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1 Twist of $\mathbb{C}S_4$

We view the Klein 4-group $T = \langle a, b \mid a^2 = b^2 = 1, ab = ba\rangle$ as a subset of $\mathbb{C}S_4$ via the following embedding: $a \mapsto (12), b \mapsto (34)$. Then $\mathbb{C}S_4$ becomes a $\mathbb{C}T$-module algebra via an action of T given by conjugation, i.e. $a \triangleright g := (12)g(12)$, $b \triangleright g = (34)g(34)$. We take the following (non-trivial) cocycle of $\mathbb{C}T$:

$$\chi = \frac{1}{2}(1 \otimes 1 + (12) \otimes 1 + 1 \otimes (34) - (12) \otimes (34))$$

We wish to investigate the structure of the twisted module algebra $(\mathbb{C}S_4)_\chi$. First recall that group algebras (over $\mathbb{C}$) are semisimple and therefore $\mathbb{C}S_4 \cong \bigoplus_{i\in I} M_{n_i}(\mathbb{C})$ for some matrix rings $M_{n_i}(\mathbb{C})$ and some index set I. Note that under this isomorphism $\bigoplus_{i\in I} M_{n_i}(\mathbb{C})$ is also a $\mathbb{C}T$-module algebra, again with T acting by conjugation, this time by certain elements of $\bigoplus_i M_{n_i}(\mathbb{C})$. Each of the matrix rings $M_{n_i}(\mathbb{C})$ can be seen to be a $\mathbb{C}T$-module algebra too. Indeed, each ring $M_{n_i}(\mathbb{C})$ is an ideal of $\bigoplus_i M_{n_i}(\mathbb{C})$, and is therefore closed under multiplication from the left or right. As the action of T on $M_{n_i}(\mathbb{C})$ is just to conjugate by some element of $\bigoplus_i M_{n_i}(\mathbb{C})$, we see this action must be closed, and therefore $M_{n_i}(\mathbb{C})$ is a $\mathbb{C}T$-submodule of $\bigoplus_i M_{n_i}(\mathbb{C})$. So each ring $M_{n_i}(\mathbb{C})$ must be a $\mathbb{C}T$-module algebra in its own right. Next we prove a general result about module algebras:

Lemma 1.1. *If A, B are H-module algebras and χ is a cocycle for H, then $(A \oplus B)_\chi = A_\chi \oplus B_\chi$.*

Proof. Certainly this equality is true at the level of vector spaces. It remains to verify the products on these two spaces coincide. If the products on A and B are denoted m_A, m_B respectively, then the product on $A \oplus B$ is $m_{A\oplus B}((a, b) \otimes (a', b')) := (m_A(a \otimes a'), m_B(b \otimes b'))$. Note $A \oplus B$ forms an H-module algebra whereby H acts diagonally, i.e. $h \triangleright (a, b) = (h \triangleright a, h \triangleright b)$. Therefore $A \oplus B$ is amenable to twisting, and the resulting product is $(m_{A\oplus B})_\chi =$

$m_{A\oplus B}(\chi^{-1} \triangleright (a,b) \otimes (a',b'))$. Suppose $\chi^{-1} = \sum \chi_1 \otimes \chi_2$ for some $\chi_1, \chi_2 \in H$, then

$$\begin{aligned}(m_{A\oplus B})_\chi &= m_{A\oplus B}((\chi_1 \triangleright a, \chi_1 \triangleright b) \otimes (\chi_2 \triangleright a', \chi_2 \triangleright b'))\\ &= (m_A(\chi_1 \triangleright a \otimes \chi_2 \triangleright a'), m_B(\chi_1 \triangleright b \otimes \chi_2 \triangleright b'))\\ &= (m_A(\chi^{-1} \triangleright (a,a'), m_B(\chi^{-1} \triangleright (b,b'))))\\ &= ((m_A)_\chi(a,a'), (m_B)_\chi(b,b')\end{aligned}$$

where this last line is the product on $A_\chi \oplus B_\chi$, as we required. □

Applying this Lemma we can deduce the following:

$$(\mathbb{C}S_4)\chi \cong (\bigoplus_i M_{n_i}(\mathbb{C}))\chi \cong \bigoplus_i M_{n_i}(\mathbb{C})_\chi \tag{1}$$

The problem of understanding $(\mathbb{C}S_4)_\chi$ therefore reduces to understanding how matrix rings change under twists.

Above we reasoned that each matrix ring $M_{n_i}(\mathbb{C})$ is a $\mathbb{C}T$-module algebra, so in particular the group T acts by automorphisms on $M_{n_i}(\mathbb{C})$, for each $i \in I$. Since every automorphism of a matrix ring is inner, we find that for each $i \in I$, we can identify T with a certain subset of invertible elements of $M_{n_i}(\mathbb{C})$ such that the action of T becomes conjugation by these elements.

Now let us consider the twisted algebra $M_{n_i}(\mathbb{C})_\chi$, which is also a $\mathbb{C}T$-module algebra since $\mathbb{C}T$ is commutative and therefore $\mathbb{C}T_\chi \cong \mathbb{C}T$. The action of $\mathbb{C}T$ on $M_{n_i}(\mathbb{C})_\chi$ is the same as that of $\mathbb{C}T$ on $M_{n_i}(\mathbb{C})$, which, by above, is given by conjugation (with respect to the product on $M_{n_i}(\mathbb{C})$).

Lemma 1.2. *If I is a 2-sided ideal of $M_{n_i}(\mathbb{C})_\chi$, then I is also a $\mathbb{C}T$-submodule of $M_{n_i}(\mathbb{C})_\chi$.*

Proof. For $t \in T,\ i \in I$ we have $t \triangleright i = \tau \cdot i \cdot \tau^{-1}$ where $\cdot$ is the product on $M_{n_i}(\mathbb{C})$ and τ is some invertible element of $M_{n_i}(\mathbb{C})$. At this point we notice $\tau \cdot i = \tau \star i$ where $\star$ is the product on $M_{n_i}(\mathbb{C})_\chi$. This is true because

$$\begin{aligned}\chi \triangleright \tau \otimes i &= \frac{1}{2}(\tau \otimes i + ((12) \triangleright \tau) \otimes i + \tau \otimes ((34) \triangleright i) - ((12) \triangleright \tau) \otimes ((34) \triangleright i))\\ &= \tau \otimes i\end{aligned}$$

since $(12) \triangleright \tau = \mu \cdot \tau \cdot \mu^{-1}$ for some $\mu \in M_{n_i}(\mathbb{C})$, where μ and τ are elements of a subgroup of $M_{n_i}(\mathbb{C})$ isomorphic to T. Since this group is commutative and every element has order 2 we see that $(12) \triangleright \tau = \tau$, from which the above follows. Therefore $\tau \star i = \ \cdot (\chi \triangleright \tau \otimes i) =$

$\cdot\, (\tau \otimes i) = \tau \cdot i$. Similarly one shows that $(\tau \cdot i) \cdot \tau^{-1} = (\tau \cdot i) \star \tau^{-1}$, and so $\tau \triangleright i = \tau \star i \star \tau^{-1}$. Finally we apply the fact I is a 2-sided ideal of $M_{n_i}(\mathbb{C})_\chi$ to deduce $\tau \triangleright i \in I$, as required. $\square$

Corollary 1.3. *If I is a 2-sided ideal of $M_{n_i}(\mathbb{C})_\chi$, then the underlying subspace of I also defines a 2-sided ideal of $M_{n_i}(\mathbb{C})$.*

Proof. Let $a \in M_{n_i}(\mathbb{C})$ and $i \in I$. Then $a \cdot i = \star(\chi \triangleright a \otimes i) = \star((\chi_1 \triangleright a) \otimes (\chi_2 \triangleright i))$ where $\chi = \sum \chi_1 \otimes \chi_2$. Now $\chi_1 \triangleright a \in M_{n_i}(\mathbb{C})$, and by Lemma 1.2 we know I is a $\mathbb{C}T$-submodule and therefore $\chi_2 \triangleright i \in I$. Since I is an ideal of $M_{n_i}(\mathbb{C})_\chi$ we deduce $\star((\chi_1 \triangleright a) \otimes (\chi_2 \triangleright i)) \in I$. So we have shown $a \cdot i \in I$, and therefore I is a left ideal of $M_{n_i}(\mathbb{C})$. Similarly one shows it is a right ideal, and therefore 2-sided. $\square$

From this corollary, and the fact that matrix rings are simple, we find that the twisted matrix rings $M_{n_i}(\mathbb{C})_\chi$ must also be simple. Additionally, since simple $\mathbb{C}$-algebras are uniquely determined (up to isomorphism) by their dimension, we deduce $M_{n_i}(\mathbb{C})_\chi \cong M_{n_i}(\mathbb{C})$. We can now continue the series of isomorphisms in (1) above,

$$(\mathbb{C}S_4)\chi \cong (\bigoplus_i M_{n_i}(\mathbb{C}))\chi \cong \bigoplus_i M_{n_i}(\mathbb{C})_\chi \cong \bigoplus_i M_{n_i}(\mathbb{C}) \cong \mathbb{C}S_4$$

2 Twisting representations of rational Cherednik algebras

2.1 Approach 1.

For a rational Cherednik algebra $H = H_c(G(m,p,n))$ with m even, and $T = (C_2)^n$, consider a representation $\rho : H \to \mathrm{End}(V)$. Now, strictly under the assumption that $\frac{m}{p}$ is even, so that $T \subseteq G(m,p,n)$, we can equip $\mathrm{End}(V)$ with a $\mathbb{C}T$-module structure via: $t \triangleright f = \rho(t) f \rho(t)^{-1}$ for $t \in T,\ f \in \mathrm{End}(V)$.

Lemma 2.1. *ρ is a $\mathbb{C}T$-module homomorphism, and additionally can be viewed as an algebra homomorphism of the twisted algebras: $\rho : H_\mathcal{F} \to End(V)_\mathcal{F}$.*

Proof. ρ is a $\mathbb{C}T$-module homomorphism since $t \triangleright \rho(h) = \rho(t)\rho(h)\rho(t)^{-1} = \rho(tht^{-1}) = \rho(t \triangleright h)$. Now let $m, m_\mathcal{F}$ denote the products on $H, H_\mathcal{F}$ respectively, and $m', m'_\mathcal{F}$ the products on $\mathrm{End}(V), \mathrm{End}(V)_\mathcal{F}$. Then $\rho(m_\mathcal{F}(h \otimes h')) = (\rho \circ m)(\mathcal{F}^{-1} \triangleright h \otimes h') = (m' \circ (\rho \otimes \rho))(\mathcal{F}^{-1} \triangleright h \otimes h') = m'(\mathcal{F}^{-1} \triangleright (\rho \otimes \rho)(h \otimes h')) = m'_\mathcal{F} \circ (\rho \otimes \rho)(h \otimes h')$, where in the second equality we apply the fact ρ is an algebra homomorphism of the untwisted algebras, and in the 3rd isomorphism we use the fact ρ is a $\mathbb{C}T$-module homomorphism. $\square$

Assuming now that V is finite-dimensional, then for each choice of basis for V we have an isomorphism $\mathrm{End}(V) \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, and as we know matrix rings are simple. Using an identical line of reasoning to that used in the proofs of Lemma 1.2 and Corollary 1.3 one can show that $\mathrm{End}(V)_{\mathcal{F}} \cong M_n(\mathbb{C})_{\mathcal{F}}$ must also be simple. Therefore $\mathrm{End}(V)_{\mathcal{F}} \cong \mathrm{End}(V')$ for some $V' \cong V$. Using this isomorphism we arrive at a new representation, of the same dimension as before, but for the twisted rational Cherednik algebra: $\rho' : H_{\mathcal{F}} \to \mathrm{End}(V')$.

2.2 Approach 2

As before we start with a representation $\rho : H \to \mathrm{End}(V)$ for $H = H_c(G(m,p,n))$. Now H is a subalgebra of $\overline{H} = H_c(G(m,1,n))$. Notice that $T \subseteq G(m,1,n)$. We can use ρ to get an induced representation for $\overline{H}$ given by: $V' = \overline{H} \otimes_H V$.

Is V' still finite dimensional? I think yes.. For groups $H \leq G$ where V is a $\mathbb{C}H$-module, we have $\dim(\mathbb{C}G \otimes_{\mathbb{C}H} V) = \frac{|G|}{|H|} \cdot \dim(V)$. Do we have for $\mathbb{C}$-algebras $S \subseteq R$ (sharing identities), where V is an S-module, that $\dim(R \otimes_S V) = \frac{\dim(R)}{\dim(S)} \cdot \dim(V)$?

Now in our case S and R are both infinite-dimensional, but one is a “finite-dimensional extension” of the other.. $H = S(V) \otimes \mathbb{C}G(m,p,n) \otimes S(V^*)$ as a vector space, so

$$\dim(H) = \dim(S(V))^2 \cdot |G(m,p,n)| = \dim(S(V))^2 \cdot \frac{m^n n!}{p}$$

Similarly $\dim(\overline{H}) = \dim(S(V))^2 \cdot m^n n!$. Is there a rigorous way to conclude $\frac{\dim(\overline{H})}{\dim(H)} = p$??

If so, then, by restriction, we have a new finite-dimensional representation ρ' of H on the space V', which is of dimension $p \dim(V)$. We can define an action of T on $\mathrm{End}(V')$ in an indentical fashion to Approach 1. Therefore it remains prove a second version of Lemma 2.1 for ρ' to get a twisted representation of a rational Cherednik algebra.

3 References