The Kochen-Specker Theorem

Part III Essay - 86

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Abstract

The Kochen-Specker Theorem states that there cannot exist non-contextual hidden variable theories. Roughly speaking, the problem of hidden variables is to imbed quantum theory within a classical theory, while non-contextuality is the property that the measured value of an observable is independent of how it was chosen to be measured. The fact that these properties of a theory are incompatible has had a long legacy within quantum foundations. In this essay we will study the Kochen-Specker Theorem and give a survey of some of the developments that have followed.

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1 Introduction

To motivate the discussion of the Kochen-Specker (KS) Theorem, we start in Sec 2 by giving the case for hidden variables, and a short introduction to the well-established hidden variable theory of Bohmian Mechanics, or Pilot Wave Theory. This is a deterministic theory that successfully reproduces the predictions of non-relativistic Quantum Mechanics. In Sec 3 we will cover the two main theorems that place constraints on possible hidden variable theories, namely the Kochen-Specker Theorem and Bell's Theorem. More specifically, we will look at Bell's assessment of von Neumann's Theorem and show how Bell subsequently strengthened it. This strengthening in fact proved the same result as Kochen-Specker, but via different means. For simplicity we give Bell's version. Bell's analysis went still further to point out that the type of hidden variables ruled out were in fact non-contextual hidden variables, leaving the case for contextual hidden variables still open. We will give a short analysis of contextuality and the options one has in order to escape Kochen-Specker (since Bohmian Mechanics certainly does!). Following this, in Sec 3.2, we prove Bell's Theorem - the more famous of the no-go theorems that manages to rule out all local hidden variable schemes. Finally, in Sec 4, we consider more modern developments that put the notion of contextuality within the framework of Topos theory. We will briefly cover the basics of this theory, and how the Kochen-Specker theorem gets rephrased in this new language. Lastly we will show how contextual generalised truth-values provide a means to avoid the Kochen-Specker theorem.

It should be noted that another key development in the history of the Kochen-Specker theorem was the Conway-Kochen theorem, or Free Will Theorem [7]. Here it was shown that if we have free will over our choices (i.e. they are not functions of the past), then so must elementary particles, up to certain assumptions. As suggested by the published rubric for the essay we choose not to investigate this, in favour of the route described above.

2 Bohmian Mechanics

Before embarking on an analysis of hidden variables it would be worthwhile explaining why they are worth thinking about. We do this by describing the measurement problem of quantum mechanics, following the formulation of Wayne Myrvold in Stanford Encyclopedia of Philosophy [24]. Hidden-variables offer a possible resolution by means of a "realist" interpretation, rather than the usual "instrumentalist" approach of the Copenhagen interpretation. Next we give a brief account of Bohmian Mechanics to motivate our discussion of the no-go theorems in Sec 3.

2.1 Motivation

Quantum mechanics is a probabilistic theory, meaning it only places statistical constraints on the results of measurements. This is in stark contrast with classical physics, and so it is unsurprising that conceptual difficulties arise when attempting to reconcile these two theories. This is encapsulated in the "measurement problem": Suppose there is a quantum system which can be prepared in at least two distinguishable states $|0\rangle_S$ and $|1\rangle_S$, and some measurement apparatus in its "ready to measure" state $|R\rangle_A$. If the measurement is a minimally disturbing one, the coupling of the system with apparatus should yield a Schrödinger evolution of the form: $|0\rangle_S|R\rangle_A \to |0\rangle_S|\text{“0”}\rangle_A$ or $|1\rangle_S|R\rangle_A \to |1\rangle_S|\text{“1”}\rangle_A$, where $|\text{“0”}\rangle_A, |\text{“1”}\rangle_A$ are the apparatus states indicating results $0, 1$ respectively.

Now take the system prepared in a superposition $|\phi\rangle_S = a|0\rangle_S + b|1\rangle_S$ for $a, b \neq 0$. Under Schrödinger evolution $|\phi\rangle_S|R\rangle_A \to a|0\rangle_S|\text{“0”}\rangle_A + b|1\rangle_S|\text{“1”}\rangle_A$. This is does not result in a specific measurement reading eigenstate like the previous evolutions did. So generally we find instrument readings do not provide definite results - obviously in contrast with our experience. The conventional way to fix this is to allow for the collapse postulate: on measurement, the deterministic Schrödinger evolution is replaced by a random collapse to the observed state. However we find a succinct criticism to this in Dürr, Goldstein and Zanghi [12, p. 4]):

> *"But doing so comes at a price. One then has to accept that quantum theory involves special rules for what happens during a measurement, rules that are in addition to, and not derivable from, the quantum rules governing all other situations. One has to accept that the notions of measurement and observation play a fundamental role in the very formulation of quantum theory, in sharp conflict with the much more plausible view that what happens during measurement and observation in a quantum universe, like everything else that happens in such a universe, is a consequence of the laws governing the behaviour of the constituents of that universe..."*

Why should an interaction of several systems, which we choose to call a "measurement", make use of different laws to those governing all other interactions? This is the conceptual difficulty of the Copenhagen interpretation. In the past physicists have left such difficulties to foundationalists and philosophers, in favour of a "shut-up and calculate" attitude. This has worked well, but with so many current avenues of research in fields such as quantum gravity, and beyond the standard model physics, there seems to be no clear conceptual direction to take. Perhaps a "realist" version of Quantum theory that describes an external reality, with observers only taking a secondary role, would clarify the direction we ought to be taking in fundamental physics. For further more on this see Döring [8], and Butterfield and Isham [18].

2.2 Bohmian Mechanics

Hidden variable theories offer one possible remedy to the situation by asserting that the wavefunction does not provide a complete description, and specification of further variables yields a "dispersion-free" state. As the name suggests the observables of such states have no statical dispersion/variance, and so take well-defined values at all times. For a more precise formulation see [20, p. 295-299]. The probabilistic nature of quantum states would appear merely as an epistemic uncertainty about the values of these hidden variables.

Bohmian mechanics is the most successful example of a hidden variable theory. There are many texts on the subject, namely Bell [1], Dürr, Goldstein and Zanghi [12], the graduate course given in Cambridge by Mike Towler [22], Bricmont [5] and Dürr and Teufel [13].

The main ingredients of the theory are as follows:

- Take a quantum system of N point-like particles with masses $m_1, \ldots, m_N$ and positions $Q_i \in \mathbb{R}^3,\ i = 1, \ldots, N$. A generic set of positions provides a point in configuation space: $q = (q_1, \ldots, q_N) \in \mathbb{R}^{3n}$. Capitals denote the *actual* positions taken by the particles.

- A complex (or spinor)-valued wavefunction over configuration space $\psi = \psi(q_1, \ldots, q_N, t)$ satisfying the Schrödinger equation:
$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

- Each particle position evolves via the "guidance equation":
$$\frac{dQ_k}{dt} = \frac{\hbar}{m_k} \mathrm{Im}\left(\frac{\nabla_k \psi}{\psi}\right)(Q, t) \tag{1}$$

Note that the evolution of the k-th particle depends on the positions of all others, no matter what their relative displacement. Hence this theory is explicitly non-local. We will see in Section 3.2 that this is a necessary feature of hidden variable theories.

So where does the guidance equation come from? Following [12, p. 30-31], from $\dot{Q} = \underline{v}^{\psi}(Q)$ we can determine the form of $\underline{v}^{\psi}$ via a minimal set of symmetry principles. First note that ψ and $c\psi$ ($c \neq 0$) are physically equivalent so we require $\underline{v}^{c\psi} = \underline{v}^{\psi}$. Next, by imposing Galilean invariance and time-reversal invariance we arrive at (1).

An alternative form of (1) can be found by expressing $\psi = |\psi| \exp(iS/\hbar)$ for $S = S(Q_1, \ldots, Q_N, t)$. We find

$$\frac{dQ}{dt} = \frac{1}{m_k} \nabla_k S$$

showing that through the guidance equation, each particle follows a trajectory along the normal to the surface of constant phase.

How does this resolve the measurement problem? The problem essentially arises out of the assumption that the wavefunction provides a complete description of the state of the system, whilst not providing any satisfactory means by which a superposition can collapse to a measured eigenstate. If one denies this assumption and asserts that a more complete description is possible - one that provides definite results on measurement - as Bohmian Mechanics manages to provide, then there is no problem.

However if we take an "effective wavefunction" describing just the system without the apparatus, we can retrieve the usual quantum behaviour (see [12, Sec 2.5]). When the two systems are decoupled from one another we find the effective wavefunction evolves via the Schrödinger equation, and by means of the *quantum equilibrium hypothesis* ($\rho = |\phi|^2$, [12, Sec 2.4]), we observe an random collapse when the systems interact.

Bohmian Mechanics has been extended to various Pilot Wave QFT's (see [21] for an accessible overview), although this is still a rather underdeveloped area. The main issue being how to interpret the inherent non-locality, a feature completely at odds with relativity. There have been recent developments in resolving this via spacetime foliations - more can be found at [11, 23].

3 The No-go Theorems

Having seen an example of a hidden variable theory, we will now look to see how ubiquitous such theories are. Through two powerful theorems we will in fact see hidden variable theories are placed under strict constraints in order to reproduce the predictions of QM.

3.1 Kochen-Specker Theorem

The Kochen-Specker (KS) Theorem has been expressed in many different forms since its original appearance in [20]. It was work by von Neumann [26], and later Jauch and Piron [19], claiming to have proved the non-existence of hidden variables that initiated the discussion. Some time later Kochen and Specker, and independently Bell [3], actually strengthened von Neumann's result. Yet this did not spell the end of hidden variables as, despite this strengthening, Bell points out that there is a loop hole: allowing hidden variables to be *contextual*. We will first cover von Neumann's theorem and discuss why Bell deemed his assumptions unreasonable. Next, we cover Bell's proof of the KS theorem since it is significantly simpler than Kochen and Specker's version. Finally we will give an account of noncontextuality and the options one has to escape the Kochen-Specker theorem.

3.1.1 Von Neumann's "Proof"

Von Neumann's theorem tells us that dispersion-free states, under certain assumptions, are impossible. It will be helpful to have an example in mind, given by Bell [3, pg. 448], in which to judge how reasonable his assumptions are.

Example Take a spin-1/2 particle given by a 2-component state vector, or spinor, ψ. Observables are given by the 2×2 -matrices $\alpha I_2 + \boldsymbol{\beta} \cdot \underline{\sigma}$, with $\alpha, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$. Their respective eigenvalues are $\alpha \pm |\boldsymbol{\beta}|$ and have expectations

$$\langle \alpha I_2 + \boldsymbol{\beta} \cdot \underline{\sigma} \rangle = \psi^T (\alpha I_2 + \boldsymbol{\beta} \cdot \underline{\sigma}) \psi$$

We attach a hidden variable scheme by defining a dispersion-free state as a pair (ψ, λ) for spinor ψ and $\lambda \in [-1/2, 1/2]$. Note we can choose a coordinate system such that $\psi = (1, 0)$, and that with respect to this system $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$.

We state that measurement of $\alpha I_2 + \boldsymbol{\beta} \cdot \underline{\sigma}$ on dispersion-free state (ψ, λ) will be the eigenvalue

$$\alpha + |\boldsymbol{\beta}| \text{sign}(\lambda |\boldsymbol{\beta}| + 1/2 |\beta_z|) \text{sign} X$$

where $X = \begin{cases} \beta_z & \text{if } \beta_z \neq 0 \\ \beta_x & \text{if } \beta_z = 0,\ \beta_x \neq 0 \\ \beta_y & \text{if } \beta_z = 0, \text{ and } \beta_x = 0 \end{cases}$ and $\text{sign}(X) = \begin{cases} +1 & \text{if } X \geq 0 \\ -1 & \text{if } X < 0 \end{cases}$

The quantum state ψ is obtained by averaging over λ. We can check this is consistent by checking that the expectation of quantum states obtained by averaging over hidden variables yields the same result as the usual expectation formula. Since the "expectation" of a dispersion-free state is just the associated eigenvalue, the expectation of a quantum state, such as $\psi = (1,0)$, can be found by averaging over the eigenvalues:

$$\begin{aligned}\langle \alpha I_2 + \boldsymbol{\beta} \cdot \underline{\sigma} \rangle &= \int_{-1/2}^{1/2} d\lambda \{\alpha + |\boldsymbol{\beta}| \text{sign}(\lambda |\boldsymbol{\beta}| + 1/2 |\beta_z|) \text{sign} X\} \\ &= \alpha + \beta_z\end{aligned}$$

which is consistent with the usual expectation formula given above, remembering $\psi = (1,0)$.

Now we give von Neumann's proof of the non-existence of hidden variables, following lectures by Butterfield [6]. Von Neumann proposes that states ϕ assign expectations to each observable. He assumes observables are represented by self-adjoint operators over an arbitrary Hilbert space, and that the states are normalised, i.e. $\phi(0) = 0$, $\phi(I) = 1$. Finally he assumes that ϕ is linear (or that expectations of linear combinations are themselves linear), i.e. they obey: $\phi(\alpha A + \beta B) = \alpha\phi(A) + \beta\phi(B)$. He then proves that the state ϕ can be identified with a density matrix ρ_ϕ, such that $\phi(A) = \text{Tr}[A \cdot \rho_\phi]$. Since density matrices cannot be dispersion-free, we deduce dispersion-free states cannot exist.

We see how this argument works more clearly in the example above. From von Neumann's linearity assumption we deduce the expectation of $\alpha I_2 + \boldsymbol{\beta} \cdot \underline{\sigma}$ for state (ψ, λ) should be linear in α and $\boldsymbol{\beta}$. But the expectation value of dispersion-free states are just the eigenvalues $\alpha \pm |\boldsymbol{\beta}|$, which is far from linear in $\boldsymbol{\beta}$. Hence there do not exist dispersion-free states.

Bell's response was to point out that the measurement of the sum of non-commuting observables cannot be obtained by summing the measurements of each observable individually. That the expectation values, even of non-commuting observables, are additive, is a very non-trivial property of Quantum Mechanics. There is no reason to assume such a non-trivial property on dispersion-free states when their only real obligation is to reproduce the predictions of quantum mechanics when they are averaged over. Note however that the expectations of the dispersion-free states should be additive for commuting operators.

3.1.2 Bell's Proof

Taking inspiration from Gleason's work, Bell followed his analysis of von Neumann with his own version of the Kochen-Specker Theorem:

Theorem For systems with Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) > 2$, non-contextual dispersion-free states cannot satisfy the condition that expectations of commuting operators are additive. (We will discuss where the non-contextuality condition comes from in the next section).

Proof For $\Phi \in \mathcal{H}$, let $P(\Phi)$ denote the projection operator onto Φ i.e.

$$P(\Phi)\psi = (\Phi, \Phi)^{-1}(\Phi, \psi)\Phi$$

If the Φ_i form a complete and orthogonal set, then $\sum_i P(\Phi_i) = 1$. Hence by the hypothesis that commuting operators have additive expectations:

$$\sum_i \langle P(\Phi_i) \rangle = 1 \tag{2}$$

Since expectation values of projectors are non-negative and any two orthogonal vectors can be regarded as members of a complete set, we get:

(**A**) If $\Phi \in \mathcal{H}$ is such that $\langle P(\Phi) \rangle = 1$ for some state, then for that state $\langle P(\psi) \rangle = 0$ for any ψ orthogonal to Φ.

Secondly, if ψ_1 and ψ_2 are an alternative orthogonal basis for the subspace spanned by Φ_1, Φ_2, then by (2), we find $\langle P(\psi_1) \rangle + \langle P(\psi_2) \rangle = \langle P(\Phi_1) \rangle + \langle P(\Phi_2) \rangle$. Since ψ_1 can be an arbitrary linear combination of Φ_1, Φ_2 and using the fact that expectations of projections are non-negative, we deduce:

(**B**) If for some state $\langle P(\Phi_1) \rangle = \langle P(\Phi_2) \rangle = 0$ for orthogonal Φ_1, Φ_2, then $\langle P(\alpha\Phi_1 + \beta\Phi_2) \rangle = 0$ for all $\forall \alpha, \beta \in \mathbb{R}$.

Now, for a given state, let $\psi, \Phi \in \mathcal{H}$ be such that

$$\langle P(\psi) \rangle = 1 \tag{3}$$

$$\langle P(\Phi) \rangle = 0 \tag{4}$$

Using what we have shown so far, we will prove that ψ and Φ cannot be arbitrarily close and hence that dispersion-free states cannot exist.

Normalising ψ (and adjusting the length of Φ correspondingly), we can express Φ as $\Phi = \psi + \epsilon\psi'$ for $\epsilon \in \mathbb{R}$ and ψ' normalized and orthogonal to ψ. Define ψ'' as any unit vector orthogonal to both ψ, ψ' (this is where we require $\dim(\mathcal{H}) > 2$). By (**A**) and 3: $\langle P(\psi') \rangle = \langle P(\psi'') \rangle = 0$. So by (**B**) and (4), $\langle P(\Phi + \gamma^{-1}\epsilon\psi'') \rangle = 0 \quad \forall \gamma \in \mathbb{R}$ and also $\langle P(-\epsilon\psi' + \gamma\epsilon\psi'') \rangle = 0$. Since the arguments in the previous equations are orthogonal we can sum them to find: $\langle P(\psi + \epsilon(\gamma + \gamma^{-1})\psi'') \rangle = 0$.

Note that if $\epsilon < 1/2$, then there exists a γ such that $\epsilon(\gamma + \gamma^{-1}) = \pm 1$. This implies

$$\langle P(\psi + \psi'') \rangle = \langle P(\psi - \psi'') \rangle = 0$$

Knowing $\psi \pm \psi''$ are orthogonal we can apply (**B**), giving us $\langle P(\psi)\rangle = 0$, which contradicts our assumption (3). Hence we must have $\epsilon > 1/2$. We have shown ψ and Φ cannot be arbitrarily close, or more specifically $|\Phi - \psi| > 1/2$ (for ψ normalised).

If dispersion-free states were to exist, then the "expectation" of projectors for these states would be either 0 or 1. Since both values must occur by (2), there must be arbitrarily close pairs ψ, Φ with differing expectation values. However we have shown they cannot be arbitrarily close - so such dispersion-free states cannot exist. □

3.1.3 Noncontextuality

Bell criticizes his proof on similar lines to his criticism of von Neumann's theorem. In particular: his assumptions, this time implicit, of dispersion-free states were unjustified.

Note that $P(\alpha\Phi_1 + \beta\Phi_2)$ does not commute with $P(\Phi_1)$ or $P(\Phi_2)$ if $\alpha, \beta \neq 0$, so condition (**B**) in the above proof non-trivially relates the results of measurements that cannot be performed simultaneously. Despite the assumption of the theorem explicitly constraining the expectations of states only for commuting operators, Bell states that (**B**) arises out of an implicit assumption that "...measurement of an observable must yield the same value independently of what other measurements may be made simultaneously." [3, pg. 451]. As before, dispersion-free states need not have this property - only their quantum mechanical averages do.

We can make this statement more explicit with the following definition from Butterfield and Isham [17]:

Definition A *global valuation* is a real-valued function V on the set of all bounded, self-adjoint operators such that:

1. $V(\hat{A}) \in \mathrm{Spec}(\hat{A})$
2. The Functional composition principle (FUNC) holds: $V(h(\hat{A})) = h(V(\hat{A}))$ for self-adjoint operators $\hat{A}, h(\hat{A})$ for some $h : \mathbb{R} \to \mathbb{R}$ (for a definition of functions of operators see Sec 4.2).

FUNC is the statement that *"...the algebraic structure of the operators should be mirrored in the algebraic structure of the possessed values of the observables"* Redhead [25, pg. 121]. If such valuations existed, each would define a dispersion-free state: meaning they would allow for the set of self-adjoint operators to be embedded in the commutative ring of real-valued functions defined over some classical state space - forming a hidden variable theory.

Now suppose you are given $\hat{A}, \hat{B}$ such that $[\hat{A}, \hat{B}] \neq 0$, with functions g, h such that an operator $\hat{C}$ can be constructed as $\hat{C} = g(\hat{A}) = h(\hat{B})$. FUNC tells us the value assigned to $\hat{C}$ by V is independent of whether we choose to measure $\hat{A}$ or

$\hat{B}$, despite the fact they require different experimental apparatus in order to be measured. This is the implicit assumption, which we term *"non-contextuality"*, that leads to (**B**), and by the Kochen-Specker theorem, to the non-existence of dispersion-free states. Since global valuations would provide a hidden variable theory of dispersion-free states, we see that the Kochen-Specker theorem asserts that there are no such global valuations.

However, returning to Bell's analysis, this non-contextuality (the independence of measuring $\hat{C}$ via $\hat{A}$ or $\hat{B}$) implied by FUNC is not a necessary requirement of dispersion-free states, so it unreasonable to use it as a condition to disprove their existence. We will see in Sec 4 that these ideas lead to a globally defined valuation with a new form of FUNC - that takes into account the context in which observables are measured.

Note:

- It can be shown that FUNC implies the sum and product rules of valuations: if $\hat{A}, \hat{B}$ commute, then $V(\hat{A}+\hat{B}) = V(\hat{A})+V(\hat{B})$ and $V(\hat{A}\hat{B}) = V(\hat{A})V(\hat{B})$. It is these, formalised into a structure called a "partial Boolean algebra", that Kochen and Specker use in their proof of the theorem.

- In Redhead ([25, pg. 120]) it is shown that property (1) above can be seen as a result of the "Value rule": If $\mathrm{Prob}(\lambda)_Q^{|\phi\rangle} = 0$ (i.e. if the probability that the value of Q equals λ, in state $|\phi\rangle$, is 0), then $V(\hat{Q})^{|\phi\rangle} \neq \lambda$. However this rule doesn't make sense for operators with continuous spectra since it implies the operator has no well-defined values. So we restrict ourselves to the discrete case.

3.1.4 Escaping Kochen-Specker

This section follows the analysis of Held [15]. For more details also see Redhead [25, Sec 5.2]. It can be shown that FUNC is derivable from four principles. Denying any of these prevents FUNC, and hence the Kochen-Specker theorem, from holding. We will give these principles and briefly consider the case for denying each one.

1. STAT FUNC: Given $\hat{A}, h(\hat{A})$ as in FUNC, then

$$\mathrm{Prob}[V(h(\hat{A})) = b] = \mathrm{Prob}[h(V(\hat{A})) = b]$$

   This is a consequence of orthodox Quantum mechanics, so we will not consider denying this!

2. Value definiteness: All the observables of a quantum system have definite values at all times.

Despite the ambition of hidden variables to provide value definiteness, one can instead compromise. By choosing a specific set of observables to take definite values—in such a way that these observables avoid the Kochen-Specker theorem by denying either of points (3) or (4) below—we get *partial value definiteness*. Bohmian Mechanics escapes the KS theorem this way; position (and any functions of position) is given definite, but contextual (see (3) below), values.

3. Noncontextuality: The value of an observable is independent of any measurement context.

   Denial of noncontextuality, or contextuality, can happen is several different ways. Firstly there is causal contextuality, whereby the value measured is the result of the system and apparatus (i.e context) interacting. Alternatively, there is ontological contextuality that requires the specification of the context for an observable to be well-defined. More detail on this can be found at [15, Sec 5.3].

4. Value realism: If there is an operationally defined real number α associated with a self-adjoint operator $\hat{A}$ (i.e. α is built from some other number that represents a physical property), and if for a given state the statistical algorithm of QM for $\hat{A}$ yields a real number β s.t. $\beta = \text{Prob}[V(\hat{A}) = \alpha]$, then there exists an observable with value α.

   This is a subtle point, but it can in fact be shown that denying this amounts to a form of contextuality [15, Sec 5.2]. So we find that denying points (3), and optionally (2), is enough to escape the Kochen-Specker theorem and allow us to construct hidden variable theories.

3.2 Bell's Theorem

Bell's Theorem [2] is the most famous of the no-go theorems - it states that *local* hidden variable theories are inconsistent with the predictions of Quantum Mechanics. This shows the non-locality of Bohmian Mechanics is not just an odd feature of the theory, but a necessary one.

We start with a pair of spin-1/2 particles prepared in a singlet spin state moving freely in opposite directions. Via Stern-Gerlach experiments one can measure the spins of each particle. Quantum mechanics states that if the spin along $\underline{a} \in \mathbb{R}^3 : |\underline{a}| = 1$, found by measuring $\underline{\sigma}_1 \cdot \underline{a}$, is 1 for particle 1, then measuring $\underline{\sigma}_2 \cdot \underline{a}$ for the second particle will always give -1. To explain this behaviour a natural supposition is that the results of the measurements are predetermined. Since wavefunctions alone cannot offer more than statistical predictions, we deduce the measurement outcomes are given by specifying extra variables, collectively denoted λ. This is essentially the EPR argument for the incompleteness of quantum theory [14], given in terms of spin by Bohm and Aharonov [4]. Next we state argument of Bells Theorem.

Let A, B denote the measurement outcomes for each particle respectively, so $A(\underline{a}, \lambda) = \pm 1$ and $B(\underline{b}, \lambda) = \pm 1$ for $|\underline{a}| = |\underline{b}| = 1$. The rest of the analysis rests of the following locality assumption: B is independent of $\underline{a}$, and A is independent of $\underline{b}$. Note that the hidden variable λ is independent of both $\underline{a}, \underline{b}$ i.e. cannot be influenced non-locally by either measurement.

If $p(\lambda)$ is the probability distribution over λ, then the expectation of the product of both measurements is

$$P(\underline{a}, \underline{b}) = \int d\lambda\, p(\lambda) A(\underline{a}, \lambda) B(\underline{b}, \lambda) \tag{5}$$

If this local hidden variable picture is consistent with quantum mechanics this expression should coincide with the quantum expectation:

$$\langle \underline{\sigma}_1 \cdot \underline{a}\ \underline{\sigma}_2 \cdot \underline{b} \rangle = -\underline{a} \cdot \underline{b} \tag{6}$$

The anti-correlation of measurements implies $A(\underline{a}, \lambda) = -B(\underline{a}, \lambda)$, hence (5) can be re-expressed as:

$$P(\underline{a}, \underline{b}) = -\int d\lambda\, p(\lambda) A(\underline{a}, \lambda) A(\underline{b}, \lambda)$$

For another $\underline{c} \in \mathbb{R}^3 : |\underline{c}| = 1$, we get:

$$\begin{aligned} P(\underline{a}, \underline{b}) - P(\underline{a}, \underline{c}) &= -\int d\lambda\, p(\lambda)[A(\underline{a}, \lambda) A(\underline{b}, \lambda) - A(\underline{a}, \lambda) A(\underline{c}, \lambda) \\ &= \int d\lambda\, p(\lambda) A(\underline{a}, \lambda) A(\underline{b}, \lambda)[A(\underline{b}, \lambda) A(\underline{c}, \lambda) - 1] \end{aligned}$$

resulting in

$$|P(\underline{a}, \underline{b}) - P(\underline{a}, \underline{c})| \leq \int d\lambda\, p(\lambda)[1 - A(\underline{b}, \lambda) A(\underline{c}, \lambda)] = 1 + P(\underline{b}, \underline{c})$$

This final inequality is Bell's infamous inequality. Note the left hand side is in general of order $|\underline{b} - \underline{c}|$ for small $|\underline{b} - \underline{c}|$. Hence the left hand side decreases linearly near its minimum at $\underline{b} = \underline{c}$. Using (6), the right hand side - predicted by quantum mechanics - would be $1 - \cos(\theta)$ for θ the angle between $\underline{a}, \underline{b}$. As θ approaches zero, the rate of change slows, resulting in a stationary point at its minimum - in contradiction with the right hand side. Hence there is no way for a local hidden variable interpretation to reproduce the behaviour of quantum mechanics. $\square$

Bell then goes further to show that (6) cannot be approximated arbitrarily well by anything of the form (5), removing any possibility for a local hidden variable interpretation.

4 Topos Theory

We saw in Sec 3.1.3 that Kochen-Specker proved the non-existence of certain global valuations. In Sections 4.1 and 4.2 we will show how the Kochen-Specker Theorem can be rephrased within the framework of Topos Theory. Next, in Sec 4.3, we describe how this new language provides contextual generalised truth-values as means for avoiding the Kochen-Specker theorem. The following analysis is based on Butterfield and Isham [17], [18].

4.1 Definitions

In order to reconstruct the Kochen-Specker Theorem we need to cover the necessary background material from Topos theory. We begin by formulating the notion of a presheaf over a poset, which will then motivate our definition of a presheaf over a general category. The basic definitions of category theory and posets can be found in 6.1.

Definition A *presheaf* X over poset C is a function that assigns a set X_p to each $p \in C$, and a map $X_{qp} : X_q \to X_p$ to each pair $p \leq q$, such that (i) $X_{pp} = \mathrm{id}_{X_p}$ and (ii) if $p \leq q \leq r$ then $X_{rp} = X_{qp} \circ X_{rq}$. In other words, a presheaf is a contravariant functor X from the category C to the category 'Set'. Or alternatively, a covariant functor from C^{op} to Set.

We can turn the set of all presheaves over poset C into the category $\mathrm{Set}^{C^{op}}$ by defining a morphism $\eta : X \to Y$ between presheaves X, Y on C as a family of natural transformations $n_p : X_p \to Y_p$ for each $p \in C$ such that for $p \leq q$: $\eta_p \circ X_{qp} = Y_{qp} \circ \eta_q$.

Definition A *subobject* of presheaf X, in the category of presheaves over a poset, is a presheaf K with morphism $i : K \to X$ such that (i) $K_p \subset X_p \ \ \forall p \in C$,
(ii) K_{qp} is the restriction of X_{qp} to $K_q \ \ \forall p \leq q$.

These definitions naturally lead onto the more general case:

Definition A *presheaf* on a small category C is a covariant functor $X : C^{\mathrm{op}} \to \mathrm{Set}$.

Again the set of such presheaves forms a category, $\mathrm{Set}^{C^{\mathrm{op}}}$, with morphisms given by natural transformations $N : X \to Y$ satisfying: for each $A \in C$, $N_A : X(A) \to Y(A)$ is such that if $f : B \to A$ in C, then $X(A) \overset{N_A}{\to} Y(A) \overset{Y(f)}{\to} Y(B)$ is equal to $X(A) \overset{X(f)}{\to} X(B) \overset{N_B}{\to} Y(B)$. Note: $\mathrm{Set}^{C^{\mathrm{op}}}$ is an example of a *Topos*.

Definition An object K is a *subobject* of X if there exists a morphism $i : K \to X$ such that $i_A : K(A) \to X(A)$ is a subset embedding $\forall A \in C$. It is helpful to think of subobjects in a general category as a generalisation of subsets in the category 'Set'.

Definition A *terminal* object in the category of presheaves is $1 : C \to \text{Set}$, where $1(A) = \{*\}\ \forall A \in C$ and if $f : B \to A$ in C, then $1(f) : \{*\} \to \{*\}$.

Definition A *global section* is a global element of a presheaf X, namely a morphism $\gamma : 1 \to X$ corresponding to a choice of $\gamma_A \in X(A)$ for each $A \in C$ such that for $f : B \to A$, $X(f)(\gamma_A) = \gamma_B$.

4.2 The Kochen-Specker theorem

With the basic definitions given, we can now go about reformulating Kochen-Specker.

Let O denote the set of all bounded, self-adjoint operators on $\mathcal{H}$ (the Hilbert space of the quantum system). Note: $\hat{A} \in O$ has a 'Spectral representation': $\hat{A} = \int_{\sigma(\hat{A})} \lambda d\hat{E}^A_\lambda$, the integral of the spectrum $\sigma(\hat{A}) \subset \mathbb{R}$ over a projection-valued measure. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, then the operator $f(\hat{A})$ is defined as $f(\hat{A}) = \int_{\sigma(\hat{A})} f(\lambda) d\hat{E}^A_\lambda$.

O forms a category, with a morphism $f_O : \hat{B} \to \hat{A}$ defined if there exists a Borel function $f : \sigma(\hat{A}) \to \mathbb{R}$ such that $\hat{B} = f(\hat{A})$. Let O_d denote the corresponding category of bounded, self-adjoint operators with discrete spectrum.

Definition The *spectral presheaf* on O_d is the contravariant functor $\Sigma : O_d \to \text{Set}$ defined as (i) $\Sigma(\hat{A}) = \sigma(\hat{A})$ the spectrum of $\hat{A}$ (ii) if $f_{O_d} : \hat{B} \to \hat{A}$ (i.e. $\hat{B} = f(\hat{A})$), then $\Sigma(f_{O_d}) : \sigma(\hat{A}) \to \sigma(\hat{B})$ is such that $\Sigma(f_{O_d})(\lambda) = f(\lambda),\ \forall \lambda \in \sigma(\hat{A})$. [This is well-defined since for operators with discrete spectra we have the following: $\sigma(f(\hat{A})) = f(\sigma(\hat{A}))$.]

A global section of the spectral presheaf $\Sigma : O_d^{\text{op}} \to \text{Set}$ is a function γ that assigns to each $\hat{A}$ a real number $\gamma_A \in \sigma(\hat{A})$, and if $\hat{B} = f(\hat{A})$ then $f(\gamma_A) = \gamma_B$. This is precisely the FUNC condition for a valuation, as defined in 3.1.3. Hence the **Kochen-Specker Theorem** can equally be expressed as: there do not exist global sections of the spectral presheaf.

4.3 Generalised Truth

Definition A *sieve* on a object A in category C is a collection of morphisms S such that if $f : A \to B$ is in S, and $g : C \to B$ is a morphism, then $f \circ g : C \to A$ is in S. (When the category is a poset, the sieve is a lower set of the element)

It is important to note that the set of all sieves over A is a Heyting algebra. The unit element is given by the *principle* sieve of A, i.e. the collection of all morphisms with domain A. The null element is given by the empty sieve and partial ordering by subset inclusion.

One of the defining properties for a category to be a topos is the existence of a object $\Omega \in \mathrm{ob}(C)$ called the *subobject classifier*. The subobject classifier provides a 1-1 correspondence between the subobjects of an object X, and the morphisms $\chi : X \to \Omega$. This is a generalisation of the fact that in the category 'Set', subsets $S \subset T$ are in 1-1 correspondence with characteristic functions $\chi^S : T \to \{0, 1\}$ with $\chi^S(x) = 1$ if $x \in S$; and $\chi^S(x) = 0$ otherwise. It can be proven that Ω is unique up to isomorphism.

Definition The *subobject classifier* of the topos $\mathrm{Set}^{C^{\mathrm{op}}}$ is the presheaf $\Omega : C \to \mathrm{Set}$ such that (i) $\Omega(A)$ is defined as the set of all sieves over A, (ii) For $f : B \to A$, $\Omega(f) : \Omega(A) \to \Omega(B)$ satisfies $\Omega(f)(S) = \{h : C \to B \,|\, f \circ h \in S\}$, which is equal to the "pull-back sieve of B from A by f".

If K is a subobject of X, the associated morphism, under the aforementioned correspondence, is $\chi^K : X \to \Omega$ such that $\chi^K_A : X(A) \to \Omega(A)$ is defined as:

$$\chi^K_A(x) = \{f : B \to A \,|\, X(f)(x) \in K(B)\} \quad \forall x \in X(A)$$

Each "stage of truth" $A \in \mathrm{ob}(C)$ provides a *context* with which to assign each $x \in X(A)$ a sieve, interpreted as a generalised truth-value in a Heyting algebra.

By means of assigning contextual and generalised truth-values to propositions, we can construct a *generalised* valuation, obeying a *generalised* functional composition principle, or FUNC, appropriate to presheaves. This is the essence for how this formalism escapes the Kochen-Specker theorem. For more detail, see [18, Sec 3.7] and [17, Sec 4].

It is worth noting that some fascinating recent developments have been attempting to address the conceptual issues eluded to at the bottom of 2.1, namely realism over instrumentalism, by harnessing the topos formalism as a new method for constructing physical theories. For more on this see Döring and Isham [10] and [9].

5 Conclusions

As we have seen, the papers that led to the Kochen-Specker theorem - and papers that followed - span the period of time from the early days of quantum theory right to the present day. From the original debates surrounding the incompleteness of quantum mechanics, to the no-go theorems that resulted in the heavy-handed dismissal of hidden-variables; right up to the present day, with efforts to realise these notions in the abstract framework of Topos theory. Still, many of the basic philosophical questions, such as the status of hidden-variables; how to resolve the measurement problem; realism vs instrumentalism; remain just as relevant today as when they were first asked. While we may not have any clear answers yet, we have gained many important insights into the underlying structure of quantum theory as a result. There is no telling, given the current climate in fundamental physics, whether these interests will remain the concern of a minority few, or become critical to mainstream efforts, but either way, the developments described above tell a fascinating story.

Before closing, I wish to present some speculative thoughts on a principle that seeks to unify the quantum and deterministic approaches to physical theories. We start by thinking of the *information* held by the state of a system in some theory as the *capacity* to make predictions about the properties of that state. The basic idea is that by weakening the notion of information held by a state, one might reasonably expect extra symmetries of the theory to appear. Or conversely, by enforcing new symmetries not present in the previous theory, we require a corresponding 'coarsening' of the predictions that can be made by states. We might be able to describe such a phenomena by means of seeking how symmetries present themselves in the topos framework, and observing how the topos, and corresponding theory structure, changes as the symmetries change.

For instance, take a physical theory that is invariant under some symmetry group of space-time transformations. We might start from a deterministic theory where states contain all the information about their future evolution - like Bohmian Mechanics, with its Galilean symmetry over space and time. The principle states that by enforcing additional space-time symmetries, such those encapsulated in the Poincaré group, necessitates a corresponding decrease in the perceived information held by a state. In other words, we require something like a quantum interpretation of states that can only offer statistical predictions about their future evolution. The resulting theory might be Quantum Field Theory. We see this not as a *new* theory, but an information-theoretic approximation. Relativity was not fundamental, but a tool for constructing a simpler picture with which to view the deterministic theory we started with. The principle "explains" why Quantum Mechanics can be merged with Relativity - the symmetry on the one hand is compatible with the information-structure on the other. This matches very closely the conclusions of recent work [16] that found, by means of information theory, that the structure of quantum theory actually dictates the corresponding structure of spacetime, rather than the usual coexistence we take for granted.

Naturally we ask how this could be applied to General Relativity, by enforcing general covariance, or diffeomorphism invariance. This is a subtler step than before, so would require an equally subtle shift in the notion of information carried by states. Whatever this new quantum framework would be, it would be fundamentally different to that of Quantum Mechanics. This would mean any attempt to find a quantum theory of gravity based on the structure of Quantum Mechanics simply could not succeed. This would agree with the suspicions stated in [8].

Clearly, such speculations are a long way off any sort of rigorous, mathematical realisation. However, if such a principle could be made rigorous, it may have interesting implications for guiding how we think about quantum gravity, and fundamental physics in general.

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6 Appendix

6.1 Categories and Posets

This section will cover some of the preliminary definitions and results required to read Section 4.

Definition A (small) *Category* $\mathfrak{C}$ consists of

- a collection of *objects* $\mathrm{ob}(\mathfrak{C})$;
- for each $A, B \in \mathrm{ob}(\mathfrak{C})$, a collection of arrows, or morphisms, from A to B (which may be the empty collection);
- an associative composition operation for morphisms such that if $f : A \to B,\ g : B \to C$, then $g \circ f : A \to C$ (associativity $\implies$ $(h \circ g) \circ f = h \circ (g \circ f)$);
- for each $A \in \mathrm{ob}(C)$ an *identity* morphism $1_A : A \to A$ satisfying for each $f : A \to B$: $f \circ 1_A = f = 1_B \circ f$.

The category being *small* indicates the collections of all objects, and morphisms, are genuine sets. We can also define the *opposite category* $\mathfrak{C}^{\mathrm{op}}$, with the same objects as those in $\mathfrak{C}$ but with arrows reversed, so $f : B \to A$ in $\mathfrak{C}$ corresponds to $f : A \to B$ in $\mathfrak{C}^{\mathrm{op}}$.

Next we define functors - the functions between categories that preserve morphisms.

Definition A *covariant functor* $\mathbf{F}$ from $\mathfrak{C}$ to $\mathfrak{D}$ is a function that assigns

- to each $\mathfrak{C}$-object A, a $\mathfrak{D}$-object $\mathbf{F}(A)$;
- to each $\mathfrak{C}$-morphism $f : B \to A$, a $\mathfrak{D}$-morphism $\mathbf{F}(f) : \mathbf{F}(B) \to \mathbf{F}(A)$;
- and such that $\mathbf{F}$ preserves identities and if $g : C \to B,\ f : B \to A$, then $\mathbf{F}(f \circ g) = \mathbf{F}(f) \circ \mathbf{F}(g)$.

Definition A *contravariant functor* $\mathbf{X}$ from $\mathfrak{C}$ to $\mathfrak{D}$ is a function that assigns

- to each $\mathfrak{C}$-object A, a $\mathfrak{D}$-object $\mathbf{X}(A)$;
- to each $\mathfrak{C}$-morphism $f : B \to A$, a $\mathfrak{D}$-morphism $\mathbf{X}(f) : \mathbf{X}(A) \to \mathbf{X}(B)$;
- and such that $\mathbf{X}$ preserves identities and if $g : C \to B,\ f : B \to A$, then $\mathbf{X}(f \circ g) = \mathbf{X}(g) \circ \mathbf{X}(f)$.

Definition A *terminal object* of a category is an object 1 such that for all objects X there is a unique morphism $X \to 1$. (In the category of sets, a terminal object is any singleton set $\{*\}$).

Definition A *global element* of an object X is any morphism $1 \to X$.

Definition A *partially-ordered set*, or poset, is a set with a partial order i.e. a binary relation $\leq$ over pairs of elements (not necessarily all) with the following properties: reflexive, anti-symmetric (i.e. $a \leq b$, $b \leq a \implies a = b$) and transitive. [Note: A poset forms a category whose objects are elements of the poset, and whose morphisms $i_{pq} : p \to q$ exist iff $p \leq q$.]

Definition A *Boolean algebra* is complemented distributive lattice (with null and unit elements). It is the algebraic model for propositional classical logic.

Definition A *Heyting algebra* is a relatively complemented distributive lattice (with null and unit elements). *Relatively complemented* is equivalent to the condition: $\forall S_1, S_2 \in \Omega$ there exists an element $S_1 \Rightarrow S_2 \in \Omega$ such that $\forall S \in \Omega$ we have

$$S \leq (S_1 \Rightarrow S_2) \iff S \wedge S_1 \leq S_2$$

A Heyting algebra is the algebraic model for propositional intuitionistic logic, which is similar to classical logic but without the law of excluded middle or double negation elimination.

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