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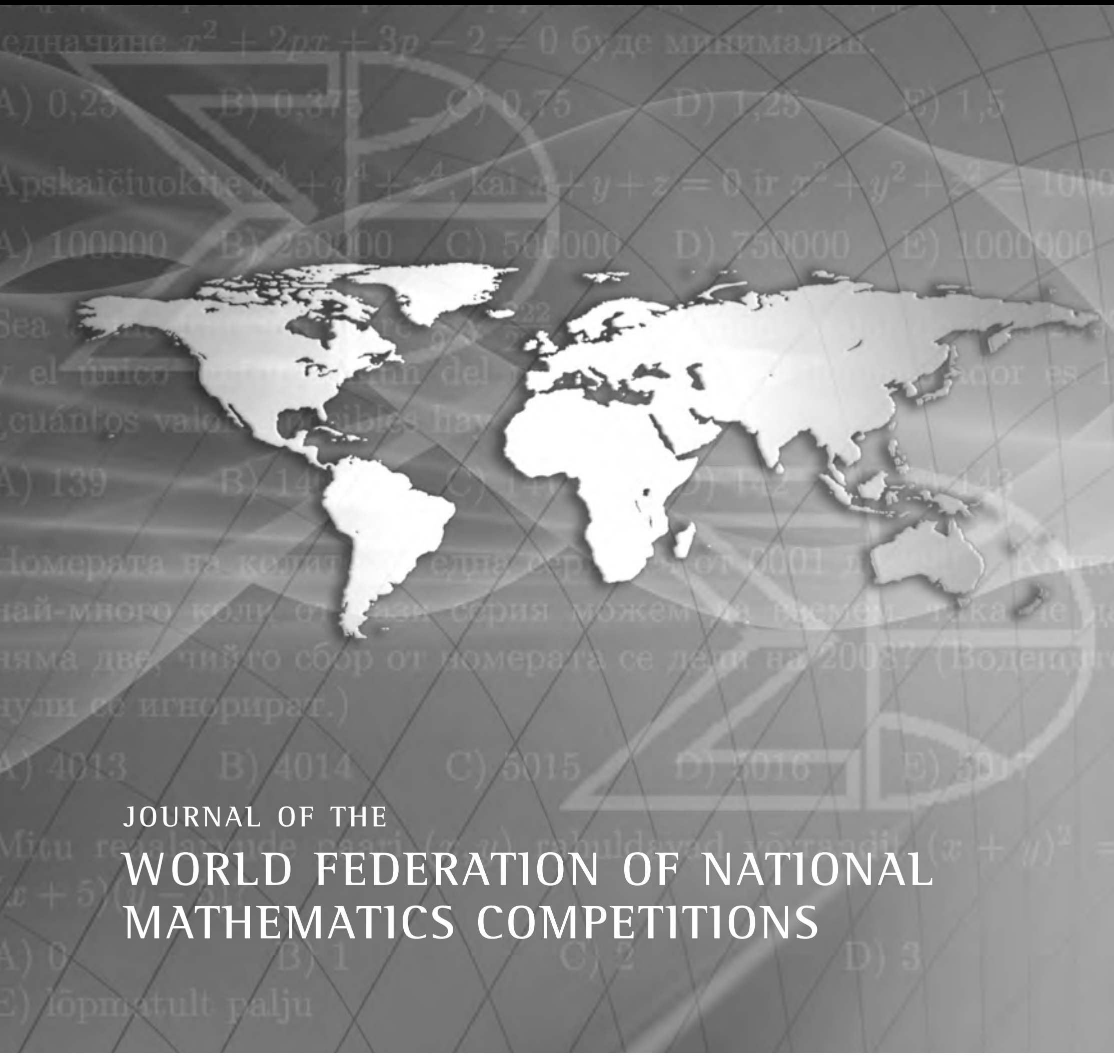

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The aims of the Federation are:

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

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From the President

Dear Fellow Federalists!

It was a true pleasure to visit with many of you at the International Congress on Mathematical Education, ICME-13, in Hamburg. As a result, speakers of our Topic Study Group (TSG) 30: *Mathematical Competitions* in collaboration with a few speakers from related topic groups, are in the late stages of producing a book *Competitions for Young Mathematicians: A Perspective from Five Continents*. This book will be published by Springer, Basel, hopefully in 2017. As the editor of the book, I have seen the submissions and believe this will be a fine book, which I warmly recommend to all of you.

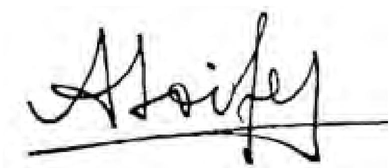
As an aside, I have just submitted my new book to Springer, New York, to be published in 2017: *The Colorado Mathematical Olympiad, The Third Decade and Further Explorations: From the Mountains of Colorado to the Peaks of Mathematics*. It continues on from the 2011 Springer book that covered the first 20 years of the Olympiad.

The Year-2016 is getting old, and will soon depart into history. At the same moment, the Baby-Year-2017 will be born.

I wish you a Happy, Successful, and Healthy New Year!

With warmest regards,

Yours always,



Alexander Soifer
President of WFNMC
December 2016

From the Editor

Welcome to *Mathematics Competitions* Vol. 29, No. 2.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefer LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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December 2016

Numbers on a circle

Kiril Bankov

Kiril Bankov is a professor of mathematics education at the University of Sofia in Bulgaria. He teaches future mathematics teachers. Bankov has been involved in mathematics competitions in Bulgaria for more than 20 years as an author of contest problems and as a member of juries. He has written many articles, made presentations, and is a co-author of books on mathematics competitions, problem solving, work with mathematically gifted students, etc. Having great experience in international mathematics education research, he works for the International Study Center of the Teacher Education and Development Study—Mathematics (TEDS-M) at Michigan State University (USA). Kiril Bankov was the Secretary of WFNMC from 2008 till 2012. In 2012 he was elected as the Senior Vice President of WFNMC.

There are a number of contest problems dealing with the following situation: several numbers are arranged on a circle and a certain admissible operation can be consecutively done a finite number of times; the task is to find conditions under which a specific final arrangement of the numbers can be obtained. The variety of these problems is determined by the different initial and final arrangements of the numbers and by the admissible operations with them. The change of some of these elements often leads to interesting generalizations. This article discusses several such examples.

1 Introduction

Life is full of operations. Many times in a day we take decisions about the series of operations to be done in order to obtain particular results. The correctness of these decisions depends on the ability to estimate the final results. Mathematics helps in modeling this reality by tasks using a particular admissible operation to transform a given situation to a different one. These problems lead to interesting generalizations by changing either the admissible operation or the initial/final situations. This article presents such examples taken from mathematics competitions in the context of arrangement of numbers on a circle.

2 Example 1

The first problem is from the 4th Austrian–Polish Mathematical Competition, 1981 [1].

Problem 1 Let $n \geq 3$ cells be arranged into a circle. Each cell can be occupied by 0 or 1. The following operation is admissible: choose any cell C occupied by a 1, change it into a 0 and simultaneously reverse the entries in the two cells adjacent to C (so that x, y become $1-x$, $1-y$).

Initially, there is a 1 in one cell and 0's elsewhere. For which values of n is it possible to obtain 0's in all cells in a finite number of admissible steps?

Solution. Denote the cells by $C_1, C_2, \ldots, C_n$. Let initially C_1 be occupied by 1 and n be such a number that after a certain number of admissible steps all cells are occupied by 0. Denote by s_i the number of operations performed in C_i and by a_i the number of the changes in C_i (i.e. when the admissible operation is performed in C_i or any of its neighbor cells C_{i-1} or C_{i+1}). Since in the final arrangement all cells are occupied by 0, then $a_1 \equiv 1 \pmod 2$ and $a_i \equiv 0 \pmod 2$ for $i \neq 1$. It is clear that $a_i \equiv s_{i-1} + s_i + s_{i+1} \pmod 2$ for $i = 1, 2, \ldots, n$. (We assume that $s_0 = s_n$ and $s_{n+1} = s_1$.)

(i) If n is divisible by 3, then

$$1 \equiv a_1 + a_4 + a_7 + \cdots + a_{n-2} \equiv \sum_{i=1}^{n} s_i \pmod 2$$

and

$$0 \equiv a_2 + a_5 + a_8 + \cdots + a_{n-1} \equiv \sum_{i=1}^{n} s_i \pmod 2,$$

which is not possible.

(ii) Let $n \equiv 1 \pmod 3$. After the performance of the admissible operation clockwise on the consecutive cells, the arrangement presented on Table 1 can be obtained.

Cell	C_1	C_2	C_3	$\dots$	C_{n-3}	C_{n-2}	C_{n-1}	C_n
Number	1	1	1	$\dots$	1	0	1	1

Table 1: Arrangement obtained when $n \equiv 1 \pmod 3$.

There are $3k$ consecutive 1's. They can be grouped in k groups by 3 and 0's can be obtained everywhere.

(iii) Let $n \equiv 2 \pmod 3$. After the performance of the admissible operation clockwise on the consecutive cells, the arrangement presented on Table 2 can be obtained.

Cell	C_1	C_2	C_3	$\dots$	C_{n-3}	C_{n-2}	C_{n-1}	C_n
Number	1	1	1	$\dots$	1	0	0	1

Table 2: Arrangement obtained when $n \equiv 2 \pmod 3$.

There are $3k$ consecutive 1's. They can be grouped in k groups by 3 and 0's can be obtained everywhere.

Therefore, it is possible to obtain 0's in all cells in a finite number of admissible steps if and only if n is not divisible by 3.

A possible change of the situation is to ask the same question under a different initial arrangement, namely, initially all cells are occupied by 1. In this case, the answer is that for any n it is possible to obtain 0's in all cells in a finite number of admissible steps. Indeed, this is obvious if $n \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, we may arrange several groups of three 1's and this way to obtain a situation when there is a 1 in one cell and 0's elsewhere; now we may apply (ii) in the above problem. If $n \equiv 2$ $\pmod 3$, we may arrange several groups of three 1's and this way to obtain a situation when there is only two neighbor 1's and 0's elsewhere; in the next step we will get a 1 in one cell and 0's elsewhere; now we may apply (iii) in the above problem.

The situation in Problem 1 generates the following:

Problem 2 Let $n \geq 3$ cells be arranged into a circle. Each cell can be occupied by 0 or 1. The following operation is admissible: choose any cell C occupied by a 1 and reverse the entries in the two cells adjacent to C (so that x, y become $1-x$, $1-y$). Initially, there is a 1 in one cell and 0's elsewhere. For which values of n is it possible to obtain 1's in all cells in a finite number of admissible steps?

The difference with problem 1 is not only in the final arrangement of the numbers (1's everywhere instead of 0's) but also that the admissible operation does not change the entry in the chosen cell C. The solution to Problem 2 is also based on a different idea. Mind that the admissible operation either does not change the number of 1's (if the entries of the two cells adjacent to C are different: 0 and 1), or change it by 2 (two more 1's if the two cells adjacent to C contain 0's, or two less 1's if they both contain 1's). Therefore, the number of 1's is always an odd number. This means that for even n it is not possible to obtain 1's in all cells. Let now n be an odd number. After the first operation three consecutive 1's can be obtained. After a certain number of admissible operations let $(2k-1)$ consecutive 1's be obtained. Now the consecutive performance of the admissible operation on the first, third, fifth, $\ldots$, $(2k-1)$-st 1's gives $(2k+1)$ consecutive 1's. This way, using induction, it is possible to obtain 1's in all cells.

Problem 2 in cases $n = 1990$ and $n = 1991$ is given on a national mathematics competition in Bulgaria, 1991 [3].

The next problem can be considered as a "reverse" of problem 2.

Problem 3 Let $n \geq 3$ cells be arranged into a circle. Each cell can be occupied by 0 or 1. The following operation is admissible: choose any cell C occupied by a 1 and reverse the entries in the two cells adjacent to C (so that x, y become $1-x$, $1-y$). Initially, there is a 1 in m cells and 0's elsewhere. For which values of m is it possible to obtain 1's in only one cell in a finite number of admissible steps?

Since the admissible operation either does not change the number or change it by 2 (see above), m must be an odd number. Now let $m = 2k+1$ be an odd number, i.e. there is a 1 in m cells and 0's elsewhere. (i) If these m cells are consecutive, we consecutively perform the admissible operation on the second, fourth, sixth,…, $2k$-th 1's and get $m-2 = 2k-1$ consecutive 1's. This way, using induction, it is possible to obtain 1's in only one cell. (ii) Now let the m cells containing 1's not be consecutive. Choose a group A of odd number (say $2s-1$) 1's. (Such a group exists, since m is an odd number.) Perform the admissible operation on the next clockwise 1's after this group A. If we continue this way (i.e. performing the admissible operation on the next clockwise 1's after group A) we will get two more 1's to group A. As a result, we decrease the groups of 1's by one at the expense of increasing the number of the consecutive 1's by two. Using this procedure, we can obtain one group of odd number of consecutive 1's. Then we can proceed as in (i). Therefore, the answer to Problem 3 is "all odd numbers".

3 Example 2

The next problem deals with a slightly different situation. The problem was given to a regional competition in Bulgaria.

Problem 4 Let $n \geq 3$ cells be arranged into a circle. Each cell contains either 1 or -1. The numbers in any two neighboring cells are multiplied, so that n products are obtained. If the sum S of these products is 0, prove that n is divisible by 4.

Solution. Denote the numbers in the cells by $a_1, a_2, \ldots, a_n$. The obtained products are $a_1a_2, a_2a_3, \ldots, a_na_1$ and they are also either 1 or -1. Since the product of these products is

$$a_1a_2a_2a_3\ldots a_na_1 = a_1^2a_2^2a_3\ldots a_n^2 > 0,$$

there is an even number of (-1)'s among the products $a_1a_2, a_2a_3, \ldots,$ a_na_1. Because $S = a_1a_2 + a_2a_3 + \cdots + a_na_1 = 0$, the number of (-1)'s in this sum is equal to the number of 1's. Therefore, the number of terms in S is an even number. But we already know that there is an even number (say $2k$) of (-1)'s among the products $a_1a_2, a_2a_3, \ldots, a_na_1$ and the same number ($2k$) is the number of 1's. Therefore, n, which is the number of the terms in S, is the sum of one and the same even number ($2k + 2k = 4k$) and is divisible by 4.

An interesting phenomena in this problem is that the reverse statement is also true, namely *if n is divisible by 4, there is an arrangement of 1's and (-1)'s in the cells, so that $S = 0$*. Certainly, consecutively write the four $\{1, 1, 1, -1\}$ several times and this gives $S = 0$.

The next step toward a possible extension is the observation that *if n is divisible by 4, then S is also divisible by 4*. This is because there is an even number of (-1)'s and also an even number of 1's among the products $a_1a_2, a_2a_3, \ldots, a_na_1$. If we allocate one and the same number of 1's (or (-1)'s) to obtain a sum of 0, the number of the remaining (-1)'s (or 1's) is divisible by 4, therefore their sum, which is S, is also divisible by 4. Similarly, the reverse statement is also true. This way we conclude that *n is divisible by 4, if and only if S is divisible by 4*.

Similar reasoning can be applied when n is not divisible by 4. The conclusion is that n and S have one and the same remainder modulo 4. Therefore the following assertion is true:

Statement In the notation in Problem 3, $n \equiv S \pmod 4$.

4 Example 3

The next problem was given in a mathematics competitions in former Yugoslavia, 1975 [2].

Problem 5 Let nine cells be arranged into a circle. Four of them are occupied by 1, the other five are occupied by 0. The following operation is admissible: draw another nine cells—one between any two of the existing cells; in these new cells write 0 if the numbers in the two neighboring existing cells are equal, and 1 if these numbers are different; then delete the existing cells. Is it possible to obtain 0's in all nine cells in a finite number of admissible steps?

Solution. Assume that in a finite number of admissible steps all nine cells contain 0's. Then in the second to the last arrangement all nine cells contain 1's. Therefore, in the previous arrangement any two neighboring cells contain different numbers, which is impossible, having nine cells.

This relatively simple problem gives birth to a variety of generalizations. There are different variations of the initial arrangements, depending on the number of cells and on the number and the positions of the initial 1's. Here is an example.

Problem 6 Let $n \geq 3$ cells be arranged into a circle. Initially, there is a 1 in one cell and 0's elsewhere. The following operation is admissible: draw another n cells—one between any two of the existing cells; in these new cells write 0 if the numbers in the two neighboring existing cells are equal, and 1 if these numbers are different, then delete the existing cells. For which values of n is it possible to obtain 0's in all cells in a finite number of admissible steps?

The conjecture is that the required values of n are all powers of 2.

The reader may try to investigate the problem with more than one initial digit 1 situated in different cells.

5 Conclusion

The circle is an amazing geometric figure. This has been known since before the beginning of recorded history. Natural circles would have been observed, such as the Moon, Sun, and a short plant stalk blowing in the wind on sand, which forms a circle shape in the sand. The circle is the basis for the wheel, which, with related inventions such as gears, makes much of modern machinery possible.

In mathematics, the study of the circle has helped inspire the development of geometry, astronomy, and calculus. Even in 1700 BC the Rhind papyrus (the best example of Egyptian mathematics) gave a method to find the area of a circular field. The result corresponds to $\frac{256}{81} \approx 3.16049\ldots$ as an approximate value of π. In 300 BC Book 3 of Euclid's Elements dealt with the properties of circles.

Besides the many interesting geometrical properties, the circle is also a closed curve. This makes it possible to consider the problems in this paper. An interesting feature of these problems is that they are not closely connected with the curriculum usually taught at school. Actually, they are not connected to any curricula because to understand the problems one does not need to possess particular mathematical knowledge. However, finding solutions needs a lot of mathematical reasoning, experience and intuition. In this respect these problems are one of the best examples of the beauty of mathematics.

The German psychologist Karl Duncker said: "Problems arise when someone has a goal for which he/she does not know a path for its achievement". From this point of view this article presents excellent examples of "problems". They are in contrast with what is usually taught at school: routine problems and exercises that are purposeful activities with a known path for its achievement.

Thanks to mathematics competitions such problems become known to students, teachers and many others that are interested in mathematics. The acquaintance and discovering of their solutions is the best way for students to get involved in sensible mathematical activities. This is also a way to present to the students the beauty of mathematical ideas and to attract them to mathematics.

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Problems on Numbers with Interesting Digits

Some problems from the Mathematical Duel Bílovec—Chorzów—Graz—Přerov presented at WFNMC Miniconference, Hamburg, Germany; July 23rd, 2016

Robert Geretschläger

Robert Geretschläger teaches at the Bundesrealgymnasium Keplerstrasse in Graz, Austria. He has been involved with all aspects of mathematics competitions in Austria for many years, among other things as leader of the Austrian IMO Team and as main organizer of the Austrian Kangaroo contest and the Mathematical Duel since 1997.

1 Introduction

For over 20 years, the schools Gymnázium Mikuláše Koperníka in Bílovec, Akademicki Zespół Szkół Ogólnokształcących in Chorzów, BRG Kepler in Graz and Gymnázium Jakuba Škody in Přerov have been holding their annual "Mathematical Duel". In this competition, students compete in three age groups (A for grades 11–12, B for grades 9–10 and C for grades 7–8) in both individual and team competitions, in which they attempt to solve olympiad-style mathematics problems. In 2014, the Mathematical Duel was granted an Erasmus+ grant by the EU, and currently the Duel is being sponsored in this way. This has broadened the scope of the competition, adding research and development aspects to the project, and bringing in the universities in Olomouc, Katowice and Graz as partners.

The structure of the competition and a bit of its history were presented in the paper *A Local International Mathematics Competition (Special Edition)*, (co-authored by Jaroslav Švrček) (*Mathematics Competitions*, Vol 18, No 1, 2005, pp. 39–51). While many of the problems used in this competition are in the classic olympiad style, there are a few differences in some cases. Some problems developed for the younger participants include the opportunity for some mathematical exploration and experimentation, often with individual points awarded for specific examples that are not yet part of a more general proof, as would normally be expected in an olympiad. Furthermore, the team competition gives the opportunity to pose questions that can be multi-level, and possibly even open-ended in some small ways.

One type of problem that has often been used in this context is discussed in this paper. Here, we consider some problems that specifically address numbers with unusual digits. This can mean numbers whose digits are all the same, or come from some limited set, or numbers in which the digits have some unusual property, as is the case with palindromic numbers. None of these problems is particularly difficult, but the level of difficulty in the collection is surprisingly varied. Tools required to solve the problems come from a surprising variety of sources, considering how similar the questions appear to be on a superficial first reading. Besides number theory, we use some tools from combinatorics and algebra, such as the pigeonhole principle or induction.

When known, the authors of the problems are named. If no name is written, the problems were my own suggestions for the competition.

Problem 1 C Team, Problem 2, 2013

We consider positive integers that are written in decimal notation using only one digit (possibly more than once), and call such numbers *uni-digit numbers.*

a) Determine a uni-digit number written with only the digit 7 that is divisible by 3.

b) Determine a uni-digit number written with only the digit 3 that is divisible by 7.

c) Determine a uni-digit number written with only the digit 5 that is divisible by 7.

d) Prove that there cannot exist a uni-digit number written with only the digit 7 that is divisible by 5.

Solution. Part a) is easily solved either by trying out small numbers of this type, or by noting that a number is divisible by 3 if the sum of its digits is. We therefore have $777 = 7 \cdot 111 = 7 \cdot 37 \cdot 3$.

b) Here, we can once again try to simply divide candidate numbers like 3, 33, 333 and so on by 7. On the other hand, we can note that $1001 = 7 \cdot 11 \cdot 13$, and therefore $333333 = 333 \cdot 1001 = 333 \cdot 7 \cdot 11 \cdot 13$.

c) As for b), we get $555555 = 555 \cdot 7 \cdot 11 \cdot 13$.

d) The last digit of any number divisible by 5 is always either 0 or 5. Any number that is divisible by 5 can therefore not be written using only the digit 7.

Problem 2 B Individual, Problem 4, 2013

We call a number that is written using only the digit 1 in decimal notation a *onesy* number, and a number using only the digit 7 in decimal notation a *sevensy* number. Determine a onesy number divisible by 7 and prove that for any sevensy number k, there always exists a onesy number m such that m is a multiple of k.

Solution. Onesy numbers are of the form $111\dots111$. We easily see that the smallest of them are not divisible by 7: 1; 11; $111 = 3 \cdot 37$; $1111 = 11 \cdot 101$.

By continuing this experimentaion, or by the same reasoning as in the previous problem, we see that a possible onesy number divisible by seven is given by $111111 = 111 \cdot 1001 = 111 \cdot 7 \cdot 11 \cdot 13$.

Sevensy numbers are of the form $k = 777\ldots777$. In order to see that there always exists a onesy multiple of any sevensy number k, we note that there exist an infinite number of onesy numbers. By the Dirichlet (pigeonhole) principle, there must therefore exist two different onesy numbers $m_1 > m_2$ with $m_1 \equiv m_2 \pmod{k}$. Writing $m_1 = 11111\ldots111$ and $m_2 = 111\ldots111$, we therefore have $m_1 - m_2 = 11\ldots1100\ldots00$.

It therefore follows that $m_1 - m_2$ is divisible by k. The number $m_1 - m_2$ can be written as $m_1 - m_2 = m \cdot 10^r$, where m is also a onesy number. Since k is certainly not divisible by 2 or 5, it follows that m must also be divisible by k, and the proof is complete. $\square$

Problem 3 B Team, Problem 1, 2005

a) A number x can be written using only the digit a both in base 8 and in base 16, i.e.

$$x = (aa\ldots a)_8 = (aa\ldots a)_{16}.$$

Determine all possible values of x.

b) Determine as many numbers x as possible that can be written in the form $x = (11\ldots1)_b$ in at least two different number systems with bases b_1 and b_2. (*author unknown*)

Solution.

a) If $(aa\ldots a)_8 = (aa\ldots a)_{16}$ holds, there exist m and n such that

$$a\cdot 16^m + a\cdot 16^{m-1} + \cdots + a\cdot 16 + a = a\cdot 8^n + a\cdot 8^{n-1} + \cdots + a\cdot 8 + a$$

holds.

This is equivalent to

$$16^m + \cdots + 16 = 8^n + \cdots + 8 \iff 16\cdot\frac{16^m - 1}{16-1} = 8\cdot\frac{8^n - 1}{8-1}$$

$$\iff 2\cdot\frac{16^m - 1}{15} = \frac{8^n - 1}{7} \iff \frac{2\cdot 16^m - 2}{15} = \frac{8^n - 1}{7}$$

$$\iff 14 \cdot 16^m - 14 = 15 \cdot 8^n - 15 \iff 15 \cdot 8^n = 14 \cdot 16^m + 1.$$

The right side is odd. Therefore, we have $n = 0$, and $m = 0$.

The only possible values of a are $a \in \{0, 1, 2, \dots, 7\}$, and we therefore have $x \in \{0, 1, 2, \dots, 7\}$. □

b) If $x = (11\dots1)_{b_1} = (11\dots1)_{b_2}$, we have $x = 1$ or $b_1, b_2 > 1$.

Assume $1 < b_1 < b_2$. We want

$$x = \sum_{i=0}^{m} b_1^i = \sum_{j=0}^{n} b_2^j \quad \text{with} \quad m > n.$$

For any $b_1 > 1$ choose $b_2 = \sum_{i=1}^{m} b_1^i$. Then

$$(11)_{b_2} = 1 \cdot b_2 + 1 \cdot b_2^0 = \sum_{i=1}^{m} b_1^i + 1 \cdot b_1^0 = \sum_{i=0}^{m} 1 \cdot b_1^i = (11\dots1)_{b_1},$$

and we have infinitely many x with the required property.

Problem 4 A Team, Problem 3, 2013

We call positive integers that are written in decimal notation using only the digits 1 and 2 *Graz numbers*. Note that 2 is a 1-digit Graz number divisible by 2^1, 12 is a 2-digit Graz number divisible by 2^2 and 112 is a 3-digit Graz number divisible by 2^3.

a) Determine the smallest 4-digit Graz number divisible by 2^4.

b) Determine an n-digit Graz number divisible by 2^n for $n > 4$.

c) Prove that there must always exist an n-digit Graz number divisible by 2^n for any positive integer n.

Solution.

a) Some 4-digit candidates are 2222, 1212, 2112. Prime decomposition of these numbers gives us

$$2222 = 2 \cdot 1111, \quad 1212 = 2^2 \cdot 303, \quad 2112 = 64 \cdot 33 = 2^6 \cdot 33,$$

and 2112 is therefore the smallest 4-digit Graz number.

b) and c) We can prove by induction that there in fact exists an n-digit Graz number for any positive integer n. Obviously, 2 is the only 1-digit Graz number, as 1 is not divisible by 2^1, but 2 is. We can therefore assume that there exists a k-digit Graz number g for some $k \geq 1$. Since g is divisible by 2^k, either $g \equiv 0 \pmod{2^{k+1}}$ or $g \equiv 2^k \pmod{2^{k+1}}$ must hold. Since $10^k \equiv 2^k \pmod{2^{k+1}}$ and $2 \cdot 10^k \equiv 0 \pmod{2^{k+1}}$, we have either $10^k + g \equiv 0 \pmod{2^{k+1}}$ or $2 \cdot 10^k + g \equiv 0 \pmod{2^{k+1}}$, and therefore the existence of an $(n-1)$-digit Graz number.

It is now easy to complete the solution. Since 112 is a 3-digit Graz number, and $112 = 16 \cdot 7$ is divisible by 16, 2112 is a 4-digit Graz number. Since $2112 = 32 \cdot 66$ is divisible by $2^5 = 32$, 22112 is a 5-digit Graz number, and the solution is complete.

Problem 5 C Individual, Problem 3, 2011

Determine the number of ten-digit numbers divisible by 4 which are written using only the digits 1 and 2. (*Józef Kalinowski*)

Solution. An example of such a number is $2212211112 = 4 \cdot 553052778$.

Any ten-digit number n divisible by 4 must end in a two-digit number divisible by 4. The last two digits of any such number written only with the digits 1 and 2 can therefore only be 12, in this order. Each of the other eight digits of the ten-digit number can be either 1 or 2. Altogether, this gives us 2^8 possibilities. There therefore exist $2^8 = 256$ ten-digit numbers with the given property.

Problem 6 B Individual, Problem 1, 2011

Let A be a six-digit positive integer which is formed using only the two digits x and y. Furthemore, let B be the six-digit integer resulting from A if all digits x are replaced by y and simultaneously all digits y are replaced by x. Prove that the sum $A + B$ is divisible by 91. (*Józef Kalinowski*)

Solution. An example of such a pair A and B is $229299 + 992922 = 1222221 = 91 \cdot 13431$. The claim certainly holds in this particular case.

In order to prove the general assertion, let $A = \overline{c_5c_4c_3c_2c_1c_0}$ and $B = \overline{d_5d_4d_3d_2d_1d_0}$, where $c_i, d_i \in \{x, y\}$, $c_i \neq d_i$ for $i = 0, 1, 2, 3, 4, 5$ and $x, y \in \{1, \ldots, 9\}$ are distinct non-zero decimal digits.

Since $c_i + d_i = x + y \neq 0$ for $i = 0, 1, 2, 3, 4, 5$ we have

$$A + B = (x + y) \cdot (10^5 + 10^4 + 10^3 + 10^2 + 10 + 1) =$$

$$= (x + y) \cdot 111111 = (x + y) \cdot 91 \cdot 1221,$$

The number $A + B$ is therefore certainly divisible by 91.

Problem 7 C Team, Problem 1, 2010

Determine the number of pairs (x, y) of decimal digits such that the positive integer in the form $\overline{xyx}$ is divisible by 3 and the positive integer in the form $\overline{yxy}$ is divisible by 4. (*author unknown*)

Solution. An example of such a pair of numbers is given by $525 = 3 \cdot 175$ and $252 = 4 \cdot 63$.

Each positive integer in the form $\overline{yxy}$ is divisible by 4 if and only if the number $\overline{xy}$ is divisible by 4 with $y \neq 0$. Hence

$$(x, y) \in \{(1; 2), (1; 6), (2; 4), (2; 8), (3; 2), (3; 6), (4; 4), (4; 8), (5; 2), \ldots$$

$$\ldots, (5; 6), (6; 4), (6; 8), (7; 2), (7; 6), (8; 4), (8; 8), (9; 2), (9; 6)\}.$$

A positive integer in the form $\overline{xyx}$ is divisible by 3 if and only if the sum of its digits is divisible by 3, i.e. $2x + y$ must be divisible by 3. After checking all possible pairs of positive integers we obtain only six possibilities:

$$(x, y) \in \{(2; 8), (3; 6), (4; 4), (5; 2), (8; 8), (9; 6)\}.$$

We therefore have 6 solutions altogether.

Problem 8 B Team, Problem 3, 2015

Determine the number of all six-digit palindromes that are divisible by seven.

[*Remark.* A six-digit palindrome is a positive integer written in the form $\overline{abccba}$ with decimal digits $a \neq 0$, b and c.] (*Pavel Calábek*)

Solution. We have

$$\overline{abccba} = 100001a + 10010b + 1100c = 7(14286a + 1430b + 157c) - (a - c).$$

Such a number is divisible by 7 iff $(a - c)$ is divisible by 7.

$a \neq 0$ and c are decimal digits, and we therefore have $-8 \leq a - c \leq 9$. The only possible values are therefore $(a - c) \in \{-7, 0, 7\}$.

For $a - c = -7$ we have $(a, c) \in \{(1, 8), (2, 9)\}$, for $a - c = 0$ we have $(a, c) \in \{(1, 1), (2, 2), \ldots, (9, 9)\}$, and finally for $a - c = 7$ we have $(a, c) \in \{(7, 0), (8, 1), (9, 2)\}$, and therefore we have 14 possibilities for the ordered pair (a, c) in total.

In all cases b is an arbitrary digit, and altogether there are therefore $14 \cdot 10 = 140$ six-digit palindromes which are divisible by 7. □

Problem 9 C Individual, Problem 3, 2012

Two positive integers are called *friends* if each is composed of the same number of digits, the digits in one are in increasing order and the digits in the other are in decreasing order, and the two numbers have no digits in common (like, for example, the numbers 147 and 952).

Solve the following problems:

a) Determine the number of all two-digit numbers that have a friend.

b) Determine the largest number that has a friend.

Solution.

a) Every two-digit number n which is composed of different digits, has its digits in increasing or decreasing order. Moreover there are at least two non-zero digits a and b different from the digits of n. It follows, that the friend of n is one of numbers $\overline{ab}$ or $\overline{ba}$.

The number of two-digit numbers with a friend is therefore equal to the number of two-digit numbers composed of different digits.

There are 90 two-digit numbers of which 9 $(11, 22, \ldots, 99)$ consist of two identical digits. There are therefore 81 two-digit numbers which have a friend. $\square$

b) If the number with a friend has k digits, its friend also has k different digits and together they have $2k$ different digits. Since there are 10 digits, the largest number with a friend has at most 5 digits.

No number begins with 0, so 0 is in a number with digits in decreasing order if $k = 5$. Moreover, if a number n with digits in increasing order has a friend k, its mirror image (that is, the number with the same digits in opposite order) is greater and has a friend (namely the mirror image of k).

The largest number with a friend has different digits in decreasing order, it has at most five digits and one of its digits is 0. The largest such number is therefore 98760 and its friend is 12345.

Problem 10 C Team, Problem 3, 2015

A *wavy number* is a number in which the digits alternately get larger and smaller (or smaller and larger) when read from left to right. (For instance, 3629263 and 84759 are wavy numbers but 45632 is not.)

a) Two five-digit wavy numbers m and n are composed of all digits from 0 to 9. (Note that the first digit of a number cannot be 0.) Determine the smallest possible value of $m + n$.

b) Determine the largest possible wavy number in which no digit occurs twice.

c) Determine a five-digit wavy number that can be expressed in the form $ab + c$, where a, b and c are all three-digit wavy numbers.

Solution.

a) The smallest possible sum is given by the expression

$$20659 + 14387 = 35046.$$

b) The largest such number is 9785634120.

c) There are many such combinations. Examples are

$$120 \cdot 142 + 231 = 17271 \quad \text{or} \quad 101 \cdot 101 + 101 = 10302.$$

Comment. This paper has also appeared in the proceedings of the MAKOS meeting, 2015, held in Zadov, Czech Republic, 1 Oct., 2015.

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Developing Problem-solving Skills

Gyula Nagy

Gyula Nagy is interested in giftedness, and quantitative, qualitative measurement of problem-solving skills, was editor-in-chief of the Hungarian Mathematical and Physical Journal for Secondary Schools. He has worked as a teacher in Szent István High School, later as professor of mathematics, 3D CAD lecturer and consultant for more than 20 years at SZIE YBL, edited projects: curriculum for math for the talented high school students, databases for gifted children and teachers, and has organized fourteen conferences for gifted children and their teachers in mathematics, physics and computer science. He has also contributed to optimization in structural engineering, geometry, and kinematics of mechanism.

This paper is based on my experience with *KöMaL* (*Középiskolai Matematikai és Fizikai Lapok*, in English: Mathematical and Physical Journal for Secondary Schools) as its editor-in-chief during the last some years. The pages of that publication provide a place for talented high school students of mathematics, physics and informatics, where they can face challenges, solve problems that fit their knowledge, compete with others of similar abilities, and develop the most, especially in critical thinking. Observations show that the students should be challenged with exercises, which they can solve only with effort. Hence, the number, difficulty and complexity of their thinking schemas grow, and after a time, they assemble into mechanisms, which help them in solving problems that are more difficult. *KöMaL* fulfills an exceptional role in offering adequate problems for that process. Its three committees involve more than twenty committed high school and university teachers who create the majority of the offered problems. The table in this paper shows how efficacy in solving problems of the journal facilitates success in the deservedly world-famous Kürschák competition and in problem solving.

1 Introduction

KöMaL has been published for 120 years in order to supplement secondary mathematics, physics and computer science education mainly for talented students. Its mission remains, as it has always been, to encourage its readers to develop a habit of analytical thinking, to offer them the intellectual pleasure of problem solving, and to help them learn how to express their thoughts accurately in writing for publication [1, 2, 4, 12, 13, 15, 29, 31, 39, 49].

Usually excellent problem solvers come from special classes. Mostly because talented students attend these classes, and furthermore because these students are led by teachers with strong qualities in analytical thinking and problem solving. The definition of talent is disputed, and we do not aim at defining it here, but we believe that the following words of Einstein in [5] describe it best:

> I have no special talents.
> I am only passionately curious.

The way a student becomes a good problem solver is a complex process, to which we would like to present a commonly known way. There definitely needs to be a good teacher who can develop the abilities of a homogeneous group in reading, writing, and counting to skill-level. In the case of an inhomogeneous group, the help of parents, other students, or a private teacher is necessary. The situation is quite the same in higher classes; the ability of the teacher to gauge the abilities of the students and to fit to it the level of difficulty of the problems is vital in the course of teaching and problem solving [28, 37, 38]. The above presented cooperation can prepare the student for matriculation, or entrance exams, and can even produce significant accomplishments on the national level. However, a qualitative leap can be experienced when the students, or adults themselves search for the problems to solve them. Then not the routine exercises, but solving the challenging problems leads to the development of thinking. Pólya presents excellent examples of such exercises in his books, mostly in the field of Mathematics [26].

In the field of thinking methodology, his book *How to Solve It* cannot easily be overemphasized. The methodologies and heuristics presented by him and in other places can be relied on as a possibility, but they do not provide a complete solution. Giving properly chosen exercises, by which the problem solver develops, to which he has to stretch himself, is very difficult. In fact, problem solving is the essence of teaching critical thinking.

2 Two media for problem solving

Kürschák: Hungarian Problem Book

Probably one of the best-known collection of problems is the work of József Kürschák, covering the problems and solutions with elaborate notes of the first national mathematics competition [6] named after Loránd Eötvös, but later renamed as the Kürschák Competition. It covered the years 1895–1928. It became Part I of *Matematikai Versenytételek* (Mathematical competition problems) [21], when it was revised [8] by György Hajós, Gyula Neukomm and János Surányi, who also published Part II of the book [9], covering the years 1929–1963. Surányi later published Part III [32] too, covering the years 1964–1987, and even Part IV [33], which covered the years 1988–1997. The English translation of Part I appeared as *Hungarian Problem Books I & II* and were the works of Elvira Rapaport [19, 20], while Part II was translated into English by Andy Liu as *Hungarian Problem Books III & IV*. (For the second volume, his co-author was Robert Barrington Leigh) [10, 11]. Parts III and IV of *Matematikai Versenytételek* are not yet published in English.

The Eötvös Competition was established by the Mathematical and Physical Society of Hungary, at the recommendation of Gyula Kőnig, under the name of "Pupils' Mathematical Competition" in 1894. This was done in honor of the Society's founder and president, the famous physicist Loránd Eötvös, who became Minister of Education that year [6, 12, 13, 16, 17, 18, 23, 27, 29, 31, 34, 35, 36, 39].

In these competitions three completely new mathematics problems were given to students. The organizing committee always paid attention to providing interesting exercises that would require only high-school

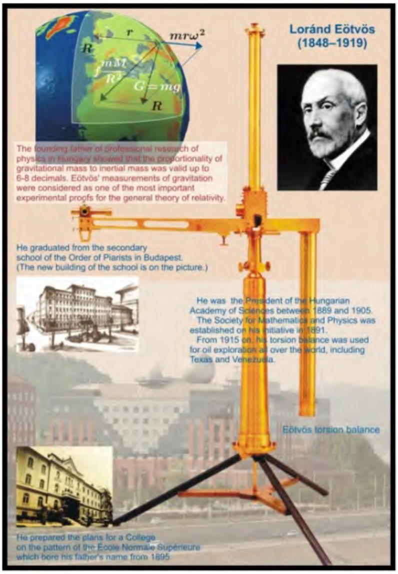

Figure 1: Loránd Eötvös (more commonly called Baron Roland von Eötvös in English literature, on a cover of the journal of *KöMaL*) and his torsion pendulum, which has been in use in oil exploration throughout the world, including Texas and Venezuela since 1915.

knowledge, but deep understanding. It can be stated without doubt that the problems were always well specified, with no ambiguity in them, and that they were accurately composed, just like the solutions given to them. The first winners were Mihály Seidner, Győző Zemplén, Lipót Fejér, Tódor Kármán [15, 27, 31, 39, 41, 42, 43].

In the years 1919–1921, when John von Neumann graduated from the Lutheran Secondary School, the mathematical competition for secondary schools did not take place because of the revolution in Hungary; but in 1918, von Neumann was permitted to sit in as an unofficial participant, and would have won the first prize [36]. Kürschák arranged that he would be tutored by the young G. Szegő; later he was also taught by M. Fekete (Fekete Mihály) and Leopold Fejér (Fejér Lipót), as well as by A. Haar (Haar Alfréd) and Frederic Riesz (Riesz Frigyes). The list of winners of the mathematical competitions before 1928 include among others L. Fejér, T. von Kármán, D. König, A. Haar, M. Riesz, G. Szegő, and E. Teller, whereas L. Szilárd won a second prize [15, 27, 31, 36, 39, 42, 43].

Figure 2: One of the contestants and some of her results in four types of competition from September 2014, the list continuous until May 2015.

Gábor Szegő said, "It would be a naïve thing to state that I taught him. . . . we met once or twice a week with von Neumann for tea, talked about mathematics and the problems that exist in set theory, integral theory and in other fields. Von Neumann promptly understood the importance of these things, and presented his own results in just one week" [16, 17, 18, 36]. Szegő also provided some guidance to von Neumann concerning the directions of mathematical development.

Many teachers would like to have such students, but naturally, many are afraid of students who are too clever. It has happened even with Pólya [26]: "There was a seminar for advanced students in Zürich that I was teaching and von Neumann was in the class. I came to a certain theorem, and I said it is not proved and it may be difficult. Von Neumann did not say anything but after five minutes, he raised his hand. When I called on him, he went to the black board and proceeded to write down the proof. After that, I was afraid of von Neumann." What can a teacher gifted with such a student do?

There is a need for *KöMaL*

What problems are most adequate for a student? Experience shows that they must be so difficult that the student can barely solve them. The role of the master is to provide such problems [22, 38]. If there are no adequate masters—and after a time, there will not be—then the student must solve many problems, among which there surely will be some that help in developing his thinking, because they are difficult enough, but still solvable by him. There are many students, who credit the hard work on the point contest of *KöMal* for their success. A typical situation was quoted in [30], when the eminent Hungarian mathematician, Kálmán Győry was asked, "Did you have anyone with whom you could discuss your ideas, solutions?" He answered: "Not really. I did not even think about going to my teachers with these type of problems. I knew I was the one who had to tackle them, and since I was really interested in these, I had a lot of fun solving such problems. Later, when I got really into them, sometimes it happened that not only during math lectures but also during other classes my mind was filled with math problems, solving them hidden under the desk".

This way our methodology is enriched, and the number of our thinking schemas, as well as their levels of difficulty and complexity grow. After a time, these schemas develop into cognitive frames, mechanisms, heuristics [7, 22, 26], which help us in solving the given problems.

József Pelikán, the coach of the Hungarian IMO team, has good memories of his experience as a participant: "In seventh grade, I finally entered the contest and turned in a few solutions ... to my father's distress, I loved playing football much better than writing up solutions... My attitude only changed when I first saw my name in print, it suddenly became real life, it was worth the effort since there was feedback ... We learned a lot from each other, and there started an intense (but always friendly) rivalry among us, concerning *KöMaL* solutions, with László Lovász, for example ... I worked really hard. I came home from school at midday, had a quick lunch, and immediately sat down to do mathematics. That continued day by day, week by week, and month by month. And the miracle occurred: We were given the opportunity (Lovász and me) to participate in the International Mathematical Olympiad as early as in grade 9." [25]. Success is grown by motivation and diligence. The feedback has become faster recently: immediately after assessment, the score appears on the list of results of the participant, and anyone can see it unless the participant blocks that option (Figure 2).

KöMaL takes an exceptional role in offering adequate problems. Its three

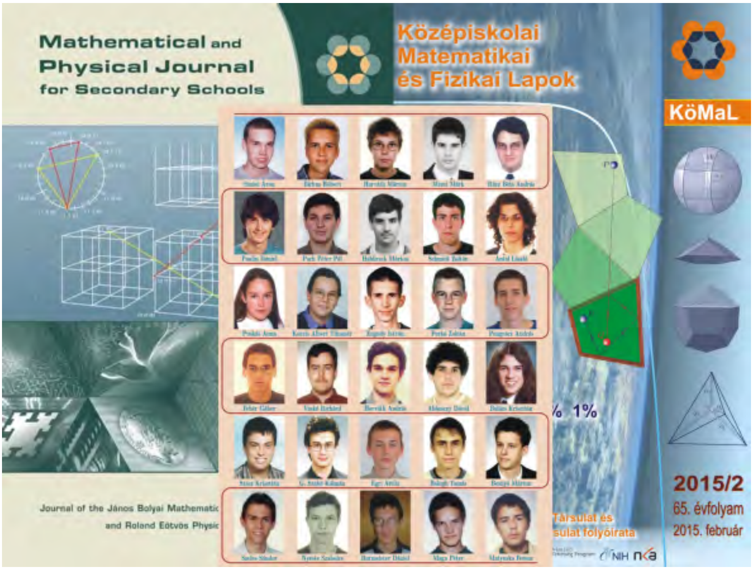

Figure 3: In the middle, the best problem solvers of the graduating class of 2015 are featured [48]. On the left an earlier cover sheet in English and on the right last year's cover from the Hungarian version are shown.

committees comprise more than twenty committed high school and university teachers who invent the majority of the offered problems. Choosing and specifying the problems is the responsibility of the appropriate committee. The coworkers and editors of the journal have been hard at work to present proper problems to high school students who crave such challenges since 1894. Our readers can send in nearly twenty problems monthly, and can follow their results on our homepage (see Figure 2).

3 Results of the *KöMaL*

Results

Figure 4 shows how efficacy in solving the problems of the journal facilitates success in the Kürschák competition [40, 41, 42, 44].

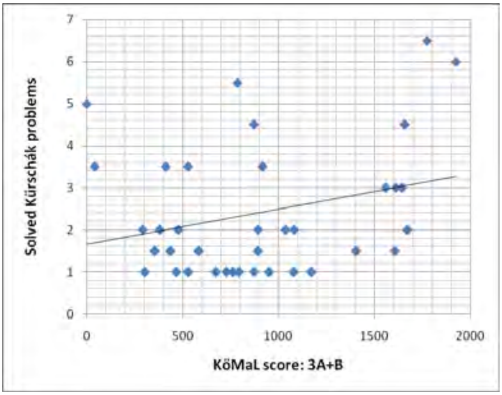

Figure 4: Accomplishment in *KöMaL* was combined into a single score against the number of Kürschák Problems solved. The resulting graph gives a good indication of the overall quality of the experience in mathematics and in problem solving.

The table depicts the points acquired by the first ten competitors of the last five years in the Kürschák competition in relation with the points acquired by the same students in the competition of *KöMaL*.

On the horizontal axis, the points acquired in the journal's A and B competitions can be seen; the points gathered in the more difficult A competition are tripled, and added to the points gathered in the easier B competition. Problems given in the A competition demand at least triple energy, and time, if the competitor can solve them at all. Experience in *KöMaL* as a predictor of the number of solved Kürschák problems was combined into a single score. The 3A+B score was used here in order to measure the quality of experience. The resulting value indicates the overall quality of experience in mathematics. The correlation analysis gives a positive but not so significant relation between them (r=0.39). However, the trendline on the graph shows the successfulness of the *KöMaL* competition. The 1000 points acquired in the competition (in the middle of the horizontal line) are in accordance with the same amount of working hours, which can lead to 2.5 solved Kürschák problems according to the trendline.

The competitors solving more than three problems attended at least two competitions, and the one acquiring 6.5 points has participated in three competitions. Participating successfully in the *KöMaL* competition is not the only key to being successful at the Kürschák competition; all components mentioned in the introduction are necessary. Hence, the introductory quote by Einstein changes to the following:

> I have a special talent,
> I am unwaveringly strenuous.

Competitors who finish with success on this competition can be regarded as the candidates, as the experts of problem solving. They fulfil the requirements of candidacy: they know and apply the needed schemas well, they have strong analytical skills, and can deliver their knowledge by expressing their thoughts accurately in writing [22].

4 Inventions

Regarding content, the editors aim to remain true to the traditions established by founders and predecessors. However, a few changes were necessary in the past few years, such as the launching of the computer science competition. Students and teachers of secondary schools face a great demand for samples of entrance exam problems. A full set of such problems are published each month, and the solutions are given in the following month.

The KöMaL homepage is continually maintained and developed according to the needs and opinions of the readers. In addition to homepage maintenance, there are lots of other administrative duties related to the journal and the point competitions [40, 44, 45, 47]. Scores achieved by solvers and subscription data needs to be recorded in databases. There is an extensive daily correspondence, the actual issue needs to be prepared for printing, the internet is searched for related content, problems and articles are checked in the archive to avoid repetitions, and the archive needs to be continually refreshed [41, 42, 46].

We make our contests as interactive as possible. KöMaL Forum is an internet site, which aims to introduce readers to mathematics, physics and computer science, the beauty of these sciences and the applications of mathematics. Of the 211 topics, one of the most visited is “Someone tell me” with 1994 comments, and maybe the most useful is “TEX—Let’s learn how to write nicely” with an easily usable practice space developed for learning TEX. The exercise book space was developed so that the signed-in competitor can choose the problem, and can immediately write their solution, even in TEX [44]. It is very important for solvers to learn, as early as possible, the rules of writing for publication, to learn how to express themselves accurately in detail, but at the same time preserving brevity and clarity. G OH Katona explained publication pressure and scientometrics in a radio show: “It is not enough to be bright; you also need to appear to be bright. I often said to my students, it is not enough to invent something smart, you also need to sell that to the public. You need to write it down nicely . . . ” [14]

The content of the *KöMaL* issues from 1994 to 2003 was published in digital form on the CD called “Aiming for the Nobel Prize”. The development of our modern internet based archive has grown out of that CD. Collecting all of the materials of *KöMaL* is still in progress. The new archive is continuously presented to the public [40, 41, 42]. It will contain more than thirty-five thousand pages, and it will be searchable by date, topic, and by names of authors and solvers. In addition to problems and articles, it will be possible to track down, through decades in the past, all those important national and international competitions that have played a role in Mathematics, Physics and Computer Science education in Hungary. At present, the issues of *KöMaL* are available at [40].

Churchill once said about the Royal Air Force:

> “Never in the history of mankind have so many owed so much to so few.”

We finish by presenting the mission of our talent-care Media with a parallel statement by Bollobás [3],

> “Never in the history of mathematics have so many owed so much to such a small journal.”

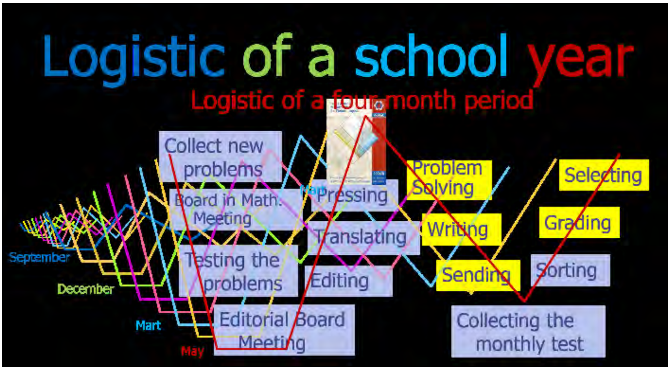

Figure 5: We can see the logistics of a four-month period in detail with red and the 4-month period 9 times a year with different colors.

5 Acknowledgements

It would be hard to highlight the names of a few eminent problem solvers from the beginning, since nearly all Hungarian mathematicians, scientists and other intellectuals used to be participants in the points competition.

In the time of Rátz, the number of problem solvers reached 200, and the current number is one order of magnitude greater. The administration needed for the competition and editorial work is presented in Figure 5.

The main graph shows parts of our work and the logistics of a four-month period. On the small graphs at the bottom part of the figure, we can see a one-year period of our work. Our contest consists of nine rounds, corresponding to the number of issues per year. “Hence I fully understand why the Kürschák Competition was easily emulated, while the *KöMaL* remained a Hungarian specialty.” So said George Berzsenyi, who was involved in organizing and initiating a variety of similar competitions in the United States.

The chief supporters of the competition and the journal with the help of the MATFUND (Hungarian High School Mathematics and Physics Foundation) include members of the Academy, and business people as

well. G. Berzsenyi, Zs. Bor, B. Bollobás, Á. Császár, K. Dobos, I. Fodor, F. Friedler, T. Földes, Á. Knuth, L. Laczkó, L. Lovász, K. Kovács, J. Pálinkás, M. Párkány, I. Szekeres and T. Vicsek. As students, almost all of them qualified among the best problem solvers.

We thank them for their support and others who help the mission of *KöMaL*.

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Preparation of 5–7 grade students for mathematics competitions: area problems

Iliana Tsvetkova

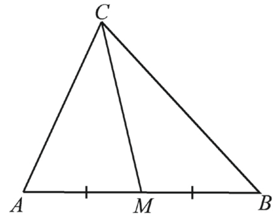
Iliana Tsvetkova is a teacher of mathematics at Sofia High School of Mathematics, Sofia, Bulgaria. She has a lot of experience working with talented mathematics students and preparing them for National and International competition. She has been a team leader during many international math competitions (PMWC, EMIC, WYMIC, WMTC, IWYMIC, Tuymaada, Jautikov olimpiad). Her students won gold, silver and bronze medals at 2004 and 2012 IMO and have won many prizes in other competitions.

Sofia High School of Mathematics is a specialized school for the preparation of mathematically talented students. As a teacher in this school I have been preparing them for competition for a long time. Geometry is one of the main areas in mathematics. To become good competitors in mathematics lower secondary students need to learn geometry a lot. In this paper I am presenting a system of problems for area that I usually use for the extracurricular work with grade 5–7 students.

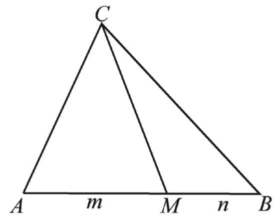
The students in lower secondary school know formulas for the area of triangles, parallelograms and trapezoids. They often use them for calculating areas. This is a routine and not very interesting activity. I am trying to motivate the young students to think using more interesting tasks. I usually present these problems when I grade 11–14 year old students for mathematics competitions and Olympiads.

The lesson begins with the formulation and proof of the statements in the basic problems (see below). These problems are not only used for the solution of the next tasks but also help the development of geometric thinking of the students. During the preparation period we use the basic problems 1–6 as theorems. However, if students need some of them for solving a problem in a competition, they have to prove the statement. This is because the basic problems, presented below, are not a part of the compulsory mathematics lessons.

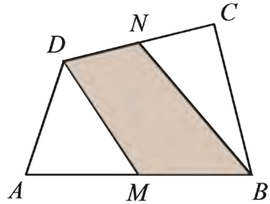
Basic problem 1.
Let CM be a median of triangle ABC. Then $S_{AMC} = S_{BMC} = \frac{1}{2}S_{ABC}$.



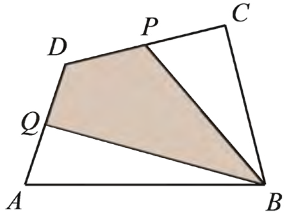
Basic problem 2.
Let M be a point of side AB of triangle ABC, such that $AM : MB = m : n$. Then $S_{AMC} : S_{BMC} = m : n$.



Basic problem 3.
In the figure, M and N are the midpoints of AB and CD, respectively. The area of the shaded quadrilateral is one-half the area of quadrilateral $ABCD$.

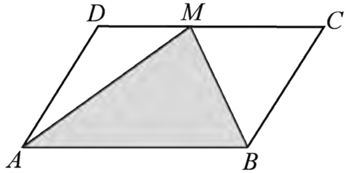


Basic problem 4.
In the figure, P and Q are midpoints of CD and AD respectively. The area of the shaded quadrilateral is one-half the area of quadrilateral $ABCD$.

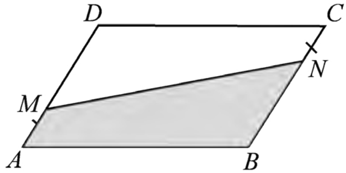


To prove the Basic problem 3 and Basic problem 4 we may construct the diagonal BD and to use Basic problem 1.

Basic problem 5.
In the figure, $ABCD$ is a parallelogram and M is a point of DC. The area of the shaded triangle is one-half the area of parallelogram $ABCD$.

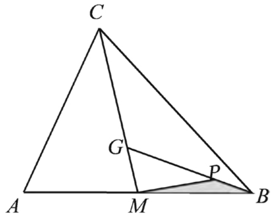


Basic problem 6.
In the figure, $ABCD$ is a parallelogram, M is a point on DA, and N is a point on AB, such that $AM = CN$. The area of the shaded quadrilateral is one-half the area of parallelogram $ABCD$.



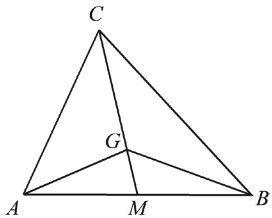
The problems that I discuss below are "Olympiad type" problems. They are usually well accepted and students enjoy working with them.

Problem 1.
In the figure, CM is a median of triangle ABC, $CG : GM = 3 : 2$ and $BP : PG = 1 : 7$. If $S_{ABC} = S$, find S_{MBP}.



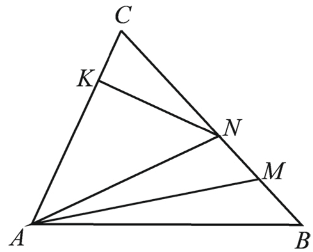
Solution. Using Basic problem 1 and twice Basic problem 2 we find $S_{MBP} = \frac{1}{35}S$.

Problem 2.
In the figure, CM is a median of triangle ABC, and G is an arbitrary point of CM. Prove that $S_{AGC} = S_{BGC}$.



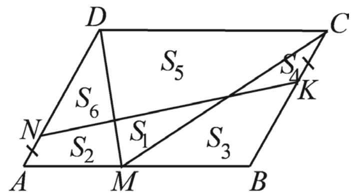
Solution. Notice that GM is median of triangle ABG. Then using Basic problem 1 twice, we obtain $S_{AGC} = S_{AMC} - S_{AMG}$ and $S_{BGC} = S_{BMC} - S_{BMG}$. This means that $S_{AGC} = S_{BGC}$.

Problem 3.
Points M and N are on side BC of the triangle ABC, point K lies on side AC such that $BM : MN : NC = 1 : 1 : 2$, and $CK : AK = 1 : 4$. If $S_{ABC} = 1$, find S_{AMNK}.



Solution. Denote $S_{NCK} = x$. From Basic problem 2 for triangle ANC we obtain that $S_{ANK} = 4x$. Segment AN is a median of triangle ABC. Therefore $S_{ANC} = S_{ABN} = 5x \Rightarrow S_{ANM} = \frac{5}{2}x, S_{ABN} = \frac{5}{2}x$. We obtain that $x + 4x + 2 \cdot \frac{5}{2}x = 1$, $x = \frac{1}{10}$. Therefore, $S_{AMNK} = S_{AMN} + S_{ANK} = \frac{5}{2}x + 4x = \frac{13}{2}x = \frac{13}{20}$.

Problem 4.
$ABCD$ is a parallelogram. Points M, K and N are on the sides AB, BC, and AD such that $AN = CK$. Show that
a) $S_5 = S_2 + S_3$,
b) $S_1 = S_4 + S_6$.

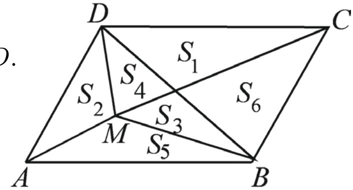


Solution. According to Basic problem 5 and Basic problem 6 we have

$$S_{DMC} = S_{ABNK} = S_{NKCD} = S_{DMA} + S_{BMC} = \frac{1}{2}S_{ABCD} \Rightarrow$$

$$\begin{aligned} &\text{a)} \quad S_5 + S_1 = S_1 + S_2 + S_3 \Rightarrow S_5 = S_2 + S_3, \\ &\text{b)} \quad S_1 + S_2 + S_3 = S_4 + S_3 + S_6 + S_2 \Rightarrow S_1 = S_4 + S_6. \end{aligned}$$

Problem 5.
Point M is inside the parallelogram $ABCD$. According to the figure, show that
a) $S_{ABM} + S_{DMC} + S_{BCD} = S_{ABCD}$,
b) $S_1 = S_2 + S_3$,
c) $S_4 + S_5 = S_6$.



Solution.

a) According to Basic problem 5 if we construct a straight line through M parallel to AB, then $S_{ABM} + S_{DMC} = \frac{1}{2}S_{ABCD}$. Also $S_{ABD} = \frac{1}{2}S_{ABCD} \Rightarrow S_{ABM} + S_{DMC} + S_{BCD} = S_{ABCD}$.

b) From $S_{ABD} = \frac{1}{2}S_{ABCD}$ and $S_{ABM} + S_{DMC} = \frac{1}{2}S_{ABCD} \Rightarrow S_1 + S_5 + S_6 = S_2 + S_3 + S_5 + S_4 \Rightarrow S_1 = S_2 + S_3$.

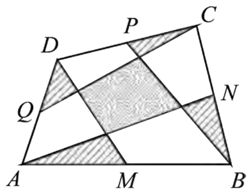
c) $S_1 + S_5 + S_4 = S_2 + S_3 + S_6 \Rightarrow S_4 + S_5 = S_6$.

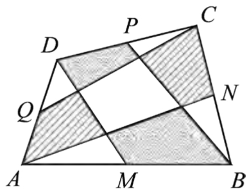
Problem 6.

In the figures, M, N, P, and Q are midpoints of sides AB, BC, CD, and DA of the convex quadrilateral $ABCD$, respectively.

a) Using the upper figure, show that the area of the regions shaded by dashes is equal to the area of the regions shaded by dots.

b) Using the lower figure, show that the area of the regions shaded by dashes is equal to the area of the regions shaded by dots.

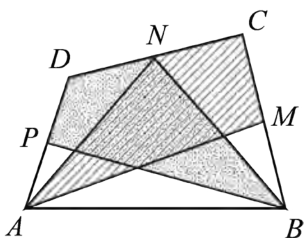




Hint. Use Basic problem 3 twice.

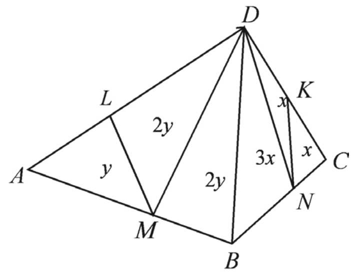
Problem 7.

In the figure, M, N, and P are the midpoints of sides BC, CD, and DA of the convex quadrilateral $ABCD$ respectively. Show that the area of the region shaded by dashes plus the area of the region shaded by dots equals the area of the quadrilateral $ABCD$.



Hint. Use Basic problem 4 twice. (Note that the central quadrilateral is shaded in both dashes and dots, i.e. its area is counted twice.)

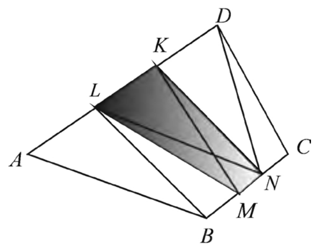
Problem 8.
Points M, N, K and L lie respectively on the sides AB, BC, CD and AD of the convex quadrilateral $ABCD$. If $AM : MB = 3 : 2$, $CN : NB = 2 : 3$, $CK = KD$ and $AL : LD = 1 : 2$, find the quotient $S_{MBNKDL} : S_{ABCD}$.



Solution. Construct the segments ML, MD, BD, DN and NK. Denote $S_{NCK} = x$ and $S_{AML} = y$. From Basic problem 2 we obtain $S_{DNK} = x$, $S_{BND} = 3x$, $S_{DML} = 2y = S_{MBD}$. Therefore,

$$\frac{S_{MBNKDL}}{S_{ABCD}} = \frac{4x + 4y}{5x + 5y} = \frac{4}{5}.$$

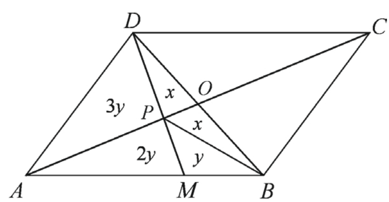
Problem 9.
On the figure, points K, L and M, N divide the opposite sides AD and BC respectively of a convex quadrilateral $ABCD$ into 3 equal segments. Prove that $S_{MNKL} = \frac{1}{3} S_{ABCD}$.



Solution. From Basic problem 2 it follows that $S_{ABL} = \frac{1}{3} S_{ABD}$ and $S_{NCD} = \frac{1}{3} S_{BCD}$, so $S_{ABL} + S_{NCD} = \frac{1}{3} S_{ABCD}$. Therefore, $S_{BNDL} = \frac{2}{3} S_{ABCD}$. From Basic problem 3 we obtain that

$$S_{MNKL} = \frac{1}{2} S_{BNDL} = \frac{1}{2} \cdot \frac{2}{3} S_{ABCD} = \frac{1}{3} S_{ABCD}.$$

Problem 10.
Let M be such a point on the side AB of parallelogram $ABCD$ that $BM : MA = 1 : 2$. The segments MD and AC intersect in P. If the area of $ABCD$ equals 1, find the area of $BCMP$.



Solution. Let O be the intersection point of AC and BD. Then O is the midpoint of both AC and BD. Denote $S_{POB} = x$ and $S_{BMP} = y$. From Basic problem 1 and Basic problem 2 it follows that $S_{POD} = x$, $S_{APM} = 2y$ and $S_{APD} = 3y$. We use again Basic problem 2 to obtain that

$$S_{AMD} = \frac{2}{3}S_{ABD} = \frac{2}{3} \cdot \frac{1}{2}S_{ABCD} = \frac{1}{3}.$$

Therefore, $5y = \frac{1}{3}$ and $y = \frac{1}{15}$. For triangle ABD: $6y + 2x = \frac{1}{2}$ and $x = \frac{1}{20}$. Finally, the area of $BCPM$ is $\frac{11}{30}$.

The ideas in this paper can be extended. Using these methods it is possible to prove that the medians in a triangle intersect in one point. Then we may prove that obtained triangles are with equal areas. Theorems of Menelaus and Cèva can also be proven using only these basic problems.

My years of experience working with these tasks show that an early introduction of students to geometric thinking and geometrical methods of solving problems develop the mathematical thinking of children, increase their interest in mathematics and contribute to the excellence in different types of competitions.

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The 57th International Mathematical Olympiad, Hong Kong, 2016

The 57th International Mathematical Olympiad (IMO) was held 6–16 July in Hong Kong. This was easily the largest IMO in history with a record number of 602 high school students from 109 countries participating. Of these, 71 were girls.

Each participating country may send a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. This Jury was chaired by Professor Kar-Ping Shum who was ably assisted by the much younger Andy Loo.[1]

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 32 problems from 121 problem proposals submitted by 40 of the participating countries from around the world. During the Jury meetings three of the shortlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. Eventually, the Jury finalised the exam questions and then made translations into the more than 50 languages required by the contestants.

The six questions that ultimately appeared on the IMO contest are described as follows.

1. An easy classical geometry problem proposed by Belgium.

2. A medium chessboard style problem proposed by Australia.

3. A difficult number theory problem with a hint of geometry proposed by Russia.

4. An easy number theory problem proposed by Luxembourg.

[1] Not to be confused with the well-known Canadian mathematician Andy Liu.

5. A medium polynomial problem proposed by Russia.

6. A difficult problem in combinatorial geometry proposed by the Czech Republic.

These six questions were posed in two exam papers held on Monday 11 July and Tuesday 12 July. Each paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

For many years now there has been an opening ceremony prior to the first day of competition. A highlight were the original music performances specially composed for the IMO by composer Dr Kwong-Chiu Mui. Following the formal speeches there was the parade of the teams and the 2016 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes, which been agreed to earlier. A local team of markers called *Coordinators* also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury.

Problem 1 turned out to be the most accessible with an average score of 5.27. At the other end, problem 3 ended up being one of the most difficult problems at the IMO averaging[2] only 0.25. Just 10 students managed to score full marks on it, while 548 students were unable to score a single point.

The medal cuts were set at 29 for gold, 22 for silver and 16 for bronze. Consequently, there were 280 (=46.5 %) medals awarded. The medal

[2]One must go back to IMO 2009 to find a problem that scored lower. Problem 6 of that year averaged 0.17 out of 7.

distributions[3] were 44 (=7.3 %) gold, 101 (=16.8 %) silver and 135 (=22.4 %) bronze. These awards were presented at the closing ceremony. Of those who did not get a medal, a further 162 contestants received an honourable mention for solving at least one question perfectly.

The following six students achieved the most excellent feat of a perfect score of 42.

Yuan Yang	China
Jaewon Choi	South Korea
Eui Cheon Hong	South Korea
Junghun Ju	South Korea
Allen Liu	United States
Yuan Yao	United States

They were given a standing ovation during the presentation of medals at the closing ceremony.

The 2016 IMO was organised by: The International Mathematical Olympiad Hong Kong Committee Limited with support from the Hong Kong University of Science and Technology and the Education Bureau of the Hong Kong SAR Government.

The 2017 IMO is scheduled to be held July 12–24 in Rio de Janeiro, Brazil. Venues for future IMOs have been secured up to 2021 as follows.

2018	Romania
2019	United Kingdom
2020	Russia
2021	United States

Much of the statistical information found in this report can also be found at the official website of the IMO `www.imo-official.org`.

[3]The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of gold, silver and bronze medals must be approximately in the ratio 1:2:3.

1 IMO Papers

First Day

Monday, July 11, 2016

Problem 1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that lines BD, FX, and ME are concurrent.

Problem 2. Find all positive integers n for which each cell of an $n \times n$ table can be filled with one of the letters *I*, *M*, and *O* in such a way that:

- in each row and each column, one-third of the entries are *I*, one-third are *M* and one-third are *O*; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one-third of the entries are *I*, one-third are *M* and one-third are *O*.

Note: The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integers (i, j) with $1 \leqslant i, j \leqslant n$. For $n > 1$, the table has $4n - 2$ *diagonals* of two types. A diagonal of the first type consists of all cells (i, j) for which $i + j$ is a constant, and a diagonal of the second type consists of all cells (i, j) for which $i - j$ is constant.

Problem 3. Let $P = A_1A_2 \ldots A_k$ be a convex polygon in the plane. The vertices A_1, A_2, …, A_k have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Second Day

Tuesday, July 12, 2016

Problem 4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

Problem 5. The equation

$$(x-1)(x-2)\cdots(x-2016) = (x-1)(x-2)\cdots(x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

Problem 6. There are $n \geqslant 2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

a) Prove that Geoff can always fulfil his wish if n is odd.

b) Prove that Geoff can never fulfil his wish if n is even.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

2 Mark Distribution by Question

Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	52	277	548	132	353	474
1	63	65	25	22	36	31
2	32	99	14	26	55	9
3	9	30	0	10	21	39
4	6	7	0	26	50	4
5	35	8	2	15	2	4
6	14	9	3	24	4	4
7	391	107	10	347	81	37
Total	602	602	602	602	602	602
Mean	5.27	2.03	0.25	4.74	1.68	0.81

3 Some Country Totals

Rank	Country	Total
1	United States of America	214
2	South Korea	207
3	China	204
4	Singapore	196
5	Taiwan	175
6	North Korea	168
7	Russia	165
7	United Kingdom	165
9	Hong Kong	161
10	Japan	156
11	Vietnam	151
12	Canada	148
12	Thailand	148
14	Hungary	145
15	Brazil	138
15	Italy	138
17	Philippines	133
18	Bulgaria	132
19	Germany	131

Rank	Country	Total
20	Indonesia	130
20	Romania	130
22	Israel	127
23	Mexico	126
24	Iran	125
25	Australia	124
25	France	124
25	Peru	124
28	Kazakhstan	122
29	Turkey	121
30	Armenia	118
30	Croatia	118
30	Ukraine	118

4 Distribution of Awards at the 2016 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	58	0	0	1	3
Algeria	41	0	0	0	2
Argentina	75	0	0	2	3
Armenia	118	0	1	4	1
Australia	124	0	2	4	0
Austria	89	0	0	3	3
Azerbaijan	79	0	0	1	4
Bangladesh	112	0	1	3	2
Belarus	112	0	1	4	1
Belgium	82	0	0	3	1
Bosnia and Herzegovina	97	0	0	4	2
Botswana	7	0	0	0	1
Brazil	138	0	5	1	0
Bulgaria	132	0	3	3	0
Cambodia	13	0	0	0	1
Canada	148	2	2	1	1
Chile	18	0	0	0	2
China	204	4	2	0	0
Colombia	63	0	0	2	1

Country	Total	Gold	Silver	Bronze	HM
Costa Rica	69	0	0	2	2
Croatia	118	0	1	4	1
Cyprus	65	0	1	0	3
Czech Republic	109	0	2	1	2
Denmark	44	0	0	0	2
Ecuador	38	0	0	0	2
Egypt	5	0	0	0	0
El Salvador	60	0	0	1	4
Estonia	67	0	0	1	3
Finland	55	0	0	0	3
France	124	0	3	2	1
Georgia	69	0	0	1	4
Germany	131	0	3	3	0
Ghana	5	0	0	0	0
Greece	84	0	0	2	3
Honduras	10	0	0	0	0
Hong Kong	161	3	2	1	0
Hungary	145	1	3	2	0
Iceland	23	0	0	0	0
India	113	0	1	5	0
Indonesia	130	0	3	3	0
Iran	125	0	3	3	0
Iraq	2	0	0	0	0
Ireland	51	0	0	0	3
Israel	127	0	3	3	0
Italy	138	1	3	0	2
Jamaica	9	0	0	0	0
Japan	156	1	4	1	0
Kazakhstan	122	1	1	3	1
Kenya	11	0	0	0	0
Kosovo	47	0	0	1	2
Kyrgyzstan	34	0	0	0	3
Laos	0	0	0	0	0
Latvia	52	0	0	0	2
Liechtenstein	2	0	0	0	0
Lithuania	84	0	0	3	3

Country	Total	Gold	Silver	Bronze	HM
Luxembourg	14	0	0	0	0
Macau	108	1	1	0	4
Macedonia (FYR)	53	0	0	0	3
Madagascar	10	0	0	0	0
Malaysia	77	0	0	2	3
Mexico	126	0	4	1	1
Moldova	65	0	0	1	3
Mongolia	115	0	2	2	2
Montenegro	24	0	1	0	0
Morocco	46	0	0	1	2
Myanmar	13	0	0	0	0
Netherlands	98	0	0	3	3
New Zealand	81	0	1	1	2
Nicaragua	45	0	0	1	3
Nigeria	24	0	0	0	1
North Korea	168	2	4	0	0
Norway	34	0	0	0	2
Pakistan	18	0	0	0	0
Paraguay	55	0	0	2	1
Peru	124	0	2	3	1
Philippines	133	2	2	0	2
Poland	102	0	2	2	2
Portugal	88	0	0	1	5
Puerto Rico	27	0	0	1	0
Romania	130	0	5	1	0
Russia	165	4	1	1	0
Saudi Arabia	104	0	0	4	2
Serbia	106	0	1	4	1
Singapore	196	4	2	0	0
Slovakia	78	0	0	2	3
Slovenia	65	0	0	0	5
South Africa	73	0	0	1	4
South Korea	207	4	2	0	0
Spain	86	0	0	2	3
Sri Lanka	63	0	0	1	4
Sweden	109	0	3	0	2

Country	**Total**	**Gold**	**Silver**	**Bronze**	**HM**
Switzerland	99	0	1	4	0
Syria	87	0	0	3	3
Taiwan	175	3	3	0	0
Tajikistan	66	0	0	0	6
Tanzania	3	0	0	0	0
Thailand	148	2	2	1	1
Trinidad and Tobago	15	0	0	0	1
Tunisia	50	0	0	0	4
Turkey	121	0	2	4	0
Turkmenistan	58	0	0	0	5
Uganda	12	0	0	0	1
Ukraine	118	0	2	4	0
United Kingdom	165	2	4	0	0
United States of America	214	6	0	0	0
Uruguay	17	0	0	1	0
Uzbekistan	47	0	0	1	2
Venezuela	29	0	0	1	1
Vietnam	151	1	4	1	0
Total (109 teams, 602 contestants)		**44**	**101**	**135**	**162**

N.B. Not all countries sent a full team of six students.

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International Mathematics Tournament of Towns Selected Problems from the Spring 2016 Tournament

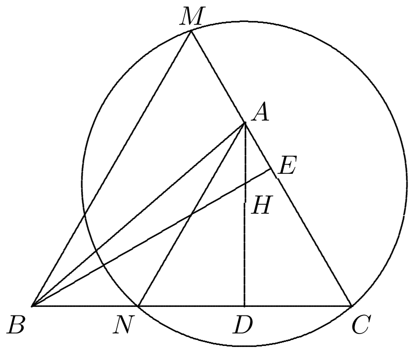
Andy Liu

1. There is at least one boy and at least one girl among twenty children in a circle. None of them is wearing more than one T-shirt. For each boy, the next child in the clockwise direction is wearing a blue T-shirt. For each girl, the next child in the counterclockwise direction is wearing a red T-shirt. Is it possible to determine the exact number of boys in the circle?

Solution. We claim that the boys and girls must alternate along the circle. Suppose to the contrary that two girls are next to each other. Then for some boy, the two children clockwise from him are both girls. The first girl must be wearing a blue T-shirt because of the boy, and a red T-shirt because of the other girl. This is a contradiction. Similarly, we cannot have two boys next to each other, and our claim is justified. It follows that the number of boys must be 10. In fact, all the boys are wearing red T-shirts and all the girls are wearing blue T-shirts.

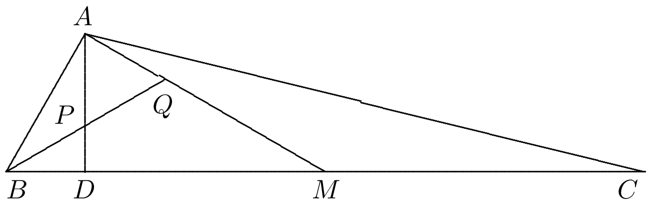
2. H is the orthocentre of triangle ABC with $\angle BCA = 60°$. The circle with centre H and passing through C cuts the lines CA and CB at M and N respectively. Prove that AN is parallel to BM or they coincide.

Solution. Suppose $CA = CB$. Then ABC is an equilateral triangle, M coincides with A and N with B. Henceforth, we assume that $CA < CB$. Then M is on the extension of CA while N is on CB. Let AD and BE be altitudes. Then each of CAD and CBE is half an equilateral triangle, so that both CAN and CBM are equilateral triangles. It follows that AN is parallel to BM.



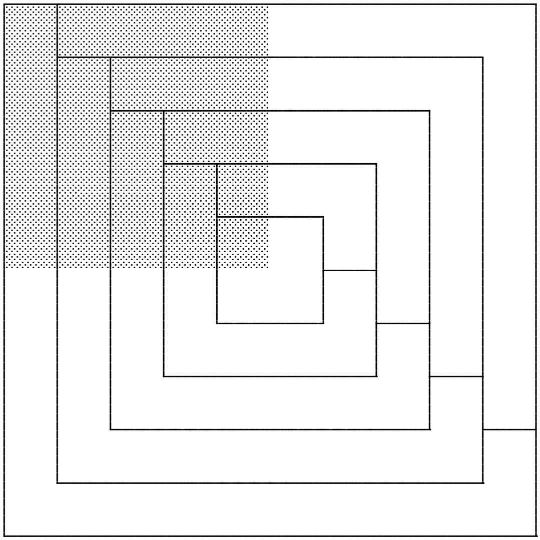
3. A median, an angle bisector and an altitude are drawn from some combination of the vertices of some triangle. Is it possible for these three lines to enclose an equilateral triangle?

Solution. This is possible. First draw a segment AB. From B, draw a ray forming a $60°$ angle with AB. From A, draw a line perpendicular to this ray, intersecting it at D. Then BAD is half an equilateral triangle. Let P be the point on AD such that $AP = 2PD$. Then BP bisects $\angle ABD$. From A, draw a line perpendicular to AB, cutting the ray at M and the extension of BP at Q. Take the point C on the ray such that $BM = MC$. In triangle ABC, AM is a median, AD is an altitude and BQ is an angle bisector, and APQ is an equilateral triangle.



4. In a 10×10 board, the 25 squares in the upper left 5×5 subboard are black while all remaining squares are white. The board is divided into a number of connected pieces of various shapes and sizes such that the number of white squares in each piece is three times the number of black squares in that piece. What is the maximum number of pieces?

Solution. Since there are only 9 black squares which share common sides with white squares, the number of pieces is at most 9. The diagram below shows that we can have 9 pieces.



5. Must a sphere pass through the midpoints of all 12 edges of a cube if it passes through at least (a) 6 of them; (b) 7 of them?

Solution. There is a sphere Ω which passes through the midpoints of all 12 edges of the cube. Stand the cube on a vertex so that the space diagonal joining it to the opposite vertex is vertical. Then the 12 points lie on three horizontal planes in a 3-6-3 distribution.

(a) The answer is "No". The 6 points in the horizontal plane in between the other two are vertices of a regular hexagon. There are infinitely many spheres other than Ω which pass through all of them.

(b) The answer is "Yes". Suppose a sphere passes through 7 of these 12 points. By the pigeonhole principle, one of the three horizontal planes must contain 3 of them. Moreover, another of the 7 points must lie on a different plane. Hence the sphere has 4 non-coplanar points in common with Ω, and must therefore coincide with Ω.

6. Do there exist integers a and b such that the equation $x^2+kax+b=0$ has no real roots, and the equation $\lfloor x^2\rfloor + kax + b = 0$ has at least one real root, where (a) $k=1$; (b) $k=2$?

Solution (by Victor Rong).

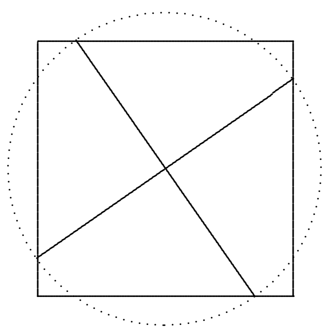
(a) We can take $a=-3$ and $b=3$. The discriminant of $x^2-3x+3=0$ is $(-3)^2-4\times 3=-3$. Hence it has no real roots. However, $\lfloor(\frac{4}{3})^2\rfloor - 3(\frac{4}{3})+3 = 1-4+3=0$.

(b) Suppose that a and b are integers such that $x^2+2ax+b=0$ has no real roots but $\lfloor x^2\rfloor + 2ax+b=0$ has at least one real root. Then $(2a)^2-4b<0$ so that $b>a^2$. Note that $2ax$ is an integer. We have a contradiction since

$$0=\lfloor x^2\rfloor+2ax+b>\lfloor x^2\rfloor+2ax+a^2=\lfloor x^2+2ax+a^2\rfloor=\lfloor(x+a)^2\rfloor\geq 0.$$

7. Dissect a 10×10 square into 100 congruent quadrilaterals which have circumcircles of diameter $\sqrt{3}$.

Solution. First divide the 10×10 square into twenty-five 2×2 squares. Draw a circle with radius $\sqrt{\frac{3}{2}}$ centred at the centre of the square, cutting its perimeter in eight points. Join four alternate points in opposite pairs by two perpendicular segments, dividing the square into four congruent quadrilaterals having two right angles. This quadrilateral is cyclic, and its diameter joins the other two vertices. Its length is indeed $\sqrt{\frac{3}{2}+\frac{3}{2}}=\sqrt{3}$.



8. On the blackboard are several monic polynomials of degree 37, with non-negative coefficients. In each move, we may replace two of them by two other monic polynomials of degree 37, such that either the sum or the product of the new pair is equal to the sum or the product, respectively, of the old pair. Coefficients are not required to be non-negative. Prove that after any finite number of moves, at least one polynomial does not have 37 distinct positive roots.

Solution (by Central Jury). We seek an invariant amidst all the changes, and this is the sum of the coefficients of the x^{36} terms of all the polynomials, which is also the sum of all 37 roots of each of them. When we replace two polynomials by two others with the same product, the combined set of roots of the old polynomials is the same as the combined set of roots of the new polynomials. When we replace two polynomials by two others with the same sum, the sum of the coefficients of their x^{36} terms is unchanged. Initially, all coefficients are non-negative. Hence this invariant is also non-negative. However, if after a finite number of moves, every polynomial has 37 distinct real roots, then this invariant must be negative. We have a contradiction.

9. There are m good batteries and $n > 2$ bad batteries. They are not distinguishable until used to light an electrical torch, the proper functioning of which requires two good batteries. What is the minimum number of attempts in order for the torch to function properly, if (a) $m = n + 1$; (b) $m = n$?

Solution (by Central Jury).

(a) The upper bound is $n + 2$. Test the $2n + 1$ batteries in pairs, leaving one off. We may as well assume that nothing works. Then each pair consists of a good battery and a bad one. Moreover, the one left off is good. Test it with both batteries in any pair, and we will have a working combination. Consider any plan with $n + 1$ attempts. Construct a graph with $2n+1$ vertices representing the batteries, and $n+1$ edges representing the attempts. Since the total degree is $2n+2$, at least one vertex V has degree 2. Choose V and either vertex in any edge not incident with V, and we have chosen at most

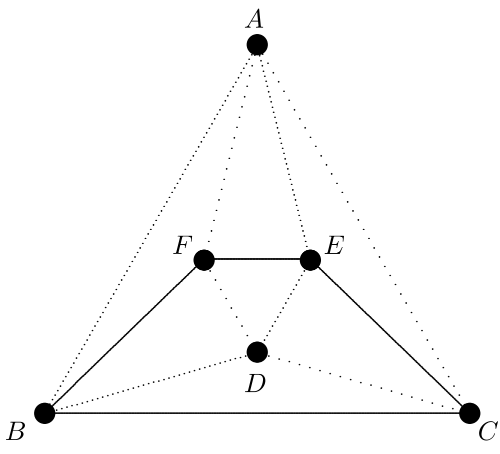
n vertices. If they represent the bad batteries, none of the $n+1$ attempts will be successful.

(b) The upper bound is $n + 3$. Test all three pairs of three batteries. We may assume that nothing works. This means that either all three are bad batteries, or two are bad and one is good. Now test the remaining $2n - 3$ batteries in pairs, leaving one off. If the initial three batteries are all bad, one pair now will work. If none of the pairs work, then the battery left off is good, and we can test it with both batteries in any pair. Thus we will find a working combination in at most $3 + (n - 2) + 2 = n + 3$ attempts. Consider any plan with $n + 2$ attempts. Construct a graph with $2n$ vertices representing the batteries, and $n + 2$ edges representing the attempts. Remove the vertex V with maximum degree, along with all edges incident with it. Since V has degree at least 2, the resulting subgraph has at most n edges. If it has at most $n - 1$ edges, we can choose either vertex of each. If it has n edges, then one of its vertices U has degree at least 2. We can then choose U along with either vertex of any edge not incident with U. In either case, we have chosen at most $n-1$ vertices of the subgraph such that every edge contains at least one of them. Along with V, we can choose n vertices to represent the bad batteries, so that none of the $n+2$ attempts will be successful.

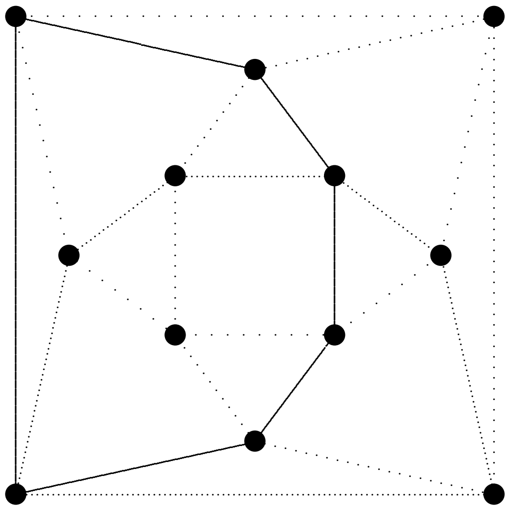
10. On a spherical planet are n great circles, each of length 1, which serve as railways. On each railway, several trains run continuously at the same positive constant speed. The trains are great arcs of the sphere but without their endpoints. If the trains never stop and never collide, what is the maximum total length of the trains, where (a) $n = 3$; (b) $n = 4$.

Solution (by Central Jury). Let P be a point of intersection of two of the railways. The trains on the two railways cannot occupy P simultaneously. Since all trains have the same uniform speed, the total length of the trains in any two railways is at most 1. If the sum of the lengths of the trains on each railway is at most $\frac{1}{2}$, then the total length is at most $\frac{1}{2}$ times the number of railways. Suppose the sum of the lengths of the trains on at least one railway exceeds $\frac{1}{2}$. Then the sum of the lengths of the trains on it and on another railway is at most 1, while the sum of the lengths of the trains on each remaining railway must be less than $\frac{1}{2}$. Hence the total length is still at most $\frac{1}{2}$ times the number of railways.

(a) We now construct an example with 3 railways on which the total length of the trains is $\frac{3}{2}$. Inscribe in the sphere a regular octahedron. Its Schlegel diagram, a planar representation of its skeleton, is shown below. Each edge represents an arc of length $\frac{1}{4}$. Note that there are three interlocking 4-cycles, $ABDE$, $BCEF$ and $CAFD$ which constitute the railways. We have six trains each of length $\frac{1}{4}$, all running counterclockwise and completing the lap in one hour. The trains on the first railway start on AB and DE, the trains on the second railway start on BC and EF, while the trains on the third railway start on CA and FD. During the first fifteen minutes, they pass through vertices B, E, C, F, A and D respectively. Hence there is no collision. The situation is symmetric in each of the remaining three blocks of fifteen minutes in the hour.



(b) We now construct an example with 4 railways on which the total length of the trains is 2. Inscribe in the sphere an cuboctahedron, an Archimedean solid in which every vertex is surrounded by an opposite pair of equilateral triangles and an opposite pair of squares. Its Schlegel diagram is shown below, where each edge represents an arc of length $\frac{1}{6}$. Note that there are four interlocking 6-cycles which constitute the railways. We have twelve trains each of length $\frac{1}{6}$, all running counterclockwise and completing the lap in one hour. The trains on the railways start on alternate edges so that each passes through a different vertex in each block of ten minutes. Hence there is no collision.



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The Martian Citizenship Quiz

Yen-Kang Fu & Te-Cheng Liu

The Martian Citizenship Quiz consists of 30 true or false questions. To pass the quiz, we have to answer all 30 questions, and all answers must be correct. If we fail, we will be told the number k of questions we have got right, but not which ones. We may attempt the same quiz any number of times. What is the minimum number of attempts in which we can guarantee ourselves Martian citizenship?

In the first test, we answer True for all 30 questions. Suppose we are told that $k = a$, then we know that a answers should be True and $30 - a$ answers should be False. If $a = 0$ or 30, there is no problem. If $k = 1$ or 29, we can sort the odd one out by a binary search. If $k = 2$ or 28, we can still use a refined binary search and keep it well under 20 attempts.

Suppose $3 \le a \le 27$. We divide the questions into three groups of 9, with 3 left over. We claim that we only need six more attempts to find the correct answers to all 9 questions in each group. We use another two tests to find the correct answer to the 28th and the 29th questions. From the value of a, we will also know the correct answer to the last question. We have used $1 + 3 \times 6 + 2 = 21$ tests so far, and we will pass the quiz on the 22nd attempt.

We now justify our claim. Let the 9 questions be 1, 2, 3, 4, 5, 6, 7, 8 and 9. In the next four tests, we change the answers for (1,2,3,8), (1,2,4,7), (1,3,4,6) and (2,3,4,5). On each attempt, we have $k = a \pm 4$, $k = a \pm 2$ or $k = a$. We consider six cases.

Case 1. We have $k = a \pm 4$ at least once.

By symmetry, we may assume that we have $k = a - 4$ for (1,2,3,8). In the fifth test, we determine the correct answer for 4. This will also yield the correct answers for 6, 7 and 8. In the sixth test, we determine the correct answer for 9.

Case 2. We have $k = a$ all four times.

The correct answers to the pair (1,5) are the same. This is also true of each of the pairs (2,6), (3,7) and (4,8). In the fifth test, we change the answers for (1,2,5). In the sixth test, we change the answers for (3,7,9). We consider two subcases.

Subcase 2(a). $k = a \pm 3$ in the fifth test.

By symmetry, we may assume that $k = a - 3$. Then 1, 2, 5 and 6 are True while 3, 4, 7 and 8 are False. We cannot have $k = a-3$ or $k = a-1$ in the sixth test. If $k = a + 1$, then 9 is true. If $k = a + 3$ instead, 9 is False.

Subcase 2(b). $k = a \pm 1$ in the fifth test.

By symmetry, we may assume that $k = a - 1$. Then 1 and 5 are true while 2 and 6 are False. In the sixth test, if $k = a - 3$, then 3, 7 and 9 are True while 4 and 8 are False. If $k = a - 1$, then 3 and 7 are True while 4, 8 and 9 are False. If $k = a+1$, 4, 8 and 9 are True while 3 and 7 are False. If $k = a + 3$, then 4 and 8 are True while 3, 7 and 9 are False.

In all subsequent cases, we do not have $k = a \pm 4$ and we have $k = a \pm 2$ at least once. By symmetry, we may assume that we have $k = a - 2$ at least once.

Case 3. We have $k = a - 2$ exactly once.

By symmetry, we assume that this occurs for (1,2,3,8). In the fifth test, we change the answers for (1,2,5). We cannot have $k = a + 3$. There are three subcases.

Subcase 3(a). $k = a - 3$.

Then 1, 2, 5 and 8 are True while 3, 4 and 7 are False. From the value of k for (1,3,4,6), we can deduce the correct answer for 6. In the sixth test, we determine the correct answer for 9.

Subcase 3(b). $k = a - 1$.

It is easy to check that 2 and 5 cannot both be True. Hence 1 is True. In the sixth test, we change the answers for (5,8,9). There are four sub-subcases.

Sub-subcase 3(b_1). $k = a - 3$.

Then 1, 3, 5, 8 and 9 are True while 2, 4 and 6 are False. From the value of k for (1,2,4,7), we can deduce the correct answer for 7.

Sub-subcase 3(b_2). $k = a - 1$.

Then 8 is True while one of 5 and 9 is True. If $k = a + 2$ for (1,2,4,7), then 1, 3, 5 and 8 are True while 2, 4, 6, 7 and 9 are False. If $k = a$ for

(1,2,4,7), then 1, 2, 8 and 9 are True while 3, 4, 5 and 7 are False. From the value of k for (1,3,4,6), we can deduce the correct answer for 6.
Sub-subcase 3(b_3). $k = a + 1$.
Then 5 is False and one of 8 and 9 is False. If $k = a$ for (2,3,4,5), then 1, 2, 3 and 9 are True while 4, 5, 6, 7 and 8 are False. If $k = a + 2$ for (2,3,4,5), then 1, 2 and 8 are True while 3, 4, 5, 7 and 9 are False. From the value of k for (1,3,4,6), we can deduce the correct answer for 6.
Sub-subcase 3(b_4). $k = a + 3$.
Then 1, 2 and 3 are True while 4, 5, 6, 7, 8 and 9 are False.
Subcase 3(c). $k = a + 1$.
One of 1 and 2 is True and the other is False. Hence 3 and 8 are True while 5 is False. Since $k > a - 1$ for both (1,3,4,6) and (2,3,4,5), 4 must be False. From the value of k for (2,3,4,5), we can determine which of 1 and 2 is True. From the values of k for (1,2,4,7) and (1,3,4,6), we can deduce the correct answers for 6 and 7. In the sixth test, we determine the correct answer for 9.

Case 4. We have $k = a - 2$ exactly twice.

By symmetry, we assume that this occurs for (1,2,3,8) and (2,3,4,5). In the fifth test, we change the answers for (1,2,5). We cannot have $k = a + 3$. There are three subcases.
Subcase 4(a). $k = a - 3$.
Then 1, 2, 3 and 5 are True while 4, 6, 7 and 8 are False.
Subcase 4(b). $k = a - 1$.
It is easy to check that 1 and 2 cannot both be False, and neither can 2 and 5. So 2 is False and 1 and 5 are True. Hence 4 and 8 are True while 3 is False. From the values of k for (1,2,4,7) and (1,3,4,6), we can deduce the correct answers for 6 and 7.
Subcase 4(c). $k = a + 1$.
It is easy to check that 1 and 2 cannot both be True, and neither can 2 and 5. So 2 is True and 1 and 5 are False. Hence 3, 4 and 8 are True while 6 and 7 are False.
In each subcase, we determine the correct answer for 9 in the sixth test.

Case 5. We have $k = a - 2$ exactly thrice.

By symmetry, we assume that this occurs for (1,2,3,8), (1,2,4,7) and (1,3,4,6). In the fifth test, we change the answers for (1,2,5). We cannot

have $k = a + 3$. There are three subcases.
Subcase 5(a). $k = a - 3$.
Then 1, 2, 3, 5 and 6 are True while 4, 7 and 8 are False.
Subcase 5(b). $k = a - 1$.
It is easy to check that 1 and 2 cannot both be True, and neither can 2 and 5. So 2 is False and 1 and 5 are True. Hence 3, 4 and 8 are True while 6 and 7 are False.
Subcase 5(c). $k = a + 1$.
It is easy to check that 1 and 2 cannot both be False, and neither can 1 and 5. So 1 is True and 2 and 5 are False. Hence 3, 4, 7 and 8 are True while 6 is False.
In each subcase, we determine the correct answer for 9 in the sixth test.

Case 6. We have $k = a - 2$ all four times.

In the fifth test, we change the answers for (1,2,5). We cannot have $k = a + 3$. There are three subcases.
Subcase 6(a). $k = a - 3$.
Then 1, 2, 5 and 6 are True. Either 3 and 7 are True while 4 and 8 are False, or the other way round. In the sixth test, we change the answers for (3,7,9). That will tell us everything.
Subcase 6(b). $k = a - 1$.
It is easy to check that 2 and 5 cannot both be True. If 1 and 2 are True, then 3 and 4 are also True while 5, 6, 7 and 8 are False. If 1 and 5 are true, then 3, 4, 7 and 8 are also True while 2 and 6 are False. In the sixth test, we change the answers for (3,7,9). That will tell us everything.
Subcase 6(c). $k = a + 1$.
It is easy to check that 1 and 2 cannot both be False, and neither can 2 and 5. So 2 is True and 1 and 5 are False. Hence 3, 4, 6, 7 and 8 are all True. In the sixth test, we determine the correct answer for 9.

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