Lie Algebras

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Abstract

Summary notes of the Lie Algebras course (MATH62112) taught by Alexander Premet at the University of Manchester in 2019. These notes are not official, or vetted by the lecturer, so any mistakes are likely my own. Sections based on my own readings separate to the course are marked with an asterisk*.

Contents

0 Notes

Sections 1 (Introduction) - 3 (Engel's & Lie's theorem) = first 50% of course + content of Sheets 1 & 2. Sections $4-9$ = second 50% of course + content of Sheets $3, 4, 5$.

1 Introduction

Definition.

- **Algebraically closed field** F is s.t. every $f \in F[t]$ has a root in F.
- The **algebraic closure** of F is the field $\hat{F}$ s.t. for every algebraically closed field F' s.t. $F \subset F'$ (which always exists) we have $F \subset \hat{F} \subset F'$ with every $f \in F[t]$ having root in $\hat{F}$.
- $\mathbb{k}$**-algebra** is a k-vector space V with bilinear product $\cdot : V \times V \to V$ (or linear map $V \otimes V \to V$). Note: bilinearity implies $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x \implies 0 \cdot x = 0$.
- **Structure constants** of algebra V with respect to basis $\{v_1, \dots, v_n\}$ are $c_{ij}^k \in \mathbb{k}$ s.t. $v_i \cdot v_j = c_{ij}^k v_k$.

  For any vector space V, one can define an algebra structure with respect to some basis by defining structure constants (wrt to that basis), and extending the multiplication by bilinearity to the rest of the vector space.
- **Algebra homomorphism** is a linear map between the underlying vector spaces, that preserves multiplication: $\alpha(x \cdot y) = \alpha(x) \cdot \beta(y)$. This is an **isomorphism** when the linear map is bijective.
- **Unital algebra** contains $1 \in A$ s.t. $a \cdot 1 = a = 1 \cdot a$. **Associative algebra** satisfies $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Commutative algebra if $x \cdot y = y \cdot x$.
- **Anti-commutative algebra** A s.t. $x \cdot x = 0 \ \forall x \in A$.

  Anti-commutativity $\implies x \cdot y = -y \cdot x \ \forall x, y$. Proof: check $(x+y) \cdot (x+y)$. In char$(\mathbb{k}) \neq 2$ the converse holds: **Anti-commutative algebra** A iff $x \cdot y = -y \cdot x \ \forall x, y$.
- **Lie algebra** is an anti-commutative $\mathbb{k}$-algebra that satisfies the Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

  Note: For checking if something is Lie algebra, by bilinearity it is sufficient to check anti-commutativity and Jacobi identity just on the algebras basis vectors.
- $a, b \in L$ **commute** if $[a,b] = 0$. (L denotes a Lie algebra)
- $X, Y \subset L$, then $[X,Y] := \mathrm{span}(\{[x,y] \ : \ x \in X, y \in Y\})$
- **Abelian Lie algebra** is any algebra A s.t. $a \cdot b = 0 \ \forall a, b \in A$.
- For associative algebra $(A, \ \cdot \ )$, then $A^{(-)} := (A, [\ ,\ ])$ with $[a,b] := a \cdot b - b \cdot a$

Lemma. *$A^{(-)}$ is a Lie algebra.*

Proof. $[a,b] = a \cdot b - b \cdot a$ anti-commutative, and expand $[a,[b,c]] + [b,[c,a]] + [c,[a,b]]$ gives Jacobi. □

Definition.

- **General linear Lie algebra** $\mathfrak{gl}_n(\mathbb{k}) := \mathrm{Mat}_n(\mathbb{k})^{(-)}$
- For vector space V, $\mathrm{End}(V)$ is the algebra of linear maps with multiplication given as composition. Then $\mathfrak{gl}(V) := \mathrm{End}(V)^{(-)}$
- **Derivation** on algebra A is a linear map (not algebra hom) $D : A \to A$ s.t. $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$. The set of derivations on A is $\mathrm{Der}(A)$.

Note: $\mathrm{Mat}_n(\Bbbk)$ and $\mathfrak{gl}(V)$ are related by the following: take basis $B = \{v_1, \dots, v_n\}$ for V. Define $e_{ij} : V \to V$ s.t. on basis vectors $e_{ij}(v_j) = v_i$, $e_{ij}(v_k) = 0$ $k \neq j$ and extend to V via linearity and define E_{ij} as the matrices with 1 in (i,j)-th position and 0 elsewhere. Then the map $M : \mathrm{End}(V) \to \mathrm{Mat}_n(\Bbbk)$, $e_{ij} \mapsto E_{ij}$ sends a linear op x to the corresponding matrix $M(x)$ wrt basis B. $M(x \circ y) = M(x) \cdot M(y)$ giving isomorphism of algebras (dependent on basis, so not unique), and hence also the isomorphism of lie algebras $\mathfrak{gl}(V) \cong \mathrm{Mat}_n(k)^{(-)}$.

Lemma. *$Der(A)$ is a subspace of $End(A)$, and a Lie subalgebra of $\mathfrak{gl}(A)$ under commutator product.*

Proof. Check $0 \in \mathrm{Der}(A)$ and if $D_1, D_2 \in \mathrm{Der}(A)$ then $\lambda_1 D_1 + \lambda_2 D_2 \in \mathrm{Der}(A)$, so subspace. Closed under commutator by Sheet 1 Q4: $[D_1, D_2] \in \mathrm{Der}(A)$. Commutator antisymmetric by defn and satisfies Jacobi since it does in $\mathrm{End}(A)^{(-)} = \mathfrak{gl}(A)$. ☐

Definition.

- **Subalgebra** $S \subset A$ is subspace s.t. $x \cdot y \in S$ $\forall x, y \in S$.
- **Lie subalgebra** $S \subset L$ of lie algebra L s.t. $[S, S] \subset S$ (i.e. lie subalgbera $\equiv$ subalgebra of lie algebra).
- **Left ideal** $I \subset A$ is subspace s.t. $x \cdot y \in I$ $\forall x \in A, y \in I$. Notation for Lie algebra L: $[L, I] \subset I$.

  Left ideal $\implies$ right ideal: For Lie alg A, $\cdot$ is anticommutative, so left ideal implies

  $$x \cdot y = -(y \cdot x) = y \cdot (-x) \in I \ \forall x \in A, y \in I \implies \forall x' = -x \in A, \forall y \in I \quad y \cdot x' \in I$$

  so is right ideal, so in fact all ideals in Lie algebras are 2-sided.
- **Simple** lie algebra is a non-abelian Lie algebra with no non-trivial ideals.
- **Special linear Lie algebra** $\mathfrak{sl}_n(\Bbbk) := \{X \in \mathfrak{gl}_n(\Bbbk) : \ \mathrm{tr}(X) = 0\}$.

  This is an ideal of $\mathfrak{gl}_n(\Bbbk)$ since its kernel of $\mathrm{tr} : \mathfrak{gl}_n(\Bbbk) \to \Bbbk$, and by rank-nullity: $\dim(\mathfrak{sl}_n(\Bbbk)) = n^2 - 1$.

Example. $\mathfrak{sl}_2(\Bbbk)$ with relations $[h, e] = 2e$, $[e, f] = h$, $[h, f] = -2f$ has "standard basis":

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Denote $b_+ = \mathrm{span}(h, e)$*, $b_- = \mathrm{span}(h, f)$, both 2-dim'l subalgebras.

Definition.

- **Lie algebra homomorphism** between L_1, L_2 is a linear map

  $$\phi : L_1 \to L_2 \text{ s.t. } \phi([x, y]) = [\phi(x), \phi(y)] \ \forall x, y \in L_1$$

  This implies for any $A, B \subset L_1$ we have equality of sets: $\phi([A, B]) = [\phi(A), \phi(B)]$.
- **Isomorphism** of Lie algebras if ϕ also bijective.
- **Automorphism** if ϕ is a isomorphism with $L_1 = L_2 = L$

*Note this may be where the Hopf Algebra $U_q(b_+)$ originates

Checking a linear map is a homomorphism it suffices to check on basis $\{x_i\}$:

$$\phi([x_i, x_j]) = [\phi(x_i), \phi(x_j)] \text{ on } 1 \leq i < j \leq n$$

Lemma. *$\phi : L_1 \to L_2$ Lie algebra hom, then $Ker(\phi)$ ideal of L_1 and $im(\phi)$ Lie subalgebra of L_2.*

Proof. As ϕ linear map certainly subspaces, and $\phi([L_1, \ker(\phi)]) = [\phi(L_1), \phi(ker(\phi))] = [\phi(L_1), 0] = 0$, and $\mathrm{im}(\phi) = \phi(L_1)$ so $[\phi(L_1), \phi(L_1)] = \phi([L_1, L_1]) \subset \phi(L_1) = \mathrm{im}(\phi)$. □

Definition.

- **Heisenberg algebras** H_n have basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n, z\}$ with $[u_i, v_j] = \delta_{ij} z$, all other commutators on basis vectors vanishing, and extending by bilinearity to H_n.
- $\mathfrak{n}_n^+ \subset \mathfrak{gl}_n(k)$ is the lie subalgebra of strictly upper triangular matrices.

  Note: $H_1 \cong \mathfrak{n}_3^+$ (See Example 1.1 in lectures).
- **L-module** for Lie algebra L is a vector space V with k-bilinear map $\cdot : L \times V \to V$ satisfying $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.
- **L-submodule** of L-module V is a subspace $W \leq V$ closed under the resticted bilinear op: $x \cdot w \in W \ \forall x \in L, w \in W$.
- **Irreducible/simple** L-module has no non-trivial submodules. For lie algebra L and vec space V, the **trivial** L-module structure has bilinear op $x \cdot v = 0 \ \forall x, v$.
  (Note every subspace of trivial module is submodule, so in this case irreducibility implies $\dim(V) = 1$.)
- **Lie algebra representation** is Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$, for vec space V.
- A representation is **finite-dimensional** when $\dim(V) < \infty$.

  Note every L-module defines a representation, and vice versa.
- **Faithful** representation s.t. $\ker(\rho) = 0$.
- **Linear lie algebra** is a lie subalgebra of $\mathfrak{gl}(V)$.

Theorem (Ado-Iwasawa)**.** *Every finite-dim Lie algebra has a faithful finite-dim representation. (i.e. every finite-dim lie algebra is isomorphism to a linear Lie algebra). (Proof non-examinable).*

Definition.

- **Center** of Lie algebra is the ideal $Z(L) := \{z \in L | \ [x, z] = 0 \ \forall x \in L\}$.
- Lie algebra is **centerless** if $Z(L) = 0$.
- **Adjoint representation** $\mathrm{ad} : L \to \mathfrak{gl}(L)$, with $x \mapsto \mathrm{ad}(x)$ s.t. $\mathrm{ad}(x)(y) := [x, y]$.
- $\mathrm{ad}(x)$ is the **adjoint endomorphism** of x.

Lemma. *The adjoint representation is indeed a representation.*

Proof. Need to show each $\text{ad}(x) \in \mathfrak{gl}(L) = \text{End}(V)^{(-)}$, i.e. they are linear: $(\text{ad}(x))(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1(\text{ad}(x))(y_1) + \lambda_2(\text{ad}(x))(y_2)$ (by bilinearity of the bracket). Also ad itself is a linear map (again via bilinearity): $(\text{ad}(\lambda_1 x_1 + \lambda_2 x_2))(y) = \lambda_1(\text{ad}(x_1))(y) + \lambda_2(\text{ad}(x_2))(y)$. Lastly ad is lie algebra hom: $\text{ad}([x,y])(z) = [\text{ad}(x), \text{ad}(y)](z)$ (using anti-comm + jacobi). □

Proposition.
- *ad(x) is derivation of L $\forall x \in L$.*
- *ker(ad)= $Z(L)$. (So for centerless L the adjoin repn is faithful).*

Proof.
- By above ad(x) linear, so just show derivation property.
- By definition/obvious.

□

Definition.
- **Inner derivation** is derivation D on L s.t. $\exists x \in L$ with $D = \text{ad}(x)$.
- ad(L) denotes the set of inner derivations.

- L abelian implies every linear map $D \in \mathfrak{gl}(V)$ is trivially a derivation. However ad(x) is 0 map $\forall x$, so only one (trivial) inner derivation. Hence in this case $\text{Der}(L) \neq \text{ad}(L)$.
- Lie algebra L is equivalent to its underlying vector space with an ad(L)-module structure. Hence ideals of $L \equiv \text{ad}(L)$-submodules. Note also L as an ad(L)-module, is the L-module corresponding to the adjoint representation (under the module - representation equivalence), hence see L simple iff L non-abelian and adjoint repn is irreducible.

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2 Nilpotent and Solvable Lie Algebras

2.1 Lower central series and derived series

Lemma. *For I, J ideals of Lie algebra L, then $I + J$ and $[I, J]$ are also ideals.*

Definition.
- **Lower central series** is the descending series of ideals $L^1 = L,\ L^{n+1} = [L, L^n]$. (These are ideals since L^2 ideal by above Lem, then recursively apply the Lem $(n-1)$-times for L^n).
- **Nilpotent** lie algebra L s.t. $L^n = 0$ for some $n \in \mathbb{N}$
- **Nilpotency class** of nilpotent lie algebra is minimum $n \in \mathbb{N}$ s.t. $L^n = 0$). Note: Heisenberg algebras H_n have nilpotency class 3.

Proposition. *Non-zero nilpotent Lie algebras L have i) $Z(L) \neq 0$, ii) if I non-zero ideal then $I \cap Z(L) \neq 0$.*

Lemma. $[L^m, L^n] \subset L^{m+n}$

Definition.

- **Derived series** is the descending chain of ideals $L^{(0)} := L,\ L^{(n+1)} = [L^{(n)}, L^{(n)}]$. (Again these are ideals by recursively applying above Lem).
- **Solvable** lie algebra s.t. $L^{(n)} = 0$ for some $n \in \mathbb{N}$.
- **Derived subalgebra** is $L^{(1)} = [L, L] = L^2 = \text{span}\{[x, y] |\ x, y \in L\}$
- **Perfect** lie algebra s.t. $L^{(1)} = L$

Lemma. *(i)* $L^{(n)} \subset L^{2^n}$ *(ii)* $(L^{(m)})^{(n)} = L^{(m+n)}$ *(iii) Every nilpotent Lie algebra is solvable.*

2.2 Factors/quotients

Definition.

- For vec space V, and subspace W, the **coset** of $v \in V$ is $v + W = \{v + w |\ w \in W\}$. Two coset are equal, $v_1 + W = v_2 + W$, iff $v_1 - v_2 \in W$.
- The **factor/quotient space** V/W is the set of cosets with ops $(v+W)+(v'+W) := (v+v')+W$ and $\lambda(v + W) := (\lambda v) + W$. This is a vector space with zero "vector": $0 + W = W$ and additive inverses $-v + W$. See Pg 12 of notes on constructing basis and showing $\dim(V/W) = \dim(V) - \dim(W)$.
- **Factor module** of L-module V by L-submodule W has underlying vector space V/W as above with $x \cdot (v + W) := (x \cdot v) + W\ \forall x \in L, v \in V$.

  Proof. Proof it is an L-module: since $x \cdot w \in W\ \forall w \in W$, have $v_1 + W = v_2 + W \implies v_1 - v_2 \in W \implies x \cdot (v_1 - v_2) \in W \implies x \cdot v_1 + W = x \cdot v_2 + W$, so L-module structure well-defined. Easy to show action is bilinear and satisfies $[x, y] \cdot (v + W) = x \cdot (y \cdot (v + W)) - y \cdot (x \cdot (v + W))$. □

- **Factor/quotient lie algebra** of lie algebra L by ideal I is vec space L/I with $[x+I, y+I] := [x, y]+I$.

  Proof. Proof this is well-defined, anticommutative and satisfies Jacobi: ... □

  Note L/I abelian iff $[L, L] \subset I$.

Theorem. *$\phi : L_1 \to L_2$ Lie algebra hom, then $L_1/ker(\phi) \cong im(L_2)$. (Showed on pg 3 that $ker(\phi)$ ideal, and $im(\phi)$ is Lie subalgebra).*

Corollary. *I, J ideals, then (i) J ideal of $I + J$ (ii) $I \cap J$ ideal of I, and (iii):*

$$I/I \cap J \cong (I + J)/J$$

2.3 Radicals and semisimplicity

Proposition. *$\phi : L \to M$ lie algebra hom. Then*

- *$\phi(L^m) = \phi(L)^m$ and $\phi(L^{(n)}) = \phi(L)^{(n)}$ $\forall m, n$*
- *L nilpotent implies im(ϕ) nilpotent*
- *L solvable implies im(ϕ) solvable*
- *I ideal of L s.t. I and L/I as Lie algebras are both solvable, then L is solvable.*
- *I, J solvable ideals then $I + J$ solvable ideal*
- *(I, J nilpotent ideals then $I + J$ nilpotent ideal - see Remark 2.4 for proof)*

Corollary. *Every finite dimensional lie algebra contains a unique maximal solvable ideal (i.e. has a radical).*

Definition. L finite dimensional, then

- The **radical**, rad(L), is the unique maximal solvable ideal of L.
- A **semisimple** lie algebra L s.t. rad(L) = 0.

Note: L semisimple iff it contains no abelian ideals (Proof in Remark 2.4)

Theorem. *L finite-dim lie algebra, then L/rad(L) semisimple.*

Very interesting note: Every simple group, excluding the 26 sporadic groups, arise as the automorphism groups of some finite-dim simple Lie algebra.

Theorem (Levi-Malcev)**.** *L finite-dim lie algebra with char(k) = 0, then*

- *exists semisimple lie subalgebra S s.t. $L = S \oplus$ rad(L)*
- *if S_1, S_2 both as above then there exists automorphism σ of L s.t. $\sigma(S_1) = S_2$.*

Proof. Non-examinable. □

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3 Engel's Theorem and Lie's Theorem

Definition.

- **Nilpotent** $a \in \text{End}(V)$ s.t. $a^n = 0$ for some $n \in \mathbb{N}$.
- For V n-dim'l, linear op $a \in \mathfrak{gl}(V)$ with λ_i are the pairwise distinct eigenvalues with multiplicities m_i (so $\sum_i m_i = n$), then **characteristic polynomial** of a is:

$$\chi_a(t) := \det(t1 - a) = \prod_{i=1}^{s}(t - \lambda_i)^{m_i}$$

- Let $q_k(t) := \prod_{i\neq k}(t-\lambda_i)^{m_i} = \chi_A(t)/(t-\lambda_k)^{m_k}$
- The **generalised eigenspace** of $A \in \mathfrak{gl}(V)$ (corresponding to the eigenvalue λ_k) is

$$V(\lambda_k) := \mathrm{Im}(q_k(A)) = \{q_k(A)(v) \mid v \in V\}$$

Lemma.
- *The* ***Cayley-Hamilton theorem*** *states $\chi_a(a) = 0$ i.e. substituting the linear op into its own characteristic poly gives the zero map.*
- *$\forall A \in \mathfrak{gl}(V)$, V decomposes into direct sum of generalised eigenspaces, i.e. $V = \bigoplus_{i=1}^{s} V(\lambda_i)$*

  Proof. See "Sec 6.1: Generalised eigenspaces". □

Lemma.
- *For nilpotent $a \in End(V)$ (i.e. $a^N = 0$ for some N), where n =dim(V), then $a^n = 0$. (Proof see lecture notes just above Lem3.1)*
- *$a \in \mathfrak{gl}(V)$ nilpotent then $ad(a) \in \mathfrak{gl}(\mathfrak{gl}(V))$ nilpotent.*

Theorem (Engel)**.** *For finite-dim vec space V and lie subalgebra $L \subset \mathfrak{gl}(V)$ s.t. every element of L is nilpotent, then $\exists v \in V\backslash\{0\}$ s.t. $x(v) = 0\ \forall x \in L$ i.e "L annihilates v".*

Definition.
- A **flag** of subspaces of n-dim'l vec space V is chain $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ s.t. $\dim(V_i) = i\ \forall i$.
- L lie subalgebra of $\mathfrak{gl}(V)$, for n-dim'l V, then L **stabilises a flag of subspaces** if $\exists$ flag $\{0\} \subset V_1 \subset \cdots \subset V_n = V$ s.t. $x(V_i) \subset V_i \quad \forall x \in L, 1 \le i \le n$.

Basis extension theorem states that every linearly independent list of vectors can be extended to a basis. Then, given a flag we can define basis $\{v_1, \dots, v_n\}$ of V s.t. $\{v_1, \dots, v_i\}$ is a basis for V_i.

Corollary. *For n-dimensional vec space V and lie subalgebra $L \subset \mathfrak{gl}(V)$ with all nilpotent elements, then there exists a flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$ s.t. $x(V_i) \subset V_{i-1}\ \forall x \in L$.*

The main practical result out of Engels theorem, and the above Corollary, is that any Lie subalgebra of $\mathfrak{gl}(V)$ with all nilpotent elements is isomorphic to a lie subalgebra of $\mathfrak{n}_n^+$.

Corollary. *L a finite-dim lie algebra, then L nilpotent iff $ad(L) \subset \mathfrak{gl}(L)$ consists of all nilpotent endomorphisms.*

Theorem (Lie's theorem)**.** *L a solvable lie algebra over algebraically closed k s.t. char(k) = 0, then every irreducible finite-dim representation of L is 1-dimensional.*

Proof. Non-examinable. □

Corollary. *k algebraically closed, char(k) = 0, V finite-dim.*
- *Any solvable lie subalgebra of $\mathfrak{gl}(V)$ stabilises a flag of V; and it's isomorphic to a lie subalgebra of $\mathfrak{b}_n^+$. (Proof: See remark 3.3)*
- *L finite-dim solvable lie algebra, then derived subalgebra $[L, L]$ nilpotent.*

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4 Cartan Criterion and Killing Form

Definition.
- The **trace** of matrix A is $\mathrm{tr}(A) := \sum_i A_{ii}$.

  Note: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, hence $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$. Also for $a \in \mathfrak{gl}(V)$, and bases B, B' of V, the matrices $M(a)_B, M(a)_{B'}$ are similar, hence have equal trace, hence the following defn is well-defined:

- For $a \in \mathfrak{gl}(V)$, $\mathrm{tr}(a) := \mathrm{tr}(M(a)_B)$ for any basis B of V.

  Note: $\mathrm{tr}(a \circ b) = \mathrm{tr}(b \circ a)$, and the fact: $\mathrm{tr}(A) = \sum$ eigenvalues of A (including multiplicities).

- A **bilinear form** on vec space V is a bilinear map $\gamma : V \times V \to k$.

- Bilinear form is **symmetric** if $\gamma(v, w) = \gamma(w, v)$, **skew-symmetric** if $\gamma(v, v) = 0$.

- For basis $S = \{v_1, \ldots, v_n\}$ of V, bilinear form γ, then the **matrix of** γ **wrt** S is $M_S(\gamma)_{ij} := \gamma(v_i, v_j)$. Such matrices for different bases are similar. γ (skew-)symmetric iff $M_S(\gamma)$ is $\forall$ bases S.

- For symmetric bilinear form γ, **radical** $\mathrm{Rad}(\gamma) := \{u \in V \mid \gamma(u, v) = 0\ \forall v \in V\}$

- Bilinear form γ **non-degenerate** if $\mathrm{Rad}(\gamma) = \{0\}$ ($\iff$ $\det(M_S(\gamma)) \neq 0\ \forall S$)

- For finite-dim Lie algebra L, **L-invariant bilinear form** $\gamma : L \times L \to k$ s.t.

$$\gamma([x, y], z) + \gamma(y, [x, z]) = 0\ \forall x, y, z \in L$$

- **Killing form** is the map $\kappa : L \times L \to k$, s.t. $\kappa(x, y) := \mathrm{tr}((\mathrm{ad}(x)) \circ (\mathrm{ad}(y)))$

Lemma. *If γ is L-invariant symmetric bilinear form, then $Rad(\gamma)$ ideal of L.*

Hence non-zero L-invariant symmetric bilinear forms on simple Lie algebras are non-denerate, since if the radical is a proper ideal, it must be $\{0\}$.

Proposition. *The Killing form is a L-invariant symmetric bilinear form on L.*

Corollary. *If Killing form on L non-degenerate, then L is semisimple Lie algebra.*

Theorem (Cartan's Criterion)**.** *V finite-dim over k alg-closed with $char(k) = 0$, L Lie subalgebra of $\mathfrak{gl}(V)$. Then L is solvable iff $tr(a \circ b) = 0\ \forall a \in L, b \in [L, L]$.*

Proof. $\implies$ direction only. ☐

Corollary. *L finite-dim Lie alg, k as above, κ Killing form on L.*

- *Then L solvable iff $[L, L] \subset Rad(\kappa)$.*
- *I ideal of L, κ_I Killing form on the Lie algebra I, then $\kappa_I = \kappa|_{I \times I}$ (i.e κ_I is the restriction).*

Theorem. *L finite-dim Lie alg, k as above, κ Killing form on L. Then L semisimple iff κ non-degenerate.*

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5 Structure of semisimple Lie algebras

Definition. For Lie algs $L_1, \dots, L_s$, the **Lie algebra direct sum** $L := L_1 \oplus \cdots \oplus L_s$ is the usual direct sum of vector spaces with componentwise addition and scalar multiplication, and for $v_1 + \cdots + v_s, w_1 + \cdots + w_s \in L$ let

$$[v_1 + \cdots + v_s, w_1 + \cdots + w_s]_L := [v_1, w_1]_1 + \cdots + [v_n, w_n]_s$$

where $[\ ,\ ]_i$ is the Lie bracket of L_i.

Since each $[\ ,\ ]_i$ is anti-comm and satisfies Jacobi, so does $[\ ,\ ]_L$, hence L is a Lie algebra. Note: Under the identification of L_i with the subspace $\{0 + \cdots + v_i + \cdots + 0 \in L | v_i \in L_i\}$, L_i is an ideal of L (hence L is not simple).

Proposition. *$L_1, \dots, L_s$ simple Lie algebas, then $L := L_1 \oplus \cdots \oplus L_s$ semisimple.*

Definition.
- For vec spaces V, W, a **pairing** is bilinear map $\psi : V \times W \to k$.
- **Left radical** of pairing ψ is $\mathrm{Rad}_l(\psi) := \{v \in V |\ \psi(v, w) = 0\ \forall w \in W\}$.
- **Right radical** of pairing ψ is $\mathrm{Rad}_r(\psi) := \{w \in W |\ \psi(v, w) = 0\ \forall v \in V\}$
- **Perfect pairing** is ψ s.t. $\mathrm{Rad}_l(\psi) = \{0\} = \mathrm{Rad}_r(\psi)$.

Proposition. *If ψ is perfect pairing of V, W then $\dim(V) = \dim(W)$.*

Theorem. *L finite-dim Lie alg, k alg-closed with char$(k) = 0$. Then L semisimple iff $\exists$ finite-dim simple Lie algs $L_1, \dots, L_s$, s.t. $L \cong L_1 \oplus \cdots \oplus L_s$.*

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6 Jordan-Chevalley decomposition

Definition.
- $A \in \mathfrak{gl}(V)$ **semisimple** if A diagonalisable (so $\exists$ basis of eigenvectors of A for V)
- The **Jordan-decomposition** of $A \in \mathfrak{gl}(V)$ is $A_s, A_n \in \mathfrak{gl}(V)$ s.t. $A = A_s + A_n$; A_s semisimple; A_n nilpotent; and $[A_s, A_n] = 0$.

Theorem. *Every $A \in \mathfrak{gl}(V)$ has a unique Jordan decomposition.*

Proof. See discussion at start of Sec 6.2 + Thm 6.1 Proof. □

Definition. A Lie subalgebra L of $\mathfrak{gl}(V)$ is **seperating** if $\forall A \in L$ then $A_s, A_n \in L$.

Proposition. *For finite-dim alg A over alg-closed k, Der(A) is a seperating Lie subalgebra of $\mathfrak{gl}(A)$.*

Lemma. *$ad(L)$ is ideal of $Der(L)$ (i.e. inner derivations are an ideal in the set of derivations).*

Corollary.
- *If $Z(L) = \{0\}$, I non-zero ideal of $Der(L)$, then $I \cap ad(L) \neq \{0\}$.*
- *L semisimple $\implies$ $Der(L)$ semisimple.*

Theorem. *L finite-dim semisimple Lie alg over alg-closed k with $char(k) = 0$. Then $Der(L) = ad(L)$ (i.e. all derivations are inner).*

Definition. The **Jordan-Chevalley decomposition** of $x \in L$ is $x_s, x_n \in L$ s.t. $\mathrm{ad}(x_s)$ semisimple, $\mathrm{ad}(x_n)$ nilpotent, and $[x_s, x_n] = 0$.

Theorem. *L semisimple Lie alg over alg-closed k with $char(k) = 0$. Then every $x \in L$ has a unique Jordan-Chevalley decomposition.*

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7 Weyl's Theorem and Representations of $sl_2(k)$

Definition. L-module V if **completely reducible** if $\exists$ irreducible L-submodules $V_1, \dots, V_n$ s.t. $V = V_1 \oplus \cdots \oplus V_n$.
Characterisation: V completely reducible iff $\forall$ L-submodules W there $\exists$ L-submodule W' s.t. $V = W \oplus W'$.

Theorem (Weyl)**.** *L finite-dim semisimple Lie alg over k s.t. $char(k) = 0$. Then every finite-dim L-module is completely reducible.*

Proof. Non-examinable. □

7.1 $\mathfrak{sl}_2(k)$

For the rest of this section let $L := \mathfrak{sl}_2(k)$. Take standard basis $\{e, h, f\}$ has relations $[h, e] = 2e$, $[e, f] = h$, $[h, f] = -2f$. Let V be an L-module (denote action of $x \in L$ on $v \in V$ as $x \cdot v$). Let ρ be corresponding representation: $\rho(x)(v) := x \cdot v$.

Definition.
- A **weight** of V is $\lambda \in k$ s.t. $\exists v \in V \backslash \{0\}$ s.t. $h \cdot v = \lambda v$ (weights = eigenvalues of $\rho(h)$).
- The **weight space** of weight λ is $V_\lambda := \{v \in V \mid h \cdot v = \lambda v\}$. (So $V_\lambda \neq \{0\} \iff \mu$ is a weight).
- A **primitive** is $v \in V \backslash \{0\}$ s.t. $e \cdot v = 0$.
- $V_{\text{prim}} := \{v \in V \mid e \cdot v = 0\} = \ker(\rho(e))$.

Lemma.
- *For $\mu \in k$, $v \in V_\mu$, then:* $\quad e \cdot v \in V_{\mu+2}, \quad f \cdot v \in V_{\mu-2}$.
- *$\exists$ weight λ of V s.t. $V_{prim} \cap V_\lambda \neq \{0\}$.*

Definition.
- A **highest weight** is a weight λ of V s.t. $V_{\text{prim}} \cap V_\lambda \neq \{0\}$
- For highest weight λ, a **highest weight vector** (of weight λ) is non-zero $v \in V_{\text{prim}} \cap V_\lambda$.
- Let v_0 a highest weight vector of weight λ, then $v_{-1} := 0$, $v_k := \frac{1}{k!} f^k \cdot v_0$.

Lemma. $\forall k \in \mathbb{Z}_{\geq 0}$

- $f \cdot v_k = (k+1)v_{k+1}$
- $h \cdot v_k = (\lambda - 2k)v_k$
- $e \cdot v_k = (\lambda - k + 1)v_{k-1}$
- *Every highest weight* $\lambda \in \mathbb{Z}_{\geq 0}$.
- *The* $v_0, v_1, \ldots, v_\lambda$ *are linearly independent, and* $v_k = 0\ \forall k > \lambda$.

Example. Constructing irreducible $\mathfrak{sl}_2(k)$-modules....

Definition. $m \in \mathbb{Z}_{\geq 0}$, $V(m) := \text{span}\{u_0, \ldots, u_m\}$. Let $u_{-1} := 0 =: u_{m+1}$, and $E, H, F \in \mathfrak{gl}(V(m))$ s.t.

$$E(u_i) = (m - i + 1)u_{i-1},\ H(u_i) = (m - 2i)u_i,\ F(u_i) = (i+1)u_{i+1}$$

Proposition. *Linear map* $\rho_m : \mathfrak{sl}_2(k) \to \mathfrak{gl}(V(m))$, $\rho_m(e) = E$, $\rho_m(h) := H$, $\rho_m(f) := F$, *is an irreducible representation of* $\mathfrak{sl}_2(k)$. *Additionally, every finite-dim irreducible* $\mathfrak{sl}_2(k)$*-module is isomorphic to* $V(m)$ *for some* $m \in \mathbb{N}$.

Corollary. *For finite-dim* $\mathfrak{sl}_2(k)$*-module* V, $\exists r_1, \ldots, r_s \in \mathbb{Z}_{\geq 0}$ *with* $r_1 \leq \cdots \leq r_s$ *s.t.* $V \cong V(r_1) \oplus \cdots \oplus V(r_s)$.

Corollary. *Finite-dim* $\mathfrak{sl}_2(k)$*-module* V*, then*

- *all weights of* V *are integers*
- λ *a weight* $\implies -\lambda$ *also is, and* $\dim(V_\lambda) = \dim(V_{-\lambda})$
- *number of irred direct summands of* V *equals* $\dim(V_{prim})$
- $\dim(V_{prim}) = \dim(V_0) + \dim(V_1)$ *with* $V_0 = \{v \in V | h \cdot v = 0\}$, $V_1 = \{v \in V | h \cdot v = v\}$.

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8 Maximal toral subalgebras and root space decomposition

L now general finite-dim semisimple Lie alg over alg-closed k, $\mathrm{char}(k) = 0$. So Killing form κ non-degenerate and every $x \in L$ has Jordan-Chevalley decomp.

Definition.

- Lie subalg T of L is **toral** if $\mathrm{ad}(t)$ semisimple $\forall t \in T$.
- **Maximal toral subalg** T s.t. for every toral subalg T' s.t. $T \subset T'$ then $T = T'$.

Lemma. *Every L (as above) has a maximal toral subalgebra, and they are abelian.*

Definition.

- The **centraliser** of $X \subset L$ is $c_L(X) := \{u \in L : [u, x] = 0 \ \forall x \in X\}$

  Note: Centralisers are Lie subalgebras.
- For maximal toral subalgebra T, and $\alpha \in T^*$, the **root space** is $L_\alpha := \{v \in L | \ [t, v] = \alpha(t)v \ \forall t \in T\}$.

  L_α subspace of L $\forall \alpha$. $\alpha = 0 \implies L_\alpha = c_L(T)$.
- **Root** of L wrt T is non-zero $\alpha \in T^*$ s.t. $L_\alpha \neq \{0\}$. Let Φ denote the set of roots.
- The **root space (Cartan) decomposition** of L wrt T is

$$L = H \oplus \sum_{\alpha \in \Phi} L_\alpha, \qquad H := c_L(T)$$

- **Root vectors** are non-zero $v \in L_\alpha$ for some $\alpha \in \Phi$ (i.e. non-zero elements of (non-zero) root spaces)

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9 *Lie groups to Lie algebras

Lie groups are the group objects in the category of smooth manifolds. *Queue subsection on category theory and defining group objects.* They are smooth manifolds with an additional group structure attached. Sophus Lie started this theory to provide an analogue for differential equations that Galois theory did for polynomials. Examples include $O(n), U(n)$, with the symplectic group $Sp(2n)$ the analogue of those for quarternions.

The smooth manifold structure of Lie groups gives them a nontrivial topology. The Lie algebra, the tangent plane at the identity element of the Lie group, is much simpler object to study. Take a linear Lie group G (i.e. has a matrix representation) then for $A, B \in G$ such that $A = 1 + \epsilon X,\ B = 1 + \epsilon Y$, for ϵ very small, so X, Y are regarded as elements of tangent space at 1. Then $AB = 1 + \epsilon(X + Y) + \epsilon^2 XY$, which for small ϵ shows multiplication in G corresponds to addition in the tangent space. Closure of group multiplication implies tangent space is vector space. Also $ABA^{-1}B^{-1} = 1 + \epsilon^2[X, Y]$ with $[X, Y] = X \cdot Y - Y \cdot X$, and $\cdot$ is matrix multiplication. So the tangent space forms an algebra under this commutator product $[\ ,\ ]$. Note the underlying matrix multiplication of elements of the tangent space is not closed, but it is closed under commutator product. For abstract Lie groups/algebras, there may not be an underlying associative multiplication generating the commutator[†]. So basically Lie groups generate Lie algebras.

Conversely, via exponentiation Lie algebras can generate connected Lie groups. Lie group multiplication can be retrieved via the Baker-Campbell-Hausdauff formula on Lie algebra elements. See "univeral covers" regarding when two connected Lie groups have isomorphic Lie algebras.

Note (very useful): For summary of the classification of simple Lie algebras via root systems/dynkin diagrams, and the construction of Weyl groups (i.e. reflection groups) for Lie algebras, see the Lie theory section of Princeton Companion to Mathematics Pgs.232-234.

[†]Note the Universal enveloping algebra is the (most general) algebra with an associative multiplication such that its commutator corresponds to that of the abstract bracket of the Lie algebra.