1 Semidirect product relations

We wish to prove that

$$\phi(g) \star \phi(\underline{x}_k) = \phi(g(\underline{x}_k)) \star \phi(g) \ \forall g \in \mu(G(m,p,n)) \tag{1}$$

We know that $\mu(G(m,p,n))$ is generated by σ_i and the elements of $T(m,p,n)$, and so far we have proven that the semidirect product relations hold for σ_i and $t_i^{(\zeta)}$. Note:

- The proof of the semidirect product relation for $t_i^{(\zeta)}$ remains valid if we replace $t_i^{(\zeta)}$ with an arbitrary $t \in T(m,p,n)$. Therefore the semidirect product relations hold for all generators of $\mu(G(m,p,n))$.
- By linearity, the semidirect product relations for $\sigma_i, t \in T(m,p,n)$ remain true if we replace $\underline{x}_k$ by any vector $v \in V$.

Assuming the validity of Section 5.8 from the preprint, i.e. that ϕ is an algebra homomorphism on the group algebra $\mathbb{C}\mu(G(m,p,n))$ so $\phi(gg') = \phi(g) \star \phi(g') \ \forall g, g' \in \mu(G(m,p,n))$, we posit that (1) does indeed hold:

Let g be an arbitrary element of $\mu(G(m,p,n))$, so $g = s \dots uv$ for some $s, \dots, u, v \in \{\sigma_i\} \cup T(m,p,n)$. Using the fact (1) holds for each of $s, \dots, u, v$, we find:

$$\begin{aligned}
\phi(g) \star \phi(\underline{x}_k) &= \phi(s \dots uv) \star \phi(\underline{x}_k) \\
&= (\phi(s) \star \dots \star \phi(u) \star \phi(v)) \star \phi(\underline{x}_k) \\
&= (\phi(s) \star \dots \star \phi(u)) \star (\phi(v(\underline{x}_k)) \star \phi(v))
\end{aligned}$$

Noting that $v(\underline{x}_k) \in V$, and the semidirect product relation holds for all elements of V, we find the above is equal to:

$$\begin{aligned}
&= \phi(s) \star \dots \star (\phi(u(v(\underline{x}_k))) \star \phi(u)) \star \phi(v) \\
&= \phi(s) \star \dots \star \phi(uv(\underline{x}_k)) \star \phi(uv) \\
&\dots \\
&= \phi(g(\underline{x}_k)) \star \phi(g)
\end{aligned}$$

as required.