \\
y_{n/2-1} &= y_{n/2-2} + \cos((n/2-2)\theta).
\end{aligned} \tag{3.4}
$$

We need to select $\theta$ so that the distance between $v_{n/2}$ and $w_{n/2-2}$ matches the other edge lengths in the polygon. Stipulating that $d(v_{n/2}, w_{n/2-2}) = d(v_0, v_2)$ requires that we find a real root of the expression

$$
\begin{aligned}
z(n,\theta) &= \cos^2((\tfrac{n}{2}-1)\theta) - 4\sin^2(\tfrac{\theta}{2}) \\
&+ \left( \frac{2\cos(\frac{n\theta}{2})\sin\theta + \sin 2\theta - 2\sin 3\theta + \sin 4\theta + \sin((\frac{n}{2}-2)\theta)}{\cos 2\theta} - \sin((\tfrac{n}{2}-1)\theta) \right)^2 .
\end{aligned}
$$

Some asymptotic analysis produces that

$$
z\left(n, \frac{\pi}{n} - \frac{3\pi^4}{n^5} + \frac{9\pi^5}{n^6} - \frac{3\pi^6}{2n^7} + \frac{9\pi^7}{4n^8} + \frac{\pi^7(2879+957\pi)}{80n^9}\right) = \frac{\pi^8}{80n^9} + O\left(\frac{1}{n^{10}}\right)
$$

and

$$
z\left(n, \frac{\pi}{n} - \frac{3\pi^4}{n^5} + \frac{9\pi^5}{n^6} - \frac{3\pi^6}{2n^7} + \frac{9\pi^7}{4n^8} + \frac{\pi^7(2881+957\pi)}{80n^9}\right) = -\frac{\pi^8}{80n^9} + O\left(\frac{1}{n^{10}}\right),
$$

so for large $n$ the expression $z(n,\theta)$ has a zero $\theta_0(n)$ satisfying

$$
\theta_0(n) = \frac{\pi}{n} - \frac{3\pi^4}{n^5} + \frac{9\pi^5}{n^6} - \frac{3\pi^6}{2n^7} + \frac{9\pi^7}{4n^8} + O\left(\frac{1}{n^9}\right).
$$

Let $W_n$ denote the polygon obtained by setting $\theta = \theta_0(n)$. Since $d(v_0, v_2) = 2\sin(\theta_0(n)/2)$, we obtain that the perimeter of $W_n$ is

$$
\begin{aligned}
L(W_n) &= 2n\sin(\theta_0(n)/2) \\
&= \pi - \frac{\pi^3}{24n^2} + \left(\frac{\pi^5}{1920} - 3\pi^4\right)\cdot\frac{1}{n^4} + \frac{9\pi^5}{n^5} - \left(9 + \frac{\pi}{40320}\right)\cdot\frac{\pi^6}{8n^6} + O\left(\frac{1}{n^7}\right).
\end{aligned}
$$

Since

$$
\overline{L}_n = 2n\sin(\pi/2n) = \pi - \frac{\pi^3}{24n^2} + \frac{\pi^5}{1920n^4} - \frac{\pi^7}{322560n^6} + O\left(\frac{1}{n^7}\right),
$$

we have

$$
\overline{L}_n - L(W_n) = \frac{3\pi^4}{n^4} - \frac{9\pi^5}{n^5} + \frac{9\pi^6}{8n^6} + O\left(\frac{1}{n^7}\right),
$$

and because

$$
L(P_n) = n\sin(\pi/n) = \pi - \frac{\pi^3}{6n^2} + O\left(\frac{1}{n^4}\right),
$$

we see that

$$
L(W_n) - L(P_n) = \frac{\pi^3}{8n^2} + O\left(\frac{1}{n^4}\right).
$$

Theorem 1.1 therefore follows once we establish that $W_n$ is convex and has diameter 1.

Our construction guarantees convexity at each vertex except $v_{n/2}$ (and $w_{n/2}$), and we can test for convexity there with a short calculation. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ denote the two vectors in $\mathbb{R}^3$ lying on the boundary of $W_n$, with direction indicating a counterclockwise orientation of the boundary, and with $\mathbf{z}_1$ pointing toward $v_{n/2}$ and $\mathbf{z}_2$ pointing away from this vertex. Then we can establish convexity at $v_{n/2}$ by checking that the cross product $\mathbf{z}_1 \times \mathbf{z}_2$ has positive $z$-component. Since $\mathbf{z}_1$ emanates from $v_{n/2-1}$ and $\mathbf{z}_2$ points toward $w_{n/2-2}$, using (3.1), (3.2), (3.3), and (3.4), we calculate

$$\det \begin{bmatrix} x_{n/2} - x_{n/2-1} & -x_{n/2-2} - x_{n/2} \\ y_{n/2} - y_{n/2-1} & y_{n/2-2} - y_{n/2} \end{bmatrix} = \frac{2\pi^3}{n^3} - \frac{3\pi^4}{n^4} + O\left(\frac{1}{n^5}\right).$$

To test that the polygon $W_n$ has unit diameter, we first obtain closed expressions for the coordinates of each of $v_{4k}$, $v_{4k+1}$, $v_{4k+2}$, and $v_{4k+3}$ in the same way we derived the formulas (3.1) and (3.2) for $v_{n/2}$. Then we test distances between pairs of vertices in the polygon which are not constrained by the construction to have unit distance. In view of Figures 7(a–c), it is perhaps not surprising that the critical case is the horizontal segment connecting $v_{n/2}$ and $w_{n/2}$. Some asymptotic analysis shows that this distance is

$$d(v_{n/2}, w_{n/2}) = -2x_{n/2} = 1 - \frac{3\pi^3}{n^3} - \frac{5\pi^5}{4n^5} + O\left(\frac{1}{n^7}\right).$$

All other pairs of vertices in the polygon which do not have unit distance are no closer than $1 - \pi^2/n^2 + O(1/n^4)$. For example, we find that

$$d(v_{4k}, v_{4k+3}) = 2\cos(\theta_0(n)) - 1 = 1 - \frac{\pi^2}{n^2} + O\left(\frac{1}{n^4}\right),$$

$$d(v_{4k+3}, v_{4k+5}) = d(v_{4k+2}, v_{4k+6}) = 1 - \frac{2\pi^2}{n^2} + O\left(\frac{1}{n^4}\right),$$

and

$$d(v_{4k+1}, v_{4k+5}) = 1 - \frac{4\pi^2}{n^2} + O\left(\frac{1}{n^4}\right)$$

for $k \geq 2$. This completes the proof of Theorem 1.1. □

Figure 8 exhibits $W_{64}$. Its perimeter is listed in Section 5, together with additional data on this and the other equilateral polygons described in this article.

## 4. Remarks on the constructions

We first remark that our constructions produce upper bounds for certain $n$ on the quantity $D(n)$ investigated by Bateman and Erdős and defined in Section 1. For example, Figure 9 shows that twelve points may be added to the interior of $W_{16}$, with none closer to any other or to a point on the boundary than the distance between two adjacent points on the boundary. By dilating $W_{16}$ so that each edge on the boundary has unit length, it follows that $D(28) \leq 5.1041460364\ldots$.

Second, it is curious that the family of polygons $W_n$ that we construct here does not appear to generalize the optimal octagon $A_8$ of Figure 5(b) in a natural way. Indeed, the octagon $W_8$ suggested by our general construction would have a single vertex of degree 3 in its skeleton, occurring at $v_1$. However, the octagon constructed with this strategy does not have unit diameter. It is displayed in Figure 10. The solid lines in the interior show the distances that exceed 1: their

FIGURE 8. The equilateral hexacontakaitetragon $W_{64}$.

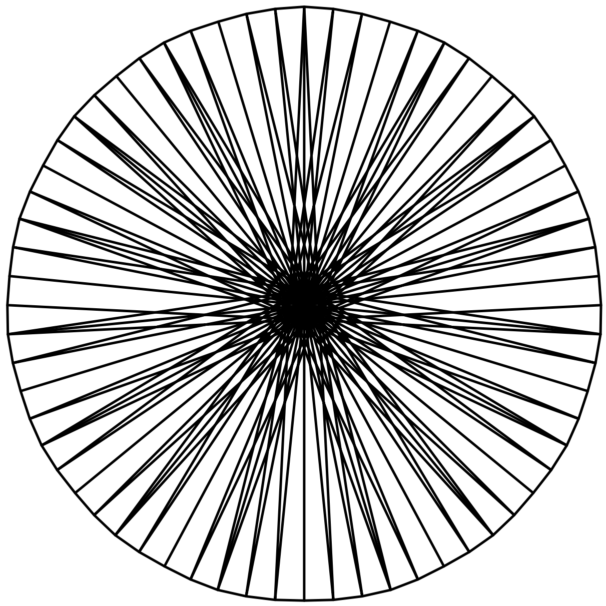
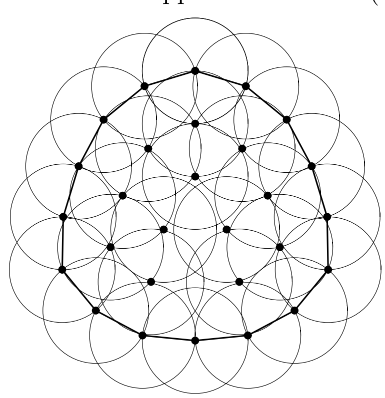


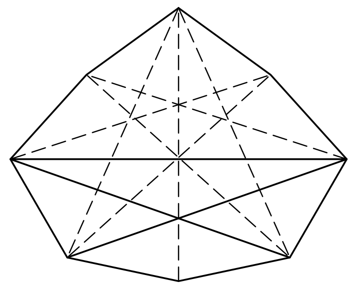
FIGURE 9. An upper bound for $D(28)$.



lengths are $1.08300\ldots$ and $1.22929\ldots$. The dashed lines show the segments of unit length mandated by the construction.

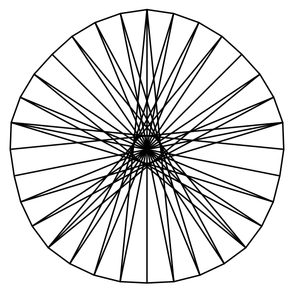
It seems quite possible then that another family of polygons might produce an improvement to Theorem 1.1. For example, one might view the skeleton of $A_8$ as one obtained by altering the skeleton of the regular octagon in a simple way, adding two additional unit distance constraints, then adjusting the positions of the vertices accordingly. Perhaps there is some similar way to perturb the vertices of a regular $2^m$-gon to produce better equilateral polygons.

FIGURE 10. The equilateral octagon $W_8$.



Indeed, a better family of polygons may be waiting to be discovered. Our experiments also revealed the triacontakaidigon we denote $X_{32}$, exhibited in Figure 11. It is convex, has unit diameter, and has just five vertices of degree 3 in its skeleton, in contrast to $W_{32}$. It has $\theta = 0.09817158704\ldots$, and its perimeter is $L(X_{32}) = 3.1402294116\ldots$, a slight improvement to $L(W_{32})$.

FIGURE 11. An improved equilateral triacontakaidigon, $X_{32}$.



## 5. Data

Table 1 displays the coordinates of the vertices of the equilateral polygons with unit diameter and large perimeter described in this article, accurate to ten digits. It lists only the coordinates $(x, y)$ with $x > 0$ when the axis of symmetry of the polygon is placed along the $y$-axis, with the vertex $v_0$ at the origin and $v_1$ at $(0, 1)$. For the convenience of the reader, we also list the coordinates of the optimal octagon $A_8$ of Figure 5(b), the triacontakaidigon $X_{32}$ of Figure 11, and the non-equilateral hexacontakaitetragon $U_{64}$ of Figure 4(d).

Table 2 shows the perimeter $L$ of the equilateral polygons described in this article, along with the upper bound $\overline{L}_n$ from (1.1). It also lists the diameter $d^*$ of the polygon when it is dilated so that each edge has unit length, for the convenience of readers interested in the dual form of the problem investigated by Vincze and Erdős, as well as the lower bound $\underline{d}_n^* = n/\overline{L}_n$ on this quantity. The data in this table are rounded to eleven or twelve decimal places.

TABLE 1. Vertices $(x, y)$ with $x > 0$ of $A_8$, $W_{16}$, $W_{32}$, $X_{32}$, $W_{64}$, and $U_{64}$.

| | | | |
|---|---|---|---|
| $A_8$ | (0.3227958396, 0.7866129109) | ($\frac{1}{2}$, 0.4426217927) | (0.3796397575, 0.07486560193) |
| $W_{16}$ | (0.1874928115, 0.9431602025) | (0.3389906788, 0.8189317309) | (0.4314562436, 0.6462052527) |
| | (0.4884695273, 0.4587651224) | (0.4921574578, 0.2628806766) | (0.3677888395, 0.1114978383) |
| | (0.1949768653, 0.01919215848) | | |
| $W_{32}$ | (0.09706721467, 0.9856023577) | (0.1894608776, 0.9525456288) | (0.2736307972, 0.9021000057) |
| | (0.3524512409, 0.8436476839) | (0.4183545875, 0.7709422151) | (0.4603165176, 0.6822374651) |
| | (0.4841678983, 0.5870510800) | (0.4985751301, 0.4899852881) | (0.4966005141, 0.3918759758) |
| | (0.4635346575, 0.2994855792) | (0.4130807192, 0.2153206436) | (0.3546206108, 0.1365059750) |
| | (0.2819086317, 0.07060981152) | (0.1931997365, 0.02865664495) | (0.09801099550, 0.004814668134) |
| $X_{32}$ | (0.09707010531, 0.9856014838) | (0.1894662351, 0.9525427812) | (0.2736378874, 0.9020942375) |
| | (0.3463506019, 0.8361944366) | (0.4122541063, 0.7634850786) | (0.4627069372, 0.6793159960) |
| | (0.4957703458, 0.5869215502) | (0.4994096993, 0.4888568892) | (0.4850062390, 0.3917875174) |
| | (0.4611579760, 0.2965972712) | (0.4191978197, 0.2078883771) | (0.3607381711, 0.1290696477) |
| | (0.2819164642, 0.07061401377) | (0.1932054330, 0.02865837584) | (0.09801397228, 0.004814961307) |
| $W_{64}$ | (0.04894923559, 0.9963893046) | (0.09730886109, 0.9879981541) | (0.1446131513, 0.9749073591) |
| | (0.1912181301, 0.9595112318) | (0.2360896181, 0.9396211751) | (0.2775569808, 0.9133624125) |
| | (0.3162508721, 0.8831655987) | (0.3534164749, 0.8511065477) | (0.3872607952, 0.8155590193) |
| | (0.4155229241, 0.7754302659) | (0.4397156773, 0.7327245815) | (0.4617838308, 0.6888832610) |
| | (0.4794485406, 0.6430899993) | (0.4902028229, 0.5952004368) | (0.4962113451, 0.5464873744) |
| | (0.4998224045, 0.4975381657) | (0.4989726982, 0.4484632965) | (0.4905811881, 0.4001037334) |
| | (0.4774900413, 0.3527995405) | (0.4620935675, 0.3061946762) | (0.4422031772, 0.2613233361) |
| | (0.4159441063, 0.2198561686) | (0.3857470047, 0.1811625019) | (0.3536876773, 0.1439971375) |
| | (0.3181398973, 0.1101530816) | (0.2780109337, 0.08189125106) | (0.2353050694, 0.05769881541) |
| | (0.1914635848, 0.03563098786) | (0.1456701918, 0.01796661860) | (0.09778054930, 0.007212692365) |
| | (0.04906744224, 0.001204532393) | | |
| $U_{64}$ | (0.04887469132, 0.9963929232) | (0.09751396918, 0.9903956651) | (0.1447486299, 0.9773221707) |
| | (0.1912860150, 0.9619500784) | (0.2350676612, 0.9399102979) | (0.2777167318, 0.9157516299) |
| | (0.3163639508, 0.8855890271) | (0.3534864689, 0.8535685839) | (0.3855136472, 0.8164380680) |
| | (0.4156821807, 0.7777822153) | (0.4398564398, 0.7351063928) | (0.4619091248, 0.6912965798) |
| | (0.4772977987, 0.6447117303) | (0.4903831516, 0.5974281916) | (0.4963899683, 0.5487219439) |
| | ($\frac{1}{2}$, 0.4997796505) | (0.4987954477, 0.4507118033) | (0.4904020086, 0.4023372657) |
| | (0.4796443421, 0.3544330037) | (0.4619700029, 0.3086121701) | (0.4420673309, 0.2637143449) |
| | (0.4157867062, 0.2222108967) | (0.3874995747, 0.1820481574) | (0.3536193559, 0.1464608463) |
| | (0.3180319532, 0.1125807238) | (0.2778524953, 0.08428134544) | (0.2363312024, 0.05799017039) |
| | (0.1913965808, 0.03807019940) | (0.1455374265, 0.02038225334) | (0.09757587842, 0.009610290373) |
| | (0.04914292594, 0.001208243511) | | |

TABLE 2. Improved equilateral polygons.

| Polygon | $L$ | $\overline{L}_n$ | $d^*$ | $\underline{d}_n^*$ |
|---|---|---|---|---|
| $A_8$ | 3.095609317477 | 3.121445152258 | 2.584305440236 | 2.562915447742 |
| $W_{16}$ | 3.134706547543 | 3.136548490546 | 5.104146036426 | 5.101148618689 |
| $W_{32}$ | 3.140133809078 | 3.140331156955 | 10.19064853462 | 10.19000812355 |
| $X_{32}$ | 3.140229411639 | 3.140331156955 | 10.19033828592 | 10.19000812355 |
| $W_{64}$ | 3.141262383605 | 3.141277250933 | 20.37397459506 | 20.37387816723 |

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