Rational Cherednik Algebras

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Abstract. The aim of this dissertation is to explore cocycles on Hopf algebras, and their applications to Rational Cherednik algebras. We show how a Hopf algebra H, and its H-module algebras A, can be twisted by a cocycle. This is central to the main result of Bazlov, Berenstein and McGaw [6] that a negative braided Cherednik algebra of a mystic reflection group is isomorphic to a twist of a Rational Cherednik algebra. We finish with a survey of the proof of this result.

Declaration

This is the author's original work unless referenced to the contrary, and no portion of the work referred to in this dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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1 Introduction

We start in Section 2 by exploring the theory of cocycles on Hopf algebras. This is followed by an introduction to Reflection groups in Section 3. The material from these first two sections will then be applied in Section 4, which looks at rational Cherednik algebras.

In particular, Section 2 extends a previous piece of work by the author that covered the basic theory of Hopf algebras. We start in Section 2.1 by briefly reminding the reader of some of these basics, before discussing several results about the structure of Hopf algebras. In Section 2.2 cocycles on Hopf algebras are introduced, following Majid [21] (Section 2.3). We show that 1-cocycles are equivalent to grouplike elements, and find an equation that characterises 2-cocycles. In the next two subsections we explore examples of cocycles on the group algebras of C_m and $C_m \times C_m$. We introduce some Hopf algebra cohomology in order to show that all 2-cocycles on $\mathbb{C}C_m$ are coboundaries. Then, having determined that $R = \frac{1}{m}\sum_{a,b} q^{ab} g^a \otimes g^b$ is a 2-cocycle on $\mathbb{C}C_m$, we investigate the problem of finding a 1-cochain ξ such that $\partial(\xi) = R$, and give ξ for the cases $m = 2$ and 3. We finish this section by proving why it is not the case for the group algebra of $C_m \times C_m$ that every 2-cocycle is a coboundary. Finally, in Section 2.3, we define the Drinfeld twist H_χ of the Hopf algebra H by a 2-cocycle χ on H. We also show how an H-module algebra can be twisted into an H_χ-module algebra. These twists are very important in the final part of this work, specifically to Theorem 4.13 in Section 4.4.

In Section 3 we introduce Reflection groups, starting with the Euclidean reflection groups (i.e. those defined over the reals). We cover the basic properties of these groups, working up to a result that establishes the relationship between Euclidean reflection groups and Coxeter systems. In Section 3.2 we move onto complex reflection groups, in particular defining the family of groups $G(m, p, n)$ that will be required in the later parts of Section 4. We also introduce some classical invariant theory. The connection between invariant theory and reflection groups can be seen through the Chevalley-Shephard-Todd theorem (Theorem 3.18) which characterises a complex reflection group as a group whose ring of invariants is a polynomial ring.

Finally in Section 4 we focus on rational Cherednik algebras. These algebras can be seen as a "rational degeneration" of the double affine Hecke algebras, introduced by Cherednik. In this dissertation we will follow Etingof and Ginzburg [12] by defining rational Cherednik algebras as a particular type of symplectic reflection algebra. In

Section 4.1 we construct the symplectic reflection algebras, before defining the rational Cherednik algebras as a special case in Section 4.2. In Section 4.3 we follow Chapter 3 of Bazlov, Berenstein and McGaw [6], in which rational Cherednik algebras over the complex reflection groups $G(m, p, n)$ are defined. We additionally define the negative braided Cherednik algebra of the mystic reflection groups $\mu(G(m, p, n))$ from Bazlov and Berenstein [3]. In the final part of this dissertation, Section 4.4, we draw the connection between rational Cherednik algebras and the cocycles on Hopf algebras introduced in the first half of this work. In particular we look at the main result from [6] that a negative braided Cherednik algebra over a mystic reflection group arises as a twist (in the sense of Section 2.3) of a rational Cherednik algebra. We highlight what has been proven to date, and briefly mention what remains to be done to prove the general case of this result.

The Appendix includes an assortment of topics that the reader may find helpful. In particular, Section 6.1 introduces some of the Category theory necessary in order to define groupoids, which are mentioned briefly in Section 2.3. We also define a universal property, since they offer additional insight to the construction of free groups and universal enveloping algebras, which are covered in the next two sections of the Appendix. Indeed in Section 6.2 we define free groups, and group presentations, since firstly, they are necessary to define the Coxeter groups in Section 3.1. But also because the definition of a group presentation helps to motivate the analogous algebra presentations, which are used to define rational Cherednik algebras. Next in Section 6.3 we define the universal enveloping algebra and discuss the Poincare-Birkoff-Witt (PBW) theorem. This is important background to understanding the analogous PBW-type theorems used in the construction of symplectic reflection algebras in Section 4.1. Finally, Section 6.4 covers some basic group cohomology, as required for the proof of Proposition 2.20.

2 Quantum Algebra

2.1 Hopf Algebras

The basic theory of Hopf algebras is assumed to be known, but for notation we recall some of the basic definitions in this section before giving several results about grouplike and primitive elements in a Hopf algebra. Unless otherwise stated, k will denote an algebraically closed field of characteristic 0.

Definition 2.1. A **Hopf algebra** is $(H, m, \eta, \triangle, \epsilon, S)$ where H is a k-vector space and the following are k-linear maps: the multiplication $m : H \otimes H \to H$, the unit $\eta : k \to H,\ \lambda \mapsto \lambda 1_H$, the coproduct $\triangle : H \to H \otimes H$, the counit $\epsilon : H \to k$, and the antipode $S : H \to H$. This data forms a Hopf algebra if the following axioms are satisfied:

- Associativity: $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$, and the unit axiom: $\forall \lambda \in k, h \in H$
$$m \circ (\eta \otimes \mathrm{id})(\lambda \otimes h) = \lambda h = m \circ (\mathrm{id} \otimes \eta)(h \otimes \lambda)$$
So (H, m, η) is an algebra.
- Coassociativity: $(\triangle \otimes \mathrm{id}) \circ \triangle = (\mathrm{id} \otimes \triangle) \circ \triangle$, and the counit axiom:
$$(\epsilon \otimes \mathrm{id}) \circ \triangle = \mathrm{id}_H = (\mathrm{id} \otimes \epsilon) \circ \triangle$$
So $(H, \triangle, \epsilon)$ is a coalgebra.
- $\triangle, \epsilon$ are algebra homomorphisms. So $(H, m, \eta, \triangle, \epsilon)$ is a bialgebra.
- Antipode axiom: $m \circ (S \otimes \mathrm{id}) \circ \triangle = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \triangle$.

Note that it is often convenient to switch between notations $m(a \otimes b)$, $a \cdot b$, or just ab, for the multiplication in a (Hopf) algebra.

Definition 2.2.
- A **Hopf algebra homomorphism** from $(H_1, m_1, \eta_1, \triangle_1, \epsilon_1, S_1)$ to $(H_2, m_2, \eta_2, \triangle_2, \epsilon_2, S_2)$ is a linear map $f : H_1 \to H_2$ satisfying:
  - $m_2 \circ (f \otimes f) = f \circ m_1$ $\quad f \circ \eta_1 = \eta_2$ $\quad$ (i.e. algebra homomorphism)
  - $(f \otimes f) \circ \triangle_1 = \triangle_2 \circ f$ $\quad \epsilon_2 \circ f = \epsilon_1$ $\quad$ (i.e. coalgebra homomorphism)
  - $S_2 \circ f = f \circ S_1$

- A **cocommutative element** $x \in H$ of a Hopf algebra H is such that $\tau \circ \triangle(x) = \triangle(x)$, where $\tau : H \otimes H \to H \otimes H,\ a \otimes b \mapsto b \otimes a$ is the "twist" map. In Sweedler notation: $x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)}$.

- A **cocommutative Hopf algebra** H is such that every element $x \in H$ is a cocommutative, i.e. $\tau \circ \triangle = \triangle$.

- A **grouplike** element is a non-zero $x \in H$ such that $\triangle(x) = x \otimes x$. Note by the counit axiom it follows that $\epsilon(x) = 1_k$ for such elements.

- A **primitive** element $x \in H$ is such that $\triangle(x) = 1 \otimes x + x \otimes 1$. By the counit axiom we have $\epsilon(x) = 0_k$ for such elements.

Note the following results from Hazewinkel, Gubareni and Kirichenko [15].

Proposition 2.3. *For Hopf algebra H,*

1. *([15], Prop 3.5.2) The set $P(H)$ of primitive elements in H forms a Lie algebra with respect to commutator product $[g, h] := g \cdot h - h \cdot g$.*

2. *The set $Grp(H)$ of grouplike elements in H forms a group.*

3. *([15], Prop 3.6.12) The elements of the group $Grp(H)$ are linearly independent.*

Proof. 1. It is a standard fact that for associative algebras H, the underlying vector space H with commutator product $[\cdot, \cdot]$ is a Lie algebra. We must then check $P(H)$ is a vector subspace of H that it is closed under $[\cdot, \cdot]$. It is a subspace since for $\lambda, \mu \in k$, $x, y \in P(H)$ then $\lambda x + \mu y$ is also a primitive element:

$$\begin{aligned}\triangle(\lambda x + \mu y) &= \lambda \triangle(x) + \mu \triangle(y) = \lambda(1 \otimes x + x \otimes 1) + \mu(1 \otimes y + y \otimes 1) \\ &= 1 \otimes (\lambda x + \mu y) + (\lambda x + \mu y) \otimes 1\end{aligned}$$

It is closed under $[\cdot, \cdot]$, i.e. for $x, y \in P(H)$ then $[x, y] \in P(H)$:

$$\begin{aligned}\triangle([x, y]) &= \triangle(xy) - \triangle(yx) \\ &= (1 \otimes x + x \otimes 1) \cdot (1 \otimes y + y \otimes 1) - (1 \otimes y + y \otimes 1) \cdot (1 \otimes x + x \otimes 1) \\ &= 1 \otimes (x \cdot y) + y \otimes x + x \otimes y + (x \cdot y) \otimes 1 - (\cdots) \\ &= [x, y] \otimes 1 + 1 \otimes [x, y]\end{aligned}$$

as required.

2. Note $\triangle(1) = 1 \otimes 1$ so $1 \in \mathrm{Grp}(H)$. For $g, h \in \mathrm{Grp}(H)$, $\triangle(gh) = \triangle(g)\triangle(h) = (g \otimes g)(h \otimes h) = (gh) \otimes (gh)$, so $\mathrm{Grp}(G)$ is closed under multiplication. Finally $S(g) \cdot g = m \circ (S \otimes \mathrm{id}) \circ \triangle(g) = \eta \circ \epsilon(g) = \eta(1_k) = 1_H$ (using antipode axiom), and similarly $g \cdot S(g) = 1$, so there are inverses in H. We need to check the inverses $S(g) \in \mathrm{Grp}(G)$, which we see by noting
$$1 \otimes 1 = \triangle(S(g)g) = \triangle(S(g)) \cdot \triangle(g) = \triangle(S(g)) \cdot (g \otimes g)$$
Multiplying from right by $S(g) \otimes S(g)$, find $\triangle(S(g)) = S(g) \otimes S(g)$.

3. Let $\mathcal{A} = \{g_1, \ldots, g_n\}$ be a minimal set of distinct linearly <u>dependent</u> elements in $\mathrm{Grp}(H)$, i.e. $\exists r_1, \ldots, r_n \in k$ not all zero such that $r_1 g_1 + \cdots + r_n g_n = 0$. Then taking coproduct of this we find $r_1 g_1 \otimes g_1 + \cdots + r_n g_n \otimes g_n = 0$. Also $0 = 0 \otimes g_1 = (r_1 g_1 + \cdots + r_n g_n) \otimes g_1$. Subtracting each of these, we find
$$g_2 \otimes r_2(g_2 - g_1) + \cdots + g_n \otimes r_n(g_n - g_1) = 0 \tag{1}$$
By minimality of the set $\mathcal{A}$, the set $\{g_2, \ldots, g_n\}$ must be linearly independent, therefore Equation (1) holds iff $r_2 = \cdots = r_n = 0$ (since $g_i - g_1 \neq 0$ for $i \neq 1$ as the g_i are distinct). Then $r_1 g_1 = 0 \implies r_1 = 0$, which contradicts our assumption not all r_i were 0. Therefore no such set $\mathcal{A}$ exists, and all elements of $\mathrm{Grp}(H)$ are linearly independent.

$\square$

Recall that for a Lie algebra $\mathfrak{g}$, its universal enveloping algebra $U(\mathfrak{g})$ (see Definition 6.11) can be given a Hopf algebra structure (see Majid [21] Example 1.5.7). We find that the set of primitive elements in $U(\mathfrak{g})$ is precisely the underlying Lie algebra: $P(U(\mathfrak{g})) = \mathfrak{g}$ ([15], Example 3.5.4). While for a group G, the group algebra kG has a Hopf algebra structure (see Example 2.12) and the set of grouplike elements satisfies: $\mathrm{Grp}(kG) = G$ ([15], Example 3.6.4). Next we give some structure theorems for cocommutative Hopf algebras.

Proposition 2.4. *1. If H is a finite-dimensional Hopf algebra, then $P(H) = 0$. Note this is false in general if $char(k) \neq 0$.*

2. Cartier-Konstant-Milnor-Moore theorem. For cocommutative Hopf algebra H with $char(k) = 0$,
$$H \cong U(P(H^1))\# k\,Grp(H)$$

where # denotes the smash product (see Definition 2.25); H^1 is an "irreducible sub-Hopf" algebra of H containing 1_H (analogous to taking connected component attached to identity for Lie groups); $U(P(H^1))$ denotes the universal enveloping algebra of the Lie algebra $P(H^1)$; and $k\,Grp(H)$ is the group algebra of $Grp(H)$.

For time we do not go into this result further, but we note that it shows how cocommutative Hopf algebras are built out of groups and Lie algebras.

3. *Every finite-dimensional cocommutative Hopf algebra is a group algebra.*

Proof. 1. The idea to this proof is to show that for a non-zero primitive element $0 \neq p \in P(H)$ the set $\{1, p, p^2, p^3, \dots\}$ is linearly independent. Hence H must be infinite-dimensional, giving a contraction. See [15] Proposition 3.5.19 for more.

2. Statement of theorem is as seen in [15] Section 3.6. For a proof, see Sweedler [25] Section 13.1.

3. Follows from the above two results. See also Andruskiewitsch [2] Theorem 1.1.

□

In the following we generalise cocommutative Hopf algebras, allowing for cocommutativity "up to conjugation":

Definition 2.5. A **quasitriangular** Hopf algebra is a pair (H, R) for Hopf algebra H and invertible $R = \sum_i R_i^{(1)} \otimes R_i^{(2)} \in H \otimes H$ (called the "quasitriangular structure"), such that for the twist map τ:

$$\tau \circ \triangle(h) = R \cdot \triangle(h) \cdot R^{-1} \quad \forall h \in H \tag{2}$$

and for $R_{12} := R \otimes 1,\ R_{23} := 1 \otimes R,\ R_{13} := \sum_i R_i^{(1)} \otimes 1 \otimes R_i^{(2)}$ then:

$$(\triangle \otimes \mathrm{id})(R) = R_{13}R_{23} \qquad (\mathrm{id} \otimes \triangle)(R) = R_{13}R_{12} \tag{3}$$

Clearly if the Hopf algebra is cocommutative, $\tau \circ \triangle = \triangle$, then equations (2) and (3) hold for $R = 1 \otimes 1$, so such algebras are trivially quasitriangular.

Proposition 2.6. *If (H, R) is a quasitriangular Hopf Algebra, then R satisfies the* ***quantum Yang-Baxter equation****: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.*

Proof. This is a corrected version of the proof given in the authors previous project work. We follow the proof of Kytölä [19] Proposition 4.9.

$$R_{12}R_{13}R_{23} = R_{12}(\triangle \otimes \mathrm{id})(R) = (R \otimes 1)(\triangle(R^{(1)}) \otimes R^{(2)}) = (R\triangle(R^{(1)})) \otimes R^{(2)}$$

$$= R\triangle(R^{(1)})R^{-1}R \otimes R^{(2)} = (R\triangle(R^{(1)})R^{-1} \otimes R^{(2)}) \cdot (R \otimes 1)$$

$$= ((\tau \circ \triangle) \otimes \mathrm{id})(R) \cdot R_{12} = ((\tau \otimes \mathrm{id}) \circ (\triangle \otimes \mathrm{id}))(R) \cdot R_{12}$$

$$= (\tau \otimes \mathrm{id})(R_{13}R_{23}) \cdot R_{12} = R_{23}R_{13}R_{12}$$

The first line above uses Equation (3), followed by noting $R_{12} = R \otimes 1$, and then performing the multiplication. The second line introduces a factor of $1 \otimes 1 = RR^{-1}$ in order to apply Equation (2) in the third line. We then expand $((\tau \circ \triangle) \otimes \mathrm{id})$ as $((\tau \otimes \mathrm{id}) \circ (\triangle \otimes \mathrm{id}))$ before applying (3) again in line 4. In the final line we apply the result: $(\tau \otimes \mathrm{id})(R_{13}R_{23}) = R_{23}R_{13}$, which can be seen to hold by simply applying definitions and computing each side and showing they are equal. □

2.2 Cocycles

Cocycles are objects that appear in the study of cohomology. Cohomology arises in many different contexts, for example in the study of topological spaces in Algebraic topology; but the techniques can also be applied to groups (see Section 6.4 on Group cohomology) and Lie algebras. In this section we define cocycles on Hopf algebras, and develop some Hopf algebra cohomology, largely following Majid [21] Section 2.3.

Let $(H, m, \eta, \triangle, \epsilon, S)$ be a Hopf algebra. Then for each $n \in \mathbb{N}$ we can define the following maps for $i = 1, \ldots, n$:

$$\triangle_i : H^{\otimes n} \to H^{\otimes(n+1)} \qquad \triangle_i := \mathrm{id}^{\otimes(i-1)} \otimes \triangle \otimes \mathrm{id}^{\otimes(n-i)}$$

and let $\triangle_0 := 1_H \otimes \mathrm{id}^{\otimes n}$, $\triangle_{n+1} = \mathrm{id}^{\otimes n} \otimes 1_H$. Also let

$$\epsilon_i : H^{\otimes n} \to H^{\otimes(n-1)}, \qquad \epsilon_i := \mathrm{id}^{\otimes(i-1)} \otimes \epsilon \otimes \mathrm{id}^{\otimes(n-i)}$$

Definition 2.7.

- An n-**cochain** is an invertible element in $H^{\otimes n}$.

- The **coboundary** of n-cochain $\chi \in H^{\otimes n}$ is the $(n+1)$-cochain

$$\partial(\chi) := \Big(\prod_{i=0}^{i \text{ even}} \triangle_i(\chi)\Big)\Big(\prod_{i=1}^{i \text{ odd}} \triangle_i(\chi^{-1})\Big) \in H^{\otimes(n+1)}$$

- An n-**cocycle** is invertible $\chi \in H^{\otimes n}$ such that $\partial(\chi) = 1 \in H^{\otimes(n+1)}$.

- An n-cocycle χ is **counital** if $\epsilon_i(\chi) = 1 \in H^{\otimes(n-1)}$ $\forall i = 1, \ldots, n$.

Example 2.8. A 1-cocycle is an invertible element $\chi \in H$ such that:

$$\begin{aligned} 1 \otimes 1 = \partial(\chi) &= \triangle_0(\chi)\triangle_2(\chi)\triangle_1(\chi^{-1}) \\ &= (1 \otimes \chi) \cdot (\chi \otimes 1) \cdot \triangle(\chi^{-1}) \\ &= (\chi \otimes \chi)\triangle(\chi^{-1}) \end{aligned}$$

Multiplying from left by $\chi^{-1} \otimes \chi^{-1}$ gives us $\triangle(\chi^{-1}) = \chi^{-1} \otimes \chi^{-1}$. Then

$$1 \otimes 1 = \triangle(1) = \triangle(\chi\chi^{-1}) = \triangle(\chi)\triangle(\chi^{-1}) = \triangle(\chi)(\chi^{-1} \otimes \chi^{-1})$$

Multiplying from right by $\chi \otimes \chi$ gives us $\triangle(\chi) = \chi \otimes \chi$, i.e. χ is group-like. Note by the counit axiom all grouplike elements are counital: $\chi = \mathrm{id}(\chi) = (\epsilon \otimes \mathrm{id}) \circ \triangle(\chi) = (\epsilon \otimes \mathrm{id})(\chi \otimes \chi) = \epsilon(\chi)\chi$, hence $\epsilon(\chi) = 1$, so 1-cocycles are automatically counital.

Example 2.9. A 2-cocycle is invertible $\chi \in H \otimes H$ such that

$$\begin{aligned} 1 \otimes 1 \otimes 1 = \partial(\chi) &= \triangle_0(\chi)\triangle_2(\chi)\triangle_1(\chi^{-1})\triangle_3(\chi^{-1}) \\ &= (1 \otimes \chi) \cdot (\mathrm{id} \otimes \triangle)(\chi) \cdot (\triangle \otimes \mathrm{id})(\chi^{-1}) \cdot (\chi^{-1} \otimes 1) \end{aligned}$$

Multiplying from right by $(\chi \otimes 1) \cdot (\triangle \otimes \mathrm{id})(\chi)$, and using the fact $(\triangle \otimes \mathrm{id})$ is an algebra homomorphism so $(\triangle \otimes \mathrm{id})(\chi^{-1}) \cdot (\triangle \otimes \mathrm{id})(\chi) = 1 \otimes 1 \otimes 1$, we find χ satisfies:

$$(\chi \otimes 1) \cdot (\triangle \otimes \mathrm{id})(\chi) = (1 \otimes \chi) \cdot (\mathrm{id} \otimes \triangle)(\chi) \tag{4}$$

which we shall refer to as the "2-cocycle equation". Next χ is counital if

$$(\epsilon \otimes \mathrm{id})(\chi) = 1 = (\mathrm{id} \otimes \epsilon)(\chi)$$

Note that $\chi = 1 \otimes 1$ clearly satisfies equation (4), so $1 \otimes 1$ is trivially a 2-cocycle on all Hopf algebras. Whereas we noted above that $1 \otimes 1$ is only a quasitriangular structure for cocommutative Hopf algebras. Hence we see not every 2-cocycle is a quasitriangular structure, although we prove below that every quasitriangular structure is a 2-cocycle. As an example, note that group algebras $\mathbb{C}G$ are cocommutative, hence $1 \otimes 1$ is both a quasitriangular structure and a 2-cocycle for these Hopf algebras.

Proposition 2.10. *Every coboundary of a 1-cochain is a 2-cocycle, i.e. for all 1-cochains ξ, we have: $\partial^2(\xi) = 1$.*

Proof. Let $\chi := \partial(\xi) = (\xi \otimes \xi) \cdot \triangle(\xi^{-1})$, then we must show χ satisfies Equation (4):

$$\begin{aligned}(\chi \otimes 1) \cdot (\triangle \otimes \mathrm{id})(\chi) &= (\xi \otimes \xi \otimes 1) \cdot (\triangle(\xi^{-1}) \otimes 1) \cdot (\triangle(\xi) \otimes \xi) \cdot (\triangle \otimes \mathrm{id})(\triangle(\xi^{-1})) \\ &= (\xi \otimes \xi \otimes 1) \cdot (1 \otimes 1 \otimes \xi) \cdot (\mathrm{id} \otimes \triangle)(\triangle(\xi^{-1})) \\ &= (\xi \otimes \xi \otimes \xi) \cdot (\mathrm{id} \otimes \triangle)(\triangle(\xi^{-1})) \\ &= (1 \otimes \xi \otimes \xi) \cdot (1 \otimes \triangle(\xi^{-1})) \cdot (\xi \otimes \triangle(\xi)) \cdot (\mathrm{id} \otimes \triangle)(\triangle(\xi^{-1})) \\ &= (1 \otimes \chi) \cdot (\mathrm{id} \otimes \triangle)(\chi)\end{aligned}$$

In the first line we use: $1 \otimes (a \cdot b) = (1 \otimes a) \cdot (1 \otimes b)$ and that $\triangle \otimes \mathrm{id}$ is an algebra homomorphism. In the next line we perform multiplication of the inner two terms (using algebra homomorphism property of $\triangle$), and use coassociativity to change the last term. We next multiply the first two terms, and in the following line expand it again. Finally use algebra homomorphism property of $\triangle \otimes \mathrm{id}$ again and the definition of χ. $\square$

Proposition 2.11. *If (H, R) is a quasitriangular Hopf algebra, then R is a counital 2-cocycle.*

Proof. We elaborate on the hints given in [21] Example 2.3.6. By Proposition 2.6, R satisfies the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, which can be reexpressed using equation (3) and $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ to give: $(R \otimes 1)(\triangle \otimes \mathrm{id})(R) = (1 \otimes R)(\mathrm{id} \otimes \triangle)(R)$. This is precisely the 2-cocycle equation (4). So every quasitriangular structure is a 2-cocycle.

We see R is also counital using Majid Proposition 2.1.2*, by applying $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})$ to equation (3): $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\triangle \otimes \mathrm{id})(R) = (((\epsilon \otimes \mathrm{id}) \circ \triangle) \otimes \mathrm{id})(R) = R$ using the counit

*I think there is an error here, as Majid states $(\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(\triangle \otimes \mathrm{id})(R) = R_{23}$, which cannot be true since the LHS is in $H \otimes H$ while the RHS is in $H \otimes H \otimes H$. Instead, by above, the LHS equals R.

axiom in the second equality. While $(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{13}R_{23}) = \sum_{i,j} R_j^{(1)}\otimes\epsilon(R_i^{(1)})R_i^{(2)}R_j^{(2)} = (1\otimes(\epsilon\otimes\mathrm{id})(R)))\cdot R$. So $(1\otimes(\epsilon\otimes\mathrm{id})(R)))\cdot R = R$, and therefore $(\epsilon\otimes\mathrm{id})(R) = 1$. Similarly it can be shown $(\mathrm{id}\otimes\epsilon)(R) = 1$, so R is a counital 2-cocycle. □

2.2.1 2-cocycles on C_m

Let $C_m = \langle g \mid g^m = 1\rangle$ be the finite cyclic group of order $m \in \mathbb{N}$. Over the next two sections we work over $k = \mathbb{C}$. For a finite group G, we recall the group algebra $\mathbb{C}G$, and its dual $\mathbb{C}G^*$, have the following Hopf algebra structures:

Example 2.12.

- $\mathbb{C}G$: is the $\mathbb{C}$-vector space with basis given by elements $g \in G$. It is an algebra with the product of basis vectors given by the group multiplication, which is extended linearly to the whole space. Then it has a Hopf algebra structure with:

$$\eta(\lambda) = \lambda 1_G,\ \triangle(g) = g\otimes g,\ \epsilon(g) = 1_k,\ S(g) = g^{-1}$$

- The dual space $\mathbb{C}G^*$ has the dual basis $\{\delta_g \mid g \in G\}$. It is a Hopf algebra with the following maps[†]: for $\phi, \mu \in \mathbb{C}G^*, \lambda \in \mathbb{C}$

$$m'(\phi\otimes\mu)(g) = \phi(g)\mu(g) \qquad \eta'(\lambda) = \lambda\sum_{g\in G}\delta_g$$

$$\epsilon'(\delta_g) = \delta_{1_G}(g) \qquad S'(\delta_g) = \delta_{g^{-1}}$$

The coproduct $\triangle'$ is defined on the basis as: $\triangle'(\delta_g) = \sum_{h\in G}\delta_h\otimes\delta_{h^{-1}g}$. Using Proposition 2.14 below, we can identify elements of $\mathbb{C}G^*\otimes\mathbb{C}G^*$ with maps $G\times G\to\mathbb{C}$, and we find: $\triangle'(\phi)(g,h) = \phi(g\cdot h)$ for $\phi\in\mathbb{C}G^*$ and $f, g \in G$.

We see that $\mathbb{C}C_m$ is cocommutative, and hence is quasitriangular with quasitriangular sructure $1\otimes 1$. However it also has the following nontrivial quasitriangular structure:

Proposition 2.13. *The following element of $\mathbb{C}C_m\otimes\mathbb{C}C_m$ is a quasitriangular structure on $\mathbb{C}C_m$:*

$$R := \frac{1}{m}\sum_{a,b=0}^{m-1} q^{ab}g^a\otimes g^b \tag{5}$$

[†]We use $'$'s to distinguish the maps on $\mathbb{C}G^*$ from those on $\mathbb{C}G$, as this will be helpful in the proof of Proposition 2.16.

when $q = e^{\frac{2\pi i\lambda}{m}}$ is a primitive m-th root of unity (i.e. if λ and m are coprime).

Proof. We follow Majid Example 2.1.6, where the result is proven when $q = e^{-2\pi i/m}$. This corresponds to $\lambda = m-1$, which we note is coprime to m. We take λ to be any integer coprime to m. Recall equation (2): $\tau \circ \triangle(h) = R \cdot \triangle(h) \cdot R^{-1}$, which holds since $\mathbb{C}C_m$ is commutative, and therefore so is the algebra structure on $\mathbb{C}C_m \otimes \mathbb{C}C_m$, and $\mathbb{C}C_m$ is also cocommutative.

As $q^m = 1$ then $q^{am} = 1$ $\forall a \in \mathbb{Z}_m$, so

$$0 = q^{am} - 1 = (q^a - 1)(1 + q^a + \cdots + q^{a(m-1)}) = (q^a - 1)\big(\sum_{b=0}^{m-1} q^{ab}\big) \tag{6}$$

Note $q^a = 1$ if $m|\lambda a$. Since λ and m are coprime, the smallest value for a when this holds is m (or 0). So for $a \in \{1, \ldots, m-1\}$, i.e. $a \neq 0$, we have $m \nmid \lambda a$, hence $q^a \neq 1$, and so by equation (6) we must have $\sum_c q^{ac} = 0$. Clearly $\sum_c q^{ac} = m$ when $a = 0$, hence:

$$\sum_{b=0}^{m-1} q^{ab} = m\delta_a(0) \tag{7}$$

We will use this to check equation (3):

$$(\triangle \otimes \mathrm{id})(R) = \frac{1}{m} \sum_{a,b} q^{ab} g^a \otimes g^a \otimes g^b$$

while $R_{13} = \frac{1}{m}\sum_{a,b} q^{ab} g^a \otimes 1 \otimes g^b$ and $R_{23} = \frac{1}{m}\sum_{c,d} q^{cd} 1 \otimes g^c \otimes g^d$, so

$$\begin{aligned}
R_{13}R_{23} &= \frac{1}{m^2} \sum_{a,b,c,d} q^{ab+cd} g^a \otimes g^c \otimes g^{b+d} \\
&= \frac{1}{m^2} \sum_{a,b,b',c} q^{b(a-c)} q^{cb'} g^a \otimes g^c \otimes g^{b'}, \qquad b' := b + d \\
&= \frac{1}{m^2} \sum_{a,b',c} \big(\sum_b q^{b(a-c)}\big) q^{cb'} g^a \otimes g^c \otimes g^{b'} \\
&= \frac{1}{m} \sum_{a,b'} q^{ab'} g^a \otimes g^a \otimes g^{b'} = (\triangle \otimes \mathrm{id})(R)
\end{aligned}$$

where we move to the final line by applying equation (7): $\sum_b q^{b(a-c)} = m\delta_{a-c}(0)$, and then summing over c. Similarly for $(\mathrm{id}\otimes\triangle)(R)$, so R is a quasitriangular structure. □

By inspecting the form of the 2-cocycle equation (4), notice χ appears the same number of times on each side of the equation, so scalings of χ will cancel out. So by Proposition

2.11 the quasitriangular structure R satisfies the 2-cocycle equation and hence so must scalar multiples $\lambda R \quad \forall \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$. However λR will not be a quasitriangular structure $\forall \lambda \in \mathbb{C}^*$. For instance it can be checked mR does not satisfy equation (3). Note that setting $q = 0$ results in $R = 0 \otimes 0$, which is not an invertible element of $\mathbb{C}C_m \otimes \mathbb{C}C_m$, so clearly this cannot be a quasitriangular structure.

We use the following result to reinterpret our nontrivial quasitriangular structure R as an element of $\hom(\mathbb{C}C_m^*, \mathbb{C}C_m)$.

Proposition 2.14. *For finite-dimensional $\mathbb{C}$-vector spaces V, W:*

$$V \otimes W \cong \hom(W^*, V)$$

Proof. Define $f : V \times W \to \hom(W^*, V)$ as $f(v, w)(\phi) := \phi(w)v$ where $\phi \in W^*$, so $\phi(w) \in \mathbb{C}$. This is bilinear, hence we have a (unique) linear map $\bar{f} : V \otimes W \to \hom(W^*, V)$, $\bar{f}(v \otimes w)(\phi) = \phi(w)v$. We show it is injective. Note $\phi(w)v = 0 \ \forall \phi \in W^*$ if $v = 0$, or w is such that $\phi(w) = 0 \ \forall \phi$. Given a basis $\{w_i\}$ of W, $w = \sum_j \lambda_j w_j$ for some $\lambda_j \in \mathbb{C}$, and $\delta_{w_i}(\sum_j \lambda_j w_j) = \lambda_i = 0 \ \forall i$ implies $w = 0$. Hence $\phi(w)(v) = 0 \ \forall \phi$ iff $v \otimes w = 0$, so $\ker(\bar{f}) = \{0\}$ i.e. $\bar{f}$ injective.

For surjectivity, take $\lambda \in \hom(W^*, V)$ and let $X = \sum_i \lambda(\delta_{w_i}) \otimes w_i \in V \otimes W$ then we have $\bar{f}(X) = \lambda$, hence $\bar{f}$ is surjective. We check this: let $\phi \in W^*$ then $\bar{f}(X)(\phi) = \sum_i \bar{f}(\lambda(\delta_{w_i}) \otimes w_i)(\phi) = \sum_i \phi(w_i)\lambda(\delta_{w_i}) = \lambda(\phi)$ using the fact $\phi = \phi(w_i)\delta_{w_i}$. Therefore $\bar{f}$ defines an isomorphism of these vector spaces. □

With $V = W = \mathbb{C}C_m$, we find that applying the map $\bar{f}$ from the proof of Proposition 2.14 to $R = \frac{1}{m}\sum_{a,b} q^{ab} g^a \otimes g^b$ yields the following element in $\hom(\mathbb{C}C_m^*, \mathbb{C}C_m)$:

Definition 2.15. The **discrete Fourier transform** for C_m is $\bar{f}(R) : \mathbb{C}C_m^* \to \mathbb{C}C_m$, such that:

$$\bar{f}(R)(\phi) = \frac{1}{m}\sum_{a,b} q^{ab}\phi(g^b)g^a \in \mathbb{C}C_m \quad \forall \phi \in \mathbb{C}C_m^*$$

Proposition 2.16. *The discrete Fourier transform $\bar{f}(R)$ is a Hopf algebra homomorphism.*

Proof. Recall $R := \frac{1}{m}\sum_{a,b=0}^{m-1} q^{ab} g^a \otimes g^b$ where $q = e^{\frac{2\pi i \lambda}{m}}$ is an m-th root of unity such that m and λ are coprime. We must check the conditions of a Hopf algebra homomorphism (see Definition 2.2) are satisfied. Firstly, by definition of $\bar{f}$, $\bar{f}(R)$ is certainly a linear map. Then for $\phi, \mu \in \mathbb{C}C_m^*$:

$$\bar{f}(R)(m'(\phi \otimes \mu)) = \frac{1}{m}\sum_{a,b} q^{ab} m'(\phi \otimes \mu)(g^b)g^a = \frac{1}{m}\sum_{a,b} q^{ab}\phi(g^b)\mu(g^b)g^a \tag{8}$$

Whereas

$$\begin{aligned}\bar{f}(R)(\phi) \cdot \bar{f}(F_q)(\mu) &= \frac{1}{m^2}\Big(\sum_{a,b} q^{ab}\phi(g^b)g^a\Big) \cdot \Big(\sum_{c,d} q^{cd}\phi(g^d)g^c\Big) \\ &= \frac{1}{m^2}\sum_{a,b,c,d} q^{ab+cd}\phi(g^b)\mu(g^d)g^{a+c}\end{aligned}$$

Introducing $a' = a + c$, we have $q^{ab+cd} = q^{a'b}q^{c(d-b)}$, and on applying $\sum_c q^{c(d-b)} = m\delta_{d-b}(0)$ (by equation (7) - it is here that we require m,λ are coprime), and summing over d, everything reduces to the same expression as the RHS of equation (8) (with dummy variable a' instead of a). So multiplication is preserved by $\bar{f}(R)$. To show this is an algebra homomorphism we must check it preserves identities:

$$\bar{f}(R)\big(\sum_c \delta_{g^c}\big) = \frac{1}{m}\sum_{a,b,c} q^{ab}\delta_{g^c}(g^b)g^a = \frac{1}{m}\sum_{a,c} q^{ac}g^a = \frac{1}{m}\sum_a (m\delta_a(0))g^a = \frac{1}{m}mg^0 = 1$$

Next we show $\bar{f}(R)$ is a coalgebra homomorphism: i.e. $(\bar{f}(R) \otimes \bar{f}(R)) \circ \triangle' = \triangle \circ \bar{f}(R)$ holds on the dual basis vectors δ_{g^c}, and hence by linearity holds everywhere. So

$$(\triangle \circ \bar{f}(R))(\delta_{g^c}) = \triangle\Big(\frac{1}{m}\sum_a q^{ac}g^a\Big) = \frac{1}{m}\sum_a q^{ac}g^a \otimes g^a \tag{9}$$

Next

$$\begin{aligned}(\bar{f}(R) \otimes \bar{f}(R)) \circ \triangle'(\delta_{g^c}) &= (\bar{f}(R) \otimes \bar{f}(R))\Big(\sum_i \delta_{g^i} \otimes \delta_{g^{c-i}}\Big) \\ &= \frac{1}{m^2}\sum_i \Big(\big(\sum_a q^{ai}g^a\big) \otimes \big(\sum_b q^{b(c-i)}g^b\big)\Big) \\ &= \frac{1}{m^2}\sum_{i,a,b} q^{ai+b(c-i)}g^a \otimes g^b\end{aligned}$$

As we have done before, note $ai + b(c-i) = bc + (a-b)i$ and $\sum_i q^{(a-b)i} = m\delta_{a-b}(0)$ so this reduces to the RHS of equation (9) as required. Next we require: $\epsilon \circ \bar{f}(R) = \epsilon'$:

$$\epsilon \circ \bar{f}(R)(\delta_{g^c}) = \epsilon\Big(\frac{1}{m}\sum_a q^{ac}g^a\Big) = \frac{1}{m}\sum_a q^{ac} = \delta_c(0)$$

But $\delta_c(0) = \delta_1(g^c) = \epsilon'(\delta_{g^c})$, so indeed $\bar{f}(R)$ is a coalgebra homomorphism.

Finally we must show $\bar{f}(R) \circ S' = S \circ \bar{f}(R)$.

$$(S \circ \bar{f}(R))(\delta_{g^c}) = S\Big(\frac{1}{m}\sum_a q^{ac}g^a\Big) = \frac{1}{m}\sum_a q^{ac}g^{m-a} = \frac{1}{m}\sum_{a'} q^{(m-a')c}g^{a'}$$
$$= \frac{1}{m}\sum_{a'} q^{-a'c}g^{a'}$$

Where we use: $a' := m - a$ and $q^{mc} = (q^m)^c = 1$. Now using the definition of S' as the dual map to S, we have $S'(\delta_{g^c})(h) = \delta_{g^c}(S(h)) = \delta_{g^c}(h^{-1}) = \delta_{(g^c)^{-1}}(h) = \delta_{g^{m-c}}(h)$. Then $\bar{f}(R)(S'(\delta_{g^c})) = \bar{f}(R)(\delta_{g^{m-c}}) = \frac{1}{m}\sum_a q^{a(m-c)}g^a = \frac{1}{m}\sum_a q^{-ac}g^a$, as required. Hence $\bar{f}(R)$ is a homomorphism of Hopf algebras. □

Proposition 2.17. *The discrete Fourier transform $\bar{f}(R)$ is bijective, and hence an isomorphism of Hopf algebras.*

Proof. Since the dimensions of $\mathbb{C}C_m^*$ and $\mathbb{C}C_m$ are equal, we just have to check that $\ker(\bar{f}(R)) = \{0\}$. For $\phi \in \mathbb{C}C_m^*$, $\bar{f}(R)(\phi) = \frac{1}{m}\sum_a \Big(\sum_b \phi(g^b)q^{ab}\Big)g^a = 0$ iff:

$$\sum_{b=0}^{m-1} \phi(g^b)q^{ab} = 0 \quad \forall a = 0, \dots, m-1 \tag{10}$$

Let Q denote the $m \times m$-matrix with (i,j)-th entries $Q_{ij} = q^{ij}$. Then equation (10) can be read as saying that the vector with i-th entry equal to $\phi(g^i)$ lies in the kernel of Q. In Garrett [13] (Sec 19.6.1) it was observed that the Q is invertible, with inverse given by $(Q^{-1})_{ij} = \frac{1}{m}q^{-ij}$, which we can check:

$$Q_{ij}Q^{-1}_{jk} = \frac{1}{m}\sum_j q^{j(i-k)} = \delta_{i-k}(0) = \delta_{ik}$$

where the second equality uses equation (7). Likewise $Q^{-1}_{ij}Q_{jk} = \delta_{ik}$. Since Q is invertible, its kernel must be $\{0\}$. Therefore $\phi(g^i) = 0\ \forall i$, and so $\phi = 0 \in \mathbb{C}C_m^*$. We find $\ker(\bar{f}(R)) = \{0\}$, and finally the discrete Fourier transform $\bar{f}(R)$ is seen to be an isomorphism. □

Since the discrete Fourier transform is an isomorphism, it maps the standard basis $\{\delta_c\}$ of $\mathbb{C}C_m^*$ into a basis of $\mathbb{C}C_m$, which we denote $\{X_c |\ c = 0, \dots, m-1\}$ where:

$$X_c := \bar{f}(R)(\delta_c) = \frac{1}{m}\sum_{a=0}^{m-1} q^{ac}g^a \tag{11}$$

We can express elements of $\mathbb{C}C_m$ with respect to either the standard basis $\{g^c\}$ or $\{X_c\}$. Note X_c satisfies $X_c^2 = X_c$ and $X_a \cdot X_b = 0$ for $a \neq b$. Using these facts it is easy to see that elements $\chi \in \mathbb{C}C_m$ are invertible iff their coefficients μ_c with respect to basis $\{X_c\}$ are non-zero, i.e. if $\chi = \sum_c \mu_c X_c$ with $\mu_c \neq 0\ \forall c$. The inverse is then given by $\chi^{-1} = \sum_c \mu_c^{-1} X_c$. We prove this now:

$$\chi\chi^{-1} = \sum_{a,b} \mu_a \mu_b^{-1} X_a X_b = \sum_a \mu_a \mu_a^{-1} X_a^2 = \sum_a X_a$$

this then equal to

$$\frac{1}{m}\sum_c (\sum_a q^{ac}) g^c = \frac{1}{m}\sum_c m\delta_c(0) g^c = g^0 = 1$$

as required.

Question: is every counital 2-cocycle on $\mathbb{C}C_m$ the coboundary of some 1-cochain?

Note in Proposition 2.10 we showed the converse holds for all Hopf algebras, i.e. every coboundary of a 1-cochain is a 2-cocycle. To answer this question we will require several ideas coming from group cohomology, which we introduce in Section 6.4 of the Appendix. In that section we define cocycles, coboundaries and the Schur multiplier $M(G)$ (also called the second cohomology group $H^2(G, \mathbb{C}^*)$) on a group G. We then show these definitions generalise to a commutative Hopf algebra H, with its second cohomology group denoted $H^2(H, \mathbb{C})$. For Hopf algebras that are dual to a group algebra, i.e. $H = \mathbb{C}G^*$, the "Hopf algebra cohomology" reduces to the group cohomology of G. Whilst this cohomological setup is sufficult to answer our question for the Hopf algebra $\mathbb{C}C_m$ (since it is commutative), we will briefly discuss Majids "non-abelian cohomology" which further generalises the definitions for arbitrary Hopf algebras. Note it is in Proposition 2.19 below that we see how this cohomological setup helps us to answer the above question.

For counital 1-cochain ξ recall $\partial(\xi) = (\xi \otimes \xi)\triangle(\xi^{-1})$. Also, by Majid Proposition 2.3.3, if χ is a counital 2-cocycle then $(\xi \otimes \xi)\chi\triangle(\xi^{-1})$ is also a counital 2-cocycle.

Definition 2.18. For arbitrary Hopf algebra H,

- Counital 2-cocycles χ, χ' are **cohomologous**, $\chi \sim \chi'$, iff there exists a counital 1-cochain ξ such that

$$\chi' = (\xi \otimes \xi)\chi\triangle(\xi^{-1}) \tag{12}$$

We see this coincides with equation (40) (in the Appendix) for cohomologous cocycles on commutative Hopf algebras, although in equation (12) the multiplication is not commutative, hence we cannot rearrange to say $\chi'\chi^{-1} = \partial(\xi)$ as we did in the commutative case. So it is not necessarily true that cohomologous cocycles differ by a coboundary in this more general setting.

Let $\chi \sim_\xi \chi'$ denote $\chi' = (\xi \otimes \xi)\chi\triangle(\xi^{-1})$. Then we check $\sim$ is an equivalence relation. It is reflexive since: $\chi \sim_1 \chi$. Also if $\chi \sim_\xi \chi'$ then $\chi' \sim_{\xi^{-1}} \chi$, so it is symmetric. Finally transitivity: $\chi \sim_\xi \chi'$ and $\chi' \sim_{\xi'} \chi''$ implies $\chi \sim_{\xi'\xi} \chi''$.

- The **non-abelian cohomology space** $H^2(H, k)$ is the set of counital 2-cocycles modulo the relation of being cohomologous.

In Proposition 2.10 we showed for arbitrary Hopf algebras that $\partial^2(\xi) = 1$ for all 1-cochains ξ. This can be extended for commutative Hopf algebras to all n-cochains λ: $\partial^2(\lambda) = 1$, or simply $\partial^2 = 1$. However, according to Majid, this property does not extend to the non-abelian setting.

The next result explains how all this relates to the question we asked above:

Proposition 2.19. *Every counital 2-cocycle on a Hopf algebra H is the coboundary of some 1-cochain on H iff $H^2(H, k)$ is trivial.*

Proof. $\Longleftarrow$: Having a trivial cohomology space means every counital 2-cocycle is cohomologous to each other, and in particular to the trivial 2-cocycle $1 \otimes 1$. So for counital 2-cocycle χ, then $1 \otimes 1 \sim \chi$ implies there exists a counital 1-cochain ξ such that $\chi = (\xi \otimes \xi)(1 \otimes 1)\triangle(\xi^{-1}) = (\xi \otimes \xi)\triangle(\xi^{-1}) = \partial(\xi)$. Hence χ is a coboundary.

$\Longrightarrow$: Take arbitrary counital 2-cocycle χ, then there exists a 1-cochain ξ such that $\chi = \partial(\xi) = (\xi \otimes \xi)\triangle(\xi^{-1})$. Multiplying on the left by $(\xi^{-1} \otimes \xi^{-1})$ and the right by $\triangle(\xi)$ we find: $(\xi^{-1} \otimes \xi^{-1})\chi\triangle(\xi) = 1 \otimes 1$. I.e. $\chi \sim 1 \otimes 1$. Note since χ is counital, so is ξ:

$$1 = (\epsilon \otimes \mathrm{id})(\chi) = (\epsilon(\xi)\xi) \cdot (\epsilon \otimes \mathrm{id}) \circ \triangle(\xi^{-1}) = \epsilon(\xi)\xi \cdot \xi^{-1} = \epsilon(\xi)$$

using the counit axiom. So we find the cohomology space $H^2(H, k)$ is trivial. □

Proposition 2.20. *Every counital 2-cocycle on $\mathbb{C}C_m$ is the coboundary of some 1-cochain on $\mathbb{C}C_m$.*

Proof. It was shown in the author's previous work that for finite abelian groups G, we have an isomorphism of Hopf algebras: $\mathbb{C}G \cong \mathbb{C}G^*$. So in particular: $\mathbb{C}C_m \cong \mathbb{C}C_m^*$. By this isomorphism we have an equality of the cohomology spaces $H^2(\mathbb{C}C_m, \mathbb{C}) = H^2(\mathbb{C}C_m^*, \mathbb{C})$. We also know (see Appendix) that cocycles and coboundaries on Hopf algebras of the form $\mathbb{C}G^*$ coincide with cocycles and coboundaries on the group G, in the sense of group cohomology. Hence we have an equality as sets of the Hopf algebra cohomology space $H^2(\mathbb{C}C_m^*, \mathbb{C})$ with the Schur multiplier $M(G)$ (or second cohomology group $H^2(C_m, \mathbb{C}^*)$). If we can prove $M(C_m) = 1$, then by these equalities we will have shown $H^2(\mathbb{C}C_m, \mathbb{C})$ is trivial, and the result will follow by Proposition 2.19.

We can prove $M(C_m) = 1$ in several ways. One appproach is by Rotman [23] Corollary 7.70, which uses the modern result of Alperin-Kuo (Theorem 7.68). For a group G define the exponent $\exp(G)$ to be the lowest common multiple of the orders of each element of G, then the result says that for $e := \exp(M(G))$ and $e' := \exp(G)$ we have ee' divides $|G|$. For cyclic groups G, $\exp(G) = |G|$, hence we see $e = 1$ and $M(G) = 1$.

However a second approach is given by Corollary 2.5 (Chapter 11) of [22], a result attributed to Schur. It says that for a finite group G generated by n elements with r relations, and for s the minimal number of generators for $M(G)$, then $r \geq n + s$. As a direct consequence, for cyclic groups such as C_m, we have $r = n = 1$, hence $s = 0$. By convention a group with minimal number of generators being 0 is the trivial group, hence $M(C_m) = 1$. $\square$

Applying Proposition 2.20, we know the quasitriangular structure $R = \frac{1}{m}\sum_{a,b} q^{ab} g^a \otimes g^b$ on $\mathbb{C}C_m$ is the coboundary of some 1-cochain ξ, i.e. $R = \partial(\xi) = (\xi \otimes \xi) \cdot \triangle(\xi^{-1})$. It is easier to instead try to solve $R\triangle(\xi) = (\xi \otimes \xi)$, noting that we must make sure to check any solutions ξ are invertible. Let $\xi := \lambda_0 1 + \lambda_1 g + \cdots + \lambda_{m-1} g^{m-1}$.

$$\begin{aligned}
R\triangle(\xi) &= \frac{1}{m}\sum_{a',b',c} q^{a'b'} \lambda_c g^{a'+c} \otimes g^{b'+c} \\
&= \frac{1}{m}\sum_{a,b,c} q^{(a-c)(b-c)} \lambda_c g^a \otimes g^b, \qquad a = a' + c,\ b = b' + c \\
(\xi \otimes \xi) &= \sum_{a,b} \lambda_a \lambda_b g^a \otimes g^b
\end{aligned}$$

So comparing coefficients of the $g^a \otimes g^b$, we find the coefficients λ_a of ξ solve the following system:

$$\lambda_a \lambda_b = \frac{1}{m} \sum_c q^{(a-c)(b-c)} \lambda_c \tag{13}$$

Note we can instead take ξ with respect to the basis X_c (see Equation (11)), so $\xi = \mu_0 X_0 + \cdots + \mu_{m-1} X^{m-1}$, and repeating the above steps we find the μ_a solve the system:

$$\sum_{c,d} q^{ac+bd} \mu_c \mu_d = \sum_{c,d} q^{(a-c)(b-c)+cd} \mu_d$$

The benefit of using the basis X_c is that it is clear whether ξ is invertible or not by simply checking all the μ_c are non-zero. However it proved easier to solve the system (13), and we give the method for cases $m = 2$ and 3. For $\mathbb{C}C_m$, each side of (13) can be rewritten as an $m \times m$-matrix using a, b as indices, and note this matrix is symmetric so we omit the values below diagonal for clarity. We give these matrices for $m = 2$ (where $q = -1$) and $m = 3$ (where we take $q = e^{\frac{2\pi i}{3}}$):

$$\begin{pmatrix} \lambda_0^2 & \lambda_0 \lambda_1 \\ & \lambda_1^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \lambda_0 + q\lambda_1 & \lambda_0 + \lambda_1 \\ & q\lambda_0 + \lambda_1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_0^2 & \lambda_0\lambda_1 & \lambda_0\lambda_2 \\ & \lambda_1^2 & \lambda_1\lambda_2 \\ & & \lambda_2^2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \lambda_0 + q\lambda_1 + q\lambda_2 & \lambda_0 + \lambda_1 + q^2\lambda_2 & \lambda_0 + q^2\lambda_1 + \lambda_2 \\ & q\lambda_0 + \lambda_1 + q\lambda_2 & q^2\lambda_0 + \lambda_1 + \lambda_2 \\ & & q\lambda_0 + q\lambda_1 + \lambda_2 \end{pmatrix} \tag{14}$$

The $m = 2$ case is easily solved directly, and one finds:

$$\xi = \frac{1+i}{2} 1 + \frac{1-i}{2} g = \frac{1}{\sqrt{2}} q^{1/4} 1 + \frac{1}{\sqrt{2}} q^{7/4} g = \frac{q^{1/4}}{\sqrt{2}} (1 + q^{3/2} g)$$

Making a change of basis, it is equivalently: $\xi = X_0 + iX_1 = X_0 + q^{1/2} X_1$.

For $m = 3$, we will find $\lambda_0, \lambda_1, \lambda_2$ by instead solving the easier problem of finding the coefficients of a polynomial whose roots are precisely $\lambda_0, \lambda_1, \lambda_2$. Suppose $x^3 + a_2 x^2 + a_1 x + a_0$ has roots $\lambda_0, \lambda_1, \lambda_2$, then it is a fact that the a_i are elementary symmetric polynomials (see Example 3.13) in the roots:

$$-a_2 = \lambda_0 + \lambda_1 + \lambda_2,\ a_1 = \lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2,\ -a_0 = \lambda_0\lambda_1\lambda_2 \tag{15}$$

It proves easier to find the a_i, and to solve the resulting polynomial for the λ_i. The sum of the upper triangular entries in (14) tells us:

$$\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2 = \frac{q^2 + 2}{3} (\lambda_0 + \lambda_1 + \lambda_2) \tag{16}$$

Hence $a_1 = \frac{q^2+2}{3}(-a_2)$. Taking the trace of (14) gives:

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = \frac{2q+1}{3}(\lambda_0 + \lambda_1 + \lambda_2) \tag{17}$$

Additionally (14) tells us: $\lambda_0\lambda_1 = \lambda_0 + \lambda_1 + q^2\lambda_2$, so: $q\lambda_0\lambda_1 = q\lambda_0 + q\lambda_1 + \lambda_2 = \lambda_2^2$. Similarly $\lambda_0^2 = q\lambda_1\lambda_2$ and $\lambda_1^2 = q\lambda_0\lambda_2$. Hence $\lambda_0^3 = \lambda_1^3 = \lambda_2^3 = q\lambda_0\lambda_1\lambda_2$, and finally:

$$\lambda_0^3 + \lambda_1^3 + \lambda_2^3 = 3q\lambda_0\lambda_1\lambda_2 \tag{18}$$

The LHS of equations (17),(18) are power-sum symmetric polynomials (p_2 and p_3 respectively, see Example 3.13), which can be expressed as polynomials in the elementary symmetric polynomials. By Newtons identities[‡]: $p_2 = e_1^2 - 2e_2$ and $p_3 = e_1^3 - 3e_2e_1 + 3e_3$, and by (15), $e_1 = -a_2$, $e_2 = a_1$ and $e_3 = -a_0$. So (17) tells us: $(-a_2)^2 - 2a_1 = \frac{2q+1}{3}(-a_2)$, while (18) provides: $(-a_2)^3 + 3a_1a_2 + 3(-a_0) = 3q(-a_0)$. Using the $a_1 = \frac{q^2+2}{3}(-a_2)$ from above, these three equations are easily solved for a_i, and we find:

$$a_2 = -1,\ a_1 = \frac{q^2+2}{3},\ a_0 = \frac{1}{6}(-1 + \frac{i}{\sqrt{3}})$$

Solving $x^3 + a_2x^2 + a_1x + a_0$, we find it has a root $\frac{-i}{\sqrt{3}} = \frac{q^{9/4}}{\sqrt{3}}$, and a repeated root: $\frac{1}{6}(3+i\sqrt{3}) = \frac{q^{1/4}}{\sqrt{3}}$. Note S_3 acts on $\{\lambda_0, \lambda_1, \lambda_2\}$ by permuting the elements. The system (14) is invariant under this action, meaning if we permute the λ_i, each equation in the system is sent to another. Hence we can assign the values $\frac{q^{9/4}}{\sqrt{3}}$, $\frac{q^{1/4}}{\sqrt{3}}$, $\frac{q^{1/4}}{\sqrt{3}}$ to $\lambda_0, \lambda_1, \lambda_2$ in any order. So take:

$$\xi = \frac{1}{\sqrt{3}}(q^{9/4}1 + q^{1/4}g + q^{1/4}g^2) = \frac{q^{1/4}}{\sqrt{3}}(q^2 1 + g + g^2)$$

Making a change of basis we find: $\xi = X_0 + q^2X_1 + q^2X_2$, where the coefficients are all non-zero, hence ξ is invertible as required.

For $m \geq 4$ it appears to be possible to find relations between power-sum symmetric polynomials p_k and elementary symmetric polynomials $\forall 2 \leq k \leq m$. One can then similarly solve for the coefficients of a polynomial whose roots are the coefficients of ξ. However in general the system is not invariant under arbitrary permutations of the λ_i, so this method only determines the set of coefficients of ξ, but not how they should be assigned to the λ_i.

[‡] e_m denotes the m-th elementary symmetric polynomial. Also note these identities were proven by Albert Girard before Newton!

2.2.2 2-cocycles on $C_m \times C_m$

Let $G := C_m \times C_m = \langle \gamma_1, \gamma_2 \mid \gamma_1^m = \gamma_2^m = 1,\ \gamma_1\gamma_2 = \gamma_2\gamma_1 \rangle$. For an m-th root of unity $q \in \mathbb{C}$, define:

$$F(q) := \sum_{a,b=0}^{m-1} q^{ab} \gamma_1^a \otimes \gamma_2^b \in \mathbb{C}G \otimes \mathbb{C}G$$

Proposition 2.21. *$F(q)$ is a 2-cocycle on $\mathbb{C}G$ for all m-th roots of unity q.*

Proof. Since γ_1, γ_2 generate every element of the group G, they also generate the basis of the group algebra $\mathbb{C}G$. We can define the map $\phi : \mathbb{C}G \to \mathbb{C}C_m \otimes \mathbb{C}C_m$ by the following: $\gamma_1 \mapsto g \otimes 1,\ \gamma_2 \mapsto 1 \otimes g$ and extending as an algebra homomorphism. This is also an isomorphism of Hopf algebras, so we identify F with its image under ϕ: $\sum_{a,b} q^{ab} g^a \otimes 1 \otimes 1 \otimes g^b$. We check this satisfies the 2-cocycle equation in $\mathbb{C}C_m \otimes \mathbb{C}C_m$:

$$(F \otimes 1)(\hat{\triangle} \otimes \mathrm{id})(F) = (1 \otimes F)(\mathrm{id} \otimes \hat{\triangle})(F)$$

where the coproduct $\hat{\triangle}$ here is the coproduct from the tensor product coalgebra structure on $\mathbb{C}C_m \otimes \mathbb{C}C_m$: $\hat{\triangle} = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\triangle \otimes \triangle)$, where $\triangle$ is the coproduct on $\mathbb{C}C_m$. Using this we have the following:

$$\begin{aligned}
(\hat{\triangle} \otimes \mathrm{id})(\gamma_1^a \otimes \gamma_2^b) &= (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\triangle \otimes \triangle)(g^a \otimes 1) \otimes (\mathrm{id} \otimes \mathrm{id})(1 \otimes g^b) \\
&= g^a \otimes 1 \otimes g^a \otimes 1 \otimes 1 \otimes g^b \\
(\mathrm{id} \otimes \hat{\triangle})(\gamma_1^c \otimes \gamma_2^d) &= g^c \otimes 1 \otimes 1 \otimes g^d \otimes 1 \otimes g^d
\end{aligned}$$

Then

$$\begin{aligned}
(F \otimes 1)(\hat{\triangle} \otimes \mathrm{id})(F) &= \sum_{\lambda,c} q^{\lambda c} g^\lambda \otimes 1 \otimes 1 \otimes g^c \otimes 1 \otimes 1 \cdot \sum_{b,d} q^{bd} g^b \otimes 1 \otimes g^b \otimes 1 \otimes 1 \otimes g^d \\
&= \sum_{\lambda,b,c,d} q^{bd+\lambda c}(g^{b+\lambda} \otimes 1 \otimes g^b \otimes g^c \otimes 1 \otimes g^d) \\
&= \sum_{a,b,c,d} q^{b(d-c)+ac}(g^a \otimes 1 \otimes g^b \otimes g^c \otimes 1 \otimes g^d)
\end{aligned}$$

where in the last line we set $a := b + \lambda$, and so in the exponent of q we substitute $a - b$

for λ. This cannot be simplied further. We expand the RHS of the 2-cocycle equation:

$$\begin{aligned}(1 \otimes F)(\mathrm{id} \otimes \hat{\triangle})(F) &= \sum_{b,\lambda} q^{b\lambda} 1 \otimes 1 \otimes g^b \otimes 1 \otimes 1 \otimes g^\lambda \cdot \sum_{a,c} q^{ac} g^a \otimes 1 \otimes 1 \otimes g^c \otimes 1 \otimes g^c \\ &= \sum_{a,b,c,\lambda} q^{b\lambda + ac} (g^a \otimes 1 \otimes g^b \otimes g^c \otimes 1 \otimes g^{\lambda + c}) \\ &= \sum_{a,b,c,d} q^{b(d-c)+ac} (g^a \otimes 1 \otimes g^b \otimes g^c \otimes 1 \otimes g^d)\end{aligned}$$

where we set $d := \lambda + c$, and replace λ with $d - c$. Hence $(F \otimes 1)(\hat{\triangle} \otimes \mathrm{id})(F) = (1 \otimes F)(\mathrm{id} \otimes \hat{\triangle})(F)$, so F is a 2-cocycle. □

By the same reasoning used in the proof of Proposition 2.20 above, as $G = C_m \times C_m$ is abelian, we have $H^2(\mathbb{C}G, \mathbb{C})$ is equal to the Schur multiplier $M(C_m \times C_m)$. By Proposition 7.1 (Chapter 10) of [22] (another result proven by Schur), for an abelian group $A \cong C_{t_1} \times \cdots \times C_{t_n}$ for $t_i \in \mathbb{N}$, then $M(A) \cong \prod_{1 \leq j < k \leq n} C_{(t_j, t_k)}$ where (t_j, t_k) denotes the greatest common divisor of the integers t_j, t_k. Hence we have $M(C_m \times C_m) \cong C_m$, and we see there are precisely m elements in $H^2(\mathbb{C}G, \mathbb{C}) = M(C_m \times C_m)$. So immediately by Proposition 2.19 we know that there are cocycles on $\mathbb{C}G$ that are not coboundaries. Also, by Proposition 2.21 the $F(q)$ are 2-cocycles on $\mathbb{C}G$, however it still remains to be shown that $F(q)$ and $F(q')$ are not cohomologous when $q \neq q'$. If this is indeed the case, then $F(q)$ would provide m distinct elements of $H^2(\mathbb{C}G, \mathbb{C})$, one for each root of unity q. Hence the $F(q)$ would characterise all cocycles up to coboundary on $\mathbb{C}G$. Additionally, if it is also true that $F(1)$ is a coboundary, and so corresponds to the identity in $H^2(\mathbb{C}G, \mathbb{C})$, then it would follow that $F(q)$ is not a coboundary for all $q \neq 1$.

2.3 Twisting

Proposition 2.22 (Majid [21], Proposition 2.3.4)**.** *A quasitriangular Hopf algebra $(H, m, \eta, \triangle, \epsilon, S, R)$ can be "twisted" by a counital 2-cocycle χ into the quasitriangular Hopf algebra $(H, m, \eta, \triangle_\chi, \epsilon, S_\chi, R_\chi)$ where*

$$\triangle_\chi(h) := \chi \cdot \triangle(h) \cdot \chi^{-1} \qquad R_\chi := \chi_{21} \cdot R \cdot \chi^{-1}$$
$$S_\chi(h) := U \cdot S(h) \cdot U^{-1} \qquad U := \sum_i \chi_i^{(1)} \cdot S(\chi_i^{(2)})$$

*and $\chi_{21} := \tau(\chi)$. We call this Hopf algebra a **Drinfeld twist**, and denote it H_χ.*

Example 2.23 (Majid [21], Example 2.3.6)**.** We can twist a quasitriangular Hopf algebra (H, R) by its quasitriangular structure R, since R is a counital 2-cocycle. The result is the "co-opposite" Hopf algebra H^{cop} given by the opposite coproduct: $\triangle^{\mathrm{cop}} := \tau \circ \triangle$.

Proof. First note by Majid Proposition 1.3.3, that given a bialgebra $(B, m, \eta, \triangle, \epsilon)$, then $(B, m, \eta, \triangle^{\mathrm{cop}}, \epsilon)$ with $\triangle^{\mathrm{cop}} := \tau \circ \triangle$ is also a bialgebra. Additionally given a Hopf algebra $(H, m, \eta, \triangle, \epsilon, S)$, then $(H, m, \eta, \triangle^{\mathrm{cop}}, \epsilon, S^{-1})$ is a Hopf algebra iff S is invertible, which we denote H^{cop}. By Majid Proposition 2.1.8, the antipode of a quasitriangular Hopf algebra H is invertible, hence we can define the co-opposite H^{cop} for such algebras.

We must now verify the twisted Hopf algebra H_R coincides with H^{cop}. Using the expressions in Proposition 2.22 above, we have $\triangle_R(h) := R\triangle(h)R^{-1} = \tau \circ \triangle(h) = \triangle^{\mathrm{cop}}(h)$, where we use equation (2) from the definition of a quasitriangular structure. Also $R_R := \tau(R) \cdot R \cdot R^{-1} = \tau(R)$, which agrees with Majid Exercise 2.1.3 which states that $\tau(R)$ is a quasitriangular structure on H^{cop}. Finally $S_R(h) := US(h)U^{-1}$ where $U := \sum_i R_i^{(1)} S(R_i^{(2)})$. So $S_R(S(h)) = US^2(h)U^{-1}$. Then by Majid Proposition 2.1.8: for $u := \sum_i S(R_i^{(2)})R_i^{(1)}$, $v := S(u) = U$, we have: $S^2(h) = uhu^{-1}$, hence: $S_R(S(h)) = vuhu^{-1}v^{-1}$. Finally Majid Corollary 2.1.9 proves that $uv = vu$ is central in H, hence $(vu)h(vu)^{-1} = (vu)(vu)^{-1}h = h$, and so $S_R \circ S = \mathrm{id}$. Similarly $S \circ S_R = \mathrm{id}$, hence $S_R = S^{-1}$ as required, and we find $H_R = H^{\mathrm{cop}}$. □

The core idea to this twisting of Hopf algebras is the following: if χ, ψ are cohomologous 2-cocycles then their twisted Hopf algebras H_χ, H_ψ are isomorphic. See Majid [21] Proposition 2.3.5 for the proof of this. We now discuss a special case of this result.

Example 2.24. Suppose 2-cocycle χ on H is a coboundary, i.e. for some 1-cochain $\xi \in H$: $\chi = \partial(\xi) = (\xi \otimes \xi)\triangle(\xi^{-1})$. In other words χ is cohomologous to $1 \otimes 1$. Then we show $H \cong H_\chi$.

$$\triangle_\chi(h) = \chi\triangle(h)\chi^{-1} = (\xi \otimes \xi)\triangle(\xi^{-1}h\xi)(\xi^{-1} \otimes \xi^{-1}) \tag{19}$$

Note the algebra parts of H and H_χ coincide, and we can define an "inner automorphism" , i.e. a linear map $\phi : H \to H_\chi$, $h \mapsto \xi h \xi^{-1}$, which is an isomorphism of the algebras (with inverse $h \mapsto \xi^{-1} h \xi$). Then equation (19) tells us that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\phi"] \arrow[d, "\triangle"] & H_\chi \arrow[d, "\triangle_\chi"] \\ H \otimes H \arrow[r, "\phi\otimes\phi"] & H_\chi \otimes H_\chi \end{tikzcd}$$

Similarly ϕ "commutes" with the counits and antipodes, so ϕ provides an isomorphism of H and H_χ as Hopf algebras. By the result mentioned above, as $H = H_{1\otimes 1}$ and $\chi \sim 1 \otimes 1$, we must have $H \cong H_\chi$.

Similarly to the ideas discussed in Bazlov and Berenstein [4] (Sec 0.7), given a quasitriangular Hopf algebra $(H, m, \eta, \triangle, \epsilon, S, R)$, a groupoid (see Definition 6.1) can be formed whose objects are isomorphism classes of Hopf algebras sharing the same underlying algebra and counit as H. There exists a morphism between the classes of $H' = (H, m, \eta, \triangle', \epsilon, S', R')$ and $H'' = (H, m, \eta, \triangle'', \epsilon, S'', R'')$ if there is a counital 2-cocycle χ on H' such that $H'' = H'_\chi$. However we have more morphisms than necessary since twists by cohomologous cocycles give Hopf algebras that are isomorphic. Since our objects are isomorphism classes of Hopf algebras, we can instead take morphisms to be the equivalence classes of cocycles modulo being cohomologous. Also since cocycles are invertible, twists also are, and our morphisms are isomorphisms. This category is therefore a groupoid. We see that the set of morphisms in this category with domain being the isomorphism class of H is precisely Majid's non-abelian cohomology space $H^2(H, k)$.

Definition 2.25. Let H be a Hopf algebra.

- A **left** H**-module** is a vector space V with an action $\triangleright$ of H on V satisfying $\forall g, h \in H, v \in A$:

$$(g \cdot h) \triangleright v = g \triangleright (h \triangleright v),\ 1_H \triangleright v = v \tag{20}$$

- H**-module algebra** is an algebra A which as a vector space is a left H-module, and additionally $\forall h \in H,\ a, b \in A$:

$$h \triangleright (a \cdot b) = \sum_i (h_{i(1)} \triangleright a) \cdot (h_{i(2)} \triangleright b) \tag{21}$$

$$h \triangleright 1_A = \epsilon(h) 1_A \tag{22}$$

where ϵ is the counit of H and $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$.

- The **smash product algebra** of Hopf algebra H and H-module algebra A is the algebra $A\#H$ with underlying vector space $A \otimes H$, unit $1_A \otimes 1_H$, and for $a, a' \in A, h, h' \in H$ the product is:

$$(a \otimes h) \cdot (a' \otimes h') = (a \cdot (h_{(1)} \triangleright a')) \otimes (h_{(2)} \cdot h')$$

Note that given an H-module algebra A, the tensor product $A \otimes A$ is naturally an $H \otimes H$-module algebra. If $\chi := \sum_i \chi_i^{(1)} \otimes \chi_i^{(2)} \in H \otimes H$, then for $a \otimes b \in A \otimes A$ we let $\chi \triangleright (a \otimes b) := \sum_i (\chi_i^{(1)} \triangleright a) \otimes (\chi_i^{(2)} \triangleright b)$. With this we can reexpress equation (21) as:

$$h \triangleright m(a \otimes b) = m(\triangle(h) \triangleright (a \otimes b)) \tag{23}$$

where m is the product in A. Similarly for $A \otimes A \otimes A$ etc.

Proposition 2.26. *[Majid [21], Proposition 2.3.8] Let χ be counital 2-cocycle of Hopf algebra H, and algebra $B = (B, m, \eta)$ be an H-module algebra. The* ***twisting*** *of B is the associative algebra $B_\chi = (B, m_\chi, \eta)$ with $m_\chi(a \otimes b) := m(\chi^{-1} \triangleright (a \otimes b))$. B_χ is also an H_χ-module algebra.*

Proof. We must check m_χ is associative:

$$\begin{aligned}
m_\chi \circ (m_\chi \otimes \mathrm{id})(a \otimes b \otimes c) &= m(\chi^{-1} \triangleright m(\chi^{-1} \triangleright a \otimes b) \otimes c) \\
&= \sum_i m\Big((\chi_i^{-1(1)} \triangleright m(\chi^{-1} \triangleright a \otimes b)) \otimes (\chi_i^{-1(2)} \triangleright c) \Big) \\
&= \sum_i m \circ (m \otimes \mathrm{id}) \Big(\triangle(\chi_i^{-1(1)}) \triangleright (\chi^{-1} \triangleright a \otimes b) \otimes (\chi_i^{-1(2)} \triangleright c) \Big) \\
&= \sum_i m \circ (m \otimes \mathrm{id}) \Big(((\triangle(\chi_i^{-1(1)}) \cdot \chi^{-1}) \triangleright a \otimes b \big) \otimes (\chi_i^{-1(2)} \triangleright c) \Big) \\
&= \sum_i m \circ (m \otimes \mathrm{id}) \Big((\triangle \otimes \mathrm{id})(\chi^{-1})(\chi^{-1} \otimes 1) \triangleright (a \otimes b \otimes c) \Big)
\end{aligned}$$

where the step between lines 2 and 3 uses equation (23), although we have $(m \otimes \mathrm{id})$ since we wish to apply id to the term $(\chi_i^{-1(2)} \triangleright c)$. Moving to line 4 we apply $h \triangleright (g \triangleright v) = (h \cdot g) \triangleright v$. Note that the 2-cocycle equation can equivalently be expressed as:

$$(\mathrm{id} \otimes \triangle)(\chi^{-1})(1 \otimes \chi^{-1}) = (\triangle \otimes \mathrm{id})(\chi^{-1})(\chi^{-1} \otimes 1)$$

Hence we can apply this to the final expression above, and also using associativity of m, we find by working backwards it equals: $m_\chi \circ (\mathrm{id} \otimes m_\chi)(a \otimes b \otimes c)$ as required. Now to show B_χ is an H_χ-module algebra:

$$\begin{aligned}
h \triangleright m_\chi(b \otimes c) &= h \triangleright m(\chi^{-1} \triangleright b \otimes c) \\
&= m(\triangle(h) \triangleright (\chi^{-1} \triangleright b \otimes c)) \\
&= m(\triangle(h)\chi^{-1} \triangleright b \otimes c) \\
&= m(\chi^{-1}\triangle_\chi(h) \triangleright b \otimes c) \\
&= m(\chi^{-1} \triangleright (\triangle_\chi(h) \triangleright b \otimes c)) \\
&= m_\chi(\triangle_\chi(h) \triangleright b \otimes c)
\end{aligned}$$

Where between lines 3/4 we use the definition of $\triangle_\chi(h) = \chi\triangle(h)\chi^{-1}$. This gives us an expression equivalent to equation (23) but with H_χ acting on B_χ. Also since the unit and counit of H and H_χ coincide equation (22) trivially holds. Hence B_χ is an H_χ-module algebra. $\square$

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3 Reflection groups

This section introduces some of the foundational material of Reflection groups and their invariant theory. We start by surveying some of the theory of Euclidean reflection groups. In Section 3.2 we define the complex reflection groups, and define the family of complex reflection groups $G(m, p, n)$ that will be used in Section 4.

3.1 Euclidean reflection groups

This section follows Chapters I-V of Kane [16]. Roughly speaking a reflection is a linear transformation on a Euclidean (i.e. real) vector space that fixes a hyperplane, and sends vectors orthogonal to this hyperplane to their negatives. Then a Euclidean reflection group is a group of linear maps that are generated by reflections.

Definition 3.1.

- **Euclidean space** is an n-dimensional $\mathbb{R}$-vector space $\mathbb{E}$ with an inner product $(\cdot,\cdot) : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ such that: (i) $(\lambda x + \mu y, z) = \lambda(x, z) + \mu(y, z)$ (ii) $(x, y) = (y, x)$ and (iii) $(x, x) \geq 0$ with $(x, x) = 0$ iff $x = 0$.

  For $0 \neq \alpha \in \mathbb{E}$, then $H_\alpha := \{x \in \mathbb{E} \mid (x, \alpha) = 0\}$ is the hyperplane orthogonal to α. Then $\mathbb{E} = \text{span}\{\alpha\} \oplus H_\alpha$.

- The **reflection** given by non-zero $\alpha \in \mathbb{E}$ is $s_\alpha : \mathbb{E} \to \mathbb{E}$ such that $s_\alpha(x) := x \ \forall x \in H_\alpha$ and $s_\alpha(\alpha) := -\alpha$. H_α is the "reflecting hyperplane" of the reflection s_α.

- **Orthogonal group** is $O(\mathbb{E}) := \{\text{linear } f : \mathbb{E} \to \mathbb{E} \mid (f(\alpha), f(\beta)) = (\alpha, \beta) \ \forall \alpha, \beta\}$.

- **Euclidean reflection group** is a subgroup $W \subset O(\mathbb{E})$ generated by reflections.

- Reflection groups $W \subset O(\mathbb{E})$ and $W' \subset O(\mathbb{E}')$ are **isomorphic** if there exists a linear isomorphism $f : \mathbb{E} \to \mathbb{E}'$ that preserves inner products: $(f(x), f(y)) = (x, y) \ \forall x, y \in \mathbb{E}$, and satisfies: $fWf^{-1} = W'$, i.e. for each $w' \in W'$ there exists some $w \in W$ such that $w' = f \circ w \circ f^{-1}$.

- Reflection group $W \subset O(\mathbb{E})$ is **reducible** if there exist reflection groups $W_1 \subset O(\mathbb{E})$, $W_2 \subset O(\mathbb{E})$ with $W \cong W_1 \times W_2$. W is **irreducible** if it is not reducible.

Proposition 3.2. *[Kane [16], Section 1.1] The following properties hold for all non-zero $\alpha \in \mathbb{E}$:*

1. $s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha \quad \forall x \in \mathbb{E}$.

2. $s_\alpha \in O(\mathbb{E})$.

3. $\det(s_\alpha) = -1$.

4. For all automorphisms $\phi \in O(\mathbb{E})$*:*

$$\phi(H_\alpha) = H_{\phi(\alpha)}, \qquad \phi \circ s_\alpha \circ \phi^{-1} = s_{\phi(\alpha)}$$

Proof. 1. For $x \in H_\alpha$, $(x,\alpha) = 0$ so $s_\alpha(x) = x$, and $\alpha - \frac{2(\alpha,\alpha)}{(\alpha,\alpha)}\alpha = -\alpha$, so this formula satisfies the conditions for s_α as above, and since $\mathbb{E} = \text{span}(\alpha) \oplus H_\alpha$ they coincide everywhere.

2. Using part (1) and the linearity of the inner product we have

$$\begin{aligned}(s_\alpha(x), s_\alpha(y)) &= (x,y) - \frac{2(x,\alpha)}{(\alpha,\alpha)}(\alpha,y) - \frac{2(y,\alpha)}{(\alpha,\alpha)}(x,\alpha) + 4\frac{(x,\alpha)(y,\alpha)}{(\alpha,\alpha)(\alpha,\alpha)}(\alpha,\alpha) \\ &= (x,y)\end{aligned}$$

3. s_α is the matrix $\text{diag}(1,\ldots,1,-1)$ with respect to a basis formed by the union of linearly independent vectors spanning H_α, with α. This has determinant -1, and recall the determinant is independent of the choice of basis.

4. If $x \in H_\alpha$ then $(\phi(x),\phi(\alpha)) = (x,\alpha) = 0$, so $\phi(H_\alpha) \subset H_{\phi(\alpha)}$. As ϕ is automorphism, $\ker(\phi) = \{0\}$, and also: $\ker(\phi|_{H_\alpha}) = \{0\}$, which implies $\dim(\phi(H_\alpha)) = \dim(H_\alpha) = n-1 = \dim(H_{\phi(\alpha)})$. Hence $\phi(H_\alpha) = H_{\phi(\alpha)}$.

Finally, $\phi(s_\alpha(\phi^{-1}(H_{\phi(\alpha)}))) = \phi(s_\alpha(\phi^{-1}(\phi(H_\alpha)))) = \phi(s_\alpha(H_\alpha)) = \phi(H_\alpha) = H_{\phi(\alpha)}$ and also $\phi(s_\alpha(\phi^{-1}(\phi(\alpha)))) = \phi(s_\alpha(\alpha)) = -\phi(\alpha)$. Similarly $s_{\phi(\alpha)}(H_{\phi(\alpha)}) = H_{\phi(\alpha)}$ and $s_{\phi(\alpha)}(\phi(\alpha)) = -\phi(\alpha)$, so the actions of $\phi \circ s_\alpha \circ \phi^{-1}$ and $s_{\phi(\alpha)}$ on coincide on all of $\mathbb{E}$, so they are equal.

□

Example 3.3 (Kane [16], Section 1.1)**.** S_n acts faithfully on $\mathbb{E} = \mathbb{R}^n$ via the action $\sigma \triangleright (x_1,\ldots,x_n) = (x_{\sigma(1)},\ldots,x_{\sigma(n)})$. Hence S_n can be identified with a subgroup of $O(\mathbb{E})$. The transposition (ij) corresponds in $O(\mathbb{E})$ to the reflection about the hyperplane $H = \{(x_1,\ldots,x_n | x_i = x_j)\}$. This hyperplane is determined by the vector with all zero entries except for 1 in i-th position and -1 in j-th position. Since S_n is generated by its transpositions, and these can be seen as reflections in $\mathbb{E}$, we find S_n is a Euclidean reflection group.

We now define the semidirect product of groups as they are used in the construction of many examples of reflection groups.

Definition 3.4.
- The **automorphism group** $\mathrm{Aut}(G)$ of group G is the set of group homomorphisms $G \to G$ with product given by composition.
- An **inner automorphism** is $\phi \in \mathrm{Aut}(G)$ such that $\phi(x) = gxg^{-1}$ for some $g \in G$. The set of inner automorphisms is denoted $\mathrm{Inn}(G)$.
- **Outer automorphism** is any $\phi \in \mathrm{Aut}(G)$ that is not an inner automorphism.
- Group $K = G \rtimes H$ is the **inner semidirect product** of subgroups $G, H \subset K$ if $K = GH$ (i.e. $\forall k \in K\ \exists g \in G, h \in H$ such that $k = gh$), $G \lhd K$ (i.e. G is normal subgroup of K) and $G \cap K = \{1\}$.
- The **outer semidirect product** of groups G, H and $\phi : H \to \mathrm{Aut}(G)$, denoted $G \rtimes_\phi H$, is the group with underlying set $G \times H$ and multiplication

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot \phi(h_1)(g_2), h_1 \cdot h_2)$$

Note the outer semidirect product $G \rtimes_\phi H$ has subgroups $G' := \{(g, 1_H) | g \in G\}$ and $H' := \{(1_G, h) | h \in H\}$ such that $G' \cong G,\ H \cong H'$. In fact $G \rtimes_\phi H$ is the inner semidirect product of G' and H'. Also

$$(1, h) \cdot (g, 1) \cdot (1, h^{-1}) = (1, h) \cdot (g\phi(1)(1), h^{-1}) = (\phi(h)(g), 1) \tag{24}$$

So if we identify each $g \in G, h \in H$ with its embedding into $G \rtimes_\phi H$ (i.e. $(g, 1)$ and $(1, h)$ resp.) then equation (24) says $\phi(h)(g) = hgh^{-1}$. We also see that an inner semidirect product $G \rtimes H$ is isomorphic to the outer semidirect product of the subgroups G and H, with $\phi(g)(h) := hgh^{-1}$. Indeed there is a bijection $\Phi : G \rtimes H \to G \rtimes_\phi H,\ gh \mapsto (g, h)$ that is also a homomorphism since:

$$(g_1 h_1) \cdot (g_2 h_2) = g_1 (h_1 g_2 h_1^{-1}) h_1 h_2 = g_1 \phi(h_1)(g_2) h_1 h_2$$

Example 3.5. Dihedral group $D_n := \mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2$ where $\phi(1)(y) = y$ and $\phi(x)(y) = y^{-1}$ where x is the generator of $\mathbb{Z}_2$ and $y \in \mathbb{Z}_n$. Regarding x and y as elements of D_n we have: $\phi(x)(y) = xyx$ (using $x = x^{-1}$). Hence we find: $xyx = y^{-1}$. We also have the relations $x^2 = 1$ and $y^n = 1$, when y is the generator of $\mathbb{Z}_n$. So we see D_n has the following presentation:

$$D_n = \langle x, y \mid x^2 = y^n = (xy)^2 = 1 \rangle$$

D_n also has a matrix representation on $\mathbb{R}^2$ given by:

$$y = \begin{pmatrix} \cos(\frac{2\pi}{m}) & -\sin(\frac{2\pi}{m}) \\ \sin(\frac{2\pi}{m}) & \cos(\frac{2\pi}{m}) \end{pmatrix} \qquad x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We see y^j are rotations by $2\pi j/m$, while $y^j x$ gives the reflections about the line making an angle $\pi j/m$ with the x-axis. Therefore D_n is the automorphism group of the regular n-sided polygon in $\mathbb{R}^2$. Note if we define $z = yx$ as the reflection about the line π/m above x-axis, then we get a second presentation $D_n = \langle x, z | \ x^2 = z^2 = (zx)^n = 1 \rangle$. This shows D_n is generated by reflections, and hence is a Euclidean reflection group.

Example 3.6. [Kane [16], Section 1.3]$\mathbb{Z}_2^n \rtimes S_n$ where S_n acts on $\mathbb{Z}_2^n$ by permuting the factors. This is the automorphism group of an n-dimensional cube. By above S_n is a Euclidean reflection group, as is $\mathbb{Z}_2^n$ when acting on $\mathbb{R}^n$ by sign changes on each coordinate. It then follows that $\mathbb{Z}_2^n \rtimes S_n$ is a Euclidean reflection group.

Next we introduce root systems, following Chapter 2 of Kane [16].

Definition 3.7. A **root system** is a finite set $\triangle \subset \mathbb{E}$ such that:

- if $\alpha \in \triangle$, then $\lambda\alpha \in \triangle$ iff $\lambda = \pm 1$
- For all $\alpha, \beta \in \triangle$ we have $s_\alpha(\beta) \in \triangle$.

The root system is **unitary** if every $\alpha \in \triangle$ has unit length, $|\alpha| = 1$.

For finite Euclidean reflection group $W \subset O(\mathbb{E})$, let $\triangle := \{\alpha | \ s_\alpha \in W, \ |\alpha| = 1\} \subset \mathbb{E}$, i.e. the set of unit vectors that are orthogonal to each of the reflecting hyperplanes of W. Then $\triangle$ is a unitary root system since firstly condition (1) is clearly satisfied. The second condition follows since $s_\alpha s_\beta s_\alpha \in W$, and by Part 4 of Proposition 3.2 $s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)}$. Also s_α is orthogonal and $|\beta| = 1$, so $|s_\alpha(\beta)| = 1$, so $s_\alpha(\beta) \in W$, as required.

Conversely, given a root system $\triangle \subset \mathbb{E}$ one can construct a finite Euclidean reflection group $W(\triangle)$, that is the group generated by the set $\{s_\alpha \mid \alpha \in \triangle\}$. We must check this gives a finite group. Condition (2) says every element of $W(\triangle)$ corresponds to a permutation of the set $\triangle$. So we have a map $\phi : W(\triangle) \to S(\triangle)$ into the permutation group on $\triangle$. Let $\mathbb{E}_\triangle \subset \mathbb{E}$ be the span of $\triangle$, and $\mathbb{E}^\triangle = \cap_{\alpha\in\triangle} H_\alpha$ is the subspace of $\mathbb{E}$ on with every element of $W(\triangle)$ acts as the identity. Then we have $\mathbb{E} = \mathbb{E}_\triangle \oplus \mathbb{E}^\triangle$. Hence if $w \in W(\triangle)$ fixes $\triangle$ (i.e. $\phi(w) = 1$), we know w automatically fixes $E^\triangle$, and since w fixes $\triangle$ it fixes $\mathbb{E}_\triangle$, and it fixes all of $\mathbb{E}$. Hence $w = 1$ and the map ϕ is injective. So ϕ is an injective map into a finite set, which means $W(\triangle)$ is finite.

Definition 3.8.

- A root system $\triangle$ is **crystallographic** if it also satisfies
$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}\ \forall \alpha,\beta \in \triangle$$

- A **Weyl group** is a finite Euclidean reflection group $W(\triangle)$ generated by a crystallographic root system $\triangle$.

- An **essential** root system $\triangle \subset \mathbb{E}$ is such that $\triangle$ spans $\mathbb{E}$. So $\mathbb{E}_\triangle = \mathbb{E}$. A reflection group is **essential** if its associated root system is essential.

  A very important fact is that there is a $1-1$ correspondence between the isomorphism classes of essential crystallographic root systems and the isomorphism classes of finite-dimensional semisimple Lie algebras over $\mathbb{C}$. See

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Note different root systems can generate the same reflection groups, i.e. $\triangle \neq \triangle'$, but $W = W(\triangle) = W(\triangle')$. In this case it is a fact that $\mathbb{E}_\triangle = \mathbb{E}_{\triangle'}$, so one can instead refer to $\mathbb{E}_W$ as the space spanned by any root system generating the reflection group W. We similarly find the decomposition $\mathbb{E} = \mathbb{E}_W \oplus \mathbb{E}^W$ for $\mathbb{E}^W := \{v \in \mathbb{E} \mid g(v) = v\ \forall g \in W\}$. We can restrict W to act just on E_W, and we find $W \subset O(E_W)$. We can now generalise the definition of an isomorphism of reflection groups from Definition 3.1:

Definition 3.9. Reflection groups $W \subset O(\mathbb{E}), W' \subset O(\mathbb{E}')$ are **stably isomorphic** if there exists a linear isomorphism $f : \mathbb{E}_W \to \mathbb{E}'_{W'}$ that preserves inner products: $(f(x), f(y)) = (x, y)\ \forall x, y \in \mathbb{E}_W$, and satisfies: $fWf^{-1} = W'$.

All Euclidean reflection groups have a Coxeter group structure. These are groups permitting a presentation (see Section 6.2 in Appendix) of a particular form. Note Björner [8] has helpful material on Coxeter groups.

Definition 3.10. • Group W is a **Coxeter group** if there exists a subset $S \subset W$ such that $W = \langle s \in S \mid (ss')^{m_{ss'}} = 1 \rangle$ with $m_{ss'} = 1$ if $s = s'$ and $m_{ss'} \in \{2, 3, \dots\} \cup \{\infty\}$ otherwise.

We have $m_{ss} = 1 \ \forall s \in S$, hence $s^2 = 1 \ \forall s \in S$. Also if $m_{ss'} = 2$ then $ss'ss' = 1$. Multiplying by $s's$ results in $ss' = s's$. Similarly $(ss')^{m_{ss'}} = (ss')\dots(ss') = 1$ and multiplying by $(s's)^{m_{ss'}}$ results in the identity $(s's)^{m_{ss'}} = 1$, so we see $m_{ss'} = m_{s's}$. Finally $m_{ss'} = \infty$ indicates no relation between s and s' is being imposed.

- **Coxeter system** is a pair (W, S), for Coxeter group W and a choice of generating set S.
- **Rank** of Coxeter system is the size of the set S.
- Finite Coxeter systems (W, S) and (W', S') are **isomorphic** if there exists group isomorphism $\phi : W \to W'$ such that ϕ maps $S = \{s_1, \dots, s_l\}$ to $S' = \{s'_1, \dots, s'_l\}$ such that $m_{ij} = m'_{ij}$ where m_{ij} (m'_{ij}) is the order of $s_i s_j$ $(s'_i s'_j)$.
- The **Coxeter graph** of the Coxeter system (G, S) is the graph with node set S, and for each $s, s' \in S$, no edge is drawn if $m_{ss'} = 2$, and an edge with label $m_{ss'}$ if $m_{ss'} \geq 3$. (Note this requires the fact: $m_{ss'} = m_{s's}$).
- A Coxeter system is **irreducible** if its Coxeter graph is connected, otherwise it is **reducible**.

Note that the presentation we gave for the Dihedral group, $D_n = \langle x, z \mid x^2 = z^2 = (zx)^n = 1 \rangle$, shows D_n is a Coxeter group where $S = \{x, z\}$ and $m = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$.

Having defined Euclidean reflection groups, and what it means for such groups to be "stably isomorphic", we see these groups are deeply related to the Coxeter groups we just defined by the following theorem:

Theorem 3.11 (Kane [16], Chapter 8)**.** *There is a* $1-1$ *correspondence between stable isomorphism classes of Euclidean reflection groups and isomorphism classes of finite Coxeter systems.*

3.2 Complex reflection groups and Invariant theory

We start by briefly introducing some invariant theory following Kane [16] Section 1.7. For a finite group $G \subset \mathrm{GL}(V)$, the action of G on V extends to an action of G on $S(V)$, the symmetric algebra over V (see Definition 6.10), by setting $g(v) := g(v_1)\dots g(v_n)\ \forall g \in G$ where $v = v_1 \dots v_n \in S(V)$ with $v_i \in V$. The **ring of invariants** is $S(V)^G := \{v \in S(V) |\ g(v) = v\ \forall g \in G\}$.

Many texts instead consider $S(V^*)$, the polyomials over V. These are polyomials in a basis for V^*, and hence can be evaluated at points in V. For $x \in V$, $y \in V^*$, let $\langle y, x\rangle \in \mathbb{C}$ denote y evaluated on x.

Proposition 3.12. *If G acts on V, then G acts on V^* by means of the following "dual" action[§]: if $g \in G, y \in V^*$ then $g(y) \in V^*$ is such that $\langle g(y), x\rangle := \langle y, g^{-1}(x)\rangle\ \forall x \in V$.*

Proof. We know $1_G(x) = x$ and $(g \cdot h)(x) = g(h(x))\ \forall x \in V$. Then $\langle 1_G(y), x\rangle = \langle y, 1_G(x)\rangle = \langle y, x\rangle$, hence $1_G(y) = y$. Also $\langle (g \cdot h)(y), x\rangle = \langle y, (h^{-1} \cdot g^{-1})(x)\rangle = \langle y, h^{-1}(g^{-1}(x))\rangle = \langle h(y), g^{-1}(x)\rangle = \langle g(h(y)), x\rangle$, so this is an action of G on V^*. $\square$

The action of G then extends to $S(V^*)$, similarly to above, so that:

$$\langle g(f), x\rangle = \langle f, g^{-1}(x)\rangle \qquad \text{for } g \in G, f \in S(V^*), x \in V$$

Then we have the ring of invariants $S(V^*)^G = \{f \in S(V^*) |\ g(f) = f\ \forall g \in G\}$.

Example 3.13. Let $G = S_n$, and V an n-dimensional $\mathbb{R}$-vector space. For basis $\{x_1, \dots, x_n\}$ of V, let S_n act on V by permuting the basis, so for $\sigma \in S_n$, $\sigma(x_i) = x_{\sigma(i)}$. The action extends to $S(V) \cong \mathbb{R}[x_1, \dots, x_n]$, and $S(V)^{S_n}$ are the "symmetric polynomials" in $\{x_1, \dots, x_n\}$. As a consequence of the "Fundamental theorem of symmetric polynomials" $S(V)^{S_n} = \mathbb{R}[e_1, \dots, e_n] \cong S(V)$ where e_k are the "elementary symmetric polynomials" defined as

$$e_k := \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \dots x_{i_k}$$

[§] Regarding the action of G on V as a representation, this dual action corresponds precisely to the dual representation of G on V^*.

So every polynomial in $S(V)^{S_n}$ can be expressed as a polynomial in the e_k. Taking $n = 3$ as an example, we see

$$e_1 := x_1 + x_2 + x_3,\ e_2 := x_1x_2 + x_1x_3 + x_2x_3,\ e_3 := x_1x_2x_3$$

Then take the symmetric polynomial $f = x_1^2 + x_2^2 + x_3^2$, which can indeed be expressed as a polynomial in the e_k, as $f = e_1^2 - 2e_2$. Note f is an example of a larger family of symmetric polynomials known as the "power-sum symmetric polynomials":

$$p_k := x_1^k + \cdots + x_n^k$$

So $f = p_2 = e_1^2 - 2e_2$. Newtons identities give p_k as polynomials in the e_k $\forall k \geq 1$.

Example 3.14. Recall the group $\mathbb{Z}_2^n \rtimes S_n$ from Examples 3.6 above. This acts on an n-dimensional $\mathbb{R}$-vector space by permuting a basis and changing signs of the coordinates. It can be shown that $S(V)^{\mathbb{Z}_2^n \rtimes S_n} = \mathbb{R}[\bar{e}_1, \ldots, \bar{e}_n]$ where:

$$\bar{e}_i := \sum_{1 \leq i_1 < \cdots < i_k \leq n} t_{i_1}^2 \ldots t_{i_k}^2$$

Both S_n and $\mathbb{Z}_2^n \rtimes S_n$ are Euclidean reflection groups, and we have just seen their invariant rings are isomorphic to polynomial algebras. We see in Theorem 3.18 below that this is not a coincidence. Before we can state this theorem we must introduce complex reflection groups. Useful material was found in Kane [16] Chapter 14, and Elvidge [11] Chapter 4.

Definition 3.15. Take V to finite-dimensional $\mathbb{C}$-vector space.

- A **complex reflection** is $s \in \mathrm{GL}(V)$ that has finite order, and which fixes a hyperplane $H \subset V$.
- A **complex reflection group** is a finite subgroup $W \subset \mathrm{GL}(V)$ generated by complex reflections.
- **Positive-definite Hermitian form** on V is a map $(\ ,\ ) : V \times V \to \mathbb{C}$ that is linear in first argument, and satisfies: $(x, y) = \overline{(y, x)}$ $\forall x, y \in V$, $(x, x) > 0$ for $x \neq 0$, and $(x, x) = 0$ for $x = 0$.

If complex reflection s has order $n \in \mathbb{N}$, then its exceptional eigenvalue must be a primitive n-th root of unity ξ. Also given a positive-definite Hermitian form $(\ , \ )'$ on V, and a complex reflection group $W \subset \mathrm{GL}(V)$, then we have another positive-definite Hermitian form $(\ , \ ) : V \times V \to \mathbb{C}$ given by:

$$(x, y) := \sum_{w \in W} (w(x), w(y))' \quad \forall x, y \in V$$

This form has the added property of being W-invariant, meaning $(w(x), w(y)) = (x, y)\ \forall w \in W$. Given such a W-invariant form $(\ , \ )$ we have the following formula for a complex reflection s (compare with Proposition 3.2 Part (1) for Euclidean reflection groups):

$$s(x) = x + (\xi - 1)\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha \quad \forall x \in V \tag{25}$$

where ξ is a primitive n-th root of unity and $\alpha \in V$ such that $s(\alpha) = \xi\alpha$. Equation (25) follows since $V = H \oplus \mathbb{C}\alpha$ (for H the reflecting hyperplane) and for $x \in H$, $(\alpha, x) = (s(\alpha), s(x)) = \xi(\alpha, x)$, which implies $(1 - \xi)(\alpha, x) = 0$. As $\xi \neq 1$, we find H and $\mathbb{C}\alpha$ are orthogonal. So finally we can verify the formula by evaluating the RHS of it on α and $x \in H$.

Example 3.16. Every Euclidean reflection group defines a complex reflection group by extending the scalars over the vector space from $\mathbb{R}$ to $\mathbb{C}$.

Example 3.17. For $m, p, n \geq 1$ with $p|m$, then the group $G(m, p, n)$ is made up of $n \times n$-matrices over $\mathbb{C}$ with exactly one non-zero entry for each row and column, and such that the non-zero entries are m-th roots of unity and the product of all non-zero entries is an m/p-th root of unity.

For an n-dimensional $\mathbb{C}$-vector space V, we can identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(\mathbb{C})$ by fixing basis $\{x_1, \dots, x_n\}$ for V. Let $C_{m/p} \subset C_m \subset \mathbb{C}^*$ be the multiplicative groups of order $\frac{m}{p}$ and m respectively, and let $\mathbb{S}_n$ be the group of permutation matrices, i.e. matrices such that each row and column contains has a single non-zero entry of 1. Finally $(C_m)^n$ denotes the group of diagonal $n \times n$-matrices with entries in C_m. Then

$$G(m, p, n) := \{wt \in (C_m)^n \rtimes \mathbb{S}_n |\ \det(t) \in C_{m/p}\} \tag{26}$$

This group is indeed a complex reflection group with complex reflections:

$$S = \{s_{ij}^{(\epsilon)} | 1 \leq i < j \leq n,\ \epsilon \in C_m\} \cup \{t_i^{\zeta} |\ 1 \leq i \leq n,\ \zeta \in C_{m/p} \backslash \{1\}\}$$

where:

$$s_{ij}^{(\epsilon)}(x_k) := \begin{cases} x_k, & k \notin \{i,j\} \\ \epsilon x_j, & k = i \\ \epsilon^{-1} x_i, & k = j \end{cases} \qquad t_i^{(\zeta)}(x_k) := \begin{cases} \zeta x_i, & k = i \\ x_k, & k \neq i \end{cases} \tag{27}$$

We can see the $s_{ij}^{(\epsilon)}$ have order 2.

Complex reflection groups were classified by Shephard-Todd in [24], and they fall into the infinite family $G(m,p,n)$, and 34 exceptional cases. Note the subfamily $G(m,m,2)$ is given by monomial 2×2-matrices with entries being m-th roots of unity such that the product of both entries is a $m/m = 1$-th root of unity, i.e. is 1. So $G(m,m,2)$ is generated by:

$$y = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \qquad x = \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix}$$

which act on $\mathbb{C}^2$. It's easy to see $y^n = x^2 = 1$ and $(xy)^2 = 1$, so $G(m,m,2) \cong D_m$.

The subfamily $G(2,1,n)$ produces the automorphism groups of the n-dimensional cubes, which we mentioned earlier was $\mathbb{Z}_2^n \rtimes S_n$. This group is also known as the "hyperoctahedral group". The still larger family $G(m,1,n)$ are the "generalised symmetric groups" $\mathbb{Z}_m \wr S_n$. See Rotman [23] Chapter 7 for a good introduction to wreath products $\wr$, and Vale [26] which looks at rational Cherednik algebras over $G(m,1,n)$.

Finally we see the importance of complex reflections to invariant theory in the following theorem:

Theorem 3.18. *[Chevalley-Shepherd-Todd Theorem] Given a finite dimensional $\mathbb{C}$-vector space V, and finite group $G \subset GL(V)$, then G is a complex reflection group iff $S(V)^G$ is a polynomial algebra.*

Proof. A more general version of this theorem can be stated, where V is over an arbitrary field k, and G is a "pseudo-reflection group" such that $\mathrm{char}(k) \nmid |G|$. See Kane [16] Chapter 18 for more. $\square$

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4 Rational Cherednik Algebras

The aim of this chapter is to introduce rational Cherednik algebras, and to survey the preprint by Bazlov, Berenstein and McGaw [6]. We start in Section 4.1 by introducing the symplectic reflection algebras, which enables us to define the rational Cherednik algebras in Section 4.2. Next in Section 4.3 we give the rational Cherednik algebra of the group $G(m, p, n)$, and also define the negative braided Cherednik algebra from Bazlov and Berenstein [3]. Finally in Section 4.4, we discuss the proof of the main result (Theorem 4.1) from Bazlov, Berenstein and McGaw [6]. This result states that the negative braided Cherednik algebra $\underline{H}_c(\mu(G(m,p,n)))$ is isomorphic to a twisting of the rational Cherednik algebra $H_c(G(m,p,n))$.

4.1 Symplectic reflection algebras

In this section we introduce symplectic reflection algebras, as this will be necessary for defining the rational Cherednik algebras in the following section. We begin by briefly mentioning the motivation for symplectic reflection algebras, before defining the symplectic reflection groups. Next we define the algebras H_θ, and discuss the "PBW property". The symplectic reflection algebras are precisely the algebras H_θ for which the PBW property is satisfied. Finally we state an important result from Etingof and Ginzburg [12] that helps us to construct symplectic reflection algbras.

There are many movations for symplectic reflection algebras, several in particular coming from algebraic geometry. We know from the previous section that a group G acting on n-dimensional vector space V also naturally acts on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, which can be regarded as the coordinate ring of V. The invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ can be regarded as the coordinate ring of a certain "orbit space" V/G, containing the orbits of G in V. As a consequence of the Shepherd-Todd theorem (see Theorem 3.18) the orbit space V/G is smooth, in a certain sense, iff G is a complex reflection group (Bellamy [7], Theorem 1.1.4). In constrast, symplectic reflection groups are not complex reflection groups, so their orbit spaces are singular. However these orbit spaces can often be understood by means of an associated symplectic reflection algebra. This is one motivation for these algebras, and more can be found at Bellamy [7]. We now introduce these objects, following Bellamy [7], Brown [9], Vale [26] and Etingof and Ginzburg [12].

Definition 4.1. • A **symplectic vector space** is a $\mathbb{C}$-vector space V with symplectic form $\omega : V \times V \to \mathbb{C}$, i.e. ω is bilinear, and:

  - non-degenerate: if $\omega(v, w) = 0 \ \forall w \in V$ then $v = 0$.
  - alternating: $\omega(v, v) = 0 \ \forall v \in V$

- For symplectic vector space V, the **symplectic linear group**

$$Sp(V) := \{\gamma \in \mathrm{GL}(V) \mid \omega(\gamma(v), \gamma(w)) = \omega(v, w) \ \forall v, w \in V\}$$

- **Symplectic reflection** is $s \in Sp(V)$ such that $\mathrm{rank}(1 - s) = 2$.

- **Symplectic reflection group** is (V, ω, G) for symplectic vector space (V, ω) and finite group $G \subset Sp(V)$ generated by a set S of symplectic reflections.

We will now construct a symplectic reflection algebra. This will require several steps, starting with recalling the smash product algebra (see Definition 2.25) $A\#H$ of a Hopf algebra H with a H-module algebra A. Next, for a symplectic reflection group (V, ω, G), we know the group algebra $\mathbb{C}G$ has a Hopf algebra structure, and we show that $T(V)$ is also a $\mathbb{C}G$-module algebra. We can then construct $T(V)\#\mathbb{C}G$, and define an algebra H_θ as a certain quotient of $T(V)\#\mathbb{C}G$. The H_θ will be our symplectic reflection algebras if they satisfy a condition known as the **PBW property**. The PBW property places a constraint on the structure of the associated graded algebra of H_θ, with respect to a certain filtration, and hence also places a constraint on the structure of H_θ too. The name PBW makes reference to the analogous Poincare-Birkoff-Witt theorem for universal enveloping algebras. The reader may want to refer to Section 6.3 of the Appendix where we discuss this result, and introduce the notions of filtrations and associated graded algebras.

The underlying vector space of the smash product algebra $A\#H$ is $A \otimes H$, with unit $1 \otimes 1$, and multiplication rule:

$$(a \otimes h) \cdot (a' \otimes h') = (a \cdot (h_{(1)} \triangleright a')) \otimes (h_{(2)} \cdot h')$$

For symplectic reflection group (V, ω, G), clearly G acts on V. Identifying V with $T^1(V)$ in the tensor algebra $T(V) = \mathbb{C} \oplus T^1(V) \oplus T^2(V) \ldots$, we can then extend the action of G to $T(V)$ as a $\mathbb{C}G$-module algebra. Meaning we set $g \triangleright 1_{\mathbb{C}} := \epsilon(g)1_{\mathbb{C}} = 1_{\mathbb{C}} \ \forall g \in G$ and $g \triangleright (v \otimes w) := (g \triangleright v) \otimes (g \triangleright w) \ \forall v, w \in T(V)$ (these are equations (21),(22) from the definition of an H-module algebra). We must check the action we have defined satisfies the conditions for $T(V)$ to be a $\mathbb{C}G$-module (i.e. satisfies equation (20)). For $v = v_1 \otimes \cdots \otimes v_n \in T^n(V)$ then

$$\begin{aligned}(g\cdot h)\triangleright v &= (g\cdot h)\triangleright v_1 \otimes \cdots \otimes (g\cdot h)\triangleright v_n = g\triangleright(h\triangleright v_1)\otimes\cdots\otimes g\triangleright(h\triangleright v_n)\\ &= g\triangleright(h\triangleright v_1\otimes\cdots\otimes h\triangleright v_n) = g\triangleright(h\triangleright v)\\ 1_G\triangleright v &= (1_G\triangleright v_1)\otimes\cdots\otimes(1_G\triangleright v_n) = v_1\otimes\cdots\otimes v_n = v\end{aligned}$$

as required, so $T(V)$ is a $\mathbb{C}G$-module algebra. So we can define the smash product $T(V)\#\mathbb{C}G$[¶]. The underlying vector space of $T(V)\#\mathbb{C}G$ is:

$$\begin{aligned}(\mathbb{C}\oplus V\oplus V^{\otimes 2}\oplus\dots)\otimes\mathbb{C}G &\cong (\mathbb{C}\otimes\mathbb{C}G)\oplus(V\otimes\mathbb{C}G)\oplus(V^{\otimes 2}\otimes\mathbb{C}G)\oplus\dots\\ &\cong \mathbb{C}G\oplus(V\otimes\mathbb{C}G)\oplus(V^{\otimes 2}\otimes\mathbb{C}G)\oplus\dots\end{aligned}\tag{28}$$

with the first isomorphism by Allufi [1] (VIII.2.1 Corollary 2.7), and the second by the fact $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}G\cong\mathbb{C}G$. Now, for a skew-symmetric bilinear map $\theta: V\times V\to\mathbb{C}G$, define I_θ to be the 2-sided ideal in $T(V)\#\mathbb{C}G$ generated by $\{x\otimes y - y\otimes x - \theta(x,y)|\ x,y\in V\}$. We identify $x\otimes y$ for $x,y\in V$ with $x\otimes y\otimes 1_G\in V^{\otimes 2}\otimes\mathbb{C}G$, and we take $\theta(x,y)$ to be in the first $\mathbb{C}G$ in the final expression of equation(28), so:

$$x\otimes y - y\otimes x - \theta(x,y)\in\mathbb{C}G\oplus\{0\}\oplus(V^{\otimes 2}\otimes\mathbb{C}G)$$

Let $H_\theta := T(V)\#\mathbb{C}G/I_\theta$. Note that for $\theta = 0$ we have $H_0 = S(V)\#\mathbb{C}G$. Next we will describe the PBW property, which if satisfied, distinguishes H_θ as a symplectic reflection algebra.

By Vale [26] (Section 1.2), there is a filtration (see Definition 6.12) $\{F_i\}$ on H_θ given by:

$$F_{-1} := \{0\},\ F_0 := \mathbb{C}G,\ F_1 := \mathbb{C}G\oplus(V\otimes\mathbb{C}G),\ F_i := (F_1)^i\ \ \forall i\geq 2$$

where the F_i are subspaces of $H_\theta = T(V)\#\mathbb{C}G/I_\theta$. The associated graded algebra is $G(H_\theta) = \bigoplus_{i=0}^{\infty} F_i/F_{i-1}$. By Bellamy [7] (Section 1.4) one can define a map $\sigma: H_\theta\to G(H_\theta)$ such that if $a\in H_\theta$ is of degree i (i.e. $a\in F_i$ but $a\notin F_{i-1}$) then $\sigma(a)$ is the image of a in F_i/F_{i-1}.

It can be shown that $x\otimes y - y\otimes x\in V^{\otimes 2}\otimes\mathbb{C}G$ is degree 2, but since $x\otimes y - y\otimes x = \theta(x,y)\in\mathbb{C}G = F_0\subset F_1$, we find that that σ maps $x\otimes y - y\otimes x$ to 0. By Vale [26] it is possible to extend $\sigma: H_\theta\to G(H_\theta)$ to an algebra homomorphism $\rho: S(V)\#\mathbb{C}G\to G(H_\theta)$ such that $v\mapsto v, g\mapsto g\ \forall v\in V, g\in G$.

[¶]Note Vale [26] and Brown [9] equivalently use the "skew group algebra" $T(V)\star G$, however we choose to follow Etingof and Ginzburg's [12] convention of the smash product algebra.

Definition 4.2.

- H_θ has the **PBW property** if ρ is an isomorphism of algebras:

$$G(H_\theta) \cong S(V)\#\mathbb{C}G$$

- A **symplectic reflection algebra** is an algebra H_θ such that PBW property holds. By Proposition 6.13, in this case, $H_\theta \cong S(V) \otimes \mathbb{C}G$ as vector spaces.

Next we give a result by Etingof and Ginzburg which describes a class of algebras that satisfy the PBW property. Take symplectic reflection group (V, ω, G) and $S \subset G$ the symplectic reflections. For each $s \in S$, define $\omega_s : V \times V \to \mathbb{C}$ as the bilinear form satisfying:

- $\omega_s(v, w) = \omega(v, w)\ \forall v, w \in \mathrm{Im}(1 - s)$.
- the radical of ω_s is $\ker(1 - s)$.

Recall the radical is the kernel of ω_s regarded as a map $V \to V^*$, with $\omega_s(v)(w) = \omega_s(v, w)$ (this is really the left radical, but for skew-symmetric forms the left and right radicals coincide). So for ω_s to have a radical of $\ker(1 - s)$ says that for each $v \in \ker(1 - s)$, $\omega_s(v, w) = 0 = \omega_s(w, v)\ \forall w \in V$. Since $V = \mathrm{Im}(1 - s) \oplus \ker(1 - s)$ these two conditions determine ω_s everywhere on $V \times V$.

Theorem 4.3. *[Theorem 1.3, [12]] Take symplectic reflection group (V, ω, G), $S \subset G$ the symplectic reflections, and $\{\omega_s : V \times V \to \mathbb{C}G|\ s \in S\}$ as above. Set $t \in \mathbb{C}$ and define $c : S \to \mathbb{C}$, $s \mapsto c_s$ such that $c_{gsg^{-1}} = c_s\ \forall s \in S, g \in G$. If bilinear $\theta : V \times V \to \mathbb{C}G$ is given by*

$$\theta(x, y) := t\omega(x, y)1_G + \sum_{s \in S} c_s \omega_s(x, y)s$$

then the algebra $H_\theta := T(V)\#\mathbb{C}G/I_\theta$ satisfies the PBW property.

Note that the θ defined above is skew-symmetric since ω and ω_s are skew-symmetric. Also the map c only makes sense if symplectic reflections are closed under conjugation, i.e. $gsg^{-1} \in S\ \forall g \in G, s \in S$. This holds for all symplectic reflection groups, but we will prove it in Proposition 4.4 for those symplectic reflection groups that are generated by a complex reflection group. We discuss in the next section how a complex reflection group generates a symplectic reflection group, as this will be necessary for defining rational Cherednik algebras.

4.2 Rational Cherednik algebras

In this section we start by showing how a complex reflection group $G \subset \mathrm{GL}(V)$ generates a symplectic reflection group on $V \oplus V^*$. We will then use this to define rational Cherednik algebras as a special case of symplectic reflection algebras. Finally we give a second definition of the rational Cherednik algebra by means of a presentation, as given in [6], and then discuss how these these two characterisations are equivalent.

By Proposition 3.12 above, a complex reflection group $G \subset \mathrm{GL}(V)$ acts on V^* by $\langle g(y), x\rangle := \langle y, g^{-1}(x)\rangle \ \forall x \in V, y \in V^*$. We also let G act diagonally on $V \oplus V^*$, meaning for $x \in V, \mu \in V^*$, let $g(x+\mu) := g(x) + g(\mu)$. It is easy to check this is an action. So G can be regarded as a subset of $GL(V)$, $GL(V^*)$ or $GL(V \oplus V^*)$. For $S \subset G$ the set of complex reflections and $s \in S$, we know $\mathrm{rank}(1-s) = 1$ on V. Define basis $\{v_1, \dots, v_n\}$ of V such that $s(v_1) \neq v_1$ whilst $s(v_i) = v_i \ \forall i \in \{2, \dots, n\}$. Take $\{\delta_{v_i}\}$ as the dual basis for V^*. Note

$$\langle s(\delta_{v_1}), v_j\rangle = \langle \delta_{v_1}, s^{-1}(v_j)\rangle = 1 \iff s^{-1}(v_j) = v_1 \iff v_j = s(v_1) \neq v_1$$

So we find $s(\delta_{v_1}) \neq \delta_{v_1}$. Also for $i \in \{2, \dots, n\}$,

$$\langle s(\delta_{v_i}), v_j\rangle = \langle \delta_{v_i}, s^{-1}(v_j)\rangle = 1 \iff s^{-1}(v_j) = v_i \iff v_j = s(v_i) = v_i \iff i = j$$

So $s(\delta_{v_i}) = \delta_{v_i} \ \forall i \in \{2, \dots, n\}$. Hence $\mathrm{rank}(1-s) = 1$ on V^*. Finally let s act on $V \oplus V^*$, then

$$\begin{aligned} s(v_1 + 0) &= s(v_1) + 0 \neq v_1 + 0 \\ s(0 + \delta_{v_1}) &= 0 + s(\delta_{v_1}) \neq 0 + \delta_{v_1} \\ s(v_i + \delta_{v_j}) &= s(v_i) + s(\delta_{v_j}) = v_i + \delta_{v_j} \qquad \forall i, j \in \{2, \dots, n\} \end{aligned}$$

So we see $\mathrm{rank}(1-s) = 2$ on $V \oplus V^*$, hence s is a symplectic reflection when regarded as an operator on $V \oplus V^*$. We can define a bilinear form ω on $V \oplus V^*$ such that for $x, y \in V, \mu, \nu \in V^*$[‖]:

$$\omega(x+\mu, y+\nu) := \langle \mu, y\rangle - \langle \nu, x\rangle \tag{29}$$

This makes $(V \oplus V^*, \omega)$ a symplectic vector space. Since the generators of G are complex reflections on V, they are symplectic reflections on $V \oplus V^*$, so we find $(V \oplus V^*, \omega, G)$ is a symplectic reflection group.

[‖]Many texts define ω with the opposite sign, i.e. $\omega(x+\mu, y+\nu) := \langle \nu, x\rangle - \langle \mu, y\rangle$. But this results later in $\kappa_c = -\theta$, rather than $\kappa_c = \theta$, as is preferable, hence why we choose ω this way.

Proposition 4.4. *For complex reflection group $G \subset GL(V)$ and complex reflections $S \subset G$, then $gsg^{-1} \in S \ \forall g \in G, s \in S$. Additionally if $(V \oplus V', \omega, G)$ is the corresponding symplectic reflection group, and S is the set of symplectic reflections, then similarly $gsg^{-1} \in S \ \forall g \in G, s \in S$.*

Proof. The elements $g \in G$ are invertible so $\text{rank}(g) = \dim(V) \ \forall g \in G$. For $s \in S$, $1 - gsg^{-1} = g(1-s)g^{-1}$. Since $\text{rank}(g) = \text{rank}(g^{-1}) = \dim(V)$ and $\text{rank}(1-s) = 1$, we find $\text{rank}((1-s)g^{-1}) = 1$ and then $\text{rank}(g(1-s)g^{-1}) = 1$. Additionally $(gsg^{-1})^n = gs^n g^{-1} = 1 \iff s^n = 1$, so the order of gsg^{-1} is equal to that of s, so it must be finite. Therefore gsg^{-1} is a complex reflection as required.

By how the symplectic reflection group $(V \oplus V', \omega, G)$ was constructed, G is clearly a complex reflection on V. Since the set of complex reflections coincides with the set of symplectic reflections, the second result follows immediately from the first. □

Let us now characterise **rational Cherednik algebras** as the algebras of the form H_θ from Theorem 4.3 when the symplectic reflection group is $(V \oplus V^*, \omega, G)$. We can in fact give a presentation of rational Cherednik algebras, analogously to the group presentations introduced in Section 6.2. For presentations of algebras, one takes the free algebra (i.e. tensor algebra) of some generating vector space V, and quotient by a 2-sided ideal generated by certain "relations", being elements in $T(V)$. If an algebra is generated by several vector spaces, one takes the tensor algebra over the direct sum of these vector spaces. Bazlov, Berenstein and McGaw [6] give the presentation of rational Cherednik algebras as follows:

Definition 4.5. Let $G \subset GL(V)$ be a complex reflection group, and $S \subset G$ the reflections in G. Define $c : S \to \mathbb{C}, s \mapsto c_s$ such that $c_{gsg^{-1}} = c_s \ \forall g \in G, s \in S$, and bilinear map $\kappa_c : V^* \times V \to \mathbb{C}G$:

$$\kappa_c(y,x) := \langle y, x\rangle 1_G + \sum_{s \in S} c_s \langle y, (1-s)(x)\rangle s \tag{30}$$

Then the **rational Cherednik algebra** $H_c(G)$ is generated by V, $\mathbb{C}G$ and V^*, subject to the relations, $\forall x, x' \in V, \ y, y' \in V^*, \ g \in G$:

$$xx' - x'x = 0 \qquad yy' - y'y = 0 \qquad yx - xy = \kappa_c(y,x) \tag{31}$$

$$gx = g(x)g \qquad yg = g \cdot g^{-1}(y) \qquad gg' = g \cdot g' \qquad 1_G = 1_{\mathbb{C}} \tag{32}$$

We briefly elaborate on the notation of this definition. Recall by Proposition 4.4 for complex reflection group G, we have $gsg^{-1} \in S\ \forall g \in G, s \in S$ as required for c to make sense. Additionally if $X := V \oplus \mathbb{C}G \oplus V^*$, then $H_c(G)$ is the quotient of the tensor algebra $T(X) = \mathbb{C} \oplus T^1(X) \oplus T^2(X) \oplus \dots$ by the 2-sided ideal I generated by the relations. The x, x', y, y', g in the relations should be regarded as elements of $T^1(X) \cong X$, i.e. $x = x \oplus 0 \oplus 0,\ g = 0 \oplus g \oplus 0,\ y = 0 \oplus 0 \oplus y$. The product in $T(X)$ is the tensor product, so for instance xx' denotes: $(x \oplus 0 \oplus 0) \otimes (x' \oplus 0 \oplus 0) \in T^2(X)$ and $gx = (0 \oplus g \oplus 0) \otimes (x \oplus 0 \oplus 0) \in T^2(X)$. The condition $gg' = g \cdot g'$ says the ideal I is generated by the following elements:

$$(g \oplus 0 \oplus 0) \otimes (g' \oplus 0 \oplus 0) - (g \cdot g') \oplus 0 \oplus 0 \in T^1(X) \oplus T^2(X)\ \forall g, g' \in G$$

where $\cdot$ is the product in G. Also the condition $1_G = 1_{\mathbb{C}}$ says I also contains

$$1_G - 1_{\mathbb{C}} = (0 \oplus 1_G \oplus 0) - 1_{\mathbb{C}} \in \mathbb{C} \oplus T^1(X)$$

The other relations can be interpreted similarly.

Our next aim is to show how the presentation in Definition 4.5 is equivalent to the original characterisation of the rational Cherednik algebra as $H_\theta = T(V \oplus V^*)\#\mathbb{C}G/I_\theta$, for I_θ generated by:

$$\{x \otimes y - y \otimes x - \theta(x, y) | x, y \in V \oplus V^*\} \tag{33}$$

and

$$\theta(x, y) = t\omega(x, y)1_G + \sum_{s \in S} c_s \omega_s(x, y)s \tag{34}$$

Note that ω, ω_s and θ are defined on $(V \oplus V^*) \times (V \oplus V^*)$. We inspect their values on the following subsets: $V \times V, V^* \times V^*, V \times V^*$ and $V^* \times V$**. By equation (29), we see $\omega = 0$ on $V \times V$ and $V^* \times V^*$. The same goes for ω_s which is either equal to ω or 0. Hence $\theta = 0$ on $V \times V$ and $V^* \times V^*$. So the set in equation (33) contains the following generators of I_θ:

$$\{x \otimes y - y \otimes x | x, y \in V\} \cup \{x \otimes y - y \otimes x | x, y \in V^*\}$$

We see that these correspond precisely to the relations $xx' - x'x = yy' - y'y = 0$ (from equation (31)) in the presentation of the rational Cherednik algebra $H_c(G)$ above.

**Where we are identifying V with $V \oplus \{0\}$ and V^* with $\{0\} \oplus V^*$.

Note ω and ω_s can only be non-zero over $V \times V^*$ and $V^* \times V$, and the same goes for θ. Also we find θ is skew-symmetric since ω and ω_s are skew-symmetric, hence:

$$x \otimes y - y \otimes x - \theta(x, y) = -(y \otimes x - x \otimes y - \theta(y, x)) \; \forall x \in V, y \in V^*$$

So we lose nothing by considering θ defined just over $V^* \times V$. Recall $\kappa_c : V^* \times V \to \mathbb{C}$ from equation (30) in the presentation of a rational Cherednik algebra. We will show next that $\theta = \kappa_c$. Then it will be clear that the relations $\{y \otimes x - x \otimes y - \theta(y, x) | x \in V, y \in V^*\}$ for H_θ correspond to the relations $yx - xy = \kappa_c$ for $x \in V,\ y \in V^*$ in $H_c(G)$.

Firstly compare κ_c in equation (30) with θ in equation (34). Clearly if they are to be equal we must set $t = 1$. Recall ω_s is equal to ω on $\mathrm{Im}(1 - s) \times \mathrm{Im}(1 - s)$ and 0 elsewhere, and by definition of ω: $\omega(y, x) = \langle y, x\rangle \; \forall x \in V, y \in V^*$. Let $\alpha_s^\vee$ be a basis for $\mathrm{Im}(1 - s)|_V$ such that if s is of order n, then $s(\alpha_s^\vee) = \xi\alpha_s^\vee$ where ξ is a primitive n-th root of unity. Recall we found in equation (25) that there exists a G-invariant positive-definite Hermitian form $(\ ,\ )$ on V such that:

$$s(x) = x + (\xi - 1)\frac{(\alpha_s^\vee, x)}{(\alpha_s^\vee, \alpha_s^\vee)}\alpha_s^\vee \quad \forall x \in V$$

We can define $\alpha_s \in V^*$ as:

$$\langle \alpha_s, x\rangle := \big(x, \frac{(1 - \xi)}{(\alpha_s^\vee, \alpha_s^\vee)}\alpha_s^\vee\big) = \frac{(1 - \xi)(x, \alpha_s^\vee)}{(\alpha_s^\vee, \alpha_s^\vee)} \quad \forall x \in V$$

We note that α_s is a basis of $\mathrm{Im}(1 - s)|_{V^*}$ such that $\langle \alpha_s, \alpha_s^\vee\rangle = 1 - \xi$. Then

$$\begin{aligned}\langle y, (1 - s)x\rangle &= \frac{(1 - \xi)}{(\alpha_s^\vee, \alpha_s^\vee)}(x, \alpha_s^\vee)\langle y, \alpha_s^\vee\rangle \\ &= \frac{(1 - \xi)}{(\alpha_s^\vee, \alpha_s^\vee)}\frac{(\alpha_s^\vee, \alpha_s^\vee)\langle \alpha_s, x\rangle}{(1 - \xi)}\langle y, \alpha_s^\vee\rangle = \langle \alpha_s, x\rangle\langle y, \alpha_s^\vee\rangle = \omega_s(y, x)\end{aligned}$$

as required, so we find $\theta = \kappa_c$.

So far we have shown how the relations in equation (31) of Definition (4.5) arise from structure of the symplectic reflection algebra H_θ. The four relations in equation (32) will be derived next.

$H_c(G)$ is defined as a quotient of $T(V \oplus \mathbb{C}G \oplus V^*)$, which is isomorphic to $T(V) \otimes T(\mathbb{C}G) \otimes T(V^*)$ by Proposition 6.2. Note the presence of $T(\mathbb{C}G)$ in $H_c(G)$. It is a fact that the quotient of $T(\mathbb{C}G)$ by the relations $1_G = 1_{\mathbb{C}}$ and $g \otimes g' = g \cdot g' \; \forall g, g' \in G$ is isomorphic to $\mathbb{C}G$. Therefore we see that these same relations arise in the presentation of $H_c(G)$ due to the presence of $\mathbb{C}G$ in $H_\theta := T(V \oplus V^*)\#\mathbb{C}G/I_\theta$.

Finally the relations $gx = g(x)g$ and $yg = gg^{-1}(y)$ for $H_c(G)$ arise from the smash product multiplication in $H_\theta = T(V \oplus V^*)\#\mathbb{C}G/I_\theta$. In particular $gx = g(x)g$ in $H_c(G)$ corresponds to fact:

$$(1_{\mathbb{C}} \otimes g) \cdot (x \otimes 1_G) = (1_{\mathbb{C}} \cdot (g \triangleright x)) \otimes (g \cdot 1_G) = g(x) \otimes g \in H_\theta$$

Finally we are able to see how each of the relations for $H_c(G)$ in Definition 4.5 arise from the structure of a rational Cherednik algebra as the symplectic reflection algebra $H_\theta := T(V \oplus V^*)\#\mathbb{C}G/I_\theta$.

Finally we can use the fact that rational Cherednik algebras satsify the PBW property (by Theorem 4.3) to deduce a bit of their structure. Indeed we know that $G(H_c(G)) \cong S(V \oplus V^*) \otimes \mathbb{C}G$ as algebras, while $H_c(G) \cong S(V \oplus V^*) \otimes \mathbb{C}G$ as vector spaces. Viewing S as a functor from vector spaces to commutative algebras, we can use the same argument used in the proof of Proposition 6.2 to find: $S(\bigoplus_i V_i) \cong \bigotimes_i S(V_i)$. So $H_c(G)$ has the following underlying vector space

$$\begin{aligned} S(V \oplus V^*) \otimes \mathbb{C}G &\cong S(V) \otimes \mathbb{C}G \otimes S(V^*) \\ &\cong \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}G \otimes \mathbb{C}[y_1, \dots, y_n] \end{aligned}$$

where x_i, y_i are bases of V, V^* respectively. Analogously to universal enveloping algebras (see Corollary 6.15) we find $H_c(G)$ has the following basis:

Theorem 4.6. *[[12]] If x_i, y_i bases of V, V^* resp, then the algebra $H_c(G)$ has basis:*

$$\{x_1^{k_1} \dots x_n^{k_n} g y_1^{l_1} \dots y_n^{l_n} |\ g \in G, k_i, l_i \in \mathbb{N}_0\ \forall i\}$$

4.3 Examples

In this section we follow Section 3 of [6]. We start by defining the rational Cherednik algebra over the complex reflection group $G(m, p, n)$. Next we define the negative braided Cherednik algebra $\underline{H}_c(\mu(G(m, p, n)))$ of a related group $\mu(G(m, p, n))$. We finish by briefly discussing how rational Cherednik algebras and negative braided Cherednik algebras are special cases of a larger family of **braided Cherednik algebras**.

In Definition 3.17 we defined $G(m, p, n)$ as a subgroup of $\mathrm{GL}_n(\mathbb{C})$, which is identified with $\mathrm{GL}(V)$ by fixing a basis $\{x_1, \dots, x_n\}$ of the $\mathbb{C}$-vector space V. Let $\{y_1, \dots, y_n\}$

denote the dual basis on V^*. The complex reflections in $G(m,p,n)$ are:

$$S = \{s_{ij}^{(\epsilon)} \mid 1 \le i < j \le n,\ \epsilon \in C_m\} \cup \{t_i^{(\zeta)} \mid 1 \le i \le n,\ \zeta \in C_{m/p} \backslash \{1\}\}$$

where $s_{ij}^{(\epsilon)}$ are given in equation (27), and $t_i^{(\zeta)}$ can be found below. It is stated in [6] that for $n \ge 3$, the set $\{s_{ij}^{(\epsilon)}\}$ forms a conjugacy class in $G(m,p,n)$, while for each $\zeta \in C_{m/p}\backslash\{1\}$ there is a conjugacy class $\{t_i^{(\zeta)} | 1 \le i \le n\}$. Clearly any map $c : S \to \mathbb{C}$ satisfying $c_{gsg^{-1}} = c_s\ \forall g \in G(m,p,n), s \in S$ must take constant values on conjugacy classes. So let $c_1 \in \mathbb{C}$ denote the value of c on $\{s_{ij}^{(\epsilon)}\}$, and c_ζ the value on $\{t_i^{(\zeta)}\}$.

Definition 4.7. For $n \ge 3$, the rational Cherednik algebra $H_c(G(m,p,n))$ has a presentation given by generators $\{x_1, \dots, x_n, y_1, \dots, y_n\} \cup G(m,p,n)$ and relations:

$$x_i x_j - x_j x_i = 0 \qquad y_i y_j - y_j y_i = 0 \qquad g x_i = g(x_i) g \qquad y_i g = g \cdot g^{-1}(y)$$

$$y_i x_j - x_j y_i = c_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(\epsilon)} \qquad y_i x_i - x_i y_i = 1 - c_1 \sum_{j \ne i} \sum_{\epsilon \in C_m} s_{ij}^{(\epsilon)} - \sum_{\zeta \in C_{m/p} \backslash \{1\}} c_\zeta t_i^{(\zeta)}$$

Additionally, we have $gg' = g \cdot g'\ \forall g, g' \in G(m,p,n)$ and $1_{G(m,p,n)} = 1_{\mathbb{C}}$.

In Example 3.13 we found that the ring of invariants $S(V)^{S_n}$ is such that $S(V)^{S_n} \cong S(V)$. Indeed by the Chevalley-Shepherd-Todd theorem, for $\mathbb{C}$-vector space V, $S(V)^G \cong S(V)$ iff G is a complex reflection group. We can generalise $S(V)$ in the following way. Let q_{ij} be a complex $n \times n$-matrix, such that $q_{ij} q_{ji} = 1$ and $q_{ii} = 1$ for each $1 \le i, j \le n$. Then define the **q-symmetric algebra** as:

$$S_q(V) := \mathbb{C}\langle x_1, \dots, x_n |\ x_i x_j = q_{ij} x_j x_i,\ i < j\rangle$$

where $x_1, \dots, x_n$ is a basis for V. In Kirkmann, Kuzmanovich and Zhang [17], a non-commutative analog of the Chevalley-Shepherd-Todd theorem was proven (Theorem 0.1), in which $S_q(V)^G$ has "finite global dimension" iff G is generated by "quasi-reflections". For the purposes of this work we do not need to go into defining what these terms mean. But we do wish to note that, unlike the classical Chevalley-Shepherd-Todd theorem, in the case that G is generated by "quasi-reflections", $S_q(V)^G$ may not be isomorphic to $S_q(V)$, but is instead isomorphic to $S_{q'}(V)$ for some other matrix q'. In Bazlov and Berenstein [5] the groups G that act on $S_q(V)$ by degree-preserving automorphisms such that $S_q(V)^G \cong S_{q'}(V)$ for some matrix q' are called **mystic reflection groups**.

Definition 4.8. For $n \geq 1$, m even, and $p|m$, define the group:

$$\mu(G(m,p,n)) := \{tw \in (C_m)^n \rtimes \mathbb{S}_n | \ \det(tw) \in C_{m/p}\}$$

The following subset of $\mu(G(m,p,n))$, called the **mystic reflections**, generates the group:

$$\underline{S} := \{\sigma_{ij}^{(\epsilon)} | 1 \leq i < j \leq n, \epsilon \in C_m\} \cup \{t_i^{(\zeta)} : 1 \leq i \leq n, \zeta \in C_{m/p} \backslash \{1\}\}$$

where

$$t_i^{(\zeta)}(x_k) := \begin{cases} \zeta x_i, & k = i \\ x_k, & k \neq i \end{cases} \qquad \sigma_{ij}^{(\epsilon)}(x_k) = \begin{cases} x_k, & k \notin \{i,j\} \\ \epsilon x_j, & k = i, \\ -\epsilon^{-1} x_i & k = j \end{cases}$$

Note that $\sigma_{ij}^{(\epsilon)}$ is of order 4. Also by Theorem 2.6 of [5], $\mu(G(m,p,n))$ is indeed a mystic reflection group, and by Theorem 2.8 of [5], the group algebras $\mathbb{C}G(m,p,n)$ and $\mathbb{C}\mu(G(m,p,n))$ are isomorphic.

Again for $n \geq 3$, the $\{\sigma_{ij}^{(\epsilon)}\}$ form a single conjugacy class in $\mu(G(m,p,n))$, while for each $\zeta \in C_{m/p} \backslash \{1\}$ there is a conjugacy class $\{t_i^{(\zeta)}\}$. So for a map $c : \underline{S} \to \mathbb{C}$ satisfying $c_{gsg^{-1}} = c_s$, let c_1 denote the value of c on $\{\sigma_{ij}^{(\epsilon)}\}$, and c_ζ the value on the conjugacy class of $t_i^{(\zeta)}$.

Definition 4.9. For $n \geq 3$ and $c : \underline{S} \to \mathbb{C}$ as above, the **negative braided Cherednik algebra** $\underline{H}_c(\mu(G(m,p,n)))$ has a presentation with generators $\{\underline{x}_1, \ldots, \underline{x}_n, \underline{y}_1, \ldots, \underline{y}_n\}$ and $g \in \mu(G(m,p,n))$, and relations: $\forall i \neq j$

$$\underline{x}_i \underline{x}_j + \underline{x}_j \underline{x}_i = 0 \qquad \underline{y}_i \underline{y}_j + \underline{y}_i \underline{y}_j = 0 \qquad g \underline{x}_i g^{-1} = g(\underline{x}_i) \qquad g \underline{y}_i g^{-1} = g(\underline{y}_i)$$

$$\underline{y}_i \underline{x}_j + \underline{x}_j \underline{y}_i = c_1 \sum_{\epsilon \in C_m} \epsilon \sigma_{ij}^{(\epsilon)} \qquad \underline{y}_i \underline{x}_i - \underline{x}_i \underline{y}_i = 1 + c_1 \sum_{j \neq i} \sum_{\epsilon \in C_m} \sigma_{ij}^{(\epsilon)} + \sum_{\zeta \in C_{m/p} \backslash \{1\}} c_\zeta t_i^{(\zeta)}$$

$$1_{\mu(G(m,p,n))} = 1_{\mathbb{C}} \qquad gg' = g \cdot g' \quad \forall g, g' \in \mu(G(m,p,n))$$

To clarify, the elements on both sides of $gg' = g \cdot g'$ are in $\underline{H}_c(\mu(G(m,p,n)))$. However the multiplication on the LHS is in $\underline{H}_c(\mu(G(m,p,n)))$, whilst the multiplication on RHS is in $\mu(G(m,p,n))$. Also note $gg' = g \cdot g'$ $\forall g, g'$ is a lot more relations than necessary to define $\underline{H}_c(\mu(G(m,p,n)))$. We only really require the relations $gg' = g \cdot g'$ for those g, g' that are generators of a presentation of $\mu(G(m,p,n))$, and where $g \cdot g'$

is given by one of the defining relations of the group. This will be importance in the proof of Theorem 4.13.

Theorem 4.10. *[[3]] $\underline{H}_c(\mu(G(m,p,n)))$ has a basis:*

$$\{\underline{x}_i^{k_1}\dots\underline{x}_n^{k_n}g\underline{y}_1^{l_1}\dots\underline{y}_n^{l_n}|\ g\in\mu(G(m,p,n)),\ k_i,l_i\in\mathbb{N}_0\ \forall i\}$$

and underling vector space $S_{\boldsymbol{-1}}(V)\otimes\mathbb{C}\mu(G(m,p,n))\otimes S_{\boldsymbol{-1}}(V^*)$, where $\boldsymbol{-1}$ denotes the $n\times n$-matrix q such that $q_{ii}=1$ and $q_{ij}=-1$ for $i\neq j$.

Note that the rational, and negative braided, Cherenik algebras are special cases of a broader family of **braided Cherednik algebras**. Braided Cherednik algebras are discussed in detail in Bazlov and Berenstein [3] (Definition 3.2). A property of these algebras is that the underlying vector space is given by $S_q(V)\otimes\mathbb{C}G\otimes S_q(V^*)$ for some matrix q. We see the rational Cherednik algebras are precisely the braided Cherednik algebras with matrices $q_{ij}=1\ \forall i,j$. Additionally in Proposition 3.6 of [3] it is proven that the negative braided Cherednik algebra $\underline{H}_c(\mu(G(m,p,n)))$ are also braided Cherednik algebra, with $q=\boldsymbol{-1}$.

4.4 Twists of rational Cherednik Algebras

The aim of this section is to summarise Section 4 of [6]. For the basis $\{x_1,\dots,x_n\}$ of V, let $\mathbb{T}_n:=\{t\in\mathrm{GL}(V)|\ t(x_i)\in\mathbb{C}x_i\ \forall i\}$ be the group of diagonal matrices. Then let $T:=T(2,1,n)=G(2,1,n)\cap\mathbb{T}_n$. This is the abelian group generated by the maps $t_i^{(-1)}$, which by definition act on V by flipping the sign of the i-th coordinate. So T is precisely $(C_2)^n$. Next define the following elements in $\mathbb{C}T\otimes\mathbb{C}T$:

$$f(-1,i,j):=\frac{1}{2}(1\otimes1+t_i^{(-1)}\otimes1+1\otimes t_j^{(-1)}-t_i^{(-1)}\otimes t_j^{(-1)})$$
$$\mathcal{F}:=\prod_{1\leq j<i\leq n}f(-1,i,j)$$

Proposition 4.11. *[Prop 4.2, [6]] $\mathcal{F}$ and $f(-1,i,j)$ are involutions. Additionally they are quasitriangular structures on $\mathbb{C}T$.*

Proof. Note $(t_i^{(-1)})^2 = 1\ \forall i$. Let t_i denote $t_i^{(-1)}$ in the following:

$$\begin{aligned} f(-1,i,j)^2 =& \frac{1}{4}(1\otimes 1 + t_i\otimes 1 + 1\otimes t_j - t_i\otimes t_j + 1\otimes 1 + t_i\otimes 1 - 1\otimes t_j + t_i\otimes t_j \\ &+ 1\otimes 1 + t_i\otimes t_j + 1\otimes t_j - t_i\otimes 1 - t_i\otimes t_j - 1\otimes t_j - t_i\otimes 1 + 1\otimes 1) \\ =& \frac{1}{4}(4\cdot 1\otimes 1) = 1\otimes 1 \end{aligned}$$

So $f(-1,i,j)$ is an involution, and since $\mathbb{C}T\otimes\mathbb{C}T$ is a commutative algebra it follows

$$\mathcal{F}^2 = \prod_{1\leq j<i\leq n} f(-1,i,j)^2 = 1\otimes 1$$

Now we show they are quasitriangular structures. Since $\mathbb{C}T$ is cocommutative, and $\mathbb{C}T\otimes\mathbb{C}T$ is a commutative algebra, $\tau\circ\triangle = R\cdot\triangle\cdot R^{-1}$ holds trivially $\forall R\in\mathbb{C}T\otimes\mathbb{C}T$. It remains to show that $f := f(-1,i,j)$ satisfies: $(\triangle\circ\mathrm{id})(f) = f_{13}f_{23}$, and $(\mathrm{id}\circ\triangle)(f) = f_{13}f_{12}$. Indeed:

$$\begin{aligned} (\triangle\circ\mathrm{id})(f) &= \frac{1}{2}(1\otimes 1\otimes 1 + t_i\otimes t_i\otimes 1 + 1\otimes 1\otimes t_j - t_1\otimes 1\otimes t_j) \\ f_{13} &= \frac{1}{2}(1\otimes 1\otimes 1 + t_i\otimes 1\otimes 1 + 1\otimes 1\otimes t_j - t_i\otimes 1\otimes t_j) \\ f_{23} &= \frac{1}{2}(1\otimes 1\otimes 1 + 1\otimes t_i\otimes 1 + 1\otimes 1\otimes t_j - 1\otimes t_i\otimes t_j) \end{aligned}$$

And it easy to check $(\triangle\circ\mathrm{id})(f) = f_{13}f_{23}$ holds. Similarly $(\mathrm{id}\circ\triangle)(f) = f_{13}f_{12}$ holds, so $f(-1,i,j)$ are quasitriangular structures on $\mathbb{C}T$. Also, by the multiplication rule in $\mathbb{C}T\otimes\mathbb{C}T$, it can be seen that: $\mathcal{F}_{13} = \prod_{1\leq j<i\leq n} f(-1,i,j)_{13}$ and $\mathcal{F}_{23} = \prod_{1\leq j<i\leq n} f(-1,i,j)_{23}$. Then

$$\begin{aligned} \mathcal{F}_{13}\mathcal{F}_{23} &= \prod_{1\leq j<i\leq n} f(-1,i,j)_{13}\cdot\prod_{1\leq l<k\leq n} f(-1,k,l)_{23} \\ &= \prod_{1\leq j<i\leq n} f(-1,i,j)_{13} f(-1,i,j)_{23} \\ &= \prod_{1\leq j<i\leq n} (\triangle\circ\mathrm{id})(f(-1,i,j)) \\ &= (\triangle\circ\mathrm{id})(\prod_{1\leq j<i\leq n} f(-1,i,j)) \\ &= (\triangle\circ\mathrm{id})(\mathcal{F}) \end{aligned}$$

The second equality holds by commutivity in $\mathbb{C}T\otimes\mathbb{C}T$, the third holds by the identities proven above, and the forth by the fact $(\triangle\otimes\mathrm{id})$ is an algebra homomorphism. Similarly $(\mathrm{id}\circ\triangle)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12}$. So $\mathcal{F}$ is a quasitriangular structure on $\mathbb{C}T$. $\square$

Since $\mathcal{F}$ is a quasitriangular structure, it is also a counital 2-cocycle on $\mathbb{C}T$ by Proposition 2.11.

Proposition 4.12. *The rational Cherednik algebra $H_c(G(m,p,n))$ is a $\mathbb{C}T$-module algebra, with the action given by: $\forall 1 \le i,j \le n,\ g \in G(m,p,n)$*

$$t_i^{(-1)} \triangleright g = t_i^{(-1)} g t_i^{(-1)} \qquad t_i^{(-1)} \triangleright x_j = t_i^{(-1)}(x_j) \qquad t_i^{(-1)} \triangleright y_j = t_i^{(-1)}(y_j)$$

Proof. See Proposition 4.4 of [6]. □

Recall in Proposition 2.26 we showed that a counital 2-cocycle χ on a Hopf algebra H can twist an H-module algebra $B = (B, m, \eta)$. The result is the H_χ-module algebra $B_\chi = (B, m_\chi, \eta)$ where $m_\chi(a \otimes b) := m(\chi^{-1} \triangleright a \otimes b)$, and H_χ is the twisted Hopf algebra defined in Proposition 2.22. By Proposition 4.11 we know $\mathcal{F}$ is a counital 2-cocycle on $\mathbb{C}T$, and by Proposition 4.12, $H_c(G(m,p,n))$ is a $\mathbb{C}T$-module algebra. Therefore we can apply Proposition 2.26 and define the algebra $H_c(G(m,p,n))_{\mathcal{F}}$ as the result of twisting $H_c(G(m,p,n))$ by $\mathcal{F}$. Let $\cdot$ and $\star$ denote multiplications in $H_c(G(m,p,n))$ and $H_c(G(m,p,n))_{\mathcal{F}}$ respectively, so

$$a \star b := \cdot(\mathcal{F} \triangleright a \otimes b)$$

Note this uses the fact $\mathcal{F} = \mathcal{F}^{-1}$ since $\mathcal{F}$ was proven to be an involution above.

We now look at the main result (Theorem 4.5) from the preprint [6]. There are two important points the reader should be aware of in regards to this result. Firstly, the statement here contains a minor correction from the original. This involves the use of the map $\underline{c}$, and more details on this can be found in the proof (of relation 4 below). Additionally, whilst studying this result it was noticed by the author that there is in fact a gap in the final stage of the proof (relation 7 below). At this point in time the proof has only been completed fully for the special case that $m = 2p$, i.e. $C_{m/p} = \{-1, 1\}$. It will be a task for the future to complete the proof in the general case.

Theorem 4.13. *[[6], Theorem 4.5] Take $n \ge 3$, m even and $p|m$.* ***For the time being, also assert*** *$m = 2p$. Then $H_c(G(m,p,n))_{\mathcal{F}}$ is isomorphic to the negative braided Cherednik algebra $\underline{H}_{\underline{c}}(\mu(G(m,p,n)))$, where $\underline{c} : \underline{S} \to \mathbb{C}$ is such that $\underline{c}_1 = c_1$ and $\underline{c}_\zeta = -c_\zeta\ \forall \zeta \in C_{m/p}$.*

In particular, the map $\phi : \underline{H}_{\underline{c}}(\mu(G(m,p,n))) \to H_c(G(m,p,n))_{\mathcal{F}}$ *with*

$$\phi(\underline{x}_i) = x_i \qquad \phi(\underline{y}_i) = y_i \qquad \phi(\sigma_{ij}^{(\epsilon)}) = -s_{ij}^{(-\epsilon)} \qquad \phi(t_i^{(\zeta)}) = t_i^{(\zeta)}$$

can be extended as an algebra homomorphism, and is an isomorphism.

Proof. The main part of this proof involves showing ϕ extends to a well-defined algebra homomorphism. In other words the relations between $\underline{x}_i, \underline{y}_i, g$ in Definition 4.9 must also be satisfied by $\phi(\underline{x}_i), \phi(\underline{y}_i), \phi(g)$ in $H_c(G(m,p,n))_{\mathcal{F}}$. These relations are listed as (1)-(7) below. If these hold, then by Propositions 4.6 and 4.10 we see the basis of $\underline{H}_{\underline{c}}(\mu(G(m,p,n)))$

$$\{\underline{x}_i^{k_1} \ldots \underline{x}_n^{k_n} g \underline{y}_1^{l_1} \ldots \underline{y}_n^{l_n} | \; g \in \mu(G(m,p,n)), \; k_i, l_i \in \mathbb{N}_0 \; \forall i\}$$

is mapped by ϕ to the basis of $H_c(G(m,p,n))$

$$\{x_1^{k_1} \ldots x_n^{k_n} g y_1^{l_1} \ldots y_n^{l_n} | \; g \in G(m,p,n), k_i, l_i \in \mathbb{N}_0 \; \forall i\}$$

since $\underline{x}_i \mapsto x_i$, $\underline{y}_i \mapsto y_i$ and generators of $\mu(G(m,p,n))$ are mapped to generators of $G(m,p,n)$. As the underlying vector space of $H_c(G(m,p,n))_{\mathcal{F}}$ is equal to that of $H_c(G(m,p,n))$, we find ϕ is a bijection of these vector spaces, and hence is an algebra isomorphism, as required.

In the following we discuss the steps involved in proving ϕ extends as an algebra homomorphism. From Definition 4.9, we see the relations we must check are as follows:

1. $\phi(\underline{x}_i) \star \phi(\underline{x}_j) + \phi(\underline{x}_j) \star \phi(\underline{x}_i) = 0 \qquad \forall 1 \leq i < j \leq n$
2. $\phi(\underline{y}_i) \star \phi(\underline{y}_j) + \phi(\underline{y}_j) \star \phi(\underline{y}_i) = 0 \qquad \forall 1 \leq i < j \leq n$
3. $\phi(\underline{y}_i) \star \phi(\underline{x}_j) + \phi(\underline{x}_j) \star \phi(\underline{y}_i) = \underline{c}_1 \sum_{\epsilon \in C_m} \epsilon \phi(\sigma_{ij}^{(\epsilon)}) \qquad \forall i \neq j$
4. $\phi(\underline{y}_i) \star \phi(\underline{x}_i) - \phi(\underline{x}_i) \star \phi(\underline{y}_i) = 1 + \underline{c}_1 \sum_{j \neq i} \sum_{\epsilon \in C_m} \phi(\sigma_{ij}^{(\epsilon)}) + \sum_{\zeta \in C_{m/p} \setminus \{1\}} \underline{c}_\zeta \phi(t_i^{(\zeta)}) \qquad \forall i$
5. $\phi(\sigma_{ij}^{(\epsilon)}) \star \phi(\underline{x}_k) = \phi(\sigma_{ij}^{(\epsilon)}(\underline{x}_k)) \star \phi(\sigma_{ij}^{(\epsilon)}) \qquad \forall 1 \leq i,j,k \leq n, \; i \neq j, \epsilon \in C_m$
6. $\phi(t_i^{(\zeta)}) \star \phi(\underline{x}_k) = \phi(t_i^{(\zeta)}(\underline{x}_k)) \star \phi(t_i^{(\zeta)}) \qquad \forall 1 \leq i \leq n, \zeta \in C_{m/p}$
7. $\phi(g \cdot g') = \phi(g) \star \phi(g') \qquad \forall g, g' \in \mu(G(m,p,n))$

We elaborate next on how each of these relations is proven. First though we give a result that will be used to prove several of the relations.

Lemma 4.14 ([6], Lemma 4.6)**.** *(1) Take* $1 \leq j < i \leq n$*. If* $a, b \in H_c(G(m,p,n))$ *are such that* $t_i^{(-1)} \triangleright a = a$*, or* $t_j^{(-1)} \triangleright b = b$*, then* $f(-1,i,j) \triangleright (a \otimes b) = a \otimes b$*.*

(2) If $a, b \in H_c(G(m,p,n))$ are such that $t_i^{(-1)} \triangleright a = a$, or $t_j^{(-1)} \triangleright b = b$, $\forall 1 \leq j < i \leq n$, then $a \star b = ab$.

Proof. (1) Suppose $t_i^{(-1)} \triangleright a = a$ but $t_j^{(-1)} \triangleright b \neq b$. Then

$$\begin{aligned} f(-1,i,j) \triangleright (a \otimes b) &= \frac{1}{2}(1 \otimes 1 + t_i \otimes 1 + 1 \otimes t_j - t_i \otimes t_j) \triangleright (a \otimes b) \\ &= \frac{1}{2}(a \otimes b + a \otimes b + a \otimes (t_j^{(-1)} \triangleright b) - a \otimes (t_j^{(-1)} \triangleright b)) = a \otimes b \end{aligned}$$

Similarly when $t_i^{(-1)} \triangleright a \neq a$ but $t_j^{(-1)} \triangleright b = b$. Clearly if $t_i^{(-1)} \triangleright a = a$ and $t_j^{(-1)} \triangleright b = b$, then $f(-1,i,j) \triangleright (a \otimes b) = a \otimes b$ too.

(2) Since $\mathcal{F} := \prod_{1 \leq j < i \leq n} f(-1,i,j)$, applying part (1): $f(-1,i,j) \triangleright (a \otimes b) = a \otimes b \; \forall 1 \leq j < i \leq n$, hence $a \star b = \cdot(\mathcal{F} \triangleright (a \otimes b)) = \cdot(a \otimes b) = ab$, as required. $\square$

Now we elaborate on some of the proofs given in [6] for the relations 1-7 listed above.

1. Take $r < s$. We find $t_i^{(-1)} \triangleright x_r \neq x_r$ iff $i = r$ and $t_j^{(-1)} \triangleright x_s \neq x_s$ iff $j = s$, so both happen iff $i = r$ and $j = s$. However for $1 \leq j < i \leq n$ and $r < s$ we cannot have $i = r$ and $j = s$, and so either $t_i^{(-1)} \triangleright x_r = x_r$ or $t_j^{(-1)} \triangleright x_s = x_s$ $\forall 1 \leq j < i \leq n$. So by Lemma 4.14 (2): $\phi(\underline{x}_r) \star \phi(\underline{x}_s) = x_r \star x_s = x_r x_s$.

For $i \neq r$ or $j \neq s$, by Lemma 4.14 (1) we find $f(-1,i,j)$ acts trivially on $x_r \otimes x_s$. Since $\mathbb{C}T$ is commutative, we can move all terms $f(-1,i,j)$ with $i \neq r$ or $j \neq s$ in the product $\mathcal{F} = \prod_{1 \leq j < i \leq n} f(-1,i,j)$ to the right, and letting them act (trivially) on $x_r \otimes x_s$, we find: $\mathcal{F} \triangleright x_r \otimes x_s = f(-1,r,s) \triangleright x_r \otimes x_s$. Using the definition of $f(-1,r,s)$ it is easy to see $f(-1,r,s) \triangleright x_r \otimes x_s = -x_s \otimes x_r$. Hence $\phi(\underline{x}_s) \star \phi(\underline{x}_r) = x_s \star x_r = -x_s x_r$, and $\phi(\underline{x}_r) \star \phi(\underline{x}_s) + \phi(\underline{x}_s) \star \phi(\underline{x}_r) = x_r x_s - x_s x_r = 0$ $\forall 1 \leq r < s \leq n$ as required.

2. Follows similarly to part 1.

3. Again by the same method used in 1 it can be shown that:

$$x_i \star y_j = \begin{cases} x_i y_j & i < j, \\ -x_i y_j & j < i \end{cases} \qquad y_i \star x_j = \begin{cases} y_i x_j & i < j, \\ -y_i x_j & j < i \end{cases} \tag{35}$$

Applying (35) with $i < j$ we have:

$$\phi(\underline{y}_i) \star \phi(\underline{x}_j) + \phi(\underline{x}_j) \star \phi(\underline{y}_i) = y_i \star x_j + x_j \star y_i = y_i x_j - x_j y_i$$

By Definition 4.7 this equals:

$$= c_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(\epsilon)} = c_1 \sum_{\epsilon \in C_m} (-\epsilon) s_{ij}^{(-\epsilon)} = c_1 \sum_{\epsilon} \epsilon \phi(\sigma_{ij}^{(\epsilon)})$$

The second equality holds since m is even, so $-\epsilon \in C_m \ \forall \epsilon \in C_m$. The final equality holds as $\phi(\sigma_{ij}^{(\epsilon)}) := -s_{ij}^{(-\epsilon)}$. The case $j < i$ is similar, hence relation 3 holds $\forall i \neq j$.

4. For all $1 \leq k < l \leq n$ we have $t_l^{(-1)} \triangleright y_i = y_i$ or $t_k^{(-1)} \triangleright x_i = x_i$ so by Lemma 4.14 (2): $y_i \star x_i = y_i x_i$ and $x_i \star y_i = x_i y_i$. Then

$$\begin{aligned} \phi(\underline{y}_i) \star \phi(\underline{x}_i) - \phi(\underline{x}_i) \star \phi(\underline{y}_i) &= y_i x_i - x_i y_i \\ &= 1 - c_1 \sum_{j \neq i} \sum_{\epsilon} s_{ij}^{(\epsilon)} - \sum_{\zeta} c_\zeta t_i^{(\zeta)} \end{aligned} \tag{36}$$

Using $\phi(\sigma_{ij}^{(\epsilon)}) := -s_{ij}^{(-\epsilon)}$, we have $\sum_\epsilon s_{ij}^{(\epsilon)} = \sum_\epsilon s_{ij}^{(-\epsilon)} = -\sum_\epsilon \phi(\sigma_{ij}^{(\epsilon)})$. Also using $\underline{c}_1 = c_1$:

$$-c_1 \sum_{j \neq i} \sum_{\epsilon} s_{ij}^{(\epsilon)} = \underline{c}_1 \sum_{j \neq i} \sum_{\epsilon} \phi(\sigma_{ij}^{(\epsilon)})$$

Note the original preprint used the same map c for the both the twisted rational Cherednik algebra and the negative braided Cherednik algebra. However it is at this point we see it is necessary to use $\underline{c}$ instead, since by $\underline{c}_\zeta = -c_\zeta$ and $\phi(t_i^{(\zeta)}) := t_i^{(\zeta)}$ we find:

$$-\sum_{\zeta} c_\zeta t_i^{(\zeta)} = +\sum_{\zeta} \underline{c}_\zeta \phi(t_i^{(\zeta)})$$

Hence (36) equals:

$$= 1 + \underline{c}_1 \sum_{j \neq i} \sum_{\epsilon} \phi(\sigma_{ij}^{(\epsilon)}) + \sum_{\zeta} \underline{c}_\zeta \phi(t_i^{(\zeta)})$$

as required.

5. See Section 4.7 of [6].

6. We wish to show

$$\phi(t_i^{(\zeta)}) \star \phi(\underline{x}_k) = \phi(t_i^{(\zeta)}(\underline{x}_k)) \star \phi(t_i^{(\zeta)}) \tag{37}$$

We know the LHS of (37) is equal to $t_i^{(\zeta)} \star x_k$. Then by the fact that $t_j^{(-1)} \triangleright t_i^{(\zeta)} = t_j^{(-1)} t_i^{(\zeta)} t_j^{(-1)} = t_i^{(\zeta)}$, the conditions for Lemma 4.14 (2) are satisfied (since $t_k^{(-1)} \triangleright t_i^{(\zeta)} = t_i^{(\zeta)} \ \forall k$). So $t_i^{(\zeta)} \star x_k = t_i^{(\zeta)} x_k$.

For the RHS of (37), by the definition of $t_i^{(\zeta)}$ and ϕ it is clear that $\phi(t_i^{(\zeta)}(\underline{x}_k)) = t_i^{(\zeta)}(x_k)$. We can again we apply Lemma 4.14 (2) to find: $t_i^{(\zeta)}(x_k) \star t_i^{(\zeta)} = t_i^{(\zeta)}(x_k) t_i^{(\zeta)}$. By the relations in $H_c(G(m,p,n))$, we have $t_i^{(\zeta)} x_k = t_i^{(\zeta)}(x_k) t_i^{(\zeta)}$. Therefore the LHS and RHS of (37) are equal.

7. In fact we do not have to prove $\phi(g \cdot g') = \phi(g) \star \phi(g')$ for all $g, g' \in \mu(G(m,p,n))$. We only need to consider the defining relations between generators of a presentation of $\mu(G(m,p,n))$. Such a presentation has been been proposed by Dr Bazlov:

Theorem 4.15. *Define the group* $T_{C,C'} := \{t \in (C_m)^n |\ \det(t) \in C_{m/p}\}$*, and let* $\sigma_{ij} := \sigma_{ij}^{(1)}$*. Then the group* $\mu(G(m,p,n)) := \{tw \in (C_m)^n \rtimes \mathbb{S}_n |\ \det(tw) \in C_{m/p}\}$ *has a presentation with generators:* $\sigma_{ij}^{(\epsilon)}$*,* $t_i^{(\epsilon)} t_j^{(\epsilon^{-1})}$*,* $t_i^{(\zeta)}$ *for* $1 \leq i, j \leq n$*,* $i \neq j$*,* $\epsilon \in C_m$ *and* $\zeta \in C_{m/p}$*, and relations:*

i) *all relations between* $t_i^{(\epsilon)} t_j^{(\epsilon^{-1})}$ *and* $t_i^{(\zeta)}$ *in* $T_{C,C'}$*.*

ii) $\sigma_{ij}^{(\epsilon)} = \sigma_{ij} t_i^{(\epsilon)} t_j^{(\epsilon^{-1})}$

iii) $\sigma_{ij} t = s_{ij}(t) \sigma_{ij}$ *where* $t \in T_{C,C'}$ *and* $s_{ij} := s_{ij}^{(1)} \in \mathbb{S}_n$ *is the permutation matrix that transposes* i *and* j*, and acts by the permutation action on* $T_{C,C'}$*.*

iv) $\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}$ *where* $\{i,j\} \cap \{k,l\} = \emptyset$*.*

v) $\sigma_{ij}^2 = t_i^{(-1)} t_j^{(-1)}$

vi) $\sigma_{ij}\sigma_{ji} = 1$

vii) $\sigma_{ij}\sigma_{jk} = \sigma_{jk}\sigma_{ki}$

Proof. This has been proven by Dr Bazlov in the case $m = 2p$, although it remains to be proven $\forall m, p$ such that $p|m$. $\square$

Using this result, the task of proving $\phi(g \cdot g') = \phi(g) \star \phi(g')\ \forall g, g'$ reduces to proving:

ii) $\phi(\sigma_{ij}^{(\epsilon)}) = \phi(\sigma_{ij}) \star (\phi(t_i^{(\epsilon)}) \star \phi(t_j^{(\epsilon^{-1})}))$

iii) $\phi(\sigma_{ij}) \star t = s_{ij}(t) \star \phi(\sigma_{ij})$

iv) $\phi(\sigma_{ij}) \star \phi(\sigma_{kl}) = \phi(\sigma_{kl}) \star \phi(\sigma_{ij})$ for $\{i,j\} \cap \{k,l\} = \emptyset$

v) $\phi(\sigma_{ij}) \star \phi(\sigma_{ij}) = t_i^{-1} t_j^{-1}$ for $i \neq j$

vi) $\phi(\sigma_{ij}) \star \phi(\sigma_{ji}) = 1$

vii) $\phi(\sigma_{ij}) \star \phi(\sigma_{jk}) = \phi(\sigma_{jk}) \star \phi(\sigma_{ki})$ for $i \neq j \neq k$, $i \neq k$.

Notice how these correspond to the relations (ii)-(vii) in Theorem 4.15. We omit the relations induced by part (i) of Theorem 4.15 since these are seen to hold automatically using the following fact: if either of a or b is in $T_{C,C'}$, then $a \star b = ab$. This can be seen to hold using $t_j^{(-1)} \triangleright t_i^{(\zeta)} = t_i^{(\zeta)}$, and applying Lemma 4.14 (2). This fact can also be used to prove (ii) and (iii). Finally the method for proving (iv)-(vii) can be found in Section 4.8 of [6].

This finishes the proof of relation 7, and in turn proves that ϕ is a well-defined algebra homomorphism. We discussed above that ϕ is a bijection, hence it is an algebra isomorphism. So finally we see that

$$\underline{H}_{\underline{c}}(\mu(G(m,p,n))) \cong H_c(G(m,p,n))_{\mathcal{F}}$$

$\square$

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5 Conclusions

We started this dissertation looking at Hopf algebras, and in particular discussing the structure of cocommutative Hopf algebras. We found cocommutativity can be weakened, giving the more general quasitriangular Hopf algebras. Quasitriangular structures were then found to be examples of 2-cocycles, which are objects that can be used to twist Hopf algebras H, and their H-module algebras. In introducing some of the theory of Hopf algebra cohomology, it was noted that this twisting procedure only generates a new Hopf algebra (i.e. one that is not isomorphic to the original) when the 2-cocycle is not a coboundary (or equivalently, it is not cohomologous to the trivial cocycle).

In order to define the Rational Cherednik algebras, and later see this twisting procedure applied to them, we would have to define the complex reflection groups. We dedicated Section 3 to giving a short survey, working up from the basics of Euclidean reflection groups to the invariant theory of complex reflection groups.

Some time was spent in Sections 4.1 and 4.2 introducing the Rational Cherednik algebras, and their origins as symplectic reflection algebras. This setup, as well as our work on Hopf algebras and Reflection groups, culminated in the final two sections, 4.3 and 4.4, in which we explored the preprint by Bazlov, Berenstein and McGaw [6]. Here we constructed the rational Cherednik algebra over the family of complex reflection groups $G(m, p, n)$, and found that these algebras are in fact $\mathbb{C}T$-module algebras for a group algebra $\mathbb{C}T$. Once armed with a cocycle $\mathcal{F}$ on $\mathbb{C}T$, we were able to apply the twisting procedure from earlier to twist these rational Cherednik algebras. We finished by investigating the result that this twisted algebra is isomorphic to the negative braided Cherednik algebras over a mystic reflection group. During the dissertation it was noticed by the author that a small change in the statement of this result was required. In particular the twisted rational Cherednik algebra with map c is isomorphic to the negative braided Cherednik algebra with map $\underline{c}$. Additionally it is believed that the more general statement of this theorem, i.e. without the requirement that $m = 2p$, should also hold. Proving this will be a natural avenue for continuing this work.

6 Appendix

6.1 Category theory

We add to this section any pieces of Category theory that offer useful, or just interesting, insights to the ideas explored in this work. Resources that were particularly helpful include Allufi [1], Goedecke [14] and Mac Lane [20].

Definition 6.1.

- A **(small) category** $\mathscr{C}$ is a set of "objects" $\text{ob}(\mathscr{C})$ and for each $A, B \in \text{ob}(\mathscr{C})$ a set of "morphisms" $\mathscr{C}(A, B)$ from A to B. For each $A \in \text{ob}(\mathscr{C})$ there is a morphism $1_A \in C(A, A)$, and all $A, B, C \in \text{ob}(\mathscr{C})$ there is composition operator $\circ : \mathscr{C}(A, B) \times \mathscr{C}(B, C) \to \mathscr{C}(A, C)$ such that for morphisms f, g, h as follows: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$ (associativity) and $\forall f \in \mathscr{C}(A, B)$ then: $1_B \circ f = f = f \circ 1_A$.

  "Small" means $\text{ob}(\mathscr{C})$ and $\mathscr{C}(A, B)$ $\forall A, B \in \text{ob}(\mathscr{C})$ are sets, rather than some larger collection or class.

- An **isomorphism** in the category $\mathscr{C}$ is a morphism $f \in \mathscr{C}(A, B)$ such that there exists a morphism $g \in \mathscr{C}(B, A)$ such that $f \circ g = \text{id}_B, \ g \circ f = \text{id}_A$.

- A **groupoid** is a small category with all morphisms being isomorphisms.

- A **group** is a groupoid with a single object.

- The **vertex groups** of a groupoid $\mathscr{C}$ are the sets $\mathscr{C}(A, A)$ for each object $A \in \text{ob}(\mathscr{C})$. By the axioms above, each such $\mathscr{C}(A, A)$ is a group.

For vector space V, let $T(V)$ be the tensor algebra over V.

Proposition 6.2. *For vector spaces $V_1, \dots, V_n$, then $T(\bigoplus_{i=1}^n V_i) \cong \bigotimes_{i=1}^n T(V_i)$.*

Proof. T can be seen as a functor from the category of vector spaces **Vect** to the category of algebras **Alg**. It is also left adjoint to the forgetful functor. By Mac Lane [20] Section V.5 Theorem 1, any functor that has a right adjoint preserves colimits, and hence in particular preserves coproducts. The coproduct in **Vect** is the direct sum, and in **Alg** it is the direct product. Hence the result follows. $\square$

6.1.1 Universal Properties

Many concepts in algebra such as quotients, products/coproducts and free objects can be formulated as "universal properties" in Category theory. In particular the free groups and free algebras used throughout this work to define presentations can be characterised by universal properties. Additionally the universal enveloping algebra introduced in Section 6.3 is also characterised by a universal property. Universal properties don't provide an explicit construction of the objects, however the fact they can be described by a universal property tells us a lot of information. We give the next definitions following Goedecke [14].

Definition 6.3.

- A **terminal (final) object** in a category $\mathscr{C}$ is an object $1 \in \mathrm{ob}(\mathscr{C})$ such that for every $A \in \mathrm{ob}(\mathscr{C})$ there exists a unique morphism $A \to 1$ in $\mathscr{C}$ (i.e. the set $\mathscr{C}(A, 1)$ is a singleton).
- An **initial object** in the category $\mathscr{C}$ is an object $0 \in \mathrm{ob}(\mathscr{C})$ such that for every $A \in \mathrm{ob}(\mathscr{C})$ there exists a unique morphism $0 \to A$ in $\mathscr{C}$ (i.e. the set $\mathscr{C}(0, A)$ is a singleton).
- An object satisfies a **universal property** if it is an initial or terminal object in some category.

Proposition 6.4. *[Allufi [1] I.5.1 Prop 5.4] Initial, or terminal, objects in $\mathscr{C}$ are unique up to unique isomorphism.*

Proof. Suppose $0, 0'$ are initial, then as 0 is initial there is unique morphism $f : 0 \to 0'$, and as $0'$ is initial there is unique morphism $g : 0' \to 0$. Again as 0 is initial, there is unique morphism $0 \to 0$, hence $gf = 1_0$, and similarly $fg = 1_{0'}$. So f is an isomorphism between 0 and $0'$, and it is unique. Similarly for terminal objects. $\square$

In the next two sections we show how free groups and universal enveloping algebras are characterised by universal properties.

6.2 Group Presentations

A presentation is a particular way of constructing a group. We will require group presentations to define Coxeter groups in Section 3.1, and to motivate the analogous algebra presentations used in Section 4. Group presentations are given by first specifying a special set of elements, called "generators", out of which every element of the group is a product. We then construct the "free" group (i.e. the group with the fewest constraints) from these generators. Finally we impose relations on these generators by taking a certain quotient of the free group. This will be discussed in more detail next, following Allufi [1] (Section II.5.2-3).

We start with a set A, known as our "alphabet", which will be the generators. Define an isomorphic set A' such that the element in A' corresponding to $a \in A$ is denoted a^{-1}. Then words on A are finite ordered lists $w = a_1 a_2 \dots a_n$, with a_i in A or A'. The set of such words is $W(A)$, with the length of a word being the number of letters in it. The empty word is also in $W(A)$. Let $w \in W(A)$, and define the "reduction" map $r : W(A) \to W(A)$ such that $r(w)$ is the result of deleting the first occurence of aa^{-1} or $a^{-1}a$ found when moving left to right through the letters of w. A "reduced word" is $w \in W(A)$ such that $r(w) = w$. Note each time a reduction is applied the length of the word decreases by 2. For w of length n, $r^{\lfloor \frac{n}{2} \rfloor}(w)$ must be a reduced word since either $r^i(w)$ becomes a reduced word for some $i < \lfloor \frac{n}{2} \rfloor$, in which case $r^{\lfloor \frac{n}{2} \rfloor}(w)$ is still a reduced word, or if this does not happen then the length of $r^{\lfloor \frac{n}{2} \rfloor}(w)$ is either 0 or 1, so again it must be a reduced word. Define $R : W(A) \to W(A)$ as $R(w) := r^{\lfloor \frac{n}{2} \rfloor}(w)$ when w is of length n. Then let $F(A)$ be the image of R, the set of reduced words on A.

Definition 6.5. The **free group** on set A is the set $F(A)$, with the product of $w, w' \in F(A)$ given by $w \cdot w' := R(ww')$ (i.e. concatenate each word and then apply any necessary reduction). This operation is associative, with identity being the empty word, and inverse of $w = a_1 \dots a_n$ given by $w^{-1} = a_n^{-1} \dots a_1^{-1}$ (which means if $a_i = a \in A$ then $a_i^{-1} = a^{-1} \in A'$ and if $a_i = a^{-1} \in A'$ then $a_i^{-1} = a \in A$).

The free group could alternatively be defined (up to isomorphism) by a certain universal property.

Definition 6.6. For set A, the category $\mathscr{F}^A$ has for objects pairs $(j : A \to G, G)$ where G is a group and j is a set function. A morphism between $(j : A \to G, G)$ and $(j' : A \to H, H)$ is a group homomorphism $\phi : G \to H$ such that $j' = \phi \circ j$.

Definition 6.7. A **free group** on set A is an initial object in the category $\mathscr{F}^A$.

So it is given by a pair $(j : A \to F(A), F(A))$ such that for all groups G and set functions $f : A \to G$, there exists a unique group homomorphism $\phi : F(A) \to G$ such that $\phi \circ j = f$.

By Proposition 6.4, the universal property defines free groups up to isomorphism. By the following result the construction of $F(A)$ above satisfies this universal property.

Proposition 6.8. *Let $F(A)$ be the free group as in Definition 6.5, and let $j : A \to F(A)$, $a \mapsto a$. Then $(j, F(A))$ is initial in the category $\mathscr{F}^A$.*

Proof. See Allufi [1] Section II.5.3 Proposition 5.2. □

Definition 6.9 (Aluffi II.8.2)**.** A **presentation** of the group G is a pair $(A, \mathscr{R})$ where A is the set of generators, $\mathscr{R} \subset F(A)$ are the relations, and for R defined to be the smallest normal subgroup of $F(A)$ containing $\mathscr{R}$, we have: $G \cong F(A)/R$.

G is **finitely presented** if it has a presentation such that A and $\mathscr{R}$ are both finite.

A presentation of G is normally denoted: $G = \langle g_i \in G \mid r_1 = \cdots = r_n = 1 \rangle$ where the $\{g_i\}$ is the generator set A, and $\{r_1, \ldots, r_n\}$ are the relations $\mathscr{R}$.

Every group is isomorphic to the quotient of some free group, hence can be expressed by a presentation. This can be seen by defining for group G a map: $\phi : F(G) \to G$ sending $g \in F(G) \mapsto g \in G$, and extending as a group homomorphism. This map is surjective, and by the first isomorphism theorem $G \cong F(G)/\ker(\phi)$. So G is isomorphic to a quotient of a free group as required. Note there can be alternative choices of generating sets A for which G is isomorphic to a quotient of $F(A)$.

6.3 Universal Enveloping Algebras and the PBW theorem

Here we recall the construction of the universal enveloping algebra of a Lie algebra from the authors previous project, and this time charactise it via a universal property using the material from Section 6.1.1. We then briefly introduce the Poincare-Birkhoff-Witt theorem in order to provide the necessary background for understanding the PBW-type theorems of rational Cherednik Algebras in Section 4.1.

The tensor algebra of vector space V is $T(V) := \bigoplus_{n=0}^{\infty} T^n(V)$ where $T^n(V) = V \otimes \cdots \otimes V$ is the n-fold tensor product of V, and $T^0(V) = k$. Also let $T_n(V) := \bigoplus_{i=0}^{n} T^i(V)$.

Definition 6.10. For vector space V, let I be the 2-sided ideal generated by the elements $x \otimes y - y \otimes x \in V \otimes V$. The **symmetric algebra** of V is $S(V) := T(V)/I$. For a basis $\{x_1, \ldots, x_n\}$ of V, then $S(V) \cong \mathbb{C}[x_1, \ldots, x_n]$.

Definition 6.11. Let $\mathfrak{g}$ be a Lie algebra, $T(\mathfrak{g})$ the tensor algebra of $\mathfrak{g}$, and I the 2-sided ideal generated by the elements $x \otimes y - y \otimes x - [x, y] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g})$. The **universal enveloping algebra** of $\mathfrak{g}$ is $U(\mathfrak{g}) := T(\mathfrak{g})/I$.

Just as we did in Definition 6.7 for the free group, for a given Lie algebra $\mathfrak{g}$ we can define a category whose objects are pairs $(\phi : \mathfrak{g} \to A, A)$ where A is a unital associative algebra and ϕ is a linear map satisfying $\phi([x, y]) = \phi(x) \cdot \phi(y) - \phi(y) \cdot \phi(x)$. Then a morphism $(\phi, A) \to (\phi', A')$ is given by an algebra homomorphism $\lambda : A \to A'$ such that $\phi' = \lambda \circ \phi$. Then we can define $U(\mathfrak{g})$ (up to isomorphism) as an initial object in this category, meaning that it is the object $(h : \mathfrak{g} \to U(\mathfrak{g}), U(\mathfrak{g}))$ such that for all objects $(\phi : \mathfrak{g} \to A, A)$ there is a unique algebra homomorphism $\hat{\lambda} : U(\mathfrak{g}) \to A$ such that $\phi = \hat{\lambda} \circ h$. Taking $U(\mathfrak{g})$ as defined in Definition 6.11, and h as the composition of the embedding map $\mathfrak{g} \to T(\mathfrak{g})$ with the quotient map $T(\mathfrak{g}) \to U(\mathfrak{g}) = T(\mathfrak{g})/I$, then it can be shown that $(U(\mathfrak{g}), h)$ satisfies this universal property.

We now introduce some of the theory of graded and filtered algebras using Bellamy [7] (Section 1.4), before giving the statement of the PBW theorem.

Definition 6.12.

- A **graded algebra** is an associative algebra A with decomposition $A = \bigoplus_{n=0}^{\infty} A_n$ for algebras A_i, such that $A_n \cdot A_m \subset A_{n+m}$.
- A **filtered algebra** A has a sequence of subspaces $\{0\} \subset F_0 \subset F_1 \subset \cdots \subset A$ such that: $A = \cup_{i=0}^{\infty} F_i$ and $F_i \cdot F_j \subset F_{i+j}$ $\forall i, j \in \mathbb{N}_0$. The set $\{F_i\}$ is called the **filtration** of A. If $a \in F_i$ and $a \notin F_{i-1}$ then a is said to be of **degree** i.

  Every graded algebra $A = \bigoplus A_n$ is a filtered algebra, with filtration given by: $F_n := \bigoplus_{i=0}^{n} A_i$. Although not every filtered algebra is a graded algebra, one can construct an "associated" graded algebra from a filtered algebra:

- For filtered algebra A with filtration F_n, the **associated graded algebra** $G(A)$ has underlying vector space $G(A) = \bigoplus_{n=0}^{\infty} G_n$ where $G_n := F_n/F_{n-1}$ $\forall n > 1$ and $G_0 = \{0\}$, and multiplication is given by: $G_n \times G_m \to G_{n+m}$ (i.e. $F_n/F_{n-1} \times F_m/F_{m-1} \to F_{n+m}/F_{n+m-1}$):

$$(x + F_{n-1}, y + F_{m-1}) \mapsto x \cdot y + F_{n+m-1} \ \forall x \in F_n, y \in F_m$$

Note a filtered algebra A is in general distinct/non-isomorphic from its associated graded algebra $G(A)$, however the following result shows they have the same underlying vector space:

Proposition 6.13. *For filtered algebra A, we have $A \cong G(A)$ as vector spaces.*

Proof. Note that for an infinite-dimensional algebra, its dimension is given by the cardinality of a basis for the underlying vector space. See Knapp [18] Chapter II.9 for the basic theory of infinite-dimensional vector spaces. In particular we use Proposition 2.18 of Knapp: two vector spaces are isomorphic iff their dimensions are of the same cardinality, and also Corollary 2.24: for U subspace of V, then $\dim(V) = \dim(U) + \dim(V/U)$. If A has filtration $\{F_i\}$, then for $n \in \mathbb{N}$: $\dim(F_n) = \dim(F_{n-1}) + \dim(G_n)$. Then $\dim(F_n) = \dim(F_{n-2}) + \dim(G_{n-1}) + \dim(G_n)$, and iterating we find

$$\dim(F_n) = \dim(F_0) + \sum_{i=0}^{n} \dim(G_i) = 0 + \sum_{i=0}^{n} \dim(G_i)$$

Finally

$$\dim(A) = \lim_{n\to\infty} \dim(F_n) = \sum_{i=0}^{\infty} \dim(G_i) = \dim(G(A))$$

Since they have the same dimension, they are isomorphic. □

Note that a graded algebra A (which is also a filtered algebra) is isomorphic as an algebra to its associated graded algebra: $A \cong G(A)$.

Returning to the universal enveloping algebra $U(\mathfrak{g})$, it is in fact a filtered algebra. If $\phi : T(\mathfrak{g}) \to U(\mathfrak{g}) = T(\mathfrak{g})/I$ is the quotient map, then let $U_n(\mathfrak{g}) := \phi(T_n(\mathfrak{g}))$, the image of $T_n(\mathfrak{g})$ in $U(\mathfrak{g})$. Then $U_n(\mathfrak{g})$ is a filtration of $U(\mathfrak{g})$, and the PBW theorem describes the structure of the associated graded algebra $G(U(\mathfrak{g}))$:

Theorem 6.14 (Poincare-Birkhoff-Witt Theorem)**.** *$G(U(\mathfrak{g})) \cong S(\mathfrak{g})$ as algebras.*

By Proposition 6.13 we have as vector spaces: $U(\mathfrak{g}) \cong G(U(\mathfrak{g}))$, and so by PBW we also have as vector spaces: $U(\mathfrak{g}) \cong G(U(\mathfrak{g})) \cong S(\mathfrak{g}) \cong \mathbb{C}[x_1, \dots, x_n]$ for basis x_i of $\mathfrak{g}$. In fact the following result shows that "monomials" in the x_i form a basis for $U(\mathfrak{g})$:

Corollary 6.15 (Dixmier [10], Theorem 2.1.11)**.** *For basis $\{x_1, \dots, x_n\}$ of $\mathfrak{g}$, then $\{x_1^{k_1} \dots x_n^{k_n} \mid k_i \in \mathbb{N}\ \forall i\}$ is a basis for $U(\mathfrak{g})$. Note $x_i^{k_i}$ denotes the k_i-fold tensor product of x_i, and $\otimes$ between the $x_i^{k_i}$ are dropped.*

6.4 Group Cohomology

Next we introduce a little group cohomology, as required for the proof of Proposition 2.20. We follow Chapter 7 of Rotman [23].

Definition 6.16.

- A **2-cocycle** on the group G is a function $f : G \times G \to \mathbb{C}^*$ such that $\forall x, y, z \in G$

$$f(1, y) = 1_{\mathbb{C}} = f(x, 1) \tag{38}$$

$$f(x, yz)f(y, z) = f(xy, z)f(x, y) \tag{39}$$

  The set of 2-cocycles is denoted $Z^2(G, \mathbb{C}^*)$, and this has a group structure under pointwise multiplciation, i.e. for cocycles f, g then $(fg)(x, y) := f(x, y)g(x, y)$. Note $\mathbb{C}^*$ can be replaced by arbitrary abelian group.

- A **coboundary** is a function $f : G \times G \to \mathbb{C}^*$ such that there exists $h : G \to \mathbb{C}^*$ with $h(1) = 1$ and $g(x, y) = h(x)h(y)h^{-1}(xy)$. Note coboundaries are 2-cocycles, and these form a subgroup of $Z^2(G, \mathbb{C}^*)$, denoted $B^2(G, \mathbb{C}^*)$.

- Cocycles f, g are **cohomologous**, denoted $f \sim g$, iff fg^{-1} is a coboundary.

- The **Schur multiplier** (or **second cohomology group**) is $M(G) = H^2(G, \mathbb{C}^*) := Z^2(G, \mathbb{C}^*)/B^2(G, \mathbb{C}^*)$, i.e. cocycles modulo coboundaries.

Recall the dual Hopf algebra $\mathbb{C}G^*$ described in Example 2.12, where elements are maps $f : G \to \mathbb{C}$, and we identify elements of $\mathbb{C}G^* \otimes \mathbb{C}G^*$ with maps $G \times G \to \mathbb{C}$. We will now show that Majid's counital 2-cocycles and coboundaries (see Definition 2.7) on the Hopf algebra $\mathbb{C}G^*$ coincide with the cocycles and coboundaries on the group G as defined above. This is mentioned in Majid Example 2.3.2.

For $f \in \mathbb{C}G^* \otimes \mathbb{C}G^*$ to be a counital 2-cocycle it must first be invertible. So as a map $f : G \times G \to \mathbb{C}$ there must be a map $g : G \times G \to \mathbb{C}$ such that $(fg)(x,y) = f(x,y)g(x,y) = 1\ \forall x,y \in G$. This can happen iff $f : G \times G \to \mathbb{C}^*$, i.e. $f \neq 0$, in which case we can set $g(x,y) = f(x,y)^{-1} \in \mathbb{C}\ \forall x,y \in G$. Recall if $\phi : G \to \mathbb{C}$, then $\triangle(\phi)(x,y) = \phi(xy)$, so for $f : G \times G \to \mathbb{C}$ we see

$$(\triangle \otimes \mathrm{id})(f)(x,y,z) = f(xy,z) \qquad (\mathrm{id} \otimes \triangle)(f)(x,y,z) = f(x,yz)$$

Also the unit in $\mathbb{C}G^*$ is $1 = \sum_{x \in G} \delta_x$, where $1(x) = 1_{\mathbb{C}}\ \forall x \in G$. So $(f \otimes 1)(x,y,z) = f(x,y)1(z) = f(x,y)$. So we see the cocycle equation (from equation 4): $x,y,z \in G$

$$(f \otimes 1) \cdot (\triangle \otimes \mathrm{id})(f)(x,y,z) = (1 \otimes f) \cdot (\mathrm{id} \otimes \triangle)(f)(x,y,z)$$

reduces to

$$f(x,y)f(xy,z) = f(y,z)f(x,yz)$$

which is precisely the group cocycle equation (39) above. Finally, since $\epsilon(\delta_x) = \delta_{1_G}(x)$, for $\phi : G \to \mathbb{C}$ we have $\epsilon(\phi) = \phi(1_G)$. Then for $f : G \times G \to \mathbb{C}$

$$(\epsilon \otimes \mathrm{id})(f)(y) = f(1_G, y) \qquad (\mathrm{id} \otimes \epsilon)(f)(x) = f(x, 1_G)$$

So the counital condition $(\epsilon \otimes \mathrm{id})(f) = 1_{\mathbb{C}} = (\mathrm{id} \otimes \epsilon)(f)$ then reduces to equation 38 above, and therefore both notions of cocycles agree. Similarly both definitions of coboundaries agree.

Now for arbitrary commutative Hopf algebra H[††], our definitions of cocycles and coboundaries are still valid, and as was done for group cocycles, we say cocycles χ, χ' on H are cohomologous if $\chi'\chi^{-1}$ is a coboundary. I.e. $\chi'\chi^{-1} = \partial(\xi) = (\xi \otimes \xi)\triangle(\xi^{-1})$ for some 1-cocycle ξ, or equivalently:

$$\chi' = (\xi \otimes \xi)\chi\triangle(\xi^{-1}) \tag{40}$$

for some ξ. Then as above, we define the $Z^2(H,\mathbb{C})$ as the set of counital 2-cocycles on H, and $B^2(H,\mathbb{C})$ the set of coboundaries, and finally the second cohomology group (for commutative Hopf algebras) will be $H^2(H,\mathbb{C}) := Z^2(H,\mathbb{C})/B^2(H,\mathbb{C})$. Since for $H = \mathbb{C}G^*$ the cocycles and coboundaries on H coincided with those on G, we see

$$H^2(\mathbb{C}G^*, \mathbb{C}) = H^2(G, \mathbb{C}^*)$$

We use this fact in the proof of Proposition 2.20 when $G = C_m$.

[††]Note $\mathbb{C}G^*$ is commutative as $\mathbb{C}G$ is cocommutative

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