SPORADIC REINHARDT POLYGONS, II

KEVIN G. HARE AND MICHAEL J. MOSSINGHOFF

ABSTRACT. A *Reinhardt polygon* is a convex n-gon that, for n not a power of 2, is optimal in three different geometric optimization problems, for example, it has maximal perimeter relative to its diameter. Some such polygons exhibit a particular periodic structure; others are termed *sporadic*. Prior work has described the periodic case completely, and has shown that sporadic Reinhardt polygons occur for all n of the form $n = pqr$ with p and q distinct odd primes and $r \geq 2$. We show that (dihedral equivalence classes of) sporadic Reinhardt polygons outnumber the periodic ones for almost all n, and find that this first occurs at $n = 105$. We also determine a formula for the number of sporadic Reinhardt polygons when $n = 2pq$ with p and q distinct odd primes.

1. INTRODUCTION

Reinhardt polygons are a class of convex polygons that are optimal in three different geometric optimization problems: they have maximal perimeter and maximal width with respect to their diameter, and maximal width relative to their perimeter [1, 2, 9]. Further, for n not a power of 2, the Reinhardt polygons with n sides are precisely the optimal convex polygons in these three problems. We describe Reinhardt polygons only briefly here; for more details we direct the reader to [5–7].

A *Reinhardt polygon* is an equilateral convex polygon P that may be inscribed in a Reuleaux polygon R in such a way that every vertex of R is a vertex of P. If we define the *skeleton* of a polygon to be its vertices, together with the line segments that connect vertices at maximal distance from one another, then it follows that the skeleton of a Reinhardt polygon with n sides contains a star polygon, where each interior angle has measure an integer multiple of π/n. For example, when n is odd, the regular n-gon is a Reuleaux polygon, and its associated skeleton is the regular star polygon with n points. A Reinhardt polygon can be described by naming the measures of each of the angles of its associated star polygon, in the order of their visitation as one traverses it edges. Since each angle is $k_i\pi/n$ for some integer k_i, and their sum is π, it follows that we may describe a Reuleaux polygon as a composition of n into an odd number of parts ℓ: $[k_1, \ldots, k_\ell]$. Further, since we consider two polygons to be equivalent if one can be obtained from the other by some combination of rotations and flips, we associate a Reinhardt polygon
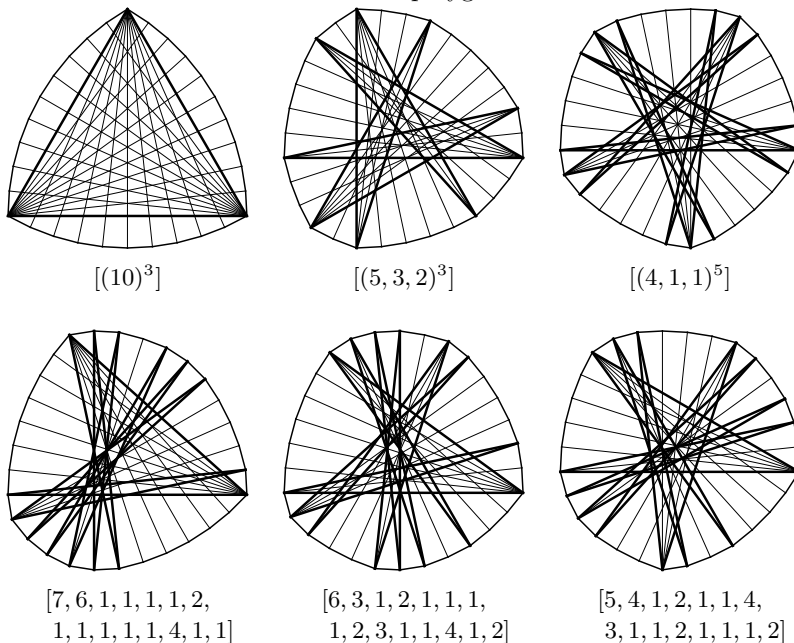
with an equivalence class of such compositions under a dihedral action. Such an equivalence class is called a *dihedral composition*.

Figure 1 exhibits six Reinhardt polygons with $n = 30$ sides, along with their associated dihedral compositions. The skeleton of each polygon is exhibited in the network of line segments within it, and the subset of each skeleton forming the associated star polygon is depicted with thicker lines. The first row of the polygons here have an evident periodic structure: for these, the associated composition is a composition for some divisor m of n into an odd number of parts, repeated n/m times, and the polygon exhibits a corresponding rotational symmetry. In these cases we abbreviate the composition by writing just the composition of m, together with n/m as an exponent to show the number of times the sequence is repeated. Reinhardt polygons with this property are called *periodic Reinhardt polygons*, and these are well understood. In [8], a formula for the number of such polygons is derived (and independently in [4] for the case of cyclic equivalence classes). If we let $E_0(n)$ denote the number of periodic Reinhardt polygons with n sides under dihedral equivalence classes, then it follows from this formula that

$$(1.1) \qquad\qquad E_0(n) = \frac{p2^{n/p}}{4n}\left(1 + o(1)\right),$$

for positive integers n with smallest odd prime divisor p. We remark that there are several additional periodic Reinhardt 30-gons, beyond the three shown in Figure 1, since $E_0(30) = 38$.

FIGURE 1. Some Reinhardt polygons with $n = 30$ sides.

$[(10)^3]$ $\qquad\qquad$ $[(5, 3, 2)^3]$ $\qquad\qquad$ $[(4, 1, 1)^5]$

$[7, 6, 1, 1, 1, 1, 2,$ $\qquad$ $[6, 3, 1, 2, 1, 1, 1,$ $\qquad$ $[5, 4, 1, 2, 1, 1, 4,$
$1, 1, 1, 1, 1, 4, 1, 1]$ $\qquad$ $1, 2, 3, 1, 1, 4, 1, 2]$ $\qquad$ $3, 1, 1, 2, 1, 1, 1, 2]$

Reinhardt [9] established a correspondence between these polygons and certain polynomials. We define a *Reinhardt polynomial* for n to be a polynomial $F(z)$ having $\deg(F) < n$, all coefficients in $\{-1, 0, 1\}$, its nonzero coefficients alternate in

sign, there are an odd number of nonzero coefficients, and the cyclotomic polynomial $\Phi_{2n}(z)$ is a factor of $F(z)$. Reinhardt in fact also required $F(0) = 1$ (and we employed this stricter definition in [5]), but it is convenient here to broaden the definition slightly. A Reinhardt polynomial $F(z) = \pm \sum_{i=0}^{\ell} (-1)^i z^{k_i}$ corresponds to the dihedral composition $[k_1-k_0, k_2-k_1, \ldots, k_\ell-k_{\ell-1}, n-k_\ell+k_0]$ of n into an odd number of parts; the condition that $\Phi_{2n}(z) \mid F(z)$ guarantees that this composition produces a closed path when constructing the star polygon. For details, see [6].

We say a Reinhardt polygon is *sporadic* if it is not periodic, and define $E_1(n)$ to be the number of sporadic Reinhardt n-gons, again using dihedral equivalence classes. For example, each of the polygons in the second row of Figure 1 is sporadic for $n = 30$. In fact, these are the only three sporadic Reinhardt triacontagons, so $E_1(30) = 3$.

In [8], the second author proved that $E_1(2^a p^b) = 0$ if p is an odd prime, $a \geq 0$, and $b \geq 1$, and computed $E_1(n)$ for a number of other values of n, in fact for all positive integers having $n - \varphi(2n) \leq 46$. Aided by that data, it was speculated there that $E_1(pq) = 0$ and that $E_1(pqr) > 0$ for distinct odd primes p and q and $r \geq 2$. Despite the fact that $E_1(n) < E_0(n)$ in each case computed there, it was also conjectured that $E_1(n) > E_0(n)$ for almost all positive integers n, and that this would first occur at $n = 105$. In [5], the authors proved the first two of these conjectures, regarding the cases $n = pq$ and $n = pqr$, and they further showed by explicit construction that a positive proportion of the Reinhardt n-gons are sporadic in the case $n = pqr$, when p and q are fixed and r is large. Some additional empirical evidence that the sporadic polygons may outnumber the periodic ones at $n = 105$ was also presented.

In this article, we describe a generalized method for constructing sporadic Reinhardt n-gons for integers of the form $n = pqr$ with p and q odd primes and $r \geq 2$. Our method allows us to prove that $E_1(n) > E_0(n)$ for almost all positive integers n, and to show that this first occurs at $n = 105$. We prove the following theorems.

Theorem 1.1. *For almost all positive integers n, the number of sporadic Reinhardt polygons exceeds the number of periodic Reinhardt polygons.*

Theorem 1.2. *The smallest integer n for which the number of sporadic Reinhardt polygons exceeds the number of periodic ones is $n = 105$.*

We also establish an exact formula for $E_1(n)$ for integers of the form $n = 2pq$, where p and q are distinct odd primes. For this, recall that the *Fermat quotient* of a prime p with base a having $p \nmid a$ is the integer $(a^{p-1} - 1)/p$.

Theorem 1.3. *For p and q distinct odd primes, the number of sporadic Reinhardt polygons with $n = 2pq$ sides is the product of the Fermat quotients with base 2 for p and q:*

$$E_1(2pq) = \frac{(2^{p-1} - 1)(2^{q-1} - 1)}{pq}.$$

Section 2 describes and verifies our new method for constructing Reinhardt polynomials for integers of the form $n = pqr$. This method depends on the choice of a composition of r into an even number of parts. Section 3 determines the number of different Reinhardt polynomials that may be produced by our method for such a fixed composition, studies how many of these correspond to sporadic Reinhardt polygons, and establishes Theorem 1.1. Section 4 describes our implementation of

this construction and the results of computations performed for a number of values of n, including $n = 105$ for establishing Theorem 1.2. Section 5 derives the formula for the case $n = 2pq$.

2. Constructing Reinhardt polynomials

Let p and q be distinct odd primes, and let r be an integer with $r \geq 2$. We describe a method for constructing a large family of Reinhardt polynomials for $n = pqr$. As in [5], by using the result of de Bruijn [3], we know that the principal ideal generated by the cyclotomic polynomial $\Phi_{2n}(z)$ in $\mathbb{Z}[z]$ is generated by $z^n + 1$ and the collection $\{\Phi_p(-z^{n/p}) : p \text{ is an odd prime and } p \mid n\}$. Thus, it suffices to construct polynomials $f_1(z)$ and $f_2(z)$ so that

$$F(z) = f_1(z)\Phi_q(-z^{pr}) + f_2(z)\Phi_p(-z^{qr})$$

has an odd number of nonzero coefficients, each ± 1, which alternate in sign. Our method developed here generalizes the construction described in [5].

2.1. **Description.** We first choose a composition $\mathbf{c} = (r_1, \ldots, r_{2m})$ of r into an even number of parts $2m$, so each r_i is a positive integer and $\sum_{i=1}^{2m} r_i = r$. We construct $f_1(z)$ as a polynomial of degree at most $pr - 1$ by describing its coefficients in p blocks of size r, which we label $A_1, \ldots, A_p$. Each block A_i is then subdivided into $2m$ sub-blocks, denoted $A_{i,j}$ with $1 \leq j \leq 2m$, according to the selected composition $\mathbf{c}$. In order to describe these components, we require some notation.

For a nonnegative integer k and $b \in \{1, -1\}$, let $S_o(k, b)$ denote the set of sequences of length k over $\{-1, 0, 1\}$ having an odd number of nonzero terms, the first of which is b, and which alternate in sign. For example, `+0-0+`, `+-+-+`, and `000+0` are all members of $S_o(5, 1)$. Likewise, let $S_e(k, b)$ denote the set of sequences of length k over $\{-1, 0, 1\}$ with an even number (possibly zero) of nonzero terms, the first of which is b (if there is a nonzero term) and alternating in sign. It is worth noting that `00000` is in both $S_e(5, 1)$ and $S_e(5, -1)$. Finally, let $Z(k)$ denote the singleton set containing only the sequence of length k consisting entirely of zeros.

We may now describe the construction of the sub-blocks $A_{i,j}$. Choose a fixed value for s from $\{-1, 1\}$. Then select

$$
(2.1) \qquad A_{i,j} \in \begin{cases} S_o(r_1 + 1, (-1)^{i+1}s), & j = 1, \\ S_e(r_j + 1, (-1)^i s), & j \geq 3 \text{ odd}, \\ Z(r_j - 1), & j \text{ even}. \end{cases}
$$

As a special case, we require that $A_{1,1}$ begins with s, not 0. The sequence of coefficients for $f_1(z)$ is created by juxtaposing these p blocks: $A_1 A_2 \ldots A_p$.

The polynomial $f_2(z)$ is also constructed relative to the composition $\mathbf{c}$ of r. In this case, we describe q blocks $B_1, \ldots, B_q$, each with size r. In the same way, each block B_i is composed of $2m$ sub-blocks, denoted $B_{i,j}$ with $1 \leq j \leq 2m$, and in this case we select

$$
(2.2) \qquad B_{i,j} \in \begin{cases} Z(r_j - 1), & j \text{ odd}, \\ S_e(r_j + 1, (-1)^i s), & j \text{ even}. \end{cases}
$$

Then we construct the sequence of coefficients for $f_2(z)$ by shifting the terms in the aggregated blocks by one position to the right, with the last element negated and then moved to the first position. If R denotes this shift-right-and-negate operator, then the coefficients of $f_2(z)$ are given in sequence by $R(B_1 B_2 \ldots B_q)$.

2.2. **Example.** Suppose $n = 120$, with $p = 3$, $q = 5$, and $r = 8$, and we choose the composition $\mathbf{c} = (1, 3, 2, 2)$, and $s = 1$. We must select $A_{1,1} = $ +0, then $A_{1,2}$ must be 00. There are four different possibilities for $A_{1,3}$: 000, -+0, -0+, or 0-+, and then $A_{1,4}$ must be 0. Suppose we select $A_{1,3} = $ -+0. Then $A_1 = $ +0|00|-+0|0, where vertical bars have been inserted between sub-blocks to aid in parsing. For A_2, the initial signs in the odd-indexed sub-blocks are inverted, so here we may choose $A_2 = $ 0-|00|+0-|0, and then for A_3 we may select 0+|00|000|0. This makes
$$f_1(z) = 1 - z^4 + z^5 - z^9 + z^{12} - z^{14} + z^{17}.$$
For $f_2(z)$, we see that $B_{i,1}$ is the empty sequence, $B_{i,2} \in S_e(4, (-1)^i)$, $B_{i,3} = $ 0, and $B_{i,4} \in S_e(3, (-1)^i)$. Suppose we choose $B_1 = $ |-+-+|0|-+0, $B_2 = $ |+-00|0|0+-, $B_3 = $ |0000|0|-0+, $B_4 = $ |+00-|0|000, and $B_5 = $ |0000|0|0-+. After applying the R operator to the combined sequence, we construct
$$f_2(z) = -1 - z + z^2 - z^3 + z^4 - z^6 + z^7 + z^9 - z^{10} + z^{15} - z^{16} - z^{22} + z^{24} + z^{25} - z^{28} - z^{39}.$$
We then compute $F(z) = f_1(z)\Phi_5(-z^{24}) + f_2(z)\Phi_3(-z^{40})$. The first 60 coefficients of each summand are then

```
+000-+00 0-00+0-0 0+000000 -000+-00 0+00-0+0 0-000000 +000-+00 0-00
--+-+0-+ 0+-0000+ -00000-0 ++00-000 0000000- ++-+-0+- 0-+0000- +000
```

Here, the coefficients are listed in blocks of size $r = 8$, with underlining showing where prior blocks were negated after multiplying $f_1(z)$ and $f_2(z)$ by $\Phi_q(-z^{pr})$ and $\Phi_p(-z^{pr})$ respectively. The last 60 coefficients of each summand for $F(z)$ are

```
+0-0 0+000000 -000+-00 0+00-0+0 0-000000 +000-+00 0-00+0-0 0+000000
00+0 --00+000 0000000+ --+-+0-+ 0+-0000+ -00000-0 ++00-000 0000000-
```

Summing both, we find the following sequence of coefficients:

```
0-+-0+-+ 00-0+0-+ -+0000-0 0+000-00 0+00-0+- +0-+-0+- +-+0-+0- +-00
+000 -000+000 -000+-0+ -0+-000+ 00-0000+ 0000-+-0 +00000-0 0+00000-
```

The polynomial $F(z)$ constructed from this list has all its coefficients in $\{-1, 0, 1\}$, its 53 nonzero coefficients alternate in sign, and $\Phi_{240}(z) \mid F(z)$. Thus, $F(z)$ is a Reinhardt polynomial for $n = 120$; its corresponding dihedral composition is

$$[1, 1, 2, 1, 1, 3, 2, 2, 1, 1, 1, 5, 3, 4, 4, 3, 2, 1, 1, 2, 1, 1, 2, 1, 1, 1,$$
$$1, 2, 1, 2, 1, 1, 3, 4, 4, 4, 4, 1, 2, 1, 2, 1, 4, 3, 5, 5, 1, 1, 2, 6, 3, 6, 2].$$

2.3. **Verifying the construction.** We claim that every polynomial $F(z)$ constructed using this method is a Reinhardt polynomial for n. Clearly $F(z)$ has $\Phi_{2n}(z)$ as a factor. Also, certainly $f_1(1) \equiv 1 \bmod 2$ and $f_2(1) \equiv 0 \bmod 2$, so $F(1) \equiv 1 \bmod 2$ since q is an odd prime. We therefore need to show that $F(z)$ has $\{-1, 0, 1\}$ coefficients, and that its nonzero coefficients alternate in sign. Clearly $f_1(z)$ and $f_2(z)$ have $\{-1, 0, 1\}$ coefficients, so we must consider how these values interact when we construct $F(z)$. We consider the coefficients of $F(z)$ in blocks of size r. Consider the nth such block with $1 \le n \le pq$, so the coefficients of $z^{(n-1)r}$, ..., z^{nr-1}, and suppose $n = i + kp$. If k is even, then the contribution to this block of coefficients from $f_1(z)\Phi_q(-z^{pr})$ is the sequence from A_i; if k is odd, then $\overline{A_i}$, the negated sequence, is used. Similarly, if $n = j + \ell q$ and ℓ is even, then for the contribution from $f_2(z)\Phi_p(-z^{qr})$, the first $r - 1$ values from B_j are used, along with the last value from B_{j-1} (unless $j = 1$, in which case it is the last value from $\overline{B_q}$), and if ℓ is odd then the negated values are used. Also, since $i + kp = j + \ell q$,

then it follows that $i+j$ and $k+\ell$ have the same parity since p and q are both odd. Thus, if $i \equiv j \bmod 2$, then either A_i is matched with (shifted) B_j, or $\overline{A_i}$ with $\overline{B_j}$; if $i \not\equiv j \bmod 2$ then A_i is paired with (shifted) $\overline{B_j}$, or $\overline{A_i}$ with B_j.

Suppose i, j, k, and ℓ are all even for a particular block n. We can exhibit the interaction of the sequences A_i and B_j in our construction for this block of size r in $F(z)$ in the following diagram. Here, $+\dots+$ denotes a sequence in $S_o(k, 1)$ for the indicated length k, and $-\dots+$ denotes a selection from $S_e(k, -1)$ for the required length. The short box at the beginning of the second line denotes the last element of the prior B block, which we note will either be 0 or -1.

(2.3)

$$
\begin{array}{cccccccc}
 & \overbrace{r_1+1} & \overbrace{r_2-1} & \overbrace{r_3+1} & \overbrace{r_4-1} & \overbrace{r_5+1} & \overbrace{r_6-1} & \\
A_i & \boxed{+\dots+} & \boxed{0} & \boxed{-\dots+} & \boxed{0} & \boxed{-\dots+} & \boxed{0} & \dots \\
B_j & \boxed{}\boxed{0} & \boxed{-\dots+} & \boxed{0} & \boxed{-\dots+} & \boxed{0} & \boxed{-\dots+} & \dots \\
 & \underbrace{r_1-1} & \underbrace{r_2+1} & \underbrace{r_3-1} & \underbrace{r_4+1} & \underbrace{r_5-1} & \underbrace{r_6+1} &
\end{array}
$$

We see that nontrivial overlaps may occur only at the boundaries of the all-zero sub-blocks, and in every case, if two nonzero values coincide, then they have opposite sign. Further, these cancellations cannot disturb the alternating sign pattern in the sum. The same diagram holds when i, j, k, and ℓ are all odd, as well as when i and k are even and j and ℓ are odd, and when i and k are odd and j and ℓ are even. The remaining four cases are similar: the corresponding diagram merely reverses the roles of $+$ and $-$. It follows that each block of r coefficients in $F(z)$ has the required alternating sign pattern. By considering the leading and trailing nonzero terms of neighboring blocks, and the possible cancellation that may occur at their boundaries, it follows that this construction always produces a Reinhardt polynomial. □

2.4. **Symmetry of the construction.** In our construction, the roles of p and q are not identical. From p, we construct our $A_{i,j}$, the first of which has the form S_o and the rest of which have the form S_e or Z. From q, we construct our $B_{i,j}$, all of which have the form S_e or Z. The next result shows that, despite this, the roles of p and q are symmetric. As a corollary of this, from a computational point of view, we may assume that $p < q$.

Lemma 2.1. *Let $f(z) = f_1(z)\Phi_q(-z^{pr}) + f_2(z)\Phi_p(-z^{qr})$ be a Reinhardt polynomial constructed as above, where $f_1(z)$ is composed of blocks A_1, A_2, ..., A_p and $f_2(z)$ is composed of blocks B_1, B_2, ..., B_q. Then there exists a dihedrally equivalent $f'(z) = f_1'(z)\Phi_p(-z^{qr}) + f_2'(z)\Phi_q(-z^{pr})$ where $f_1'(z)$ is composed of blocks A_1', A_2', ..., A_q' and $f_2'(z)$ is composed of blocks B_1', B_2', ..., B_p'.*

Proof. Let $\mathbf{c} = (r_1, r_2, \dots, r_{2m})$. Select sub-blocks $A_{i,j}$ and $B_{i,j}$ for $1 \le j \le 2m$ as in (2.1) and (2.2) to form the blocks A_1, ..., A_p and B_1, ..., B_q. For each A_i, notice that the first nonzero term is $(-1)^{i+1}s$, and that there are an odd number of alternating nonzero terms. For the first term of each A_i, add $(-1)^i s$. That is, if the first term is $(-1)^{i+1}s$, then it is changed to 0, and if it is 0 then it is changed to $(-1)^i s$.) Call these modified blocks $\widetilde{A}_i$. We see that each $\widetilde{A}_i$ has an even number of alternating nonzero terms.

For each B_i, notice that the last nonzero term, if it exists, is $(-1)^{i+1}s$, and that there are an even number of alternating nonzero terms. For the last term of each

B_i, add $(-1)^i s$. That is, if the last term is $(-1)^{i+1} s$, then it is changed to 0, and if it is 0, then it is changed to $(-1)^i s$. Call these modified blocks $\widetilde{B}_i$. We see that these $\widetilde{B}_i$ have an odd number of alternating nonzero terms.

Since the changes from B_i to $\widetilde{B}_i$ precisely balance the alterations from A_i to $\widetilde{A}_i$, we note that $f(z) = \widetilde{f}(z)$. We now have

$$\widetilde{A}_{i,j} \in \begin{cases} S_e(r_j + 1, (-1)^i s), & j \text{ odd}, \\ Z(r_j - 1), & j \text{ even}. \end{cases}$$

and

$$\widetilde{B}_{i,j} \in \begin{cases} Z(r_j - 1), & j \text{ odd}, \\ S_e(r_j + 1, (-1)^i s), & j < 2m \text{ even}, \\ S_o(r_j + 1, (-1)^i s), & j = 2m. \end{cases}$$

Let $\mathrm{Reverse}(v_1 v_2 \ldots v_{2m}) = v_{2m} v_{2m-1} \ldots v_1$. Define $A'_{i,j} = -\mathrm{Reverse}(\widetilde{B}_{i,2m+1-j})$ and $B'_{i,j} = -\mathrm{Reverse}(\widetilde{A}_{i,2m+1-j})$. Then A'_i and B'_i have the form

$$A'_{i,j} \in \begin{cases} S_o(r_{2m} + 1, (-1)^{i+1} s), & j = 1, \\ S_e(r_{2m+1-j} + 1, (-1)^i s), & j \geq 3 \text{ odd}, \\ Z(r_{2m+1-j} - 1), & j \text{ even}, \end{cases}$$

and

$$B'_{i,j} \in \begin{cases} Z(r_{2m+1-j} - 1), & j \text{ odd}, \\ S_e(r_{2m+1-j} + 1, (-1)^i s), & j \text{ even}, \end{cases}$$

as required. Here $\mathbf{c}' = (r_{2m}, r_{2m-1}, \ldots, r_1)$. This new polynomial thus produces a Reinhardt polygon in the same dihedral equivalence class as that of $f(z)$. $\qquad \square$

For example, consider $n = 45 = 3 \cdot 3 \cdot 5$, with the partition $\mathbf{c} = (1, 2)$. We use

$$A_1 = \text{+00} \qquad A_2 = \text{0-0} \qquad A_3 = \text{0+0}$$

and

$$B_1 = \text{-+0} \qquad B_2 = \text{+0-} \qquad B_3 = \text{000} \qquad B_4 = \text{0+-} \qquad B_5 = \text{000}.$$

Changing the first term of each A_i and the last term of each B_i produces

$$\widetilde{A}_1 = \text{000} \qquad \widetilde{A}_2 = \text{+-0} \qquad \widetilde{A}_3 = \text{-+0}$$

and

$$\widetilde{B}_1 = \text{-+-} \qquad \widetilde{B}_2 = \text{+00} \qquad \widetilde{B}_3 = \text{00-} \qquad \widetilde{B}_4 = \text{0+0} \qquad \widetilde{B}_5 = \text{00-}.$$

Reversing and changing the sign of each $\widetilde{A}_i$ and $\widetilde{B}_i$ gives

$$B'_1 = \text{0-+} \qquad B'_2 = \text{0+-} \qquad B'_3 = \text{000}$$

and

$$A'_1 = \text{+00} \qquad A'_2 = \text{0-0} \qquad A'_3 = \text{+00} \qquad A'_4 = \text{00-} \qquad A'_5 = \text{+-+},$$

which has the desired form.

2.5. **Remarks.** First, we note that this method generalizes the construction employed in [5] for creating Reinhardt polygons. In effect, just one composition of r was employed in that prior article, $\mathbf{c} = (1, r-1)$, so that each nonzero block of one of the polynomials in that construction was always +- or -+. The rotated coefficient at the end of each B_i was also required to be 0.

Second, it may seem that we could generalize our construction further by employing more sub-blocks with an odd number of nonzero terms. For example, when i is even, we might allow selecting $B_{i,2}$ from $S_o(r_2 + 1, -1)$ and then $A_{i,3}$ from $S_o(r_3 + 1, 1)$ (contrast with (2.3)), in addition to what is allowed in our construction. However, any polynomials we can create with this allowance can also be realized with the more restricted construction, by employing the overlapping position at the end of $B_{i,2}$ and the beginning of $A_{i,3}$ to change the parity in both sequences. It follows that we may alter the parity of the number of nonzero values selected in both $A_{i,j}$ and $B_{i,j+1}$ (or both $A_{i,j}$ and $B_{i,j-1}$) without changing the the set of possible Reinhardt polynomials that may be constructed. Since $F(z)$ must have an odd number of nonzero terms, we must specify that an odd number of nontrivial sub-blocks among the $A_{i,j}$ or $B_{i,j}$ are chosen to have an odd number of terms, for fixed i. Due to the parity switches enabled by the overlaps between sub-blocks, we may assume without loss of generality that for fixed i exactly one sub-block among the $A_{i,j}$ and $B_{i,j}$ is drawn with an odd number of terms, and we select the set $A_{i,1}$ in our construction. The constraint $A_{1,1}(1) \neq 0$ also prevents duplicate polynomials from being generated, due to the effect of the rotated element from the end of B_q.

3. Counting Reinhardt polygons

We determine the number of distinct Reinhardt polynomials for $n = pqr$ that may be produced by our construction, relative to a fixed composition of r into an even number of parts. We require one definition first: we say a sequence $v_1, \ldots, v_m$ is *d-periodic* if $d \mid n$ and $v_k = -v_{k+d}$ for each index k with $1 \leq k \leq m - d$.

Theorem 3.1. *Suppose $n = pqr$ with p and q distinct odd primes and $r \geq 2$. Let $\mathbf{c} = (r_1, \ldots, r_{2m})$ be a fixed composition of r, and let $r_e = \sum_{k=1}^{m} r_{2k}$ and $r_o = r - r_e$. Then our construction produces $2^{r_o p + r_e q}$ distinct Reinhardt polynomials for n.*

Proof. We first show that each Reinhardt polynomial constructed in this way is unique, for fixed composition $\mathbf{c} = (r_1, \ldots, r_{2m})$. We then count the number of such Reinhardt polynomials. Given $n = pqr$ and $\mathbf{c} = (r_1, \ldots r_{2m})$ as in the statement of the theorem. Suppose that $A_1, \ldots, A_p$ and $B_1, \ldots, B_q$ are selected according to the construction using the initial sign s, and these give rise to the polynomials $f_1(z)$ and $f_2(z)$, respectively. Let $g_1(z) = f_1(z)\Phi_q(-z^{pr})$ and $g_2(z) = f_2(z)\Phi_p(-z^{qr})$. Similarly, suppose $A_1^*, \ldots, A_p^*$ and $B_1^*, \ldots, B_q^*$ are constructed with initial sign s^*, producing polynomials $g_1^*(z)$ and $g_2^*(z)$. Suppose that $g_1(z) + g_2(z) = g_1^*(z) + g_2^*(z)$. Then $g_1(z) - g_1^*(z) = g_2^*(z) - g_2(z)$, and the left side is pr-periodic and the right side is qr-periodic, so both sides are r-periodic. It suffices then to investigate when $A_1 - A_1^* = R(B_1^* - B_1)$. For this, note first that many terms of A_1 and A_1^* are 0 (the ones in the sub-blocks of even index), so the corresponding positions must match B_1 and B_1^*. Likewise, required zeros in B_1 and B_1^* allow us to deduce matching elements in A_1 and A_1^*. This leaves only the positions with index $1 + \sum_{i=1}^{k} r_i$ with $0 \leq k < 2m$. Let $C(k)$ denote the kth term of the sequence C. If $s = s^*$, then

$A_{1,1}(1) = A_{1,1}^*(1) = s$, and the alternating sign requirement, together with the parity constraint on $A_{1,1}$, ensures that $A_{1,1}(r_1 + 1) = A_{1,1}^*(r_1 + 1)$, and therefore $B_{1,2}(1) = B_{1,2}^*(1)$. In the same way, we find that $B_{1,2}(r_2 + 1) = B_{1,2}^*(r_2 + 1)$, and thus $A_{1,3}(1) = A_{1,3}^*(1)$. Continuing in this way, we find that $A_1 = A_1^*$ and $B_1 = B_1^*$.

If $s \neq s^*$, then the fact that $A_{1,1}$ and $A_{1,1}^*$ begin with opposite signs, followed by $r_1 - 1$ matching values, implies that these values must all be 0, due to the alternating sign requirement. By the parity condition, we then conclude that $A_{1,1}(r_1 + 1) = A_{1,1}^*(r_1 + 1) = 0$ as well. Then $B_{1,2}(1) = B_{1,2}^*(1)$, and these must be 0 as well since $s \neq s^*$, and in the same way one finds that both $B_{1,2}$ and $B_{1,2}^*$ are entirely 0. Continuing, one deduces that $A_1(k) = B_1(k) = 0$ for $2 \leq k \leq r$, but then $A_1(1) - A_1^*(1) = 2s$, while $-(B_r^*(1) - B_r(1)) = 0$, a contradiction.

It remains to count the number of different ways to construct a polynomial $F(z)$ for n, for a fixed composition $\mathbf{c}$. Since $|S_e(k,t)| = |S_o(k,t)| = 2^{k-1}$ for any positive integer k and fixed $t \in \{-1, 1\}$, there are 2^{r_j} ways to select $A_{i,j}$ when j is odd, except for $A_{1,1}$, for which the number is 2^{r_1-1}. Then there are 2^{r_j} ways to select $B_{i,j}$ when j is even, for fixed s. Therefore, there are 2^{r_o-1} ways to construct A_1, then 2^{r_o} ways for each subsequent block A_i, and 2^{r_e} ways for each block B_i. Since the blocks $A_1, \ldots, A_p$ and $B_1, \ldots, B_q$ are constructed independently, and there are two choices for s, the formula follows. $\square$

For convenience, we say a Reinhardt polynomial is sporadic if it corresponds to a sporadic Reinhardt polygon. We next determine a lower bound on the number of sporadic Reinhardt polynomials produced by our method for a fixed composition of r into an even number of parts.

Theorem 3.2. *Suppose $n = pqr$ with p and q distinct odd primes and $r \geq 2$, let $\mathbf{c} = (r_1, \ldots, r_{2m})$ be a fixed composition of r, and let r_e and r_o be defined as above. If p and q are the only odd prime divisors of n, then this construction produces exactly $2^r(2^{r_o(p-1)} - 1)(2^{r_e(q-1)} - 1)$ different sporadic Reinhardt polynomials. Otherwise, let t denote the smallest odd prime divisor of n besides p and q, and set $U = \lfloor 2^{(2mpq+(r_o-m)p+(r_e-m)q)/t} \rfloor$. Then this construction produces at least $2^r(2^{r_o(p-1)} - 1)(2^{r_e(q-1)} - 1) - U$ different sporadic Reinhardt polynomials.*

Proof. Suppose that $F(z) = \sum_{k=1}^n v_k z^{k-1}$ is a Reinhardt polynomial constructed by using this method, with $n = pqr$ and $\mathbf{c}$ as in the statement of the theorem. Suppose further that $f_1(z)$ and $f_2(z)$ are constructed to build $F(z)$, with sequences $A_1, \ldots, A_p$ corresponding to $f_1(z)$ and sequences $B_1, \ldots, B_q$ for $f_2(z)$. Let $g_1(z) = f_1(z)\Phi_q(-z^{pr})$ and $g_2(z) = f_2(z)\Phi_p(-z^{qr})$, so that $F(z) = g_1(z) + g_2(z)$. Suppose further that $F(z)$ corresponds to a periodic Reinhardt polygon, so there exists a positive integer $d \mid n$ so that $v_k = -v_{k+d}$ for $1 \leq k \leq n - d$. Let $b = n/d$, and let ℓ be the number of nonzero v_k with $1 \leq k \leq d$. We see that there are $b\ell$ nonzero v_k for $1 \leq k \leq n$ by periodicity, and since this number is odd, then so are b and ℓ. By replacing d with an odd multiple of it if necessary, we may assume that b is an odd prime. We consider three cases.

First, suppose $b = p$, so that $d = qr$. Since $F(z)$ and $g_2(z)$ are both qr-periodic, it follows that $g_1(z)$ is as well. Thus, for each i with $1 \leq i \leq p$, we have that $A_i = A_{i+2kq}$ and $A_i = -A_{i+(2k+1)q}$ for $1 \leq k \leq (p-1)/2$, where the indices are taken modulo p (using the residue system $\{1, \ldots, p\}$). It follows that $A_i = (-1)^{i+j}A_j$ for $1 \leq i \leq j \leq p$, and thus there are 2^{r_o} ways to construct $g_1(z)$, and thus $2^{r_o+r_e q}$ such polynomials $F(z)$. Second, suppose $b = q$. In the same way, we conclude that

$B_i = (-1)^{i+j} B_j$ for $1 \leq i \leq j \leq q$, and we find that there are $2^{r_e + r_o p}$ ways to construct $F(z)$ in this case. It should be noted that it is possible for $F(z)$ to be both pr- and qr-periodic. We will account for this later.

Third, suppose $b \notin \{p, q\}$, so $r = bc$ for some positive integer c, and suppose further that $F(z)$ is neither pr-periodic nor qr-periodic. Assume that $r_i \geq 2$ for some i. Let j be an index such that v_j is in both $A_{k,i}$ and $B_{k,i}$ for some k, i.e., j is not in an overlapping section. We know that such a j exists since $r_i \geq 2$. Assume for now that i is odd. Then we see that $v_j \neq 0$ if and only if the corresponding term in $A_{k,i}$ is nonzero, and further the corresponding term in $B_{k,i}$ is zero. We then see that $v_j = (-1)^k v_{j+prk} = (-1)^k v_{j+pbck}$ for all k by construction, as $g_1(z)$ is pr-periodic. Further, by the assumption that $F(z)$ is pqc-periodic, we see that $v_j = (-1)^k v_{j+pqck}$ for all k. As $\gcd(b, q) = 1$ we then see that $v_j = (-1)^k v_{j+pck}$ for all k. This adds considerable additional structure to a polynomial having this property. In a similar fashion, if i is even, we find that $v_j = (-1)^k v_{j+qck}$ for all k.

We may now bound the number of cases where $b \notin \{p, q\}$. Let j be an index such that $v_j \neq 0$ is in both $A_{k,i}$ and $B_{k,i}$ for some k with i odd. We see in a block of size r that there are at most $r_o - m$ of these cases. At most $pc/r \leq p/t$ of these blocks of size r are needed to completely determine these v_j, since $b = r/c \geq t$. Hence we have at most $2^{(r_o - m)p/t}$ choices for these v_j. Similarly, we have at most $2^{(r_e - m)q/t}$ choices for the v_j where the corresponding i are even. Last, for those nonzero v_j with j lying in an overlap, we observe that there are at most $2m$ such indices in a block of length r, and since the v_j are pqc-periodic, we find that there are at most $2^{2mpq/t}$ choices for the v_j in this category. This produces the upper bound U for the total number of cases where $b \notin \{p, q\}$.

Since there are 2^r polynomials constructed that are both pr-periodic and qr-periodic, by using Theorem 3.1 we conclude that if p and q are the only odd prime divisors of n, then the number of polynomials that may be constructed by our method which correspond to sporadic Reinhardt polygons is precisely

$$2^{r_o p + r_e q} - 2^{r_o + r_e q} - 2^{r_e + r_o p} + 2^r = 2^r (2^{r_o(p-1)} - 1)(2^{r_e(q-1)} - 1),$$

and otherwise we obtain the lower bound $2^r (2^{r_o(p-1)} - 1)(2^{r_e(q-1)} - 1) - U$. $\qquad \square$

Next, we use this result to determine a lower bound for the number of sporadic Reinhardt n-gons which are produced by this construction for a fixed composition, and which are distinct under the dihedral action. For this, we define the *period* of a composition $\mathbf{c}$, denoted $\pi(\mathbf{c})$, as the smallest positive integer k for which $L^k(\mathbf{c}) = \mathbf{c}$, where L denotes the action of cyclically shifting the elements of a composition to the left by one position.

Corollary 3.3. *Suppose $n = pqr$ with p and q distinct odd primes, $r \geq 2$, $\mathbf{c}$ is a fixed composition of r into an even number of parts, and r_e, r_o, and U are defined as above. Set $U = 0$ if p and q are the only odd prime divisors of n, and let $v = \sum_{i=1}^{\pi(\mathbf{c})} r_i$. Then*

$$(3.1) \qquad E_1(n) \geq \frac{v}{r} \left(2^{r-2} \cdot \frac{2^{r_o(p-1)} - 1}{p} \cdot \frac{2^{r_e(q-1)} - 1}{q} - \frac{U}{4pq} \right).$$

Proof. Each of the polynomials from Theorem 3.2 corresponding to a sporadic Reinhardt polygon has n different possible cyclic shifts, each of which is another Reinhardt polynomial, but shifting by v positions to the left produces another

polynomial in the family that we construct, so we must divide by pqr/v to account for cyclic symmetries. We must also divide by 2 to account for the fact that $F(z)$ and $-F(z)$ correspond to the same Reinhardt polygon, and we divide by another factor of 2 so that we are sure to place $F(z)$ and its reversal $z^n F(1/z)$ in the same equivalence class. This accounts for flips in the dihedral action. It is possible that some polynomials $F(z)$ that we produce have a palindromic coefficient pattern (such a polynomial is often called *reciprocal*), but this only makes our lower bound on $E_1(n)$ more conservative. The bound follows. $\square$

We remark that the number of (n/b)-periodic polynomials that are neither pr-periodic nor qr-periodic is overestimated by the quantity U of Theorem 3.2. For example, if $n = 3 \cdot 5 \cdot 7$ with $p = 5$, $q = 7$, and $r = t = 3$, and $\mathbf{c} = (1, 2)$, then our upper bound U for the number of 35-periodic polynomials constructible in this way is

$$U = \lfloor 2^{(2 \cdot 5 \cdot 7 + (1-1) \cdot 5 + (2-1) \cdot 7)/3} \rfloor = \lfloor 2^{77/3} \rfloor = 53\,264\,340.$$

The actual number of 35-periodic polynomials constructible with these parameters is 0.

It is possible for the number of (n/b)-periodic polynomials to be positive. Let $n = 210 = 2 \cdot 3 \cdot 5 \cdot 7$ with $p = 3$, $q = 7$, and $r = 10$, and let $\mathbf{c} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. We select

$$A_1 = \texttt{-0 +- +- +- 00} \qquad A_2 = \texttt{0+ -+ 00 00 00} \qquad A_3 = \texttt{-0 00 00 +- +-}$$

and

$$B_1 = \texttt{0 00 00 00 +- 0} \qquad B_2 = \texttt{0 -+ -+ 00 00 0} \qquad B_3 = \texttt{0 00 +- +- +- 0}$$

$$B_4 = \texttt{0 -+ -+ 00 -+ -} \qquad B_5 = \texttt{+ 00 +- +- +- 0} \qquad B_6 = \texttt{0 00 00 00 -+ -}$$

$$B_7 = \texttt{+ 00 +- 00 00 0}.$$

From these sequences we obtain a Reinhardt polynomial with period 42, which we designate by $f(z)\Phi_5(-z^{42})$, where the coefficients of $f(z)$ are

$$\texttt{+0-+-+-0+00000-00000+00-+-000+-+000-+0-+-+}.$$

In addition, we remark that reciprocal polynomials do in fact arise in our method, although they are rather rare. Figure 2 exhibits the polygons for three of the 48 essentially different reciprocal polynomials that may be constructed for $n = 45$. One may observe the evident reflective symmetry in each.

FIGURE 2. Some sporadic Reinhardt polygons with $n = 45$ sides having a reflective symmetry.

$[2, 1, 1, 4, 1, 2, 1, 1, 9,$
$9, 1, 1, 2, 1, 4, 1, 1, 2, 1]$

$[2, 2, 2, 5, 2, 1, 6, 2,$
$2, 6, 1, 2, 5, 2, 2, 2, 1]$

$[2, 1, 1, 1, 2, 1, 2, 1, 1, 7, 1, 2,$
$2, 1, 7, 1, 1, 2, 1, 2, 1, 1, 1, 2, 1]$
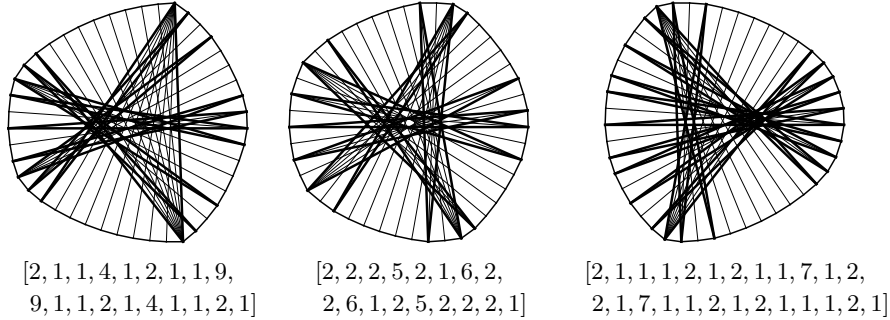
Next, we prove that there are almost always more sporadic Reinhardt polygons with n sides than periodic ones.

Proof of Theorem 1.1. From the Prime Number Theorem, the density of positive integers n having the form $n = 2^a p^b$, with p an odd prime, $a \geq 1$, and $b \geq 0$, or $n = 2^a pq$, with $a \geq 0$ and p and q distinct odd primes, is 0. We may assume then that n has the form $n = pqr$, with p and q the two smallest distinct odd prime divisors of n, and that r has at least one odd prime divisor. Choose the composition $\mathbf{c} = (1, r-1)$ of r, so that $v = r$ in Corollary 3.3. Since the smallest odd prime divisor of r is clearly at least 3, from (3.1) have

$$E_1(n) \geq \frac{2^r(2^{p-1} - 1)(2^{(q-1)(r-1)} - 1) - W}{4pq}$$

where

$$W = 2^{(2mpq + (r_o - m)p + (r_e - m)q)/3} = 2^{(2pq + (r-2)q)/3}.$$

Next, since $W \ll 2^r(2^{p-1} - 1)(2^{(q-1)(r-1)} - 1)$ as $r \to \infty$, for fixed p and q and $n = pqr$ we find that

$$E_1(n) \gg \frac{2^{n/p}}{4pq} \cdot (2^{p-1} - 1)(2^{1-q} - 2^{r(1-q)}),$$

and using (1.1), we conclude that

$$\frac{E_1(n)}{E_0(n)} > \frac{r(2^{p-1} - 1)}{p2^{q-1}}(1 + o(1))$$

as r grows large. It follows that $E_1(n) > E_0(n)$ for all such n with $r > cp2^{q-p}$, for some positive constant c. We complete the proof by considering the density of positive integers $n = pqr$ having bounded gap between their two smallest odd prime divisors.

Clearly, the density of positive integers that are not divisible by a fixed prime p is $1 - 1/p$, so the density of integers that have no odd prime divisor less than m is

$$\prod_{\substack{3 \leq p \leq m \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = \frac{2 + o(1)}{\log m}.$$

It follows that the density of positive integers that are divisible by exactly one odd prime $p < m$ is

$$\sum_{\substack{3 \leq p \leq m \\ p \text{ prime}}} \frac{1}{p} \cdot \frac{1}{1 - 1/p} \cdot \frac{2}{\log m}(1 + o(1)) = \frac{2 \log \log m}{\log m}(1 + o(1)).$$

Thus, the density of integers having at least two distinct odd prime divisors less than m is $1 - \frac{2 \log \log m}{\log m}(1 + o(1))$, and any such integer n has $q - p < m$ if p and q are the two smallest odd prime divisors of n. We conclude that the density of positive integers $n = pqr$ whose two smallest odd prime divisors satisfy $q - p < m$ and for which $r > cm2^m$ approaches 1 as $m \to \infty$. □

4. COMPUTATIONS

We implemented the construction described in Section 2 in `Maple`, and used it to create a large number of sporadic Reinhardt polygons for various values of n. Given a qualifying integer n, then for each choice of distinct odd primes p and q dividing n, we set $r = n/pq$, and then for each composition of r into an even number of parts, we compute all of the associated Reinhardt polynomials arising from our method. Each polynomial constructed is normalized with respect to sign and with respect to its dihedral equivalence class, and all of the distinct Reinhardt polygons produced are recorded.

Let $\hat{E}_1(n)$ denote the number of sporadic Reinhardt polygons with n sides up to dihedral equivalence that may be constructed by using the method of [5], and let $\ddot{E}_1(n)$ denote the number that can be built by using the more general method developed here. In [8], the exact value of $E_1(n)$ was computed for 24 different values of n that exhibit sporadic examples, and in [5] it was shown that $\hat{E}_1(n) = E_1(n)$ for nineteen of these values. Using our new method, we find that $\ddot{E}_1(n) = E_1(n)$ for the remaining five values. It should be noted that $\ddot{E}_1(105) \neq E_1(105)$, although the exact value for $E_1(105)$ is not known. The results for these five values are summarized in Table 1.

TABLE 1. Number of sporadic Reinhardt polygons constructed.

n	Factorization	r	$\hat{E}_1(n)$	$\ddot{E}_1(n)$	$E_1(n)$
60	$2^2 \cdot 3 \cdot 5$	4	3 492	4 392	4 392
75	$3 \cdot 5^2$	5	107 400	153 660	153 660
84	$2^2 \cdot 3 \cdot 7$	4	150 444	161 028	161 028
90	$2 \cdot 3^2 \cdot 5$	6	3 371 568	5 385 768	5 385 768
140	$2^2 \cdot 5 \cdot 7$	4	478 548	633 528	633 528

The nineteen other integers n for which the value of $E_1(n)$ was computed precisely in [5] all have the form $n = pqr$ with $r \in \{2, 3\}$, and fourteen of these have $r = 2$. We compute $E_1(n)$ for five additional values of n of the form $n = 2pq$, and summarize these results in Table 2. The values newly computed here are marked with an asterisk. For each of the nineteen integers n recorded here, one may verify that $E_1(n)$ is the product of the Fermat quotients for p and q with base 2, as claimed in Theorem 1.3. This formula is proved in the next section.

TABLE 2. Values of $E_1(2pq)$ (new values marked with *).

p	$q = 5$	7	11	13	17	19	23
3	3	9	93	315	3 855	13 797	182 361*
5		27	279	945	11 565	41 391	547 083*
7			837	2 835	34 695	124 173*	
11				29 295	358 515*	1 283 121*	

We close this section by considering the case $n = 105$, beginning with a proof of the second theorem announced in the introduction.

Proof of Theorem 1.2. In [8], it was shown that the number of periodic Reinhardt 105-gons is $E_0(105) = 245\,518\,324$, and the method of [5] showed that $E_1(105) \geq$

126 714 582. Using the current construction, we compute $\ddot{E}_1(105) = 211\,752\,810$. In addition, the methods of [8] allow us to count a number of additional qualifying polygons. Let $E_1(n, m)$ denote the number of sporadic Reinhardt polygons with n sides whose corresponding dihedral composition has largest part m. The values of $E_1(105, m)$ for $m = 2$ and for $m \geq 12$ were computed in [8], and the sum of these values is 12 978 294. Of these, we find that 6 394 732 cannot be generated by the method of Section 2. We therefore need more than 27 million additional sporadic polygons to show that $E_1(105) > E_0(105)$, and we find these by computing $E_1(105, m)$ for a few additional values of m. By enumerating the dihedral compositions of 105 with an odd number of parts and largest part m, and removing those with a periodic structure, we compute $E_1(105, 11) = 9\,194\,314$, $E_1(105, 10) = 15\,188\,197$, $E_1(105, 9) = 22\,135\,902$, and $E_1(105, 8) = 34\,641\,634$. Of these 81 160 047 polygons, 31 449 744 cannot be generated by our construction. We conclude that $E_1(105) \geq 249\,597\,286 > E_0(105)$. Finally, there are only thirteen positive integers $n < 105$ having the form $n = pqr$ with p and q distinct odd primes and $r \geq 2$: they are $n = 30, 42, 45, 60, 63, 66, 70, 75, 78, 84, 90, 99$, and 102. Table 1 of [8] verifies that $E_0(n) > E_1(n)$ for each of these values. $\qquad\square$

Because our construction finds 59.8% of all of the sporadic Reinhardt 105-gons whose corresponding composition has restricted largest part in the ranges we considered, we might expect the true value of $E_1(105)$ to be over 350 million.

Last, we remark that it may be possible to generalize our method for constructing Reinhardt polynomials by accounting for additional odd prime factors. For example, let $F(z) = f_3(z)\Phi_3(-z^{35}) + f_5(z)\Phi_5(-z^{21}) + f_7(z)\Phi_7(-z^{15})$, where the coefficients of $f_3(z)$, $f_5(z)$, and $f_7(z)$ are

$$f_3(z) : \texttt{0000000+-+0000-000+-+0000-+-0+0-00+,}$$
$$f_5(z) : \texttt{+000-000+-000++--+-+0,}$$
$$f_7(z) : \texttt{-000+00-00+00-0.}$$

The coefficient sequence for $F(z)$ is then

$$\texttt{0000000000+00000-+0-+-+00-+-+00-00+-+0-+0000-+-000+-+}$$
$$\texttt{-+-+-0+-0+-00+00-+-+-0+-00+-+000-00+-00+-+-+-+000-0+}$$

and so $F(z)$ is a Reinhardt polynomial for $n = 105$. However, $F(z)$ cannot be generated by using our algorithm. To see this, suppose $p = 3$ and $q = 5$, and suppose that $F(z) = A(z)\Phi_3(-z^{35}) + B(z)\Phi_5(-z^{21})$, with $A(z) = \sum_{k=0}^{34} a_k z^k$ and $B(z) = \sum_{k=0}^{21} b_k z^k$. By observing the constant term and the coefficients of z^{14}, z^{42}, and z^{70} in $F(z)$ (respectively 0, 0, 0, and 1), we find $a_0 + b_0 = 0$, $a_7 + b_7 = 0$, $-a_7 + b_0 = 0$, and $a_0 - b_7 = 1$. Subtracting the fourth equation from the first produces $b_0 + b_7 = -1$, but adding the second and third equations yields $b_0 + b_7 = 0$, a contradiction. Similar analyses show that there are no decompositions using only $f_3(z)$ and $f_7(z)$, or using only $f_5(z)$ and $f_7(z)$.

5. The case $n = 2pq$

We show that our method constructs all Reinhardt polygons when $n = 2pq$. To establish this, we first require a generalization of [5, Lemma 2.2], which is the case $k = 0$ of the following statement.

Lemma 5.1. *Let $n = 2^k pq$ with p and q distinct odd primes and $k \geq 0$, and suppose that $F(z)$ is a Reinhardt polynomial for n. Then there exist polynomials $f_1(z)$ and $f_2(z)$ with integer coefficients, $\deg(f_1) < 2^k p$, $\deg(f_2) < 2^k q$, and*

$$F(z) = f_1(z)\Phi_q(-z^{n/q}) + f_2(z)\Phi_p(-z^{n/p}).$$

Further, we may choose $f_1(z)$ and $f_2(z)$ to have all their coefficients in $\{-1, 0, 1\}$.

We omit the proof, as it is very similar to the one presented in [5], and may be obtained in essence by replacing each occurrence of z when it appears as an argument of a cyclotomic polynomial in that proof with z^{2^k}. (The roles of p and q have also been interchanged for convenience here.)

Proof of Theorem 1.3. Let $n = 2pq$ with p and q distinct odd primes. Clearly, $\mathbf{c} = (1, 1)$ is the only possible composition of $r = 2$ into an even number of parts, and Corollary 3.3 produces $E_1(n) \geq (2^{p-1} - 1)(2^{q-1} - 1)/pq$. Let $F(z)$ be a Reinhardt polynomial for n. We need to show that there exist sequences $A_1, \ldots, A_p$ over $\{-1, 0, 1\}$, each with length 2 and selected according to (2.1), and sequences $B_1, \ldots, B_q$ over $\{-1, 0, 1\}$, each with length 2 and selected according to (2.2), so that $F(z)$ is created by using these sequences in our method. By Lemma 5.1, we may assume that $F(z) = f_1(z)\Phi_q(-z^{2p}) + f_2(z)\Phi_p(-z^{2q})$, with $\deg(f_1) < 2p$, $\deg(f_2) < 2q$, and where each nonzero coefficient of both $f_1(z)$ and $f_2(z)$ is ± 1. Without loss of generality, we may assume that $F(0) = 1$, $f_1(0) = 1$, and $f_2(0) = 0$. Let $A_1, \ldots, A_p$ denote the consecutive pairs of coefficients from $f_1(z)$, and let $B_1, \ldots, B_q$ be pairs from $\{-1, 0, 1\}$ so that $R(B_1 \ldots B_q)$ are the coefficients of $f_2(z)$ in order.

Finally, by Lemma 2.1, there are no additional sporadic Reinhardt polynomials introduced by reversing the roles of p and q in the construction. $\qquad\square$

References

[1] C. Audet, P. Hansen, and F. Messine, *Isoperimetric polygons of maximum width*, Discrete Comput. Geom. **41** (2009), no. 1, 45–60. MR2470069 (2009m:51031)

[2] A. Bezdek and F. Fodor, *On convex polygons of maximal width*, Arch. Math. (Basel) **74** (2000), no. 1, 75–80. MR1728365 (2000m:52025)

[3] N. G. de Bruijn, *On the factorization of cyclic groups*, Nederl. Akad. Wetensch. Proc. Ser. A. **15** (1953), 370–377.

[4] S. Gashkov, *Inequalities for convex polygons and Reinhardt polygons*, Mat. Prosveshchenye (3) **11** (2007), 91–103. (Russian).

[5] K. G. Hare and M. J. Mossinghoff, *Sporadic Reinhardt polygons*, Discrete Comput. Geom. **49** (2013), no. 3, 540–557. MR3038529

[6] M. J. Mossinghoff, *A \$1 problem*, Amer. Math. Monthly **113** (2006), no. 5, 385–402. MR2225472 (2006m:51021)

[7] _____, *Isodiametric problems for polygons*, Discrete Comput. Geom. **36** (2006), no. 2, 363–379. MR2252109 (2007i:52014)

[8] _____, *Enumerating isodiametric and isoperimetric polygons*, J. Combin. Theory Ser. A **118** (2011), no. 6, 1801–1815. MR2793611

[9] K. Reinhardt, *Extremale Polygone gegebenen Durchmessers*, Jahresber. Deutsch. Math.-Verein. **31** (1922), 251–270.

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1.
E-mail address: `kghare@uwaterloo.ca`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, DAVIDSON COLLEGE, DAVIDSON, NC 28036 USA.
E-mail address: `mimossinghoff@davidson.edu`