The Martian Citizenship Quiz

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The Martian Citizenship Quiz consists of 30 true or false questions. To pass the quiz, we have to answer all 30 questions, and all answers must be correct. If we fail, we will be told the number k of questions we have got right, but not which ones. We may attempt the same quiz any number of times. What is the minimum number of attempts in which we can guarantee ourselves Martian citizenship?

In the first test, we answer True for all 30 questions. Suppose we are told that $k = a$, then we know that a answers should be True and $30 - a$ answers should be False. If $a = 0$ or 30, there is no problem. If $k = 1$ or 29, we can sort the odd one out by a binary search. If $k = 2$ or 28, we can still use a refined binary search and keep it well under 20 attempts.

Suppose $3 \le a \le 27$. We divide the questions into three groups of 9, with 3 left over. We claim that we only need six more attempts to find the correct answers to all 9 questions in each group. We use another two tests to find the correct answer to the 28th and the 29th questions. From the value of a, we will also know the correct answer to the last question. We have used $1 + 3 \times 6 + 2 = 21$ tests so far, and we will pass the quiz on the 22nd attempt.

We now justify our claim. Let the 9 questions be 1, 2, 3, 4, 5, 6, 7, 8 and 9. In the next four tests, we change the answers for (1,2,3,8), (1,2,4,7), (1,3,4,6) and (2,3,4,5). On each attempt, we have $k = a \pm 4$, $k = a \pm 2$ or $k = a$. We consider six cases.

Case 1. We have $k = a \pm 4$ at least once.

By symmetry, we may assume that we have $k = a - 4$ for (1,2,3,8). In the fifth test, we determine the correct answer for 4. This will also yield the correct answers for 6, 7 and 8. In the sixth test, we determine the correct answer for 9.

Case 2. We have $k = a$ all four times.

The correct answers to the pair (1,5) are the same. This is also true of each of the pairs (2,6), (3,7) and (4,8). In the fifth test, we change the answers for (1,2,5). In the sixth test, we change the answers for (3,7,9). We consider two subcases.

Subcase 2(a). $k = a \pm 3$ in the fifth test.

By symmetry, we may assume that $k = a - 3$. Then 1, 2, 5 and 6 are True while 3, 4, 7 and 8 are False. We cannot have $k = a-3$ or $k = a-1$ in the sixth test. If $k = a + 1$, then 9 is true. If $k = a + 3$ instead, 9 is False.

Subcase 2(b). $k = a \pm 1$ in the fifth test.

By symmetry, we may assume that $k = a - 1$. Then 1 and 5 are true while 2 and 6 are False. In the sixth test, if $k = a - 3$, then 3, 7 and 9 are True while 4 and 8 are False. If $k = a - 1$, then 3 and 7 are True while 4, 8 and 9 are False. If $k = a+1$, 4, 8 and 9 are True while 3 and 7 are False. If $k = a + 3$, then 4 and 8 are True while 3, 7 and 9 are False.

In all subsequent cases, we do not have $k = a \pm 4$ and we have $k = a \pm 2$ at least once. By symmetry, we may assume that we have $k = a - 2$ at least once.

Case 3. We have $k = a - 2$ exactly once.

By symmetry, we assume that this occurs for (1,2,3,8). In the fifth test, we change the answers for (1,2,5). We cannot have $k = a+3$. There are three subcases.

Subcase 3(a). $k = a - 3$.

Then 1, 2, 5 and 8 are True while 3, 4 and 7 are False. From the value of k for (1,3,4,6), we can deduce the correct answer for 6. In the sixth test, we determine the correct answer for 9.

Subcase 3(b). $k = a - 1$.

It is easy to check that 2 and 5 cannot both be True. Hence 1 is True. In the sixth test, we change the answers for (5,8,9). There are four sub-subcases.

Sub-subcase 3(b_1). $k = a - 3$.

Then 1, 3, 5, 8 and 9 are True while 2, 4 and 6 are False. From the value of k for (1,2,4,7), we can deduce the correct answer for 7.

Sub-subcase 3(b_2). $k = a - 1$.

Then 8 is True while one of 5 and 9 is True. If $k = a + 2$ for (1,2,4,7), then 1, 3, 5 and 8 are True while 2, 4, 6, 7 and 9 are False. If $k = a$ for

(1,2,4,7), then 1, 2, 8 and 9 are True while 3, 4, 5 and 7 are False. From the value of k for (1,3,4,6), we can deduce the correct answer for 6.
Sub-subcase 3(b_3). $k = a + 1$.
Then 5 is False and one of 8 and 9 is False. If $k = a$ for (2,3,4,5), then 1, 2, 3 and 9 are True while 4, 5, 6, 7 and 8 are False. If $k = a + 2$ for (2,3,4,5), then 1, 2 and 8 are True while 3, 4, 5, 7 and 9 are False. From the value of k for (1,3,4,6), we can deduce the correct answer for 6.
Sub-subcase 3(b_4). $k = a + 3$.
Then 1, 2 and 3 are True while 4, 5, 6, 7, 8 and 9 are False.
Subcase 3(c). $k = a + 1$.
One of 1 and 2 is True and the other is False. Hence 3 and 8 are True while 5 is False. Since $k > a - 1$ for both (1,3,4,6) and (2,3,4,5), 4 must be False. From the value of k for (2,3,4,5), we can determine which of 1 and 2 is True. From the values of k for (1,2,4,7) and (1,3,4,6), we can deduce the correct answers for 6 and 7. In the sixth test, we determine the correct answer for 9.

Case 4. We have $k = a - 2$ exactly twice.

By symmetry, we assume that this occurs for (1,2,3,8) and (2,3,4,5). In the fifth test, we change the answers for (1,2,5). We cannot have $k = a + 3$. There are three subcases.
Subcase 4(a). $k = a - 3$.
Then 1, 2, 3 and 5 are True while 4, 6, 7 and 8 are False.
Subcase 4(b). $k = a - 1$.
It is easy to check that 1 and 2 cannot both be False, and neither can 2 and 5. So 2 is False and 1 and 5 are True. Hence 4 and 8 are True while 3 is False. From the values of k for (1,2,4,7) and (1,3,4,6), we can deduce the correct answers for 6 and 7.
Subcase 4(c). $k = a + 1$.
It is easy to check that 1 and 2 cannot both be True, and neither can 2 and 5. So 2 is True and 1 and 5 are False. Hence 3, 4 and 8 are True while 6 and 7 are False.
In each subcase, we determine the correct answer for 9 in the sixth test.

Case 5. We have $k = a - 2$ exactly thrice.

By symmetry, we assume that this occurs for (1,2,3,8), (1,2,4,7) and (1,3,4,6). In the fifth test, we change the answers for (1,2,5). We cannot

have $k = a + 3$. There are three subcases.
Subcase 5(a). $k = a - 3$.
Then 1, 2, 3, 5 and 6 are True while 4, 7 and 8 are False.
Subcase 5(b). $k = a - 1$.
It is easy to check that 1 and 2 cannot both be True, and neither can 2 and 5. So 2 is False and 1 and 5 are True. Hence 3, 4 and 8 are True while 6 and 7 are False.
Subcase 5(c). $k = a + 1$.
It is easy to check that 1 and 2 cannot both be False, and neither can 1 and 5. So 1 is True and 2 and 5 are False. Hence 3, 4, 7 and 8 are True while 6 is False.
In each subcase, we determine the correct answer for 9 in the sixth test.

Case 6. We have $k = a - 2$ all four times.

In the fifth test, we change the answers for (1,2,5). We cannot have $k = a + 3$. There are three subcases.
Subcase 6(a). $k = a - 3$.
Then 1, 2, 5 and 6 are True. Either 3 and 7 are True while 4 and 8 are False, or the other way round. In the sixth test, we change the answers for (3,7,9). That will tell us everything.
Subcase 6(b). $k = a - 1$.
It is easy to check that 2 and 5 cannot both be True. If 1 and 2 are True, then 3 and 4 are also True while 5, 6, 7 and 8 are False. If 1 and 5 are true, then 3, 4, 7 and 8 are also True while 2 and 6 are False. In the sixth test, we change the answers for (3,7,9). That will tell us everything.
Subcase 6(c). $k = a + 1$.
It is easy to check that 1 and 2 cannot both be False, and neither can 2 and 5. So 2 is True and 1 and 5 are False. Hence 3, 4, 6, 7 and 8 are all True. In the sixth test, we determine the correct answer for 9.

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