Quantum Groups and Drinfeld Doubles

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Abstract. This is the Semester 2 MSc Project on Hopf Algebras, supervised by Dr Yuri Bazlov. The question of concern is whether all finite-dimensional semisimple Hopf algebras are isomorphic, as algebras, to a group algebra. Note that over a field of characteristic 0, group algebras of finite groups are semisimple, and so cannot be isomorphic to any Hopf algebra that is not also semisimple. In this essay we restrict ourselves to the case that the Hopf algebra is the Drinfeld (quantum) double $D(G)$ of a finite group G, taking in particular the case $G = S_3$.

Contents

1 Introduction

We begin with an introduction to the theory of Hopf algebras, with sections on the basic definitions and examples of the subject, followed by a discussion of the dual Hopf algebras. With this background, in Section 2.4 we are able to define the main object of the essay, the Drinfeld double (also known as the quantum double) of a group algebra. We briefly try to motivate this object by discussing its quasitriangular structure, before giving the conditions under which it is semisimple as an algebra. Finally in Section 2.4.1 we answer the essay question for the special case of the Drinfeld double of an abelian group.

In Section 3 we build the Representation theory required in the final parts of the essay. In particular we discuss a key result for tackling the essay question, the Artin-Wedderburn theorem, and how it can be used to determine an isomorphism of the Drinfeld double with a group algebra. In Section 3.2 we introduce induced representations, which are a core tool used in classifying the simple $D(G)$-modules in Section 4. Finally in Section 3.2 we look at the similarities between the representation theory of groups and Hopf algebras, one of the important motivations for studying Hopf algebras.

In Section 4 we discuss the classification of the simple $D(G)$-modules, following Witherspoon [16]. The dimensions of these objects make the essential data we require to determine if there exists a group H such that $D(G) \cong kH$ as algebras. In Section 5 we compute these simple modules explicitly for the case $G = S_3$, and finish in Section 5.2 by answering the essay question for $D(S_3)$.

A central object throughout this essay is the tensor product, so we have included a short introduction to this in Appendix 6.1. Note also that many of the ideas in this essay benefit from being framed within the context of Category theory, so we have included in Appendix 6.2 some basic definitions and results on this.

2 Hopf Algebras

2.1 Definitions

We start by introducing Hopf algebras, following closely the texts of Majid [9, 10] and notes by Chua [3]. Note throughout this essay k will denote a field of characteristic 0, and tensor products will be taken over the field k unless indicated otherwise. For a brief introduction to tensor products see Appendix 6.1.

Definition 2.1. A **unital algebra** A over field k, is a vector space $(A, +; k)$ with an associative, bilinear product $\cdot : A \times A \to A$ with unit 1_A such that $a \cdot 1_A = a = 1_A \cdot a$. By using tensor products we can give the product and unit in the following equivalent forms:

- the **product** is a linear map $m : A \otimes A \to A$. Associativity is given by:

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$$

as maps $A \otimes A \otimes A \to A$, where $\mathrm{id} : A \to A$ is the identity linear map. This can be expressed in the following commutative diagram:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

- the **unit** is the linear map $\eta : k \to A,\ \lambda \mapsto \lambda 1_A$ where the axiom making 1_A a unit is equivalent to η satisfying the commutative diagram:

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"] & A \otimes A \arrow[d, "m"] & A \otimes k \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

In other words $m \circ (\eta \otimes \mathrm{id}) : k \otimes A \to A$ is equivalent to the natural isomorphism: $k \otimes A \to A$ where $\lambda \otimes a \mapsto \lambda a$. So

$$(m \circ (\eta \otimes \mathrm{id}))(\lambda \otimes a) = m(\eta(\lambda) \otimes a) = \eta(\lambda) \cdot a = \lambda 1_A \cdot a = \lambda a \quad \forall \lambda \in k, a \in A$$

Similarly for $m \circ (\mathrm{id} \otimes \eta)$.

By expressing the associativity and unit axioms as commuting diagrams, we motivate the following definition that arises by reversing the direction of the arrows:

Definition 2.2. A **coalgebra** C is a vector space $(C, +; k)$ with linear map $\triangle : C \to C \otimes C$ called the **coproduct**, satisfying **coassociativity**: $(\triangle \otimes \mathrm{id}) \circ \triangle = (\mathrm{id} \otimes \triangle) \circ \triangle$, as maps $C \to C \otimes C \otimes C$, i.e.

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "\mathrm{id}\otimes\Delta"'] \\ C \otimes C \arrow[u, "\Delta\otimes\mathrm{id}"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"] \end{tikzcd}$$

and linear map $\epsilon : C \to k$, the **counit**, satisfying: $(\epsilon \otimes \mathrm{id}) \circ \triangle = \mathrm{id}_C = (\mathrm{id} \otimes \epsilon) \circ \triangle$, given by

$$\begin{tikzcd} k \otimes C & C \otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & C \otimes k \\ & C \arrow[ul, "\cong"] \arrow[u, "\mu"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

The coproduct can be expressed explicitly as $\triangle(c) = \sum_i c_{i(1)} \otimes c_{i(2)}$. Using this notation to expand the left hand side of the coassociativity axiom gives:

$$(\triangle \otimes \mathrm{id}) \circ \triangle(c) = \sum_i \triangle(c_{i(1)}) \otimes \mathrm{id}(c_{i(2)}) = \sum_i \sum_j c_{i(1)j(1)} \otimes c_{i(1)j(2)} \otimes c_{i(2)}$$

Elaborating on this formula, we have used the fact that the tensor map $(\triangle \otimes \mathrm{id})$ acts linearly on the sum created by the first coproduct $\triangle$. The $\triangle$ in $\triangle \otimes \mathrm{id}$ then acts on the first factor $c_{i(1)}$ and id acts on the second factor $c_{i(2)}$, and we then use the fact tensor product distributes over addition.

It is helpful to use **Sweedler notation** for the coproduct which drops the $\sum$ and the summation index i. So we instead write: $\triangle(c) = c_{(1)} \otimes c_{(2)}$. Then we have $((\triangle \otimes \mathrm{id}) \circ \triangle)(c) = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$ (where we suppress both $\sum$'s and both summation indices i, j). Coassociativity can then be expressed as

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$$

In this notation the counit axiom can be reexpressed as: $\epsilon(c_{(1)})c_{(2)} = c = c_{(1)}\epsilon(c_{(2)})\ \forall c \in C$ (where again $\sum$ is omitted but implied here — from now on I won't point this out).

Definition 2.3. A **bialgebra** is $(B, m, \eta, \triangle, \epsilon)$ where (B, m, η) is an algebra, $(B, \triangle, \epsilon)$ is a coalgebra, and $\triangle, \epsilon$ are algebra homomorphisms.

In other words $\triangle(gh) = \triangle(g)\triangle(h)$ — with multiplication on the right hand side in the tensor product algebra $B \otimes B$ (see Definition 6.3 in Appendix). Note this expression hides a lot of details — in its expanded form it says:

$$\triangle \circ m = (m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\triangle \otimes \triangle) \tag{1}$$

for the **twist map** $\tau : B \otimes B \to B \otimes B,\ v \otimes w \mapsto w \otimes v$. Note that $(m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$ is just the product in $B \otimes B$. The second part of the definition of an algebra homomorphism gives $\triangle(1) = 1 \otimes 1$, and similarly $\epsilon(gh) = \epsilon(g)\epsilon(h),\ \epsilon(1_B) = 1_k$.

Definition 2.4. Take coalgebras $(C, \triangle_1, \epsilon_1), (D, \triangle_2, \epsilon_2)$, and let $\phi : k \otimes k \to k$ be the natural isomorphism defined as: $\phi(\lambda \otimes \mu) = \lambda\mu$, $\phi^{-1}(\lambda) = 1 \otimes \lambda = \lambda \otimes 1$.

- A **homomorphism of coalgebras** is a linear map $f : C \to D$ such that

$$\triangle_2 \circ f = (f \otimes f) \circ \triangle_1 \qquad \epsilon_2 \circ f = \epsilon_1 \tag{2}$$

- The **tensor product coalgebra** is the vector space $C \otimes D$ with coproduct and counit:

$$\triangle_{C\otimes D}(c \otimes d) := (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\triangle_1 \otimes \triangle_2)(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \quad (3)$$

$$\epsilon_{C\otimes D} = \phi \circ (\epsilon_1 \otimes \epsilon_2)$$

Note this is analogous to the tensor product algebra in Definition 6.3.

With these definitions, it then follows from Equation (1) that the condition that $\triangle$ is an algebra homomorphism in the definition of a bialgebra is equivalent to the product m being a coalgebra homomorphism, i.e. satisfying Equation(2), with the tensor product coalgebra structure on $B \otimes B$. Equally the counit ϵ being an algebra homomorphism is equivalent to the unit η being a coalgebra homomorphism.

Definition 2.5. A **Hopf algebra** H is a bialgebra $(H, m, \eta, \triangle, \epsilon)$ with linear map $S : H \to H$, the **antipode**, satisfying the **antipode axiom**:

$$S(h_{(1)})h_{(2)} = \epsilon(h)1_H = h_{(1)}S(h_{(2)}), \ \forall h \in H$$

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\triangle"] \arrow[rr, "\epsilon"] \arrow[dr, "\triangle"'] & & k \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

Note the tensor product of Hopf algebras H_1, H_2 can be made into a Hopf algebra with the tensor product algebra and coalgebra structures as defined already, and antipode defined as $S_{H_1\otimes H_2} := S_{H_1} \otimes S_{H_2}$.

We give some basic properties of Hopf Algebras, and their antipodes, before moving onto the next section where we see some examples of these objects.

Lemma 2.6. *The antipode of a Hopf algebra is unique.*

Proof. Take antipodes S_1, S_2. The first equality of Equation (4) below follows by applying S_1 to both sides of the counit axiom: $h = h_{(1)}\epsilon(h_{(2)})$ and using the linearity of S_1. The remaining equalities follow by multiplying by 1_H, and finally applying the antipode axiom for S_2 to the expression $\epsilon(h_{(2)})1_H$:

$$S_1(h) = S_1(h_{(1)})\epsilon(h_{(2)}) = S_1(h_{(1)})\epsilon(h_{(2)})1_H = S_1(h_{(1)})h_{(2)(1)}S_2(h_{(2)(2)}) \tag{4}$$

Note the final expression is the result of the map $m \circ (m \otimes \mathrm{id}) \circ (S_1 \otimes \mathrm{id} \otimes S_2)$ applied to $h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(1)}$. Using coassociativity we can apply this map to the equivalent expression $h_{(1)(1)} \otimes h_{(1)(2)} \otimes h_{(2)}$, so Equation (4) equals:

$$S_1(h_{(1)(1)})h_{(1)(2)}S_2(h_{(2)}) = \epsilon(h_{(1)})1_H S_2(h_{(2)}) = S_2(h)$$

with final equalities via the antipode axiom for S_1 applied to $S_1(h_{(1)(1)})h_{(1)(2)}$, using linearity of S_2, and finally the counit axiom again. □

Lemma 2.7. *Basic properties of the antipode:*

1. $S(hg) = S(g)S(h)$, $S(1) = 1$ *i.e. the antipode is an anti-algebra homomorphism. Note this implies* S^2 *is an algebra homomorphism.*

2. $(S \otimes S) \circ \triangle = \tau \circ \triangle \circ S$, *for* τ *the twist map, and* $\epsilon \circ S = \epsilon$. *Comparing with the definition of a coalgebra homomorphism in Equation (2), we see this as saying* S *is anti-coalgebra homomorphism.*

Proof. 1. Applying $(\tau \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\triangle \otimes \triangle)$ to $g \otimes h \in H \otimes H$ we get: $h_{(1)} \otimes g_{(1)} \otimes g_{(2)} \otimes h_{(2)}$. We next apply each side of Equation (1) (i.e. the algebra homomorphism property) to the first two terms $h_{(1)} \otimes g_{(1)}$, giving us:

$$(h_{(1)}g_{(1)})_{(1)} \otimes (h_{(1)}g_{(1)})_{(2)} \otimes g_{(2)} \otimes h_{(2)} = (h_{(1)(1)}g_{(1)(1)}) \otimes (h_{(1)(2)}g_{(1)(2)}) \otimes g_{(2)} \otimes h_{(2)}$$

Then applying $(m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (S \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})$ to both sides of this gives:

$$S((h_{(1)}g_{(1)})_{(1)})(h_{(1)}g_{(1)})_{(2)} \otimes g_{(2)} \otimes h_{(2)} = S(h_{(1)(1)}g_{(1)(1)})h_{(1)(2)}g_{(1)(2)} \otimes g_{(2)} \otimes h_{(2)} \tag{5}$$

Using the antipode axiom on $h_{(1)}g_{(1)}$ the left hand side becomes:

$$\epsilon(h_{(1)}g_{(1)})1_H \otimes g_{(2)} \otimes h_{(2)} = 1 \otimes g \otimes h \tag{6}$$

with the second equality using ϵ is an algebra homomorphism and the counit axiom. Hence applying the map $(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes \mathrm{id})$ to the right hand sides of Equations (5) and (6), gives the first equality below:

$$\begin{aligned} S(g) \otimes h &= S(h_{(1)(1)}g_{(1)(1)})h_{(1)(2)}g_{(1)(2)}S(g_{(2)}) \otimes h_{(2)} \\ &= S(h_{(1)(1)}g_{(1)})h_{(1)(2)}g_{(2)(1)}S(g_{(2)(2)}) \otimes h_{(2)} \\ &= S(h_{(1)(1)}g_{(1)})h_{(1)(2)}\epsilon(g_{(2)})1_H \otimes h_{(2)} \\ &= S(h_{(1)(1)}g)h_{(1)(2)} \otimes h_{(2)} \end{aligned}$$

The second equality follows by applying coassociativity, similarly to how we did in the proof of Lemma 2.6 (i.e. using $g_{(1)(1)}g_{(1)(2)}g_{(2)} = g_{(1)}g_{(2)(1)}g_{(2)(2)}$). Then we use the antipode axiom on $g_{(2)}$, followed by the counit axiom along with the linearity of S. Finally applying $m \circ (\mathrm{id} \otimes S)$ to both sides of the above gives us

$$\begin{aligned} S(g)S(h) &= S(h_{(1)(1)}g)h_{(1)(2)}S(h_{(2)}) \\ &= S(h_{(1)}g)h_{(2)(1)}S(h_{(2)(2)}) \\ &= S(h_{(1)}g)\epsilon(h_{(2)})1_H \\ &= S(hg) \end{aligned}$$

With the final three equalities following from coassociativity, antipode axiom and then counit axiom. We show $S(1) = 1$ by applying the antipode axiom $S(h_{(1)})h_{(2)} = \epsilon(h)1_H$ with $h = 1$. Using the fact that $\triangle, \epsilon$ are both algebra homomorphisms we have $\triangle(1) = 1 \otimes 1,\ \epsilon(1) = 1$, so then the result follows.

2. For proof see the discussion at the end of Proof 1.3.1 in Majid [10]. □

2.2 Examples

Next we look at some examples of Hopf algebras. The main object we require in the following sections is the group algebra kG of a finite group G, which we define first. Also in the next section we will look at the Hopf algebra structure on the dual of the group algebra.

Following this we briefly discuss the tensor algebra and the universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$. We finish the section with the quantum group, $U_q(b_+)$. Note that these last three examples are infinite-dimensional, so strictly speaking lie outside the scope of the main essay question. However I include them as they are important examples within the broader topic of Hopf algebras. We give their construction more informally, but provide references where more details can be found.

Example 2.8. For finite group G, the **group algebra** kG, has the usual algebra structure with basis elements given by $g \in G$, coefficients in k, and product between basis elements given by the group product, extended linearly to kG. Define the following maps on basis elements, and extend each linearly to kG:

- coproduct $\triangle(g) := g \otimes g$
- counit $\epsilon(g) := 1_k\ \forall g \in G$, i.e on a general element: $\epsilon\big(\sum_g \lambda_g(g)\big) = \sum_g \lambda_g$
- antipode $S(g) := g^{-1}$.

Proof. Collating all the definitions in the preceding section together, what needs to be checked to show this is a Hopf algebra? The coalgebra structure requires the coassociativity and counit axioms hold, the bialgebra structure requires that $\triangle, \epsilon$ be algebra homomorphisms, and finally a Hopf algebra requires the antipode axiom be satisfied.

All these axioms can be reformulated as requiring the vanishing of some linear map, for instance for coassociativity we require: $(\triangle\otimes\mathrm{id})\circ\triangle-(\mathrm{id}\otimes\triangle)\circ\triangle$ be the zero map. Hence by showing the axiom holds on basis vectors, by linearity we get that the axiom holds on all of kG. Coassociativity holds since $((\triangle\otimes\mathrm{id})\circ\triangle)(g) = g\otimes g\otimes g = ((\mathrm{id}\otimes\triangle)\circ\triangle)$. The counit axiom: $((\epsilon \otimes \mathrm{id}) \circ \triangle)(g) = (\epsilon \otimes \mathrm{id})(g \otimes g) = \epsilon(g)g = 1_k \cdot g = g$ as required. Also $\triangle(gh) = (gh) \otimes (gh) = (g \otimes g) \cdot (h \otimes h) = \triangle(g)\triangle(h)$, and clearly $\triangle(1) = 1 \otimes 1$. Similarly ϵ is an algebra homomorphism. Finally for the antipode: $(m \circ (S \otimes \mathrm{id}) \circ \triangle)(g) = m(S(g) \otimes g) = g^{-1}g = 1_G = (m \circ (\mathrm{id} \otimes S) \circ \triangle)(g)$, while $(\eta \circ \epsilon)(g) = \eta(1_k) = 1_G$. So kG is indeed a Hopf algebra. □

Example 2.9. For vector space V, let $T^n(V) := V\otimes\cdots\otimes V$, the n-fold tensor product of V. Then the **tensor algebra** of V is the vector space:

$$T(V) := \bigoplus_{n=0}^{\infty} T^n(V) = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

with multiplication given by the tensor product:

$$T^k(V) \otimes T^l(V) \to T^{k+l}(V),\ (v \otimes \cdots \otimes w) \cdot (x \otimes \cdots \otimes y) := (v \otimes \cdots \otimes w \otimes x \otimes \cdots \otimes y)$$

which we can extend linearly to $T(V)$. Then on the subspace V of $T(V)$ define:

$$\triangle(v) := v \otimes 1 + 1 \otimes v,\ \epsilon(v) := 0,\ S(v) := -v$$

Also on the subspace k of $T(V)$, let

$$\triangle(1_k) := 1_k \otimes 1_k,\ \epsilon(\lambda) := \lambda,\ S(\lambda) := 1\ \forall \lambda \in k$$

$\triangle, \epsilon$ extend to all of $T(V)$ as algebra homomorphisms. In particular: $\epsilon(x) = 0\ \forall x \in \bigoplus_{n=1} T^n(V)$. Whereas S extends as an anti-algebra homomorphism, so it can be shown: $S(v_1 \otimes \cdots \otimes v_n) = (-1)^n v_n \otimes \cdots \otimes v_1$. Explicitly describing the extension of $\triangle$ to $T(V)$ brings us to into the intricacies of gradings of $T(V) \otimes T(V)$ and shuffle products, which takes us quite far from the objective of this essay. Note this can be done, and does indeed give $T(V)$ a Hopf algebra structure for which the reader can find proofs of this in Theorem 111.2.4 and Lemma 111.3.6 of Kassle [7].

Example 2.10. Using the above we can construct the universal enveloping algebra. To quote the article [15] "*The idea of the universal enveloping algebra is to embed a Lie algebra $\mathfrak{g}$ into an associative algebra A with identity in such a way that the abstract bracket operation in $\mathfrak{g}$ corresponds to the commutator $xy - yx \in A$. There may be many ways to make such an embedding, but there is one "largest" such A, called the universal enveloping algebra of $\mathfrak{g}$*".

For finite-dimensional Lie algebra $\mathfrak{g}$, we can take the tensor algebra of its underlying vector space: $T(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$ Since this is an algebra with respect to the tensor product multiplication we can define left, right and 2-sided ideals generated by a subset in the usual way of ring theory. In particular we use the set

$$\{\mu \otimes \nu - \nu \otimes \mu - [\mu, \nu] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \mid \mu, \nu \in \mathfrak{g}\}$$

to generate a 2-sided ideal I containing linear combinations of elements of the form:

$$a \otimes (\mu \otimes \nu - \nu \otimes \mu - [\mu, \nu]) \otimes b \qquad \text{for } a, b \in T(\mathfrak{g}), \mu, \nu \in \mathfrak{g}$$

Then the **universal enveloping algebra** is $U(\mathfrak{g}) := T(\mathfrak{g})/I$. It is generated by 1, and a basis of $\mathfrak{g}$, and inherits the multiplication of the tensor algebra subject to the relations given by $[\mu, \nu] = \mu \otimes \nu - \nu \otimes \mu \ \forall \mu, \nu \in \mathfrak{g}$.

As was done for the tensor algebra, we define the comultiplication, counit and antipode on the subspace $\mathfrak{g} \subset U(\mathfrak{g})$ as:

$$\triangle(\mu) = \mu \otimes 1 + 1 \otimes \mu, \ \epsilon(\mu) = 0, \ S(\mu) = -\mu$$

Then we extend $\triangle, \epsilon$ as algebra homomorphisms, and S as an anti-algebra homomorphism, to the rest of $U(\mathfrak{g})$. Similarly to the case of $T(V)$, this gives $U(\mathfrak{g})$ a Hopf algebra structure. See Chapter 3 of Majid [10] for more on this.

Example 2.11. The Hopf Algebra $U_q(b_+)$, for invertible $q \in k$, has generators: $\{1, X, g, g^{-1}\}$ with maps defined on generators as:

- product: $gg^{-1} = 1 = g^{-1}g, \ Xg = qgX, \ Xg^{-1} = q^{-1}g^{-1}X$
- coproduct: $\triangle X = X \otimes 1 + g \otimes X, \ \triangle g = g \otimes g, \ \triangle g^{-1} = g^{-1} \otimes g^{-1}$
- counit: $\epsilon(X) = 0, \ \epsilon(g) = 1 = \epsilon(g^{-1})$
- antipode: $S(X) = -g^{-1}X, \ S(g) = g^{-1}, \ S(g^{-1}) = g$

What it means for $U_q(b_+)$ to be generated by $1, X, g, g^{-1}$ is that we construct the tensor algebra over the vector space with basis elements $1, X, g, g^{-1}$, just as we did above when we defined $U(\mathfrak{g})$ by first taking $T(\mathfrak{g})$. Then, as before, we quotient by an ideal, I, generated by the relations determined by the products above. However, unlike the case of $U(\mathfrak{g})$, the coproduct, counit and antipode are not inherited from those of the tensor algebra, they satisfy different relations. However we still must extend $\triangle, \epsilon$ as algebra homomorphisms, and S as an anti-algebra homomorphism, to the whole algebra. Note that S, as defined on the generators $1, X, g, g^{-1}$, can be extended to the whole tensor algebra, but to extend to a quotient of the tensor algebra we require that $S(I) \subset I$. In particular we check for generators i of I that we have $S(i) \in I$:

$$\begin{aligned}
S(gg^{-1}-1) &= S(g^{-1})S(g) - S(1) = gg^{-1} - 1 \in I \\
S(gX - qXg) &= S(X)S(g) - qS(g)S(X) = (-g^{-1}X)g^{-1} - qg^{-1}(-g^{-1}X) \\
&= -qg^{-2}X + qg^{-2}X = 0 \in I
\end{aligned}$$

To extend the coproduct $\triangle$ as an algebra homomorphism requires compatibility between the product, coproduct and Equation (1). We check this compatibility for a particular pair of generators. Note $gX = q^{-1}Xg$ so we should find $\triangle(gX) = \triangle(q^{-1}Xg)$.

$$\begin{aligned}
\triangle(gX) &= \triangle(g) \cdot \triangle(X) = (g \otimes g) \cdot (X \otimes 1 + g \otimes X) \\
&= gX \otimes g + g^2 \otimes gX \\
\triangle(q^{-1}Xg) &= q^{-1}\triangle(X)\triangle(g) = q^{-1}(X \otimes 1 + g \otimes X) \cdot (g \otimes g) \\
&= q^{-1}(Xg \otimes g + g^2 \otimes Xg) = q^{-1}(qgX \otimes g + g^2 \otimes qgX) \\
&= gX \otimes g + g^2 \otimes gX
\end{aligned}$$

as required. Also for the antipode, we must check $S(hg) = S(g)S(h)$ is compatible with the product, and finally that it satisfies the antipode axioms on the generators of H. Since $\triangle, \epsilon$ are algebra homomorphisms, if S satisfies antipode axiom on generators, it satisfies it on H. For more details we refer the reader to Example 1.3.2 of Majid [10], and the paragraph immediately following it, which discusses how this example fits in with the broader theory of quantum groups.

Definition 2.12. For τ, the twist map, define the following:

- A (Hopf) algebra is **commutative** if $m \circ \tau = m$
- A Hopf algebra/coalgebra is **cocommutative** if $\tau \circ \triangle = \triangle$
- A **quantum group** is a noncommutative noncocommutative Hopf algebra.

We see that the product and coproduct of $U_q(b_+)$ is neither commutative nor cocommutative on the generators, hence is an example of a quantum group.

Note: it is possible to lose cocommutativity, whilst still retaining some of its good behavior, by means of a **quasitriangular** structure — given in Definition 2.15. Whilst there is no strict definition of a quantum group, it is often assumed to have the additional condition of being quasitriangular.

2.3 Dual structures

Now we show if C is a coalgebra, then we can define an algebra structure on the dual vector space C^*. Likewise, in the finite-dimensional case, for an algebra A, we can define coalgebra structure on A^*.

Recall from linear algebra, for a linear map $\alpha : V \to W$, the dual map is $\alpha^* : W^* \to V^*,\ \alpha^*(\phi) := \phi \circ \alpha$. Also, for a general vector space C, we have $C^* \otimes C^* \subset (C \otimes C)^*$, but this inclusion becomes an equality when C is finite-dimensional. Since much of the following requires this to be an equality, it is for this reason we restrict to finite-dimensional Hopf algebras.

Applying this dual map construction to the coproduct and counit of a coalgebra C, generates maps $\triangle^* : C^* \otimes C^* \to C^*$ and $\epsilon^* : k \to C^*$. Using coassociativity of $\triangle$ we find $\triangle^*$ is associative: let $\phi, \mu, \tau \in C^*,\ h \in C$ then

$$\begin{aligned}(\triangle^* \circ (\triangle^* \otimes \mathrm{id}))(\phi \otimes \mu \otimes \tau)(h) &= \triangle^*(((\phi \otimes \mu) \circ \triangle) \otimes \tau)(h)\\ &= ((((\phi \otimes \mu) \circ \triangle) \otimes \tau) \circ \triangle)(h)\\ &= \phi(h_{(1)(1)})\mu(h_{(1)(2)})\tau(h_{(2)})\end{aligned}$$

and using coassociativity the right hand side equals

$$\phi(h_{(1)})\mu(h_{(2)(1)})\tau(h_{(2)(2)}) = (\phi \otimes ((\mu \otimes \tau) \circ \triangle)) \circ \triangle(h) = (\triangle^* \circ (\mathrm{id} \otimes \triangle^*))(\phi \otimes \mu \otimes \tau)(h)$$

as required. Similarly the counit axiom for ϵ implies the unit axiom for ϵ^* holds, hence we have an algebra structure on the dual space C^* (see Theorem 3.34 of Kytölä [8] for more). Going the other way, for an algebra A that is crucially finite-dimensional, the dual maps $m^* : A^* \to A^* \otimes A^*$, $\eta^* : A^* \to k$ define a coproduct and counit, hence making A^* a coalgebra.

Note also that if C is cocommutative then C^* will be commutative: i.e. for $\phi, \mu \in C^*$

$$\begin{aligned}\triangle^*(\phi \otimes \mu) = (\phi \otimes \mu) \circ \triangle \quad &= (\phi \otimes \mu) \circ (\tau \circ \triangle)\\ &= ((\phi \otimes \mu) \circ \tau) \circ \triangle = (\mu \otimes \phi) \circ \triangle = \triangle^*(\mu \otimes \phi)\end{aligned}$$

and vice versa, if A is commutative, A^* will be cocommutative.

Finally then for a finite-dimensional Hopf algebra H, the dual space H^* is a bialgebra by above. Looking at the commuting diagram for the antipode axiom, we see that taking the dual results in reversing the direction of all the arrows, and what results is an identical diagram depicting the antipode axiom for H^*. Hence the dual map of the antipode, S^*, is an antipode for H^*. So the dual of a Hopf algebra is still a Hopf algebra.

Example 2.13. The dual $(kG)^*$ to the group algebra kG for finite group G is given by dual vector space with:

- The product is generated by the coproduct $\triangle$ on kG, as $\triangle^* : (kG)^* \otimes (kG)^* \to (kG)^*$ such that:

$$\triangle^*(\phi \otimes \mu) := (\phi \otimes \mu) \circ \triangle, \; \triangle^*(\phi \otimes \mu)(g) = \phi(g)\mu(g)$$

  with multiplication on the RHS in the field k.

- The coproduct for $(kG)^*$ is $m^* : (kG)^* \to (kG)^* \otimes (kG)^*$ such that $m^*(\phi) := \phi \circ m$ i.e. $m^*(\phi)(g \otimes h) = \phi(g \cdot h)$. We give the coproduct explicitly on basis $\{\delta_g\}$ for $(kG)^*$ dual to basis $\{g\}$ of kG: let $h_1, h_2 \in G$, then

$$\triangle(\delta_g)(h_1 \otimes h_2) = \delta_g(h_1 \cdot h_2) = \begin{cases} 1 & \text{if } h_1 \cdot h_2 = g, \\ 0 & \text{otherwise.} \end{cases}$$

  In other words

$$\triangle(\delta_g) = \sum_{\{(h_1,h_2) \in G \times G \; : \; h_1 \cdot h_2 = g\}} \delta_{h_1} \otimes \delta_{h_2} \tag{7}$$

  Note when G is non-abelian, kG is non-commutative cocommutative, and hence by above, $(kG)^*$ is commutative non-cocommutative.

- The unit, $\epsilon^* : k \to (kG)^*, \; \epsilon^*(\lambda)(g) := \lambda\epsilon(g) = \lambda$. So

$$\epsilon^*(\lambda) = \lambda \sum_g \delta_g$$

- The counit, $\eta^* : (kG)^* \to k^*, \; \eta^*(\delta_g)(\lambda) := (\delta_g \circ \eta)(\lambda) = \lambda\delta_g(1_G)$

- Antipode $S^* : (kG)^* \to (kG)^*, \; S^*(\delta_g)(h) := \delta_g(S(h)) = \delta_g(h^{-1}) = \delta_{g^{-1}}(h)$

2.4 Drinfeld double

This section approximately follows the definitions of Section 1 of Witherspoon [16] and Section 4 of Chua [3]. Before we define the main object of this essay, the Drinfeld double, we must define a particular action of the group G on $(kG)^*$: for $g \in G, \phi \in (kG)^*$, let $\phi^g(x) := \phi(gxg^{-1})$, then define the action as:

$$g \triangleright \phi := \phi^{g^{-1}} \in (kG)^* \qquad \text{so: } (g \triangleright \phi)(x) = \phi^{g^{-1}}(x) := \phi(g^{-1}xg)$$

Definition 2.14. Then the **Drinfeld double** (or **quantum double**) of finite group G has underlying vector space $D(G) := (kG)^* \otimes kG$, with basis: $\{\delta_g \otimes h \mid g, h \in G\}$. The product on this basis is

$$(\delta_g \otimes h) \cdot (\delta_{g'} \otimes h') := \delta_g \delta_{g'}^{h^{-1}} \otimes hh' = \delta_g \delta_{hg'h^{-1}} \otimes hh'$$

Note the unit is $1_{D(G)} = \sum_{g'} \delta_{g'} \otimes 1$. The coproduct is given by:

$$\triangle(\delta_g \otimes h) := \sum_{\{(g_1,g_2) \in G \times G \ \mid \ g_1 g_2 = g\}} \delta_{g_1} \otimes h \otimes \delta_{g_2} \otimes h \equiv \sum_{x \in G} \delta_x \otimes h \otimes \delta_{x^{-1}g} \otimes h$$

The counit:

$$\epsilon(\delta_g \otimes h) := \delta_1(g)$$

and antipode:

$$S(\delta_g \otimes h) := \delta_{h^{-1}g^{-1}h} \otimes h^{-1}$$

Proof. We show that the antipode axiom holds, i.e.

$$m \circ (S \otimes \mathrm{id}) \circ \triangle = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \triangle$$

Firstly $(\eta \circ \epsilon)(\delta_g \otimes h) = \eta(\delta_1(g)) = \delta_1(g) \sum_{g'} \delta_{g'} \otimes 1$. Then

$$\begin{aligned}(m \circ (S \otimes \mathrm{id}) \circ \triangle)(\delta_g \otimes h) &= (m \circ (S \otimes \mathrm{id}))(\sum_{x \in G} (\delta_x \otimes h) \otimes (\delta_{x^{-1}g} \otimes h)) \\ &= m(\sum_{x \in G} (\delta_{h^{-1}x^{-1}h} \otimes h^{-1}) \otimes (\delta_{x^{-1}g} \otimes h)) \\ &= \sum_{x \in G} \delta_{h^{-1}x^{-1}h} \delta_{h^{-1}x^{-1}gh} \otimes h^{-1}h\end{aligned}$$

The terms in this sum are non-zero $\iff h^{-1}x^{-1}h = h^{-1}x^{-1}gh \iff g = 1$, so we can set $g = 1$ while multiplying by a factor of $\delta_1(g)$. Since $G = \{h^{-1}x^{-1}h \mid x \in G\}$ we can make a change of variable to $g' = h^{-1}x^{-1}h$ and sum over g' instead, giving us the result of $\delta_1(g) \sum_{g'} \delta_{g'} \otimes 1$ that we required. Showing $m \circ (\mathrm{id} \otimes S) \circ \triangle$ is similar.

It remains to check that $\triangle, \epsilon$ are indeed algebra homomorphisms, and that S is an anti-algebra homomorphism. Also the coassociativity and counit axioms hold. Note this is a special case of the Drinfeld double $D(H)$ for a general Hopf algebra H. See Proposition 8.1 of Majid [9] for the general definition, and full proof that it is indeed a Hopf algebra. □

Where does $D(G)$ come from? The algebra structure of $D(G)$ is an example of a smash product algebra (see Definition 3.3), which generalises the semidirect product for groups. The coalgebra structure is precisely the tensor product coalgebra structure of $(kG)^*$ and kG, so the coproduct comes from

$$\triangle_{D(G)} := (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\triangle_{(kG)^*} \otimes \triangle_{kG})$$

using Equation (3) with $\triangle_{(kG)^*}$, $\triangle_{kG}$ denoting the coproduct of $(kG)^*$ (see Equation (7)) and kG respectively.

Recall in Definition 2.12 we mentioned that cocommutativity could be weakened by means of a "quasitriangular" structure. Next we briefly define what this is, and how it relates to the Drinfeld double.

Definition 2.15. A **quasitriangular Hopf algebra** is (H, R) for Hopf algebra H and $R = \sum_i R_i^{(1)} \otimes R_i^{(2)} \in H \otimes H$ such that:

- R is invertible with respect to the multiplication in the tensor product algebra $H \otimes H$.
- Cocommutative up to conjugation by R:
$$\tau \circ \triangle(h) = R \cdot \triangle(h) \cdot R^{-1} \; \forall h \in H \tag{8}$$
- Let
$$R_{ij} := \sum_i 1 \otimes \cdots \otimes R_i^{(1)} \otimes 1 \otimes \cdots \otimes R_i^{(2)} \otimes \cdots \otimes 1 \in H \otimes \cdots \otimes H$$
then
$$(\triangle \circ \mathrm{id})(R) = R_{13} \cdot R_{23} \text{ and } (\mathrm{id} \otimes \triangle)(R) = R_{13} \cdot R_{12} \tag{9}$$
with $R_{ij} \in H \otimes H \otimes H$, and the multiplication $\cdot$ in $H \otimes H \otimes H$ taken analogously to that of $H \otimes H$.

Proposition 2.16. *$D(G)$ is a quasitriangular Hopf algebra with*
$$R = \sum_{g \in G} (\delta_g \otimes 1) \otimes (1 \otimes g) \text{ and } R^{-1} = \sum_{g \in G} (\delta_g \otimes 1) \otimes (1 \otimes g^{-1})$$

Proof. We check Equation (8) above holds. The left hand side is:
$$(\tau \circ \triangle)(\delta_g \otimes h) = \sum_{x \in G} (\delta_{x^{-1}g} \otimes h) \otimes (\delta_x \otimes h) \tag{10}$$

While the right hand side is:

$$
\begin{aligned}
R\triangle(\delta_g \otimes h)R^{-1} &= R(\sum_{x\in G}\delta_x \otimes h \otimes \delta_{x^{-1}g} \otimes h)\cdot(\sum_{g'\in G}(\delta_{g'} \otimes 1)\otimes(1\otimes g'^{-1})) \\
&= R\sum_x\sum_{g'}(\delta_x \otimes h)\cdot(\delta_{g'}\otimes 1)\otimes(\delta_{x^{-1}g}\otimes h)\cdot(1\otimes g'^{-1}) \\
&= R\sum_x\sum_{g'}(\delta_x\delta_{hg'h^{-1}}\otimes h)\otimes(\delta_{x^{-1}g}\otimes hg'^{-1}) \\
&= \sum_{g''\in G}\sum_x\sum_{g'}(\delta_{g''}\otimes 1)\otimes(1\otimes g'')\cdot(\delta_x\delta_{hg'h^{-1}}\otimes h)\otimes(\delta_{x^{-1}g}\otimes hg'^{-1}) \\
&= \sum_{g''}\sum_x\sum_{g'}(\delta_{g''}\otimes 1)\cdot(\delta_x\delta_{hg'h^{-1}}\otimes h)\otimes(1\otimes g'')\cdot(\delta_{x^{-1}g}\otimes hg'^{-1}) \\
&= \sum_{g''}\sum_x\sum_{g'}(\delta_{g''}\delta_x\delta_{hg'h^{-1}}\otimes h)\otimes(\delta_{g''x^{-1}gg''^{-1}}\otimes g''hg'^{-1})
\end{aligned}
$$

We see the summands are non-zero iff $g'' = x = hg'h^{-1}$, in which case: $g''hg'^{-1} = (hg'h^{-1})hg'^{-1} = h$, resulting in the above collapsing to:

$$\sum_x(\delta_x \otimes h)\otimes(\delta_{gx^{-1}}\otimes h)$$

Making a change of variables to $y = gx^{-1}$ puts this in same form as Equation (10), as required. See Proposition 8.2 of Majid [9] for the proof that $D(H)$ satisfies Equation (9) for a general Hopf algebra H. □

Note $D(H)$ is quasitriangular for arbitrary Hopf algebra H, so we see the Drinfeld double as a useful way of constructing quasitriangular Hopf algebras. The following result isn't required for later parts of the essay, but we include it as it is the starting point of many interesting applications of Hopf algebra theory to statistical mechanics and knot theory:

Proposition 2.17. *If (H, R) is a quasitriangular Hopf Algebra, then R satisfies the* ***abstract quantum Yang-Baxter equation****:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Proof. We roughly follow the proof of Prop 4.9 of Kytölä [8], although we add more detail. The first line below uses Equation (9), followed by noting $R_{12} = R\otimes 1$, and then performing the multiplication. The second line introduces a factor of $1 = RR^{-1}$ in order to apply Equation (8). In the final line we expand $((\tau\circ\triangle)\otimes\mathrm{id})$ as $((\tau\otimes\mathrm{id})\circ(\triangle\otimes\mathrm{id}))$, apply (9) again.

$$R_{12}R_{13}R_{23} = R_{12}(\triangle \otimes \mathrm{id})(R) = (R \otimes 1)(\triangle(R^{(1)}) \otimes R^{(2)}) = (R\triangle(R^{(1)})) \otimes R^{(2)}$$

$$= R\triangle(R^{(1)})R^{-1}R \otimes R^{(2)} = (R\triangle(R^{(1)})R^{-1} \otimes R^{(2)}) \cdot (R \otimes 1) = ((\tau \circ \triangle) \otimes \mathrm{id})(R) \cdot R_{12}$$

$$= ((\tau \otimes \mathrm{id}) \circ (\triangle \otimes \mathrm{id}))(R) \cdot R_{12} = (\tau \otimes \mathrm{id})(R_{13}R_{23}) \cdot R_{12} = R_{23}R_{13}R_{12}$$

□

Note: The dimension of both kG and $(kG)^*$ is the order of the group $|G|$, so we clearly have

$$\dim(D(G)) = |G|^2$$

A further constraint we will place on our Hopf algebras, is that they are semisimple as algebras. This is necessary since we wish to determine those Hopf algebras which are isomorphic, as algebras, to group algebras. By Maschke's theorem, the group algebra of a finite group in $\mathrm{char}(k) = 0$ is semisimple. Therefore if our Hopf algebra is not semisimple, clearly no such isomorphism can exist. So, in investigating the Drinfeld double of finite groups, we need to know under what conditions such Hopf algebras are semisimple. By Proposition 1.2 of Witherspoon [16] we have the result we need:

Proposition 2.18. *As an algebra, $D(G)$ is semisimple if and only if the characteristic of the field does not divide the order of the group: $\mathrm{char}(k) \nmid |G|$.*

Proof. See discussion in Proposition 1.2 of Witherspoon [16]. For the analogous result, with an explicit proof, of the semisimplicity of $D(H)$ for a general Hopf algebra H, see Corollary 10.3.13 of Montgomery [11]. □

So we find for the case that we are concerned with, where G is finite and $\mathrm{char}(k) = 0$, that $D(G)$ is semisimple. We can finally describe the main object of the essay, $D(S_3)$, as the 36-dimensional vector space $(kS_3)^* \otimes kS_3$, with Hopf algebra structure given as above, which is also semisimple as an algebra.

2.4.1 Abelian groups

What is $D(G)$ when G is abelian? The action of G on $(kG)^*$ is trivial in this case:

$$\phi^g(x) := \phi(gxg^{-1}) = \phi(gg^{-1}x) = \phi(x)$$

Therefore the algebra structure of $D(G)$ comes from the usual tensor product algebra structure of $(kG)^*$ and kG (as in Definition 6.3). To further describe $(kG)^*$ we require the following definition:

Definition 2.19. For finite abelian group G, the **character group** $\hat{G}$ (or **Pontryagin dual**) is the set of characters on G, i.e. group homomorphisms $G \to k^* = k \backslash \{0\}$, which forms a group under pointwise multiplication, i.e. for $f, f' \in \hat{G}$ then $(f \cdot f')(g) := f(g)f'(g)$.

Now by Corollary 1.5.6 of Majid [10] we have $(kG)^* \cong k\hat{G}$ (as Hopf algebras), for $\hat{G}$ the character group of G. Also by Theorem 3.11 of Conrad [2] we find for finite abelian groups G, that G is isomorphic to $\hat{G}$. Therefore we find $(kG)^* \cong k\hat{G} \cong kG$, so, in particular as algebras:

$$D(G) \cong kG \otimes kG \cong k(G \times G)$$

with the last term being the group algebra* of the product group $G \times G$, with basis elements given by $\{(g, h) : \ g, h \in G\}$. Note the second isomorphism follows by defining a linear map $\phi : kG \otimes kG \to k(G \times G)$ between basis vectors $g \otimes h \mapsto (g, h)$ (hence an isomorphism of vector spaces), and it easy see that with the tensor product algebra structure on $kG \otimes kG$ this map is an algebra homomorphism also.

So answering the main question of the essay for the class of Hopf algebras that are the Drinfeld double of a finite abelian group G, they are indeed isomorphic to a group algebra, in particular the group algebra of $G \times G$.

---

3 Representation theory

Next we recall some basic facts from Representation theory that we will require in what follows. Firstly, every group representation $\rho : G \to \mathrm{GL}(V)$ induces a (left) kG-module structure on V via the action of kG on V:

$$(\sum_g \lambda_g g) \cdot v := \sum_g \lambda_g(\rho(g)(v))$$

Conversely, a kG-module V gives a representation by simply restricting the action of the group algebra to the group elements. In this way, subrepresentations are equivalent to kG-submodules and irreducibility of a representation is equivalent to the corresponding kG-module being simple. For more this see Section 3.2 of Sengupta [13].

Note we can make the above more formal using Category theory. See Example 6.8 where we discuss the isomorphism of the category of kG-modules with the category of representations over G.

*This is an abuse of notation since $k(G \times G)$ could refer to the group **function** algebra of $G \times G$, however here we mean it to be the group algebra.

Next one can take the group algebra kG as a left kG-module over itself. This then corresponds to the **left regular representation** of G. We know from Representation theory that every simple kG-module (i.e. irreducible representation) U is isomorphic to a kG-submodule of the regular representation kG. Additionally the multiplicity k of U occurring in kG is equal to its dimension, i.e. $k = \dim(U)$.

Definition 3.1. If $\{U_i | \ i = 1, \dots, n\}$ is the complete list of simple kG-modules, then the **isotypic component** of kG of type U_i is $\mathrm{Iso}_{U_i}(kG) := U_{i,1} \oplus \cdots \oplus U_{i,k_i}$ with $U_{i,j} \cong U_i$ and $k_i = \dim(U_i)$.

Then we have $kG = \mathrm{Iso}_{U_1}(kG) \oplus \cdots \oplus \mathrm{Iso}_{U_n}(kG)$, and using this we find the sum of the squares of the degrees of the irreducible representations equals the order of the group:

$$|G| = \dim(kG) = \sum_i \dim(\mathrm{Iso}_{U_i}(kG)) = \sum_i k_i \dim(U_i) = \sum_i k_i^2 \tag{11}$$

Now, if there exists some group H such that as algebras, $D(S_3) \cong kH$, then because we know $D(S_3)$ has dimension 36, we have: $36 = \dim(D(S_3)) = \dim(kH) = |H|$, so our group must be of order 36. Secondly, under this isomorphism, the simple $D(S_3)$-modules will correspond also to the simple kH-modules, and therefore to the irreducible representations of H. We will see later how to characterise the simple $D(S_3)$-modules, and in particular will compute their dimension. We also know from data [6] the degrees of all the irreducible representations of groups of order 36. Hence all that will remain is to compare these values to find a (possible) match!

But why is this enough? By Artin-Wedderburn, every semisimple left Artinian ring is isomorphic to the direct sum of matrix rings $M_{n_i}(D_i)$, where the integers n_i are unique, and the division rings D_i are unique up to isomorphism. A finite dimensional k-algebra is certainly an Artinian ring, and we know $D(G)$ and kH are semisimple. It can also be shown that for algebraically closed k, the $D_i = k \ \forall i$. Now applying the theorem to $D(S_3)$, the integers n_i will correspond to the dimensions of the simple $D(S_3)$-modules. Applying the theorem to kH, we find integers n'_i, which will correspond to the degrees of the irreducible representations of H. Hence by the uniqueness part of Artin-Wedderburn, finding a match between the $\{n_i\}, \{n'_i\}$ implies that, as algebras, $D(S_3) \cong kH$.

3.1 Induced representations

Next we introduce the notion of an induced representation, as this will be required in the characterisation of irreducible G-equivariant vector bundles in Lemma 4.3. We follow the discussion given in Fulton and Harris [5].

Take a representation $\rho : G \to \mathrm{GL}(V)$ of group G, and a subgroup $H \subset G$. Then let $\mathrm{Res}^G_H(V)$ denote (the vector space of) the representation of H given by restricting the above representation: $\rho|_H : H \to \mathrm{GL}(V)$. Take a subspace $W \subset V$. We can "translate" W by $g \in G$ as: $g \cdot W := \{g \cdot w | \ w \in W\}$. Also W is "H-invariant" if $h \cdot W \subset W \ \forall h \in H$[†], or in other words W is a subrepresentation of $\mathrm{Res}^G_H(V)$. For such an H-invariant W, if we take two elements $g' = gh', g'' = gh''$ of the same left coset gH, their translated subspaces are equal:

$$g' \cdot W = (gh') \cdot W = g \cdot (h' \cdot W) = g \cdot W = g \cdot (h'' \cdot W) = (gh'') \cdot W = g'' \cdot W \qquad (12)$$

Hence, for H-invariant W, we can translate subspaces instead by cosets $\sigma \in G/H$, denoted $\sigma \cdot W$.

Definition 3.2. Then V **is induced by** W, denoted $V = \mathrm{Ind}^G_H(W)$, if:

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W$$

Conversely, it can be shown that given a group G with subgroup H, and a representation W of H, there exists a unique (up to isomorphism) representation V of G such that V is induced by W, i.e. $V = \mathrm{Ind}^G_H(W)$. We show how this is constructed. First let $\{g_\sigma | \ \sigma \in G/H\}$ be a complete set of representatives of the left cosets of H. Let $W^\sigma := \{g_\sigma w \mid w \in W\}$ denote a copy of W, one for each $\sigma \in G/H$, with $g_\sigma w$ just notation for a vector in W^σ. Note since $\{g_\sigma\}$ is a complete set of representatives, $\forall g \in G \ \exists \tau \in G/H, h \in H$ such that $g \cdot g_\sigma = g_\tau \cdot h$. Then we can define $V := \bigoplus_{\sigma \in G/H} W^\sigma$, with action on the elements $g_\sigma w$ of each summand W^σ given by:

$$g \cdot (g_\sigma w) := g_\tau (h \cdot w) \in W^\tau \qquad \text{for } g \cdot g_\sigma = g_\tau \cdot h$$

where $h \cdot w \in W$ denotes h acting on w via the representation we began with. Extending this action linearly to V, it is then easily checked: $g' \cdot (g \cdot (g_\sigma w)) = (g' \cdot g) \cdot (g_\sigma w)$, so V is indeed a representation of G.

We will require an alternative way of viewing the induced representation. To do this let us first introduce, for a general ring R, the notion of the tensor product of a right R-module M with a left R-module N. Roughly speaking it is the abelian group $M \otimes_R N$ of linear combinations of symbols $m \otimes n$ for $m \in M, n \in N$, subject to the relations $(m+m') \otimes n = m \otimes n + m' \otimes n$, $m \otimes (n+n') = m \otimes n + m \otimes n'$ and $(m \cdot r) \otimes n = m \otimes (r \cdot n)$. See Skorobogatov [14] for more detail on this. We then see that the group algebra kG is a right kH-module under multiplication from the right, while the representation W of H is a left kH-module. We claim

$$\mathrm{Ind}^G_H(W) = kG \otimes_{kH} W$$

[†]Note this is equivalent to $h \cdot W = W$ since $w = 1 \cdot w = (hh^{-1}) \cdot w = h \cdot (h^{-1} \cdot w) \in h \cdot W$, so $W \subset h \cdot W \subset W$. We use this in Equation (12).

Using our set of representatives of the cosets $\{g_\sigma\}$ from above, for each $g \in G$ there exists a unique $\sigma \in G/H$ and $h \in H$ such that $g = g_\sigma \cdot h$. For a pure tensor $g \otimes w \in kG \otimes_{kH} W$, we have $g \otimes w = (g_\sigma \cdot h) \otimes w = g_\sigma \otimes (h \cdot w)$, which lies in $\{g_\sigma \otimes w | \ w \in W\}$. This space is just a copy of W, and we see it coincides precisely with the $W^\sigma = \{g_\sigma w\}$ from above. We define an action of G on $kG \otimes_{kH} W$ as

$$g \cdot (g_\sigma \otimes w) := (g \cdot g_\sigma) \otimes w = (g_\tau \cdot h) \otimes w = g_\tau \otimes (h \cdot w)$$

which coincides with the action on $\bigoplus_\sigma W^\sigma$ we defined above. Hence we see these as equivalent characterisations of the induced representation.

3.2 Representations of Hopf algebras

Now for some motivation for Hopf algebras, roughly following Section 3.5 of Kytölä [8]. In group representation theory we can define the trivial representation for a group, and for a representation ρ we can define its dual ρ^*. Also for representations ρ_{V_1}, ρ_{V_2} we can define their tensor product $\rho_{V_1 \otimes V_2}$. However for representations of algebras, i.e. algebra homomorphisms $p : A \to \mathrm{End}(V)$, there isn't a canonical way of defining these things, except when the algebra has the extra data making it a bialgebra (or Hopf algebra if you want duals too):

- Trivial representation of group G over field k is $p : G \to k, \ p(g) = 1_k \ \forall g$.

  We see in the definition of a bialgebra that the counit $\epsilon : H \to k$ must be an algebra homomorphism, which means it is also a representation of H:

  $$p : H \to \mathrm{End}(k) \cong k, \ h \cdot \lambda = \epsilon(h)\lambda \quad \forall \lambda \in k$$

  In the case that H is a group algebra, we set the counit such that $\epsilon(g) = 1 \ \forall g$, meaning it coincides with the trivial representation for groups.

- Tensor product of $p_{V_1} : G \to GL(V_1), \ p_{V_2} : G \to GL(V_2)$ is $p : G \to GL(V_1 \otimes V_2)$,

  $$p(g)(v_1 \otimes v_2) = (p_{V_1}(g)v_1) \otimes (p_{V_2}(g)v_2)$$

  For Hopf algebras, we can use coproduct to define tensor products of representations $p_{V_1} : H \to \mathrm{End}(V_1), \ p_{V_2} : H \to \mathrm{End}(V_2)$, then $p : H \to \mathrm{End}(V_1 \otimes V_2)$:

  $$p(h) := (p_{V_1} \otimes p_{V_2}) \circ \triangle$$

  Again since $\triangle$ is an algebra homomorphism, and the tensor product of algebra homomorphisms is an algebra homomorphism, this composition also is. For H a group algebra, then $\triangle : g \mapsto g \otimes g$ tells us the tensor product representation of group algebras coincides with the tensor product of group representations.

- Dual representation to $p : G \to GL(V)$, is

$$p^* : G \to GL(V^*),\ p^*(g)(\phi)(v) := \phi(p(g^{-1})v)$$

i.e. a representation over the dual space of V. For $p : H \to \mathrm{End}(V)$, then define $p^* : H \to \mathrm{End}(V^*)$ such that

$$p^*(h)(\phi)(v) := \phi(p(S(h))v)$$

Finally this also coincides with the dual representation for groups when H is a group algebra using $S(g) = g^{-1}$.

So we can motivate Hopf algebras as the necessary objects for expanding the notions of trivial, dual and tensor representations from groups to algebras. This idea is made more rigorous using Category theory — see Example 6.9 for a look at how for a Hopf algebra H, the category of H-modules is a **strict monoidal category**.

Note in the above, each representation of the Hopf algebra H is really just a representation of the underlying algebra. Such a representation $\rho : H \to \mathrm{End}(V)$, is equivalently described as a **(left)** H**-module** on the vector space V. It convenient to denote $\rho(h)(v)$ as just $h \triangleright v$. Then the axioms of an algebra homomorphism are just

$$(gh) \triangleright v = g \triangleright (h \triangleright v), \quad 1_H \triangleright v = v \quad \forall g, h \in H, v \in V \tag{13}$$

We next look at how to define the action of a Hopf algebra on an algebra, rather than just a vector space as above:

Definition 3.3.

- For Hopf algebra H and algebra A, A is a H**-module algebra** if A is a left H-module, i.e. satisfies Equation (13), and additionally satisfies:

$$h \triangleright 1_A = \epsilon(h)1_A \qquad h \triangleright (a \cdot b) = (h_{(1)} \triangleright a) \cdot (h_{(2)} \triangleright b)$$

- For Hopf algebra H and A an H-module algebra, then the **smash product algebra** $A\#H$ (which Majid calls the **left cross product algebra** $A \rtimes H$) is the vector space $A \otimes H$, with unit $1 \otimes 1$ and product:

$$(a \otimes h) \cdot (a' \otimes h') := (a \cdot (h_{(1)} \triangleright a')) \otimes (h_{(2)} \cdot h')$$

We can use this to describe the algebra part of the Drinfeld double. Take $H = kG$ and $A = (kG)^*$ as an algebra. Note $(kG)^*$ can be made into a kG-module algebra via the action we defined in Section 2.4: $g \triangleright \phi := \phi^{g^{-1}}$, where $\phi^{g^{-1}}(x) := \phi(g^{-1}xg)$. Then we see the product of the Drinfeld double $D(G)$ as:

$$(\delta_g \otimes h) \cdot (\delta'_g \otimes h') = \delta_g \delta_{hg'h^{-1}} \otimes hh' = \delta_g \delta_{g'}^{h^{-1}} \otimes hh' = \delta_g(h \triangleright \delta_{g'}) \otimes hh'$$

where $\triangle(h) = h \otimes h = h_{(1)} \otimes h_{(2)}$, we see this is precisely the product of the smash product algebra $(kG)^*\#kG$.

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4 The classification of simple $D(G)$-modules

In this section we follow Chapter 2 of Witherspoon [16], in which the simple $D(G)$-modules are classified by introducing the category of so called G-equivariant vector bundles. We classify the irreducible objects in this category, before establishing an equivalence of this category with the category of $D(G)$-modules. See also the discussion of this classification in Section 3.2 of Bakalov and Kirillov [1].

Take finite group G, then recall the following definitions from group theory: a **finite right G-set** is a finite set X with a right action $X \times G \to X, (x, g) \mapsto x^g$, such that $x^{1_G} = x \ \forall x \in X$, and $x^{gh} = (x^g)^h \ \forall g, h \in G, x \in X$. The **$G$-orbit** of $x \in X$, is $x \cdot G := \{x^g \in X : \ g \in G\}$, and the **stabilizer** of $x \in X$ is $G_x := \{g \in G : \ x^g = x\}$.

Definition 4.1. A **G-equivariant k-vector bundle** on G-set X is a collection of finite-dimensional vector spaces $U := \{U_x\}_{x \in X}$ with a representation of G on $\bigoplus_{x \in X} U_x$ satisfying

$$U_x \cdot g = U_{x^g}$$

Unpacking this definition a bit, note the action of G on $\bigoplus_{x \in X} U_x$ is from the right. Denote the representation as $\rho : G \to GL(\bigoplus_{x \in X} U_x)$ so that $\rho(g)(U_x) = U_{g^x}$, we see: $\rho(gh)(U_x) = U_{x^{gh}} = U_{(x^g)^h} = \rho(h)(\rho(g)(U_x))$. In other words ρ is a group anti-homomorphism, so could be called a **right** representation. This makes sense since we want compatibility between the representation and the **right** G-set. Just as "left" representations are also left kG-modules, we see the above just as putting a right kG-module structure on $\bigoplus_{x \in X} U_x$.

Definition 4.2.

- The **x-component/fibre** of the vector bundle $U := \{U_x\}_{x \in X}$ is the vector space U_x.
- A **morphism** of G-equivariant vector bundles U and V on X is a (right) kG-module homomorphism $f : \bigoplus_{x \in X} U_x \to \bigoplus_{x \in X} V_x$ that preserves fibres: $f(U_x) \subset V_x$.
- **Direct sum** of G-equivariant vector bundles U and V on X has x-components given by $U_x \oplus V_x$ for each $x \in X$, with the action given by: $(U_x \oplus V_x) \cdot g = U_{x^g} \oplus V_{x^g}$.

Lemma 4.3. *Classification of the irreducible G-equivariant vector bundles on X.*

Proof. Given a vector bundle $U = \{U_x\}_{x \in X}$, consider the set of G-orbits on X, $\{\sigma = x \cdot G\}$, where $x \in X$ is a representative of the orbit σ. Union of all these orbits covers X: $\cup_\sigma \sigma = X$. Then for $x \in X$, and orbit $\sigma = x \cdot G$, define:

$$U_\sigma := \bigoplus_{x' \in \sigma} U_{x'}$$

Note we are identifying each vector space U_x with its corresponding subspace in $\bigoplus_{x\in X} U_x$ (i.e. vectors with 0's in all entries besides the x-component's entry, which takes values in U_x). Then similarly U_σ is the subspace of $\bigoplus_{x\in X} U_x$ with 0's in all the entries corresponding to the x-components outside of the orbit $x \in X\backslash\sigma$. Then given U_σ is a subspace, is it also a kG-submodule? Since $\sigma = x \cdot G = \{x^g : \ g \in G\} \subset X$ and $U_x \cdot g = U_{x^g}$ we see $U_\sigma \cdot g = U_\sigma$, so it is indeed a right kG-submodule of $\bigoplus_{x\in X} U_x$. Hence we can decompose $\bigoplus_x U_x$ as the direct sum of kG-submodules as:

$$\bigoplus_\sigma U_\sigma$$

Also the U_σ can be seen as a collection of vector spaces over X as: $\{U_x\}_{x\in\sigma} \cup \{0\}_{x\in X\backslash\sigma}$, with a trivial vector space $\{0\}$ over each $x \in X\backslash\sigma$. Hence we can see these U_σ are also G-equivariant vector bundles on X.

We wish to characterise these U_σ. Note each U_x may be regarded as a kG_x-module, for G_x the stabilizer subgroup in G of $x \in X$, by restricting the action of kG. Note for every $x' \in \sigma = x \cdot G \ \ \exists g \in G$ such that $x' = x^g$, hence $U_{x'} = U_{x^g} = U_x \cdot g$. So $U_{x'}$ is determined by U_x for every such x' in the same orbit as x. Hence U_σ is also determined by U_x. Also note that by the Orbit-stabilizer theorem there is a bijection between the $\sigma = x \cdot G$ and right cosets of G/G_x. So we may reexpress $U_\sigma := \bigoplus_{x'\in\sigma} U_{x'}$ as:

$$U_\sigma = \bigoplus_{\alpha\in G/G_x} U_x \cdot g_\alpha$$

for representatives g_α of each right coset $\alpha \in G/G_x$. This coincides with the expression directly above using the fact that the action on $x \in X$ of different representatives for the same coset of G/G_x generates the same element of the orbit: in order words if $g = h \cdot g_\alpha$ for $h \in G_x$ then $x^g = x^{h\cdot g_\alpha} = (x^h)^{g_\alpha} = x^{g_\alpha}$. We then have U_σ in the form of Definition 3.2, albeit a right handed version, so we find U_σ is in fact an induced representation: $\mathrm{Ind}_{G_x}^G(U_x)$.

Under what conditions are these U_σ irreducible representations of G, and hence the irreducible G-equivariant vector bundles? If U_x is a reducible representation of G_x then U_σ is a reducible representation of G. So we certainly require U_x be an irreducible representation of G_x. Crucially by the added property we have for vector bundles that $U_x \cdot g = U_{x^g}$ and how we defined U_σ as a direct sum over the elements of an orbit, we find the irreducibility of U_x is also sufficient for U_σ to be irreducible.

Using the alternative characterisation of the induced representation, we note $U_\sigma = kG \otimes_{kG_x} U_x$. Hence we can define each irreducible G-equivariant vector bundle by (a representative x of) each orbit, and an irreducible kG_x-module U_x. □

Definition 4.4. For a general set X, define the **function algebra** $k(X) = \{f : X \to k\}$ as the algebra under pointwise addition and multiplication. This has a standard basis $\{\delta_x | \ x \in X\}$.

When X is the group G we retrieve the group function algebra $k(G) = (kG)^*$, which we know also has a Hopf algebra structure.

Definition 4.5. When X is a G-set, let $D_X(G) := k(X)\#kG$ be the smash product algebra, with the action of G on $k(X)$ as $f^g(x) := f(x^{g^{-1}})$.

See that for $X = G$, with the conjugation action on itself, we get $D_G(G) = D(G)$, the Drinfeld double.

Definition 4.6.
- Denote the category of finite-dimensional $D_X(G)$-modules as mod$D_X(G)$, with morphisms given by $D_X(G)$-module homomorphisms.
- Denote the category of G-equivariant vector bundles on the G-set X as vect(X, G), with morphisms as given in Definition 4.2.

Then by Lemma 2.1 of Witherspoon, if U is a $D_X(G)$-module, then $\{U \cdot \delta_x\}_{x\in X}$ is a G-equivariant vector bundle on X. Conversely, if $\{U_x\}_{x\in X}$ is a vector bundle, then $\bigoplus_{x\in X} U_x$ is a right $D_X(G)$-module, where the action of $\delta_x \otimes g$ on $u \in \bigoplus_x U_x$ (with components $u_x \in U_x$) is given by: $u \cdot (\delta_x \otimes g) := u_x \cdot g \in U_{x^g}$. This provides the part of the functors mapping objects to objects, between the categories mod$D_X(G)$ and vect(X, G). Maps between the morphisms can also be defined, and hence we gain functors between mod$D_X(G)$ and vect(X, G). Finally in Theorem 2.2 of Witherspoon it is shown that these functors provide an "equivalence of the categories", in the sense of Definition 6.4 (see Appendix on Category theory).

Under these functors we see that the irreducible/simple $D_X(G)$-modules correspond precisely with the irreducible G-equivariant vector bundles on X, and hence can be characterised in the way that we showed in Lemma 4.3. Also recall in the special case $X = G$, regarded as a G-set with the conjugation action, $D_G(G)$ is the Drinfeld double $D(G)$. So using the Lemma, the simple $D(G)$-modules are characterised by:

- a representative $x \in G$ for each G-orbit, which in the case of the conjugation action is precisely a conjugacy class.
- an irreducible representation of the stabilizer group G_x, which again in the case of the conjugation action, is the centralizer $C(x) = \{g \in G |\ gx = xg\}$.

5 Is $D(S_3)$ isomorphic to a group algebra?

5.1 Computing the simple $D(S_3)$-modules

In this section we compute the dimensions of the simple $D(S_3)$-modules. By above, these objects are characterised by (a representative of) each conjugacy class, and an irreducible representation of the centralizer of that representative element. For S_3, let us denote the conjugacy classes as:

$$a_1 = \{1\},\ a_2 = \{(12), (13), (23)\},\ a_3 = \{(123), (132)\}$$

Firstly the centraliser of a_1, $C(a_1)$, is the whole group S_3. Also the centralizer of every element is a subgroup of S_3, so by Lagranges theorem it must have an order of $1, 2, 3, 6$. For distinct a, b, c: $(a\ b)(a\ b\ c) = (b\ c)$ while $(a\ b\ c)(a\ b) = (a\ c)$. Hence the centralizer of any non-identity element cannot be all of S_3. Note the centralizer of any element contains the cyclic subgroup generated by that element. So with these facts and Lagranges theorem we find the centralizer of any nontrivial element of S_3 is precisely the cyclic subgroup generated by that element. So $C((12)) = \{1, (12)\} \cong \mathbb{Z}_2$, and similarly for other choices of representative from the conjugacy class a_2. Finally $C((123)) = \{1, (123), (132)\} \cong \mathbb{Z}_3$.

From representation theory we know S_3 has 3 irreducible representations. In particular the 1-dimensional trivial and sign representations T_1, S_1, and a 2-dimensional representation we denote W_1 (we use the subscript 1 to denote these are representations of the centralizer of elements from the conjugacy class a_1, i.e. S_3). $\mathbb{Z}_2$ has irreducible representations given by the 1-dimensional trivial and sign representations T_2, S_2 (for a_2). In the case $\mathrm{char}(k) = 0$ and k contains a primitive cube root of unity, $\mathbb{Z}_3$ has three 1-dimensional irreducible representations which we denote as T_3, V_3, V_3'.

Using what we just found in the last section we can index the simple $D(S_3)$-modules with a conjugacy class a_i and irreducible representation j, which we denote $U_{(i,j)}$. We give all such simple modules below:

$$U_{(1,T)} = \bigoplus_{x\in a_1} T_1,\ U_{(1,S)} = \bigoplus_{x\in a_1} S_1,\ U_{(1,W)} = \bigoplus_{x\in a_1} W_1$$
$$U_{(2,T)} = \bigoplus_{x\in a_2} T_2,\ U_{(2,S)} = \bigoplus_{x\in a_2} S_2$$
$$U_{(3,T)} = \bigoplus_{x\in a_3} T_3,\ U_{(3,V)} = \bigoplus_{x\in a_3} V_3,\ U_{(3,V')} = \bigoplus_{x\in a_3} V_3'$$

Now we can compute their dimensions:

$$\dim(U_{(1,T)}) = 1,\ \dim(U_{(1,S)}) = 1,\ \dim(U_{(1,W)}) = 2$$
$$\dim(U_{(2,T)}) = 3,\ \dim(U_{(2,S)}) = 3$$
$$\dim(U_{(3,T)}) = 2,\ \dim(U_{(3,V)}) = 2,\ \dim(U_{(3,V')}) = 2$$

The computation explicitly for $\dim(U_{(2,T)})$: since $|a_2| = 3$ and $\dim(T_2) = 1$, we have $\dim(U_{(2,T)}) = 3 \cdot \dim(T_2) = 3 \cdot 1 = 3$. The others are similar. Hence we see there are 2 simple $D(S_3)$-modules of dimension 1, 4 such modules of dimension 2, and 2 such modules of dimension 3. This information is attached to the final row of the table below.

Recall from Section 3 we discussed that the Artin-Wedderburn theorem applies to $D(S_3)$, and the size $\{n_i\}$ of the matrix rings $M_{n_i}(k)$ in the matrix ring decomposition correspond with the dimensions of the simple $D(S_3)$-modules, which we have just found. If our computations are correct we must find $\sum_i n_i^2 = \dim(D(S_3)) = 36$, which we now check: $1^2 + 1^2 + 2^2 + 3^2 + 3^2 + 2^2 + 2^2 + 2^2 = 36$, confirming what we hoped. Now by the discussion in Section 3, in particular using the Artin-Wedderburn theorem, all we require to estabish the isomorphism is a match between the dimensions $\{n_i\}$ with the degrees of the irreducible representations of groups order 36.

5.2 Comparing with the groups of order 36

We copy part of the data from [6] into the table below (with the dimensions of our simple $D(S_3)$-modules appended to the final row). This shows the number of distinct irreducible representations — up to isomorphism — there are of each degree, for groups of order 36. Since it is possible for two distinct groups of the same order to have the same numbers of distinct irreducible representations of each degree, the final column gives the number of such groups with those combinations of numbers. So by summing the last column we see there are: 4+2+2+1+4+1 = 14 groups of order 36. In Equation (11) we showed the sum of the squares of the degrees of the irreducible representations equals the order of the group. Checking this for the data in the 6-th row of the table, for instance, which is $(4, 4, 0, 1)$, we find: $4(1^2)+4(2^2)+0(3^2)+1(4^2) = 4+16+16 = 36$ as required.

Irreps of degree 1	Irreps of degree 2	Irreps of degree 3	Irreps of degree 4	Number of groups with these degrees of irreps
36	0	0	0	4
12	6	0	0	2
9	0	3	0	2
4	0	0	2	1
4	8	0	0	4
4	4	0	1	1
2	4	2	0	0

We put the multiplicities of the dimensions of the simple $D(S_3)$-modules in the final row. Comparing against the rows above it, we see there is no match. So we have found that for the smallest non-abelian group S_3, the answer to the question we asked at the start: is there a group H for which $D(S_3)$ is isomorphic, as an algebra, to its group algebra kH, is sadly: no.

An obvious avenue for further research would be to continue testing this question for more non-abelian groups. With additional data, perhaps found with a program such as GAP, it might become apparent how to refine the question to some subset of the non-abelian groups for which the answer is positive.

6 Appendix

6.1 Tensor products

Definition 6.1. For vector spaces V, W over field k, the **free vector space** over $V \times W$, denoted $F(V \times W)$, is the direct sum of copies of k indexed by $V \times W$:

$$\bigoplus_{(v,w)\in V\times W} k$$

This is a vector space whose elements are just finite, formal linear combinations

$$\lambda_1(v_1, w_1) + \cdots + \lambda_n(v_n, w_n),\ \ \lambda_i \in k,\ \ (v_i, w_i) \in V \times W$$

Definition 6.2. The **tensor product** of vector spaces V, W is

$$V \otimes W := F(V \times W)/\sim$$

for $\sim$ the equivalence relation given by

- $\lambda(v, w) \sim (\lambda v, w) \sim (v, \lambda w)$
- $(v + x, w) \sim (v, w) + (x, w)$, and $(v, w + y) \sim (v, w) + (v, y)$

Elements of $V \otimes W$ are denoted $v \otimes w$. In this notation the relations above give:

- $\lambda(v \otimes w) = \lambda v \otimes w = v \otimes \lambda w$
- $(v + x) \otimes w = v \otimes w + x \otimes w$, and $v \otimes (w + y) = v \otimes w + v \otimes y$

Since the tensor product is also a vector space, it is an abelian group over addition: $v \otimes w + x \otimes y = x \otimes y + v \otimes w$.

Note: $k \otimes V \cong V$ with the isomorphism given by linear maps $\lambda \otimes v \mapsto \lambda v$, and inverse $v \mapsto 1 \otimes v$.

The key property we require of tensor products is that they are characterised by the following universal property of bilinear maps: The tensor product $V \otimes W$ with bilinear map $\otimes : V \times W \to V \otimes W$ is defined such that for each bilinear map $\widetilde{m} : V \times W \to Z$ there exists a unique linear map $m : V \otimes W \to Z$ such that $\widetilde{m}(v, w) = m(\otimes(v, w)) = m(v \otimes w)$.

Definition 6.3. For algebras $(A, m_1, \eta_1), (B, m_2, \eta_2)$, the **tensor product algebra** is given by the tensor product of A and B as vector spaces, $A \otimes B$, with product given by: $(a \otimes c) \cdot (b \otimes d) = (ab \otimes cd) \quad \forall a, b \in A\ \forall c, d \in B$, and unit $1_{A \otimes B} = 1_A \otimes 1_B$.

Using the notation introduced in Definition 2.1 for an algebra: recall the **twist map** as $\tau : H \otimes H \to H \otimes H,\ v \otimes w \mapsto w \otimes v$, and let $\phi : k \otimes k \to k$ be the natural isomorphism defined as: $\phi(\lambda \otimes \mu) = \lambda\mu,\ \phi^{-1}(\lambda) = 1 \otimes \lambda$. Then the product and unit are defined as

$$m_{A \otimes B} : A \otimes B \otimes A \otimes B \to A \otimes B,\ m_{A \otimes B} := (m_1 \otimes m_2) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$$

$$\eta_{A \otimes B} : k \to A \otimes B,\ \eta_{A \otimes B} := (\eta_1 \otimes \eta_2) \circ \phi^{-1}$$

6.2 Category theory

We first provide some introductory definitions:

Definition 6.4.
- A (small) **category** is made up of a set $\mathrm{ob}(C)$ of **objects**, with a set $\mathrm{hom}(C)$ of **morphisms**, being arrows $f : a \to b$ such that $a, b \in \mathrm{ob}(C)$. Denote $\mathrm{hom}(a, b)$ as the subset of morphisms from a to b. There is additionally a binary operation $\mathrm{hom}(a, b) \times \mathrm{hom}(b, c) \to \mathrm{hom}(a, c),\ (f, g) \mapsto g \circ f$ corresponding to the composition of morphisms. These satisfy "associativity": $h \circ (g \circ f) = (h \circ g) \circ f$ and "identity axiom": $\forall X \in \mathrm{ob}(C)\ \exists 1_X : X \to X \in \mathrm{hom}(X, X)$ such that $\forall f \in \mathrm{hom}(A, X), g \in \mathrm{hom}(X, B)$ we have $1_X \circ f = f,\ g \circ 1_X = g$.

- A morphism $f : A \to B$ is an **isomorphism** if there exists $g : B \to A$ such that $f \circ g = 1_B,\ g \circ f = 1_A$.

- For categories C, D, a **functor** is a map $F : C \to D$ that sends $X \in \mathrm{ob}(C)$ to $F(X) \in \mathrm{ob}(D)$ and sends morphism $f : X \to Y$ in C to the morphism $F(f) : F(X) \to F(Y)$ such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and for morphisms in C, $f : X \to Y, g : Y \to Z$, we have $F(g \circ f) = F(g) \circ F(f)$.

- For functors $F, G : C \to D$ a **natural transformation** associates to each $X \in \mathrm{ob}(C)$ a morphism in D, $\eta_X : F(X) \to G(X)$, such that for each morphism in C,

$f : X \to Y$ we get the following commutative diagram:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

- A **natural isomorphism** is a natural transformation such that for every $X \in \mathrm{ob}(C)$ the morphism η_X in D is an isomorphism.

- An **equivalence of categories** C, D is made up functors $F : C \to D,\ G : D \to C$ and natural isomorphisms $\epsilon : F \circ G \to 1_D,\ \eta : 1_C \to G \circ F$, where $1_C, 1_D$ denote the identity functors on C, D respectively. The idea is that F, G are "mutually inverse up to isomorphism".

  Or the stronger relation between categories,

- An **isomorphism of categories** C, D has functors $F : C \to D,\ G : D \to C$ such that $F \circ G = 1_D$ and $G \circ F = 1_C$.

- For categories C, D the **product** category is denoted $C \times D$ with objects given by pairs (A, B) for $A \in \mathrm{ob}(C), B \in \mathrm{ob}(D)$, arrows between (A_1, B_1) and (A_2, B_2) as a pair (f, g) for $f \in \hom(A_1, A_2), g \in \hom(B_1, B_2)$ with composition componentwise and for each object (A, B) an identity morphism: $1_{(A,B)} := (1_A, 1_B)$.

- A **bifunctor** is a functor with a domain being a product category.

Definition 6.5. A **monoidal category** is a category C with

- bifunctor $\otimes : C \times C \to C$ called the **tensor product**

- for functors $(- \otimes -) \otimes -,\ - \otimes (- \otimes -) : C \times C \times C \to C$, a natural isomorphism $a : (- \otimes -) \otimes - \ \to\ - \otimes (- \otimes -)$ called the **associator**. So in particular: $\forall X, Y, Z \in \mathrm{ob}(C)$ we have an isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$.

- an object $1 \in \mathrm{ob}(C)$ called the **identity/unit object**

- two natural isomorphisms, called the **left/right unitors** λ, ρ between the functors $(1 \otimes -), (- \otimes 1) : C \to C$ and the identity functor $(-) : C \to C$ (so for $X \in \mathrm{ob}(C)$ we have isomorphisms $\lambda_X : 1 \otimes X \to X,\ \rho_X : X \otimes 1 \to X$).

such that the following axioms hold:

- the "unit axiom":

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

- the "pentagon axiom": (see commutative diagram below)

(Pentagon axiom)

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes U \arrow[r, "a_{X,Y,Z} \otimes 1_U"] \arrow[d, "a_{X \otimes Y,Z,U}"'] & (X \otimes (Y \otimes Z)) \otimes U \arrow[r, "a_{X,Y \otimes Z,U}"] & X \otimes ((Y \otimes Z) \otimes U) \arrow[d, "1_X \otimes a_{Y,Z,U}"] \\
(X \otimes Y) \otimes (Z \otimes U) \arrow[rr, "a_{X,Y,Z \otimes U}"'] & & X \otimes (Y \otimes (Z \otimes U))
\end{tikzcd}$$

Definition 6.6. A **strict** monoidal category is a monoidal category for which the natural isomorphisms a, λ, ρ are identity morphisms.

Notation: For group G its category of representations is denoted: Rep(G), while the category of kG-modules is: kG-Mod, and for Hopf algebra H, its category of H-modules is: H-Mod.

Example 6.7. Rep(G) forms a monoidal category.

Proof. See Example 2.3.4 of Etingof [4] □

Example 6.8. There is an isomorphism of categories between Rep(G) and kG-Mod.

In the start of Section 3 we explain how the representations of G can equivalently be given as kG-modules, and vice versa. This gives us the part of the functors between Rep(G) and kG-Mod mapping objects to objects. To establish the isomorphism we must also show how the functors map the morphisms between each category, and finally show that the mappings of objects (and morphisms) from each category to the other, and then back again, results in the same exact object (and morphism).

Example 6.9. For a Hopf algebra H, H-Mod is a strict monoidal category.

Proof. See Schauenburg [12]. □

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7 References

[1] B. Bakalov and A. A. Kirillov. *Lectures on Tensor Categories and Modular Functors*. University lecture series. American Mathematical Society, 2001.

[2] K. Conrad. Characters of finite abelian groups. https://kconrad.math.uconn.edu/blurbs/grouptheory/charthy.pdf. Expository paper.

[3] D. Chua. Algebras. http://dec41.user.srcf.net/notes/III_L/algebras.pdf, 2017. Lecture notes, Accessed: 03/04/2019.

[4] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor Categories*. Mathematical Surveys and Monographs. American Mathematical Society, 2016.

[5] W. Fulton and J. Harris. *Representation theory : a first course*. Graduate texts in mathematics : readings in mathematics ; 129. Springer-Verlag, New York, c1991.

[6] Groupprops contributors. Linear representation theory of groups of order 36. https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_groups_of_order_36. Accessed: 10/02/2019.

[7] C. Kassel. *Quantum Groups*. Graduate Texts in Mathematics ; 155. Springer Publishing Company, Incorporated, 2012.

[8] K. Kytölä. Introduction to hopf algebras and representations. Lecture notes, University of Helsinki, Spring 2011.

[9] S. Majid. *A quantum groups primer*. London Mathematical Society lecture note series ; 292. Cambridge University Press, Cambridge, 2002.

[10] S. Majid. *Foundations of quantum group theory*. Cambridge University Press, Cambridge, c1995.

[11] S. Montgomery. Hopf algebras and their actions on rings. In *Hopf algebras and their actions on rings*, Regional conference series in mathematics ; no. 82, Providence, R.I, c1993. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society.

[12] P. Schauenburg. Hopf algebra extensions and monoidal categories. In S. Montgomery and H.J. Schneider, editors, *New Directions in Hopf Algebras*, volume 43 of *Mathematical Sciences Research Institute Publications*, pages 321–381. Cambridge University Press, 2002.

[13] A. N. Sengupta. *Representing finite groups a semisimple introduction*. Springer New York, New York, c2012.

[14] A. Skorobogatov. Algebra IV. Lecture notes, Imperial College London, 2017.

[15] Wikipedia contributors. Universal enveloping algebra — Wikipedia, the free encyclopedia, 2019. [Online; accessed 22-April-2019].

[16] S.J. Witherspoon. The representation ring of the quantum double of a finite group. *Journal of Algebra*, 179(1):305 – 329, 1996.