Twists of rational Cherednik algebras

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Introduction

Where do rational Cherednik algebras come from?

- degeneration of the double affine Hecke algebra.
  Applications: combinatorics and representation theory.
- special case of the symplectic reflection algebras.
  Applications: algebraic geometry, deformation theory

There are several similarities between rational Cherednik algebras and the universal enveloping algebra $U(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}$:

- both are infinite-dimensional, noncommutative associative algebras
- For Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ we have triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. Similarly for rational Cherednik algebras:

$$H_{t,c} = S(V) \otimes \mathbb{C}W \otimes S(V^*)$$

- The positive/negative Borel subalgebras of $\mathfrak{g}$ are $\mathfrak{b}_\pm := \mathfrak{n}_\pm \oplus \mathfrak{h}$. The subalgebras $S(V^*) \rtimes \mathbb{C}W$ and $S(V) \rtimes \mathbb{C}W$ of $H_{t,c}$ are the analogues of the enveloping algebras $U(\mathfrak{b}_\pm)$ respectively.

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Take $\mathbb{C}$-vector space V and let W be a finite subgroup of $\mathrm{GL}(V)$.

Definition

- A **complex reflection** is an element $s \in W$ such that $\mathrm{rank}(s - \mathrm{id}_V) = 1$. I.e. s fixes a hyperplane in V.

  Let S be the set of complex reflections in W.
- W is a **complex reflection group** if it is generated by S.
- For complex reflection group W, V is the **reflection representation** of W.

Real reflection groups defined analogously on $\mathbb{R}$-vector spaces, however the reflections for real reflection groups always have order 2, whereas complex reflections can in general have any order $n \geq 2$.

Examples (Real reflection groups)

- Symmetry groups of all regular polygons
- Weyl groups associated to root systems/semisimple Lie algebras.

Notes

- Real reflection groups are "essentially"[a] in 1-1 correspondence with Coxeter groups. Recall Coxeter groups are groups with a presentation of the form

$$W = \langle s \in S \mid (ss')^{m_{ss'}} = 1 \ \forall s, s' \in S \rangle$$

  where S is a finite set of generators, and
  - $m_{ss} = 1 \ \forall s \in S \implies s^2 = 1$ so s are the abstraction of reflections.
  - $m_{ss'} \in \{2, 3, \dots\} \cup \{\infty\}$ where ∞ is used to imply there are no relations between the generators s, s'.
- Every real reflection/Coxeter group on $\mathbb{R}$-vector space V defines a complex reflection group on the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.

[a] Precise statement: There is a 1-1 correspondence between stable isomorphism classes of real reflection groups and isomorphism classes of finite Coxeter systems.

Complex reflection groups were classified by Shephard & Todd (1954) into an infinite family of groups $G(m, p, n)$ for parameters $m, p, n \in \mathbb{N}$ such that $p|m$, plus 34 exceptional cases.

The group $G(m, p, n)$ can be defined, wrt to a particular choice of basis for V, as the group of complex $n \times n$-matrices that are:

- monomial, i.e. one nonzero entry to each row and column
- the nonzero entries are all m-th roots of unity
- the product of all nonzero entries is a $\frac{m}{p}$-th root of unity

Examples

$G(m,1,n) \cong (C_m)^n \rtimes S_n$, where C_m is the cyclic group of order m and S_n permutes the terms in the direct product $(C_m)^n$.

- $G(1,1,n) \cong S_n$
- $G(2,1,n) =$ Weyl groups of type B_n. Lets elaborate on $G(2,1,2)$. This is the symmetry group of the square, and has the following generators:

$$s_{12} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ i.e. reflection about the line y=x}$$

$$t_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ i.e. reflection about the line x=0}$$

$$t_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ i.e. reflection about the line y=0}$$

Complex reflection groups and Invariant theory

Take $\mathbb{C}$-vector space V and finite subgroup $G \leq \mathrm{GL}(V)$, then note:

- action of G on V naturally extends to algebra endomorphisms of $S(V)$ via: $g \triangleright v_1 \dots v_n = (g \triangleright v_1) \dots (g \triangleright v_n)$ for $v_i \in V$.
- the invariants $S(V)^G := \{p \in S(V) | \ g \triangleright p = p\}$ form a subalgebra of $S(V)$.

Theorem (Chevalley-Shephard-Todd)

For V and G as above, the invariant subalgebra $S(V)^G$ is a polynomial ring if and only if G is a complex reflection group.

Example

Take $G(1,1,n) = S_n$. For basis $x_1, \dots, x_n$ of V, S_n acts on V by permuting the basis vectors. This action extends to $S(V) = \mathbb{C}[x_1, \dots, x_n]$. Define the elementary symmetric polynomials as follows:

$$e_k := \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \dots x_{j_k}$$

For $n = 3$:

$$e_1 = x_1 + x_2 + x_3 \qquad e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \qquad e_3 = x_1 x_2 x_3$$

We find invariants form a polynomial ring in the elementary symmetric polynomials: $S(V)^{S_n} = \mathbb{C}[e_1, \dots, e_n]$.

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Take a complex reflection group $W \leq \mathrm{GL}(V)$, then for each $s \in S \subseteq W$, let $\alpha_s \in V^*, \alpha_s^\vee \in V$ be defined such that:

$$\mathrm{Im}(s - \mathrm{id}_V) = \mathrm{span}_{\mathbb{C}}\{\alpha_s^\vee\} \qquad \mathrm{Im}(s - \mathrm{id}_{V^*}) = \mathrm{span}_{\mathbb{C}}\{\alpha_s\}$$

Definition

Take a set of parameters given by $t \in \mathbb{C}$ and function $c : S \to \mathbb{C}$ such that $c(s) = c(gsg^{-1})$. Then the **rational Cherednik algebra** $H_{t,c}(W)$ is defined as the quotient of $T(V \oplus V^*) \rtimes \mathbb{C}W$ by the ideal generated by the following relations: $\forall x, x' \in V, y, y' \in V^*$

$$[x, x'] = 0 = [y, y'] \qquad [y, x] = t\langle x, y\rangle + \sum_{s \in S} c(s)\langle x, \alpha_s\rangle\langle\alpha_s^\vee, y\rangle s$$

Rational Cherednik algebras are “flat deformations”: meaning as vector spaces we have $H_{t,c}(W) = S(V) \otimes \mathbb{C}W \otimes S(V^*)$.

Example

Recall $W = G(2,1,2)$ is the Weyl group of type B_2, with complex reflections s_{12}, t_1, t_2. Note s_{12} and $\{t_1, t_2\}$ lie in distinct conjugacy classes of W, so let $c_1 := c(s_{12})$ and $c_{-1} := c(t_1) = c(t_2)$. If $V \cong \mathbb{C}^2$ has basis x_1, x_2 and V^* has dual basis y_1, y_2 then $H_{t,c}(W)$ is the algebra generated V, V^*, W subject to relations:

$$[x_i, x_j] = 0 = [y_i, y_j] \qquad gx_i = g(x_i)g \qquad gy_i = g(y_i)g$$
$$[y_i, x_i] = t - 2c_{-1}t_i \qquad [y_i, x_j] = 2c_1 s_{ij}$$

For $\lambda \in \mathbb{C}^\times$, the mapping $x \mapsto \lambda x, y \mapsto \lambda y, w \mapsto w$ induces an isomorphism $H_{t,c} \cong H_{\lambda^2 t, \lambda^2 c}$. Hence for $t \neq 0$, $H_{t,c} \cong H_{1,\frac{c}{t}}$. So there are two cases to consider: $t = 0$ or $t = 1$. When $c = 0$, the two cases are:

$$H_{0,0} = S(V \oplus V^*) \rtimes \mathbb{C}W \qquad H_{1,0} = A_n \rtimes \mathbb{C}W$$

where A_n is the Weyl algebra. Recall A_n is defined as

$$\begin{aligned} A_n &= T(V \oplus V^*)/\langle [x, x'] = 0 = [y, y'],\ [y, x] = \langle x, y\rangle\ \forall x, x' \in V, y, y' \in V^* \rangle \\ &= \mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle / \langle [x_i, x_j] = 0 = [y_i, y_j],\ [y_i, x_j] = \delta_{ij}\ \forall i, j\rangle \end{aligned}$$

Lets create a representation of $H_{1,0} = A_n \rtimes \mathbb{C}W$:

Firstly, the Weyl algebra

$$A_n = \mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle / \langle [x_i, x_j] = 0 = [y_i, y_j],\ [y_i, x_j] = \delta_{ij}\ \forall i, j \rangle$$

has a representation as a subalgebra of $\mathrm{End}(S(V)) = \mathrm{End}(\mathbb{C}[x_1, \ldots, x_n])$:

- $x_i \in A_n$ acts on $f \in \mathbb{C}[x_1, \ldots, x_n]$ by multiplication, i.e. $x_i \triangleright f := x_i \cdot f$,
- $y_i \in A_n$ acts on $f \in \mathbb{C}[x_1, \ldots, x_n]$ by partial differentiation, i.e. $y_i \triangleright f := \frac{\partial f}{\partial x_i}$.

The relation $[y_i, x_j] = \delta_{ij}$ becomes $\partial_i \cdot x_j - x_j \cdot \partial_i = \delta_{ij}$, which we can see holds by letting it act on some $f \in \mathbb{C}[x_1, \ldots, x_n]$.

Recall W acts on V, and this action extends to algebra endomorphisms of $S(V)$. So we can identify A_n and $\mathbb{C}W$ with subalgebras of $\mathrm{End}(\mathbb{C}[x_1, \ldots, x_n])$. The subalgebra of $\mathrm{End}(\mathbb{C}[x_1, \ldots, x_n])$ generated by both of these subalgebras respects the structure of $H_{1,0}$, so we have a representation of $H_{1,0}$ on $S(V)$.

Miraculously we have a similar situation when $c \neq 0$...

Theorem (The Dunkl/polynomial representation)

There is a representation of $H_{1,c}$ on $S(V)$ where $x \in V$ act by multiplication, $y \in V^*$ act via Dunkl operators:

$$D_y := \partial_y + \sum_{s \in S} c(s) \langle \alpha_s^\vee, y \rangle {\alpha_s^\vee}^{-1}(s - 1)$$

and W acts by the natural representation described above.

Dunkl operators are very special deformations of partial derivatives with the property that they still commute, i.e. $[D_x, D_{x'}] = 0$.

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Let us deform, or "quantize", the constructions from the previous chapter. We start with symmetric algebra: Take $\mathbb{C}$-vector space V with basis $x_1, \ldots, x_n$,

Definition

The **skew symmetric algebra** $S_q(V)$ is the $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n$ subject to relations $x_i x_j = q_{ij} x_j x_i \; \forall 1 \leq i, j \leq n$, where $q = (q_{ij})$ is an $n \times n$-matrix such that $q_{ii} = 1$ and $q_{ij} q_{ji} = 1$.

Examples

- Let $\mathbf{1}$ denote the matrix with all entries equal 1. Then $S_{\mathbf{1}}(V) = S(V)$.
- Let $-\mathbf{1}$ denote the matrix with all off-diagonal entries equal to -1. Then $S_{-\mathbf{1}}(V)$ is the anti-commutative analogue of a polynomial ring, i.e. $x_i x_j = -x_j x_i \; \forall i \neq j$.

Recall Chevalley-Shephard-Todd:
$S(V)^G$ is a polynomial ring (i.e. $S(V)^G \cong S(V)$) iff G is a complex reflection group.

Theorem (Kirkman, Kuzmanovich, Zhang; 2008)

*Let G be a finite group acting by degree-preserving endomorphisms on $S_q(V)$. Then $S_q(V)^G \cong S_{q'}(V)$ for some q' iff G is generated by "quasi-reflections". We call such groups **mystic reflection groups**.*

Rational Cherednik algebras a) built from complex reflection groups W
b) have triangular decomposition as $H_{t,c} = S(V) \otimes \mathbb{C}W \otimes S(V^*)$.

Definition

For mystic reflection group G as above, a **braided Cherednik algebra** is an algebra $\underline{H}$ generated by G, V, V^* such that

- $\underline{H}$ has triangular decomposition as $\underline{H} = S_q(V) \otimes \mathbb{C}G \otimes S_q(V^*)$, and subalgebras $S_q(V) \rtimes \mathbb{C}G, S_q(V^*) \rtimes \mathbb{C}G$,
- $y_j x_i - q_{ij} x_i y_j \in \mathbb{C}G$ for x_i basis of V, y_i dual basis of V^*.

Recall the Dunkl representation of rational Cherednik algebras $H_{t,c}$, where $y_i \in H_{t,c}$ identified with certain deformations of partial derivatives, called Dunkl operators D_i, with the special property that $[D_i, D_j] = 0$.

Similarly, for skew symmetric algebra $S_q(V)$, we have "braided partial derivatives":

$$\underline{\partial}_i(x_1^{a_1} \dots x_i^{a_i} \dots x_n^{a_n}) = a_i q_{1,i}^{a_1} \dots q_{i-1,i}^{a_i-1} x_1^{a_1} \dots x_i^{a_i-1} \dots x_n^{a_n}$$

which skew-commute: $\underline{\partial}_i \underline{\partial}_j = q_{ij} \underline{\partial}_j \underline{\partial}_i$, and satisfy: $\underline{\partial}_i x_j - q_{ij} x_j \underline{\partial}_i = \delta_{ij}$.

Theorem (Bazlov, Berenstein; 2008)

Each braided Cherednik algebra has a representation on $S_q(V)$, via "braided Dunkl operators" $\underline{\nabla}_i$ which deform the braided partial derivatives, but still skew-commute, i.e. $\underline{\nabla}_i \underline{\nabla}_j = q_{ij} \underline{\nabla}_j \underline{\nabla}_i$.

There is an especially important class of braided Cherednik algebras, called the "negative braided Cherednik algebras". These are the anticommutative analogues of rational Cherednik algebras, in the sense that they have triangular decomposition $S_{-\mathbf{1}}(V) \otimes \mathbb{C}G \otimes S_{-\mathbf{1}}(V^*)$.

Notes

- General braided Cherednik algebras are "braided products" of rational Cherednik algebras and negative braided Cherednik algebras.
- Each complex reflection group G generates rational Cherednik algebra $H_{t,c}(G)$. If $G = G(m, p, n)$ with $\frac{m}{p}$ even, then there is a corresponding mystic reflection group $\mu(G)$, generating a negative braided Cherednik algebra $\underline{H}_{t,c}(\mu(G))$.

  The new result: Negative braided Cherednik algebras can be obtained as "Drinfeld twists" of rational Cherednik algebras...

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For finite group Γ, take Γ-graded algebra A, i.e. $A = \bigoplus_{g\in\Gamma} A_g$ where $A_g \cdot A_h \subset A_{g\cdot h}\ \forall g, h \in \Gamma$.

We wish to define a new product $*$ on A: take a function $\gamma : \Gamma \times \Gamma \to \mathbb{C}^\times$, then let

$$a *_\gamma b := \gamma(g, h) a \cdot b \text{ where } a \in A_g,\ \ b \in A_h$$

It turns out $*_\gamma$ defines an associative product iff γ is a 2-cocycle (with coefficients in $\mathbb{C}^\times$), i.e. $\gamma \in Z^2(\Gamma, \mathbb{C}^\times)$. In this case, the new algebra $A_\gamma = (A, *_\gamma)$ is called the **cocycle/Drinfeld twist**.

When γ and μ are cohomologous cocycles, their products $*_\gamma$ and $*_\mu$ define isomorphic algebras. So such twists of A are determined by elements of the second cohomology group $H^2(\Gamma, \mathbb{C}^\times) := Z^2(\Gamma, \mathbb{C}^\times)/B^2(\Gamma, \mathbb{C}^\times)$.

Note: When Γ is abelian and Γ acts on algebra A by algebra homomorphisms, then A is Γ-graded.

The group $T = (C_2)^n$ acts by algebra homomorphisms on the rational Cherednik algebras $H_{1,c}(G(m,p,n))$ when $\frac{m}{p}$ even, so we can twist by cocycles in $Z^2(T, \mathbb{C}^\times)$.

Theorem (Preprint; Bazlov, Berenstein, McGaw, Jones-Healey)

For $m, p, n \in \mathbb{N}$ such that $\frac{m}{p}$ is even, there is a cocycle $\mathcal{F} \in Z^2(T, \mathbb{C}^\times)$ such that:

$$\underline{H}_{1,\underline{c}}(\mu(G(m,p,n))) \cong H_{1,c}(G(m,p,n))_{\mathcal{F}}$$

where $\underline{c}_1 = c_1, \underline{c}_\zeta = -c_\zeta \ \forall \zeta \in C_{\frac{m}{p}}$.

Much work has gone into understanding when rational Cherednik algebras possess finite-dimensional representations. We find the following:

Corollary

If the rational Cherednik algebra above has finite-dimensional representations, then so too does its corresponding negative braided Cherednik algebra.

Future work: In the $t = 0$ case, twist "baby Verma modules". Investigate connections with KZ functor.