

Twists of quantum Drinfeld Hecke algebras

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Abstract

In the preprint [2] it was shown that certain rational and braided Cherednik algebras are related via a Drinfeld twist. We compare this to results from Naidu and Witherspoon [6] which show quantum Drinfeld Hecke algebras, which generalise the rational and braided Cherednik algebras, arise as certain deformations of their associated graded algebras. In particular we show that if a pair of quantum Drinfeld Hecke algebras are related by a Drinfeld twist then their associated graded algebras, which are isomorphic to skew group algebras, are related by the same twist. Conversely, if two skew group algebras are related by a Drinfeld twist, then it is possible to preserve the twisting result when the algebras are deformed by certain “equivariant” Hochschild 2-cocycles into quantum Drinfeld Hecke algebras.

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1 Introduction

1.1 Preliminaries

In this section we review several definitions and results from Naidu and Witherspoon [6] and the preprint [2], before outlining our new results in Section 1.2. Proofs will be the subject of the remainder of the article. Unless otherwise stated assume all objects in this text are \mathbb{C} -linear, with tensor products also taken over \mathbb{C} .

Given a \mathbb{C} -vector space V with basis v_1, \dots, v_n and an $n \times n$ -matrix $q = (q_{ij})$ such that $q_{ii} = 1, q_{ij}q_{ji} = 1$, we define a **skew symmetric algebra** as $S_q(V) := \mathbb{C}\langle v_1, \dots, v_n \mid v_i v_j = q_{ij} v_j v_i \ \forall i, j \rangle$, and a **skew exterior algebra** as $\bigwedge_q(V) := \mathbb{C}\langle v_1, \dots, v_n \mid v_i v_j = -q_{ij} v_j v_i \rangle$. Now let G be a group that is equipped with an action on V that extends to an action by algebra automorphisms on $S_q(V)$.

Definition 1.1. Let $\kappa : V \times V \rightarrow \mathbb{C}G$ be a bilinear map such that $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$, and define

$$H_{q,\kappa} := T(V) \rtimes G / \langle v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) \rangle$$

Recall $T(V) \rtimes G$ has underlying vector space $T(V) \otimes \mathbb{C}G$, identity $1_{\mathbb{C}} \otimes 1_G$, and product: $(a \otimes g) \cdot (a' \otimes g') = (a \cdot (g \triangleright a')) \otimes gg'$ for $a, a' \in T(V)$, $g, g' \in G$, and the action of G on $T(V)$ is given by naturally extending the action of G on V . With v_i in degree 1 and $g \in G$ in degree 0, $H_{q,\kappa}$ is a filtered algebra. $H_{q,\kappa}$ is called a **quantum Drinfeld Hecke algebra** if its associated graded algebra is isomorphic to the skew group algebra $S_q(V) \rtimes G$. Levandovsky and Shepler [3] give the conditions on κ that ensure $H_{q,\kappa}$ is a quantum Drinfeld Hecke algebra.

Drinfeld Hecke algebras (also known as graded Hecke algebras) arise in the special case $q_{ij} = 1 \ \forall i, j$. These include the symplectic reflection algebras, and hence all rational Cherednik algebras too. Of import to us is that the more general *quantum* Drinfeld Hecke algebras include the *braided* Cherednik algebras, a class of algebras introduced by Bazlov and Berenstein in [1]. One example of such an algebra is the “negative braided Cherednik algebra”, which is, in a sense, the anticommutative analogue of a rational Cherednik algebra. An explicit characterisation of the negative braided Cherednik algebra as a quantum Drinfeld Hecke algebra was given in [6] (Remark 7.4).

In Naidu and Witherspoon [6] the quantum Drinfeld Hecke algebras were characterised as deformations of the skew group algebras of the form $S_q(V) \rtimes G$. For a \mathbb{C} -algebra R , a **deformation over $\mathbb{C}[t]$** is an associative $\mathbb{C}[t]$ -algebra with underlying vector space $\mathbb{C}[t] \otimes_{\mathbb{C}} R$ and product:

$$r * s = rs + \mu_1(r \otimes s)t + \mu_2(r \otimes s)t^2 + \dots$$

for \mathbb{C} -linear maps $\mu_i : R \otimes R \rightarrow R$, $i \in \mathbb{N}$, extended linearly over $\mathbb{C}[t]$. It turns out that associativity of $*$ implies μ_1 is a Hochschild 2-cocycle (see Definition 3.1). A “quantum Drinfeld Hecke algebra over $\mathbb{C}[t]$ ” will be $\mathbb{C}[t]$ -linear analogue which specialises at $t = 1$ to the quantum Drinfeld Hecke algebras of Definition 1.1.

Theorem 1.2 ([6], Theorem 2.2). *The quantum Drinfeld Hecke algebras over $\mathbb{C}[t]$ are precisely those deformations over $\mathbb{C}[t]$ of $S_q(V) \rtimes G$ where $\deg(\mu_i) = -2i$.*

Note that $\deg(\mu_i) = -2i$ means that the map $\mu_i : (S_q(V) \rtimes G)^{\otimes 2} \rightarrow S_q(V) \rtimes G$ is a map of graded algebras which sends elements of degree j to elements of degree $j - 2i$. By this theorem we can associate a Hochschild 2-cocycle μ_1 (which is a map of degree -2) to each quantum Drinfeld Hecke algebra.

Definition 1.3. [6] A **constant Hochschild 2-cocycle** is a Hochschild 2-cocycle of a skew group algebra $S_q(V) \rtimes G$ that is degree -2 as a map of graded algebras.

The following result shows that each constant Hochschild 2-cocycles conversely gives rise to a quantum Drinfeld Hecke algebra:

Theorem 1.4 ([6], Theorem 4.4, 4.6). *When the action of G on V extends to an action on $\bigwedge_q(V)$ by algebra automorphisms, each constant Hochschild 2-cocycle α of $S_q(V) \rtimes G$ lifts to deformation of $S_q(V) \rtimes G$ over $\mathbb{C}[t]$ which is isomorphic to a quantum Drinfeld Hecke algebra Q_α with associated graded algebra isomorphic to $S_q(V) \rtimes G$.*

Finally, let us we recall the main theorem from the new preprint.

Theorem 1.5 (Bazlov, Berenstein, Jones-Healey, McGaw [2]). *For complex reflection group $G = G(m, p, n)$ with m even and reflection representation V , the corresponding rational Cherednik algebra $H_c(G)$ is a $\mathbb{C}T$ -module algebra for $T := (C_2)^n$. Additionally there is a cocycle μ of $\mathbb{C}T$ such that the Drinfeld twist of $H_c(G)$ by μ is isomorphic to the negative braided Cherednik algebra $\underline{H}_c(\mu(G))$.*

1.2 Statements of new results

Here we outline our new results as a series of three claims, whose precise statements and proofs will be given in the next sections.

Claim 1: if two quantum Drinfeld Hecke algebras are related by Drinfeld twist, then their associated graded algebras are related by the same twist.

Using Theorem 1.5 to provide an example of quantum Drinfeld Hecke algebras related by a twist we can deduce from the claim that:

$$(S(V \oplus V^*) \rtimes G)_\mu = S_{-1}(V \oplus V^*) \rtimes \mu(G) \quad (1)$$

Proving Claim 1 is the subject of Section 2. Next we consider the opposite direction, taking a pair of skew group algebras related by twist - as in (1) - and use these to produce a pair of quantum Drinfeld Hecke algebras related by a twist. In particular, suppose we have skew group algebras $A = S_q(V) \rtimes G$ and $B = S_{q'}(V') \rtimes G'$, which are:

- related by a Drinfeld twist, i.e. A is an H -module algebra for some Hopf algebra H and there exists a counital 2-cocycle μ of H such that $A_\mu \cong B$,
- and such that G and G' act by algebra automorphisms on $\bigwedge_q(V)$ and $\bigwedge_{q'}(V')$ respectively (this allows us to apply Theorem 1.4 in order to lift Hochschild 2-cocycles to a quantum Drinfeld Hecke algebras).

Claim 2: the cocycle μ of H induces an isomorphism between the space of H -equivariant Hochschild 2-cocycles of A (see Definition 3.2) and the space of H_μ -equivariant Hochschild 2-cocycles for B , i.e. we have a map

$$\mathrm{HH}_{\triangleright}^2(A) \xrightarrow{\sim} \mathrm{HH}_{\triangleright}^2(B), [\alpha] \mapsto [\alpha_\mu] \quad (2)$$

Additionally if α is constant (as in Definition 1.3), then α_μ is constant. We prove this in Section 3.

Claim 3: Each H -equivariant Hochschild 2-cocycle α of A can be used to “lift” the twisting result $A_\mu \cong B$ to a twisting result between quantum Drinfeld Hecke algebras. In particular we have $(Q_\alpha)_\mu \cong Q_{\alpha_\mu}$, where α_μ is given via (2) and Q_α, Q_{α_μ} are quantum Drinfeld Hecke algebras deforming A and B respectively via Theorem 1.4.

This claim can be illustrated as a commuting (up to isomorphism) diagram of twists and deformations:

$$\begin{array}{ccc} Q_\alpha & \xrightarrow{\text{Drinfeld twist by } \mu} & (Q_\alpha)_\mu \cong Q_{\alpha_\mu} \\ \uparrow \text{deformation by } \alpha & & \uparrow \text{deformation by } \alpha_\mu \\ S_q(V) \rtimes G & \xrightarrow{\text{Drinfeld twist by } \mu} & S_{q'}(V') \rtimes G' \end{array}$$

It should be noted that none of the arrows in this diagram are algebra homomorphisms. This relationship between Drinfeld twists and deformations adds a new perspective to where the twisting result between rational and braided Cherednik algebras in [2] comes from. We can see it as arising as a deformation of a more fundamental twisting result between underlying skew group algebras.

This suggests a strategy for finding further quantum Drinfeld Hecke algebras (or braided Cherednik algebras) that are related by a Drinfeld twist - look for skew group algebras related by twist, and deform via suitable Hochschild 2-cocycles. We apply this strategy in Section 5 to find a new example of quantum Drinfeld Hecke algebras related by twist.

2 Twists of associated graded algebras

In this section we address Claim 1 of the Introduction. First we recall some definitions and properties regarding filtered algebras and their associated graded algebras. A **filtered algebra** Q has a filtration $\{F_\bullet\}$, i.e. subspaces $\{0\} = F_{-1} \subset F_0 \subset F_1 \subset \dots \subset Q$ such that $\cup_{i=0} F_i = Q$ and $F_i \cdot F_j \subseteq F_{i+j}$. The **associated graded algebra** of Q is given by the space $\text{gr}(Q) := \bigoplus_{i \geq 0} F_i/F_{i-1}$ with product:

$$F_i/F_{i-1} \times F_j/F_{j-1} \rightarrow F_{i+j}/F_{i+j-1}, (x + F_{i-1}, y + F_{j-1}) \mapsto xy + F_{i+j-1} \quad (3)$$

Lemma 2.1. *An (iso)morphism of filtered algebras $\phi : Q \rightarrow Q'$ induces an (iso)morphism between their associated graded algebras $\text{gr}(\phi) : \text{gr}(Q) \rightarrow \text{gr}(Q')$.*

Proof. Suppose the filtered algebras Q, Q' have filtrations $\{F_\bullet\}, \{G_\bullet\}$ respectively. A morphism of these algebras is an algebra homomorphism $\phi : Q \rightarrow Q'$ satisfying $\phi(F_i) \subseteq G_i \forall i$. Let $\text{gr}(\phi) : \text{gr}(Q) \rightarrow \text{gr}(Q')$ be defined in the following way: for each $i \geq 0$ consider the maps $\text{gr}(\phi)^i : F_i/F_{i-1} \rightarrow G_i/G_{i-1}$, $v + F_{i-1} \mapsto \phi(v) + G_{i-1}$, and let $\text{gr}(\phi) := \bigoplus_{i \geq 0} \text{gr}(\phi)^i$. Each $\text{gr}(\phi)^i$ is well-defined: if $v + F_{i-1} = v' + F_{i-1}$ then $v - v' \in F_{i-1}$ and $\phi(v - v') = \phi(v) - \phi(v') \in G_{i-1}$, hence $\phi(v) + G_{i-1} = \phi(v') + G_{i-1}$. Clearly $\text{gr}(\phi)$ is a linear map. Finally it is an algebra homomorphism: for $a = x + F_{i-1}, b = y + F_{j-1}$ then applying (3) we find $\text{gr}(\phi)(ab) = \text{gr}(\phi)(xy + F_{i+j-1}) = \phi(xy) + G_{i+j-1} = \phi(x)\phi(y) + G_{i+j-1} = \text{gr}(\phi)(a)\text{gr}(\phi)(b)$, as required.

If ϕ is additionally an isomorphism of filtered algebras, i.e. it has an inverse ϕ^{-1} that satisfies $\phi^{-1}(G_i) \subseteq F_i$, then we have algebra homomorphisms $\text{gr}(\phi) : \text{gr}(Q) \rightarrow \text{gr}(Q')$ and $\text{gr}(\phi^{-1}) : \text{gr}(Q') \rightarrow \text{gr}(Q)$. Now $\text{gr}(\phi)\text{gr}(\phi^{-1})(x + G_{i-1}) = \text{gr}(\phi)(\phi^{-1}(x) + F_{i-1}) = \phi(\phi^{-1}(x)) + G_{i-1} = x + G_{i-1}$, and similarly $\text{gr}(\phi^{-1})\text{gr}(\phi) = \text{id}_{\text{gr}(Q)}$. Hence $\text{gr}(\phi^{-1}) = \text{gr}(\phi)^{-1}$, and we deduce $\text{gr}(\phi)$ is an algebra isomorphism. \square

Lemma 2.2. *Suppose Q is a filtered algebra and an H -module algebra for a Hopf algebra H acting by degree-preserving endomorphisms. Additionally let μ be a 2-cocycle on H . Then*

1. *the associated graded algebra $\text{gr}(Q)$ has a natural H -module algebra structure.*
2. *the Drinfeld twist Q_μ inherits the structure of a filtered algebra from Q .*

Proof. (1). Suppose Q has filtration $\{F_\bullet\}$, so $\text{gr}(Q) = \bigoplus_i F_i/F_{i-1}$. Since the action of H is degree-preserving we have $h \triangleright x \in F_i \forall h \in H, x \in F_i$. Therefore we can define an action of H on F_i/F_{i-1} as $h \triangleright x + F_{i-1} = (h \triangleright x) + F_{i-1}$. This indeed makes each subspace F_i/F_{i-1} an H -module: $(gh) \triangleright v + F_{i-1} = ((gh) \triangleright v) + F_{i-1} = (g \triangleright (h \triangleright v)) + F_{i-1} = g \triangleright (h \triangleright (v + F_{i-1}))$ and $1 \triangleright (v + F_{i-1}) = (1 \triangleright v) + F_{i-1} = v + F_{i-1}$. As a direct sum of H -modules $\text{gr}(Q)$ is itself an H -module. It is additionally an H -module algebra:

- The unit of $\text{gr}(Q)$ is given by $1 + F_{-1}$ where 1 is the unit of Q which is degree 0 in the filtration on Q . So: $g \triangleright 1 + F_{-1} = (g \triangleright 1) + F_{-1} = (\epsilon(g)1) + F_{-1} = \epsilon(g)(1 + F_{-1})$, since Q is an H -module algebra so $h \triangleright 1 = \epsilon(h)1$ where ϵ is the counit of H .
- Let Δ be the coproduct on H , and in Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)} \forall h \in H$. Since Q is an H -module algebra, $h \triangleright vw = (h_{(1)} \triangleright v) \cdot (h_{(2)} \triangleright w)$. Finally,

$$\begin{aligned} h \triangleright (v + F_{i-1}) \cdot (w + F_{j-1}) &= h \triangleright (vw + F_{i+j-1}) = (h \triangleright vw) + F_{i+j-1} \\ &= (h_{(1)} \triangleright v) \cdot (h_{(2)} \triangleright w) + F_{i+j-1} \\ &= \left((h_{(1)} \triangleright v) + F_{i-1} \right) \cdot \left((h_{(2)} \triangleright w) + F_{j-1} \right) \\ &= \left(h_{(1)} \triangleright (v + F_{i-1}) \right) \cdot \left(h_{(2)} \triangleright (w + F_{j-1}) \right) \end{aligned}$$

Hence all conditions are checked, and $\text{gr}(Q)$ is an H -module algebra.

(2). Q_μ has same underlying vector space as Q , so the filtration $\{F_\bullet\}$ on Q is also a filtration on the vector space of Q_μ . If m denotes product on Q and $v \in F_i, w \in F_j$, then we know $m(v \otimes w) \in F_{i+j}$. Since H acts by degree preserving homomorphisms we see $m_\mu(v \otimes w) := m(\mu^{-1} \triangleright v \otimes w) \in F_{i+j}$, as required. \square

Theorem 2.3. *Let Q, \underline{Q} be quantum Drinfeld Hecke algebras whose associated graded algebras are isomorphic to $S_q(V) \rtimes G$ and $S_{q'}(V') \rtimes G'$ respectively. Suppose further Q is an H -module algebra for some Hopf algebra H acting by degree-preserving homomorphisms on Q , and that there is a 2-cocycle μ of H such that the Drinfeld twist Q_μ is isomorphic, as a filtered algebra, to \underline{Q} . (with filtration on Q_μ given by Lemma 2.2 (2)). Then the algebras $S_q(V) \rtimes G$ and $S_{q'}(V') \rtimes G'$ are related by the same twist, i.e.*

$$(S_q(V) \rtimes G)_\mu \cong S_{q'}(V') \rtimes G' \quad (4)$$

Proof. By Lemma 2.2(1) $\text{gr}(Q)$ is an H -module algebra, and so $S_q(V) \rtimes G$ also is. Therefore we can consider the Drinfeld twist of $S_q(V) \rtimes G$ by the cocycle μ . We prove (4) via the following sequence of isomorphisms:

$$S_{q'}(V') \rtimes G' \cong \text{gr}(\underline{Q}) \cong \text{gr}(Q_\mu) \cong \text{gr}(Q)_\mu \cong (S_q(V) \rtimes G)_\mu \quad (5)$$

The first and last of these isomorphisms follow by definition of Q and \underline{Q} respectively. The second isomorphism follows by applying Lemma 2.1 with the fact $\underline{Q} \cong Q_\mu$ as filtered algebras (recall Q_μ was shown to have a filtered structure in Lemma 2.2(2)).

Finally we tackle the third isomorphism of (5). Since Q_μ and Q share the same underlying vector space and filtration, the algebras $\text{gr}(Q_\mu)$ and $\text{gr}(Q)$ also share the same underlying vector space. This implies $\text{gr}(Q_\mu)$ and $\text{gr}(Q)_\mu$ share the same vector space too. We show the products on $\text{gr}(Q_\mu)$ and $\text{gr}(Q)_\mu$ coincide, deducing these algebras are in fact equal. If m_μ is the product on Q_μ , the product on $\text{gr}(Q_\mu)$ is given, via (3), by

$$\text{gr}(m_\mu)(v + F_{i-1} \otimes w + F_{j-1}) = m_\mu(v \otimes w) + F_{i+j-1}$$

If the product on $\text{gr}(Q)$ is given similarly by $\text{gr}(m)(v + F_{i-1} \otimes w + F_{j-1}) = m(v \otimes w) + F_{i+j-1}$, we see the product for $\text{gr}(Q)_\mu$ is:

$$\begin{aligned} \text{gr}(m)_\mu(v + F_{i-1} \otimes w + F_{j-1}) &:= \text{gr}(m)(\mu^{-1} \triangleright v + F_{i-1} \otimes w + F_{j-1}) \\ &= \text{gr}(m) \left(\sum_k \mu_k \triangleright (v + F_{i-1}) \otimes \mu'_k \triangleright (w + F_{j-1}) \right) \\ &= \text{gr}(m) \left(\sum_k (\mu_k \triangleright v) + F_{i-1} \otimes (\mu'_k \triangleright w) + F_{j-1} \right) \\ &= \sum_k m((\mu_k \triangleright v) \otimes (\mu'_k \triangleright w)) + F_{i+j-1} \\ &= m(\mu^{-1} \triangleright v \otimes w) + F_{i+j-1} \\ &= \text{gr}(m_\mu)(v + F_{i-1} \otimes w + F_{j-1}) \end{aligned}$$

where $\mu^{-1} = \sum_k \mu_k \otimes \mu'_k$ for some $\mu_k, \mu'_k \in H$. \square

Corollary 2.4. *The associated graded algebras of the rational and negative braided Cherednik algebras $H_c(G), \underline{H}_c(\mu(G))$ in Theorem 1.5 are related by a Drinfeld twist:*

$$(S(V \oplus V^*) \rtimes G)_\mu \cong S_{-1}(V \oplus V^*) \rtimes \mu(G) \quad (6)$$

Proof. This result follows immediately via Theorems 1.5 and 2.3, however we first must actually check the conditions of Theorem 2.3 are satisfied in order for us to apply it. Firstly the algebras $H_c(G), \underline{H}_c(\mu(G))$ are indeed quantum Drinfeld Hecke algebras with associated graded algebras isomorphic to $S(V \oplus V^*) \rtimes G, S_{-1}(V \oplus V^*) \rtimes \mu(G)$ respectively. By Theorem 1.5 we know $H_c(G)$ is a $\mathbb{C}T$ -module algebra, but it remains to check the action is degree-preserving with respect to the filtration on $H_c(G)$. It is enough to check the action preserves the degree of the generators of $H_c(G)$, which are the elements of the group G (degree 0), the basis vectors x_i of V and y_i of V^* (degree 1). For $t \in T$ we have $t \triangleright g := tgt \in G$, so $\deg(t \triangleright g) = 0 = \deg(g)$. Also, $t \triangleright x_j = \pm x_j \in V$ and $t \triangleright y_j = \pm y_j \in V^*$, so $\deg(t \triangleright x_j) = 1 = \deg(x_j)$ as required, and similarly for y_j . So the action is indeed degree-preserving.

Finally we must check the algebra isomorphism $\phi : \underline{H}_c(\mu(G)) \rightarrow (H_c(G))_\mu$ in the proof of Theorem 1.5 is an isomorphism of filtered algebras. $\underline{H}_c(\mu(G))$ is filtered with elements of $\mu(G)$ in degree 0 and elements $\underline{x}_i, \underline{y}_i$ in degree 1. Also ϕ maps $\underline{x}_i \mapsto x_i, \underline{y}_i \mapsto y_i$ and $\mathbb{C}\mu(G) \mapsto \mathbb{C}G$, so it is degree-preserving on the generators of $\underline{H}_c(\mu(G))$, and therefore a morphism of filtered algebras. Likewise the inverse ϕ^{-1} would be degree-preserving on generators, so ϕ is isomorphism of filtered algebras. \square

3 Equivariant Hochschild 2-cocycles

In this section we prove Claim 2 from the Introduction. We begin with some definitions.

Definition 3.1. A **Hochschild 2-cocycle** of an algebra A is given by a linear map $\alpha : A \otimes A \rightarrow A$ such that $a\alpha(b \otimes c) + \alpha(a \otimes bc) = \alpha(ab \otimes c) + \alpha(a \otimes b)c \forall a, b, c \in A$. Denoting the product on A by $m : A \otimes A \rightarrow A$, we can reexpress this condition as the following equality of maps $A \otimes A \otimes A \rightarrow A$:

$$m \circ (\text{id} \otimes \alpha) + \alpha \circ (\text{id} \otimes m) = m \circ (\alpha \otimes \text{id}) + \alpha \circ (m \otimes \text{id}) \quad (7)$$

Additionally, a **2-coboundary** is a Hochschild 2-cocycle α such that

$$\alpha(a \otimes b) = m(a \otimes \beta(b)) - \beta(m(a \otimes b)) + m(\beta(a) \otimes b) \quad (8)$$

for some linear map $\beta : A \rightarrow A$.

When A is also an H -module algebra for some Hopf algebra H (i.e. an algebra object in $H\text{-Mod}$) we will want to distinguish those Hochschild 2-cocycles that are compatible with the action of H :

Definition 3.2. An **H -equivariant Hochschild 2-cocycle** is a Hochschild 2-cocycle α which commutes with the action of H , i.e.

$$h \triangleright \alpha(a \otimes b) = \alpha(\triangle(h) \triangleright a \otimes b) \quad \forall h \in H, a, b \in A \quad (9)$$

In other words α is an H -module homomorphism with respect to the natural H -module structure on $A \otimes A$. For a group algebra $H = \mathbb{C}G$, for example, this says $g \triangleright \alpha(a \otimes b) = \alpha((g \triangleright a) \otimes (g \triangleright b)) \forall g \in G$.

Definition 3.3. An **H -equivariant Hochschild 2-coboundary** is a 2-coboundary α satisfying (8) for some linear map $\beta : A \rightarrow A$ which is also an H -module homomorphism, i.e. $h \triangleright \beta(a) = \beta(h \triangleright a) \forall h \in H, a \in A$.

Notice that every H -equivariant 2-coboundary is also H -equivariant as a 2-cocycle, i.e. in the sense of Definition 3.2. However a 2-coboundary that is H -equivariant as a 2-cocycle, i.e. in the sense of Definition 3.2, is not necessarily an H -equivariant 2-coboundary in the sense of Definition 3.3.

An obvious equivalence relation can be defined on the H -equivariant Hochschild 2-cocycles whereby two cocycles are related to each other if their difference is an H -equivariant 2-coboundary (in the sense of Definition 3.3). We briefly check this is indeed an equivalence relation:

- Reflexivity: $\alpha \sim \alpha$ since the difference $\alpha - \alpha = 0$ is the trivial 2-coboundary given by $\beta = 0$, where β trivially commutes with the action of H , as required.
- Symmetry: If $\alpha \sim \alpha'$, then $\alpha - \alpha'$ is a 2-coboundary given by some H -module homomorphism β . Therefore $\alpha' - \alpha$ is also a 2-coboundary via the homomorphism using $-\beta$, which also commutes with H .
- Transitivity: Suppose $\alpha \sim \alpha'$, where $\alpha - \alpha'$ is a coboundary given by H -module homomorphism β , and $\alpha' \sim \alpha''$ is similarly given by a homomorphism γ . Then $\alpha - \alpha''$ is a coboundary given by the homomorphism $\beta + \gamma$, so we have $\alpha \sim \alpha''$.

Let $\text{HH}_{\triangleright}^2(A)$ denote the vector space of equivalence classes of H -equivariant Hochschild 2-cocycles on A .

Theorem 3.4. *For H -module algebra A , suppose α is an H -equivariant Hochschild 2-cocycle of A and μ is a counital 2-cocycle of H . Let H_μ and A_μ denote the Drinfeld twists of H and A by μ respectively. Then the following is an H_μ -equivariant Hochschild 2-cocycle of A_μ :*

$$\alpha_\mu : A_\mu \otimes A_\mu \rightarrow A_\mu, \quad a \otimes b \mapsto \alpha(\mu^{-1} \triangleright a \otimes b)$$

Additionally, if α is an H -equivariant 2-coboundary of A then α_μ is an H_μ -equivariant 2-coboundary of A_μ . Therefore we have a well-defined map $\text{HH}_{\triangleright}^2(A) \rightarrow \text{HH}_{\triangleright}^2(A_\mu)$, $[\alpha] \mapsto [\alpha_\mu]$ between equivalence classes of equivariant cocycles. This is an isomorphism since Drinfeld twist by μ^{-1} induces the inverse map.

Proof. Let m denote the product on A , and by assumption, α satisfies (7). Firstly we show α_μ is a Hochschild 2-cocycle of A_μ , which is the algebra with product defined as: $m_\mu(a \otimes b) := m(\mu^{-1} \triangleright a \otimes b)$. Suppose $\mu^{-1} = \sum \mu_1 \otimes \mu_2$ for some $\mu_1, \mu_2 \in H$ (for better readability we omit the summation indices). In analogy to (7), α_μ is a Hochschild 2-cocycle of A_μ if,

$$m_\mu \circ (\text{id} \otimes \alpha_\mu) + \alpha_\mu \circ (\text{id} \otimes m_\mu) = m_\mu \circ (\alpha_\mu \otimes \text{id}) + \alpha_\mu \circ (m_\mu \otimes \text{id}) \quad (10)$$

We check this by showing the results of each side applied to an arbitrary element $a \otimes b \otimes c$ coincide. We start by applying the LHS of (10) to $a \otimes b \otimes c$:

$$\begin{aligned} (m_\mu \circ (\text{id} \otimes \alpha_\mu) + \alpha_\mu \circ (\text{id} \otimes m_\mu))(a \otimes b \otimes c) &= m(\mu^{-1} \triangleright \alpha(\mu^{-1} \triangleright a \otimes b) \otimes c) + \alpha(\mu^{-1} \triangleright m(\mu^{-1} \triangleright a \otimes b) \otimes c) \\ &= m(\mu_1 \triangleright \alpha(\mu^{-1} \triangleright a \otimes b) \otimes (\mu_2 \triangleright c)) + \alpha(\mu_1 \triangleright m(\mu^{-1} \triangleright a \otimes b) \otimes (\mu_2 \triangleright c)) \\ &= m(\alpha(\Delta(\mu_1)\mu^{-1} \triangleright a \otimes b) \otimes (\mu_2 \triangleright c)) + \alpha(m(\Delta(\mu_1)\mu^{-1} \triangleright a \otimes b) \otimes (\mu_2 \triangleright c)) \end{aligned}$$

where in the final line we apply the H -equivariance of α , and the fact A is an H -module algebra so m commutes with the action of H in exactly the same way as α . This now becomes:

$$\begin{aligned} &= (m \circ (\alpha \otimes \text{id}) + \alpha \circ (m \otimes \text{id}))[\Delta(\mu_1)\mu^{-1} \triangleright (a \otimes b) \otimes (\mu_2 \triangleright c)] \\ &= (m \circ (\alpha \otimes \text{id}) + \alpha \circ (m \otimes \text{id}))[(\Delta \otimes \text{id})(\mu^{-1})(\mu^{-1} \otimes 1) \triangleright a \otimes b \otimes c] \\ &= (m \circ (\text{id} \otimes \alpha) + \alpha \circ (\text{id} \otimes m))[(\text{id} \otimes \Delta)(\mu^{-1})(1 \otimes \mu^{-1}) \triangleright a \otimes b \otimes c] \end{aligned} \quad (11)$$

where in the final line we apply (7) and use the fact that μ being a counital 2-cocycle of H implies:

$$(\text{id} \otimes \Delta)(\mu^{-1})(1 \otimes \mu^{-1}) = (\Delta \otimes \text{id})(\mu^{-1})(\mu^{-1} \otimes 1)$$

At this point one takes (11) and essentially does the opposite of all the previous steps in order get an expression written in terms of m_μ and α_μ again. Doing this, we arrive at $(m_\mu \circ (\alpha_\mu \otimes \text{id}) + \alpha_\mu \circ (m_\mu \otimes \text{id}))(a \otimes b \otimes c)$ (i.e. the RHS of (10) applied to $a \otimes b \otimes c$), precisely as required. So α_μ is indeed a Hochschild 2-cocycle of A_μ .

Next we check α_μ is H_μ -equivariant. The coproduct on H_μ is $\Delta_\mu(h) := \mu \cdot \Delta(h) \cdot \mu^{-1}$, so

$$\begin{aligned} h \triangleright \alpha_\mu(a \otimes b) &= h \triangleright \alpha(\mu^{-1} \triangleright a \otimes b) \\ &= \alpha(\Delta(h)\mu^{-1} \triangleright a \otimes b) \\ &= \alpha(\mu^{-1}\Delta_\mu(h) \triangleright a \otimes b) \\ &= \alpha_\mu(\Delta_\mu(h) \triangleright a \otimes b) \end{aligned}$$

as required.

For the final part, suppose α is an H -equivariant 2-coboundary, i.e. it satisfies (8) for some $\beta : A \rightarrow A$ which commutes with action of H . Then, if $\mu^{-1} = \sum_i \mu_i \otimes \mu'_i$ for some $\mu_i, \mu'_i \in H$,

$$\begin{aligned} \alpha_\mu(a \otimes b) &= \alpha(\mu^{-1} \triangleright a \otimes b) \\ &= \sum_i \alpha((\mu_i \triangleright a) \otimes (\mu'_i \triangleright b)) \\ &= \sum_i m(\mu_i \triangleright a \otimes \beta(\mu'_i \triangleright b)) - \beta \circ m(\mu_i \triangleright a \otimes \mu'_i \triangleright b) + m(\beta(\mu_i \triangleright a) \otimes \mu'_i \triangleright b) \\ &= m_\mu(a \otimes \beta(b)) - \beta \circ m_\mu(a \otimes b) + m_\mu(\beta(a) \otimes b) \end{aligned}$$

where the final equality applies the fact β commutes with action of H . So indeed α_μ satisfies (8) using the same map β as for α , here regarded as a linear map $A_\mu \rightarrow A_\mu$ (recall A_μ has the same underlying vector space as A). Also since H_μ and H have same underlying vector space and the action of H_μ on A_μ is the same as that of H on A , β also commutes under the action of H_μ . So α_μ is an H_μ -equivariant 2-coboundary as required. \square

By this theorem we deduce if algebras A and B are related via a Drinfeld twist then there is an isomorphism between the spaces of equivariant Hochschild 2-cocycles on the two algebras. Next we check the property of being a *constant* Hochschild 2-cocycle (see Definition 1.3) is preserved under this isomorphism:

Corollary 3.5. *Let $A = S_q(V) \rtimes G$ be an H -module algebra for some Hopf algebra H acting by degree-preserving homomorphisms. Suppose μ is also a counital 2-cocycle of H , and α is a constant and H -equivariant Hochschild 2-cocycle of A . Then α_μ is a constant Hochschild 2-cocycle of A_μ .*

Proof. Note $\alpha_\mu(a \otimes b) := \alpha(\mu^{-1} \triangleright a \otimes b)$, and since H acts by degree-preserving homomorphisms on A and α is a degree -2 map with respect to the grading on A (which coincides with the grading on A_μ by Lemma 2.2(2)), α_μ must also be a degree -2 map, and therefore a constant cocycle. \square

4 Lifting twists to quantum Drinfeld Hecke algebras

In this section we tackle Claim 3 from the Introduction. We begin in Section 4.1 by formulating the claim precisely, before proving it in Section 4.2.

4.1 Setting the scene

Let us start by supposing we have the following data:

- $A = S_q(V) \rtimes G$, which is an H -module algebra for a Hopf algebra H acting by degree-preserving endomorphisms,
- $B = S_{q'}(V') \rtimes G'$,
- a counital 2-cocycle μ of H ,
- a constant H -equivariant Hochschild 2-cocycle α of A , which, by Theorem 3.4, defines a constant H_μ -equivariant Hochschild 2-cocycle α_μ on A_μ ,

satisfying the following conditions:

- A and B are related by a Drinfeld twist by μ , i.e. $A_\mu \cong B$; where this is additionally an isomorphism of graded algebras,
- the action of G on V extends to an action on $\bigwedge_q(V)$ by algebra automorphisms so that, by Theorem 1.4, the cocycle α defines a quantum Drinfeld Hecke algebra Q_α satisfying $\text{gr}(Q_\alpha) \cong A$,
- the action of G' on V' extends to an action on $\bigwedge_{q'}(V')$ by algebra automorphisms, which similarly means α_μ (interpreted now as a Hochschild 2-cocycle for B using the fact $A_\mu \cong B$) defines a quantum Drinfeld Hecke algebra Q_{α_μ} satisfying $\text{gr}(Q_{\alpha_\mu}) \cong B$.

Claim 3 can now be stated in two parts:

- (a) Q_α is an H -module algebra, and is therefore amenable to twisting by the cocycle μ .
- (b) The twist of Q_α by μ is isomorphic to the algebra Q_{α_μ} .

We prove both parts of the claim in Section 4.2. However we make several further assumptions, since the claim is not likely to hold in the current level of generality:

- let $H = \mathbb{C}T$, the group algebra of a subgroup T of G . This is indeed a Hopf algebra with coproduct $\Delta(t) = t \otimes t \ \forall t \in T \subset \mathbb{C}T$.
- the action of $H = \mathbb{C}T$ on $A = S_q(V) \rtimes G$ is induced (in a way made explicit shortly) by the following actions of H on V and $\mathbb{C}G$ respectively:
 - Since T is a subgroup of G , the action of $\mathbb{C}G$ on V restricts to give an action of $H = \mathbb{C}T$ on V . We assume that the set of subspaces spanned by each basis vector $\Omega = \{V_i := \mathbb{C}v_i \mid i \in [n]\}$ forms a system of imprimitivity with respect to this action of H . This means that $t(V_i) \in \Omega \ \forall t \in T, i \in [n]$, or more simply: $t(v_i) = \lambda v_j$ for some $\lambda \in \mathbb{C}, j \in [n]$. This action of H on V extends to make the tensor algebra $T(V)$ an H -module algebra. Furthermore we assume that $q_{ij} = q_{kl}$ whenever $\exists t \in T$ such that $t(V_i) = V_k$ and $t(V_j) = V_l$.
 - $H = \mathbb{C}T$ acts on $\mathbb{C}G$ via the adjoint action: $t \triangleright g := tgt^{-1} \ \forall t \in T, g \in G$. By Majid [4] (Proposition 2.7) this makes $\mathbb{C}G$ an H -module algebra.

It is quite non-trivial how the two actions of H on V and $\mathbb{C}G$ respectively induce an action of H on $S_q(V) \rtimes G$, so we explain this in detail next. The first step is to apply the following lemma with $A = T(V)$ to deduce that $T(V) \rtimes G$ is an H -module algebra.

Lemma 4.1. *Suppose A is a $\mathbb{C}G$ -module algebra, and $A \rtimes G$ is the smash product algebra. If T is a subgroup of G , then $A \rtimes G$ has a $\mathbb{C}T$ -module algebra structure.*

Proof. First note that on restricting the action of $\mathbb{C}G$ on A , A naturally forms a $\mathbb{C}T$ -module algebra. Additionally we can make $\mathbb{C}G$ a $\mathbb{C}T$ -module algebra via the adjoint action $t \triangleright g = tgt^{-1}$ (see Majid [4] Proposition 2.7). Since $\mathbb{C}T$ is a Hopf algebra we can tensor product the two $\mathbb{C}T$ -modules A and $\mathbb{C}G$ together to get a $\mathbb{C}T$ -module structure on $A \otimes \mathbb{C}G$, with T acting diagonally on the tensor components. It remains to check this action is compatible with the algebra structure of $A \rtimes G$. Recall the product on $A \rtimes G$ is given by: $(a \otimes g) \cdot (a' \otimes g') = (a \cdot (g \triangleright a')) \otimes gg'$. Now

$$\begin{aligned}
t \triangleright (a \otimes g) \cdot (a' \otimes g') &= t \triangleright (a \cdot (g \triangleright a')) \otimes gg' \\
&= t \triangleright (a \cdot (g \triangleright a')) \otimes t \triangleright gg' \\
&= [(t \triangleright a) \cdot (tg \triangleright a')] \otimes tgg't^{-1} \\
&= (t \triangleright a) \cdot [(t \triangleright g) \triangleright (t \triangleright a')] \otimes (t \triangleright g) \cdot (t \triangleright g') \\
&= [(t \triangleright a) \otimes (t \triangleright g)] \cdot [(t \triangleright a') \otimes (t \triangleright g')] \\
&= (t \triangleright (a \otimes g)) \cdot (t \triangleright (a' \otimes g'))
\end{aligned}$$

Finally, $t \triangleright 1_{A \rtimes G} = t \triangleright 1_A \otimes 1_G = t \triangleright 1_A \otimes t \triangleright 1_G = 1_A \otimes 1_G = 1_{A \rtimes G}$. \square

Via the following Lemma we can use the action of $\mathbb{C}T$ on $T(V) \rtimes G$ to induce an action on $S_q(V) \rtimes G$,

Lemma 4.2. *Let A be an H -module algebra.*

1. *If I is an ideal and H -submodule of A then A/I is an H -module algebra.*
2. *Suppose A is also a filtered algebra and I is any ideal of A . Then A/I is also a filtered algebra. Additionally, if the action of H is degree-preserving with respect to the filtration on A , then the induced action on A/I is degree-preserving.*

Proof. 1. It is easy to check to the axioms, for instance the product rule for module algebras follows by:

$$h \triangleright (a + I) \cdot (b + I) = h \triangleright (ab + I) = (h \triangleright ab) + I = (h \triangleright a) \cdot (h \triangleright b) + I = h \triangleright (a + I) \cdot h \triangleright (b + I).$$

2. If A has filtration $\{F_\bullet\}$, then A/I can easily be shown to have filtration $\{\frac{F_\bullet + I}{I}\}$. Clearly then if the action on A is degree-preserving, i.e. $h \triangleright F_i \subseteq F_i$, then the action on A/I is degree-preserving. Indeed suppose $v \in F_i + I$, then we wish to check $h \triangleright v + I \in F_i + I$. We have $h \triangleright (v + I) = (h \triangleright v) + I$, and $h \triangleright v \in F_i + I$ using the fact I is a submodule and the action is degree-preserving on A . \square

By Lemma 4.2(1), if the 2-sided ideal $I := \langle v_i \otimes v_j - q_{ij}v_j \otimes v_i \rangle$ is an H -submodule of $T(V) \rtimes G$, then $S_q(V) \rtimes G = T(V) \rtimes G/I$ will be an H -module algebra, as we require. Showing I is an H -submodule amounts to showing the action of H on I is closed when acting on the generators of I . Recall that as part of our assumptions earlier we had that for each $t \in T$, $t \triangleright v_i = \lambda v_k$, $t \triangleright v_j = \mu v_l$ for some $\lambda, \mu \in \mathbb{C}, k, l \in [n]$. Therefore $t \triangleright v_i \otimes v_j - q_{ij}v_j \otimes v_i = \lambda\mu(v_k \otimes v_l - q_{ij}v_l \otimes v_k)$. This is seen to be just another (rescaled) generator of I , under the crucial assumption that $q_{ij} = q_{kl}$. This is the reason for our assumption that $q_{ij} = q_{kl}$ whenever $\exists t \in T$ such that $t(V_i) = V_k$ and $t(V_j) = V_l$. Therefore $S_q(V) \rtimes G$ is an $H = \mathbb{C}T$ -module algebra. Also, by construction, the action of H on $T(V) \rtimes \mathbb{C}G$ is degree-preserving, and so by Lemma 4.2(2), so is the action on $S_q(V) \rtimes G$.

4.2 Proving Claim 3

With the scene finally set, we prove the first part of Claim 3:

Proposition 4.3. *Suppose $H = \mathbb{C}T$ and $A = S_q(V) \rtimes G$ are as above, and α is a constant Hochschild 2-cocycle of A lifting to quantum Drinfeld Hecke algebra Q_α . If α is H -equivariant then Q_α is an H -module algebra.*

Proof. The quantum Drinfeld Hecke algebra that arises from α is given by

$$Q_\alpha := T(V) \rtimes G/J, \quad J := \langle v_i \otimes v_j - q_{ij}v_j \otimes v_i - \kappa(v_i, v_j) \rangle$$

where $\kappa(v_i, v_j) = \sum_{g \in G} \kappa_g(v_i, v_j) = \alpha(v_i \otimes v_j - q_{ij}v_j \otimes v_i)$ (see [6], proof of Theorem 4.4). By Lemma 4.1, $T(V) \rtimes G$ is an H -module algebra, so we just need to check the ideal J is an H -submodule of $T(V) \rtimes G$ for the result to follow by Lemma 4.2(1). We check the action is closed on the generators of J ,

$$t \triangleright [v_i \otimes v_j - q_{ij}v_j \otimes v_i - \alpha(v_i \otimes v_j - q_{ij}v_j \otimes v_i)] \in J \quad \forall t \in T \quad (12)$$

Now we must be careful since there are two copies of the expression $v_i \otimes v_j - q_{ij}v_j \otimes v_i$ appearing above, but they are shorthand for two different things. The first copy represents $v_i \otimes v_j \otimes 1_G - q_{ij}v_j \otimes v_i \otimes 1_G \in V^{\otimes 2} \otimes \mathbb{C}G \subseteq T(V) \rtimes G$, whilst the second copy represents $(v_i \otimes 1_G) \otimes (v_j \otimes 1_G) - q_{ij}(v_j \otimes 1_G) \otimes (v_i \otimes 1_G) \in A \otimes A$, where $A = S_q(V) \rtimes G$ and the v_i, v_j are now elements of $S_q(V)$.

Recall $H = \mathbb{C}T$ acts diagonally on $T(V) \rtimes G$, and via the adjoint action on $\mathbb{C}G$, so

$$\begin{aligned} t \triangleright (v_i \otimes v_j \otimes 1_G - q_{ij}v_j \otimes v_i \otimes 1_G) &= (t \triangleright v_i) \otimes (t \triangleright v_j) \otimes t 1_G t^{-1} - q_{ij}(t \triangleright v_j) \otimes (t \triangleright v_i) \otimes t 1_G t^{-1} \\ &= \lambda \mu(v_k \otimes v_l \otimes 1_G - q_{ij}v_l \otimes v_k \otimes 1_G) \\ &= \lambda \mu(v_k \otimes v_l \otimes 1_G - q_{kl}v_l \otimes v_k \otimes 1_G) \end{aligned}$$

using the discussion just before the proposition where we assumed $t \triangleright v_i = \lambda v_k$ and $t \triangleright v_j = \mu v_l$ for some $\lambda, \mu \in \mathbb{C}, k, l \in [n]$ and $q_{ij} = q_{kl}$. Next we apply the fact α is H -equivariant,

$$\begin{aligned} t \triangleright \alpha(v_i \otimes v_j - q_{ij}v_j \otimes v_i) &= \alpha\left(\Delta(t) \triangleright [(v_i \otimes 1_G) \otimes (v_j \otimes 1_G) - q_{ij}(v_j \otimes 1_G) \otimes (v_i \otimes 1_G)]\right) \\ &= \alpha\left([t \triangleright (v_i \otimes 1_G)] \otimes [t \triangleright (v_j \otimes 1_G)] - q_{ij}[t \triangleright (v_j \otimes 1_G)] \otimes [t \triangleright (v_i \otimes 1_G)]\right) \\ &= \lambda \mu \alpha\left((v_k \otimes 1_G) \otimes (v_l \otimes 1_G) - q_{ij}(v_l \otimes 1_G) \otimes (v_k \otimes 1_G)\right) \end{aligned}$$

Now we see that the expression in (12) equates to $\lambda \mu(v_k \otimes v_l - q_{kl}v_l \otimes v_k - \alpha(v_k \otimes v_l - q_{kl}v_l \otimes v_k))$, i.e. another (rescaled) generator of J , as required. So the action of H on the generators of J is closed, J is an H -submodule, and it follows that Q_α is an H -module algebra. \square

Now that we know Q_α is an H -module algebra, it makes sense to twist it by the cocycle μ of H . We denote the resulting algebra $(Q_\alpha)_\mu$. The next theorem proves the final part of Claim 3.

Theorem 4.4. $(Q_\alpha)_\mu \cong Q_{\alpha_\mu}$.

Proof. We use the following elementary fact to prove the theorem: if $\phi : C \rightarrow D$ is an algebra isomorphism and $J \trianglelefteq C$ is an ideal, then the quotient map $\hat{\phi}_J : C/J \rightarrow D/\phi(J), c + J \mapsto \phi(c) + \phi(J)$ is an algebra isomorphism. We will take $C = T(V') \rtimes G'$ and $D = (T(V) \rtimes G)_\mu$, and show for a certain choice of ideal J the resulting quotient map $\hat{\phi}_J$ provides the desired isomorphism $Q_{\alpha_\mu} \xrightarrow{\sim} (Q_\alpha)_\mu$.

4.2.1. Quotient algebras. First we wish to characterise the (twisted) skew group algebras $S_{q'}(V') \rtimes G'$ and $(S_q(V) \rtimes G)_\mu$, and their respective (twisted) deformations Q_{α_μ} , $(Q_\alpha)_\mu$, as quotient algebras.

Note there exists, by assumption, an isomorphism $\psi : S_{q'}(V') \rtimes G' \xrightarrow{\sim} (S_q(V) \rtimes G)_\mu$ of graded algebras. Now the algebras $S_{q'}(V') \rtimes G'$ and Q_{α_μ} are defined as quotients of $T(V') \rtimes G'$ by the ideals $I' = \langle v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i \rangle$ and $J' = \langle v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i - \alpha'(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i) \rangle$ respectively. Here $\alpha' := \psi^{-1} \circ \alpha_\mu \circ (\psi \otimes \psi)$ is the Hochschild 2-cocycle on $S_{q'}(V') \rtimes G'$ induced by α_μ and ψ .

We wish to also characterise $(S_q(V) \rtimes G)_\mu$ and $(Q_\alpha)_\mu$ as quotient algebras, in particular of $(T(V) \rtimes G)_\mu$. Consider the ideal $J = \langle v_i \otimes v_j - q_{ij} v_j \otimes v_i - \alpha(v_i \otimes v_j - q_{ij} v_j \otimes v_i) \rangle$ of $T(V) \rtimes G$. The underlying space of J is also an ideal of the algebra $(T(V) \rtimes G)_\mu$. Indeed, let m and m_μ denote the products on $T(V) \rtimes G$ and $(T(V) \rtimes G)_\mu$ respectively, then we show that $m_\mu(r \otimes x) \in J$ and $m_\mu(x \otimes r) \in J \forall r \in (T(V) \rtimes G)_\mu, x \in J$. If $\mu^{-1} = \sum \mu_1 \triangleright \mu_2$ then

$$m_\mu(r \otimes x) = m(\mu^{-1} \triangleright r \otimes x) = m((\mu_1 \triangleright r) \otimes (\mu_2 \triangleright x)) \quad (13)$$

and, by the proof of Proposition 4.3, J is an H -submodule, so $\mu_2 \triangleright x \in J$. Since J is an ideal of $T(V) \rtimes G$ (i.e. with respect to the product m) it follows that (13) is in J as required. Similarly one shows $m_\mu(x \otimes r) \in J$, so that J is a 2-sided ideal of $(T(V) \rtimes G)_\mu$. Next we show $(Q_\alpha)_\mu := (T(V) \rtimes G/J)_\mu = (T(V) \rtimes G)_\mu/J$.

As Drinfeld twists do not change the underlying vector space of an algebra, this equality holds at the level of vector spaces. It remains to check the products coincide. Use $[m]$ and $[m_\mu]$ to denote the products on $Q_\alpha = T(V) \rtimes G/J$ and $(T(V) \rtimes G)_\mu/J$ respectively. So for $[a], [b] \in Q_\alpha$ given by some representatives $a, b \in T(V) \rtimes G$, we have: $[m]([a] \otimes [b]) := [m(a \otimes b)]$ and $[m_\mu]([a] \otimes [b]) := [m_\mu(a \otimes b)]$. The product on $(Q_\alpha)_\mu$ is therefore

$$[m]_\mu([a] \otimes [b]) = [m](\mu^{-1} \triangleright [a] \otimes [b]) = [m]((\mu_1 \triangleright [a]) \otimes (\mu_2 \triangleright [b])) \quad (14)$$

and at this point we employ the fact that the action of H on Q_α was induced by an action of H on $T(V) \rtimes G$, so $\mu_1 \triangleright [a] = [\mu_1 \triangleright a]$ and $\mu_2 \triangleright [b] = [\mu_2 \triangleright b]$. Then (14) becomes

$$[m]([\mu_1 \triangleright a] \otimes [\mu_2 \triangleright b]) = [m]((\mu_1 \triangleright a) \otimes (\mu_2 \triangleright b)) = [m_\mu(a \otimes b)] = [m_\mu]([a] \otimes [b])$$

with the last expression being precisely the product on $(T(V) \rtimes G)_\mu/J$. This finishes the proof that $(Q_\alpha)_\mu = (T(V) \rtimes G)_\mu/J$. One can very similarly show that $I := \langle v_i \otimes v_j - q_{ij} v_j \otimes v_i \rangle$ is an ideal of $(T(V) \rtimes G)_\mu$, and therefore that $(S_q(V) \rtimes G)_\mu = (T(V) \rtimes G)_\mu/I$.

4.2.2. The isomorphism ϕ . In this section we construct the isomorphism $\phi : T(V') \rtimes G' \xrightarrow{\sim} (T(V) \rtimes G)_\mu$ required to apply the elementary fact mentioned at the start of the proof. We begin by characterising both sides of ϕ as algebra factorisations, in the sense of Majid [4] (Definition 21.3):

Definition 4.5. An **algebra factorisation** is an algebra X and subalgebras $A, B \subseteq X$ such that the product map on X defines a linear isomorphism $A \otimes B \cong X$.

By [4] (Proposition 21.4) if X has an algebra factorisation into A, B , then there exists a linear map $\tau : B \otimes A \rightarrow A \otimes B$ such that $X \cong A \underline{\otimes} B$ as algebras, where $A \underline{\otimes} B$ has underlying vector space $A \otimes B$, unit $1 \otimes 1$, and product $(a \otimes b) \cdot (c \otimes d) = a\tau(b \otimes c)d, \forall a, c \in A, b, d \in B$. More explicitly, the product on $A \underline{\otimes} B$ is $m_X = (m_A \otimes m_B) \circ (\text{id}_A \otimes \tau \otimes \text{id}_B)$ where m_A, m_B are the products on A, B respectively (given by restricting the product on X). We see from the form of m_X this is a natural generalisation of the usual tensor product algebra structure on $A \otimes B$ in which τ is just the usual swap map.

In [5] (Theorem 7.2.3) we find an explicit description of τ . If $i : A \hookrightarrow X, j : B \hookrightarrow X$ are embeddings of the subalgebras, and \cdot is the product on X , then by definition of X being an algebra factorisation $\cdot \circ (i \otimes j)$ is a linear isomorphism. Therefore there is a well-defined linear map τ such that $\cdot \circ (j \otimes i) = \cdot \circ (i \otimes j) \circ \tau$. We

see τ essentially multiplies elements of B with elements of A , and then expresses the result in the canonical form of elements in $A \otimes B$.

It is well-known that semidirect products are the archetypal examples of algebra factorisations. Indeed it is easy to check that $T(V') \rtimes G'$ has an algebra factorisation into $T(V')$ and $\mathbb{C}G'$ with induced swap map

$$\begin{aligned}\tau' : \mathbb{C}G' \otimes T(V') &\rightarrow T(V') \otimes \mathbb{C}G', \\ g' \otimes v'_1 \otimes \cdots \otimes v'_k &\mapsto (g' \triangleright v') \otimes \cdots \otimes (g' \triangleright v'_k) \otimes g'\end{aligned}\tag{15}$$

Additionally $(T(V) \rtimes G)_\mu$ has an algebra factorisation into $T(V)_\mu$ and $\mathbb{C}G_\mu$. We can see this by noting firstly that $T(V)_\mu$ and $\mathbb{C}G_\mu$ are subalgebras of $(T(V) \rtimes G)_\mu$. Let $i : T(V)_\mu \hookrightarrow (T(V) \rtimes G)_\mu, v \mapsto v \otimes 1_G$ and $j : \mathbb{C}G_\mu \hookrightarrow (T(V) \rtimes G)_\mu, g \mapsto 1 \otimes g$ be the natural embeddings of each of these subalgebras. Also let m and m_μ denote the products on $T(V) \rtimes G$ and $(T(V) \rtimes G)_\mu$ respectively. Then $(T(V) \rtimes G)_\mu$ will have an algebra factorisation into $T(V)_\mu$ and $\mathbb{C}G_\mu$ if $m_\mu \circ (i \otimes j)$ is a linear isomorphism. Here we note $m_\mu \circ (i \otimes j) = m \circ (\mu^{-1} \triangleright) \circ (i \otimes j) = m \circ (i \otimes j) \circ (\mu^{-1} \triangleright)$, where $\mu^{-1} \triangleright$ is the isomorphism on $(T(V) \otimes \mathbb{C}G)^{\otimes 2}$ given by acting by μ^{-1} , and in the second equality we apply the fact that i and j are $\mathbb{C}T$ -module homomorphisms. It is again clear $T(V) \rtimes G$ has an algebra factorisation into $T(V)$ and $\mathbb{C}G$, and this makes $m \circ (i \otimes j)$ a linear isomorphism on noting that i and j also define embeddings $T(V) \hookrightarrow T(V) \rtimes G$ and $\mathbb{C}G \hookrightarrow T(V) \rtimes \mathbb{C}G$. Therefore $m_\mu \circ (i \otimes j)$ is a composition of two linear isomorphisms and must be a linear isomorphism itself.

Now that we know that $(T(V) \rtimes G)_\mu$ has an algebra factorisation, we wish to find the map τ such that $(T(V) \rtimes G)_\mu = T(V)_\mu \otimes \mathbb{C}G_\mu$. As a linear map $\tau : \mathbb{C}G \otimes T(V) \rightarrow T(V) \otimes \mathbb{C}G$ first sends $g \otimes v_1 \otimes \cdots \otimes v_k$ to the element $(1 \otimes g) \otimes (v_1 \otimes \cdots \otimes v_k \otimes 1_G) \in (T(V) \otimes \mathbb{C}G)^{\otimes 2}$, and then takes the product in $(T(V) \rtimes G)_\mu$,

$$\begin{aligned}\tau(g \otimes v_1 \otimes \cdots \otimes v_k) &= m_\mu((1 \otimes g) \otimes (v_1 \otimes \cdots \otimes v_k \otimes 1_G)) \\ &= m(\mu^{-1} \triangleright (1 \otimes g) \otimes (v_1 \otimes \cdots \otimes v_k \otimes 1_G)) \\ &= m\left(\sum_i [1 \otimes (\mu_{(1)i} \triangleright g)] \otimes [(\mu_{(2)i} \triangleright v_1) \otimes \cdots \otimes (\mu_{(2)i} \triangleright v_k) \otimes 1_G]\right) \\ &= \sum_i ((\mu_{(1)i} \triangleright g) \triangleright [(\mu_{(2)i} \triangleright v_1) \otimes \cdots \otimes (\mu_{(2)i} \triangleright v_k)]) \otimes (\mu_{(1)i} \triangleright g) \\ &\in T(V) \otimes \mathbb{C}G\end{aligned}\tag{16}$$

where we let $\mu^{-1} = \sum_i \mu_{(1)i} \otimes \mu_{(2)i}$ for some $\mu_{(1)i}, \mu_{(2)i} \in \mathbb{C}T$.

Next we wish to understand under what conditions two algebra factorisations are isomorphic. Suppose we have two algebra factorisations, X into A, B , and X' into A', B' , so that $X' \cong A' \otimes B'$ and $X \cong A \otimes B$. Further suppose we have algebra homomorphisms $\phi_1 : A' \rightarrow A$ and $\phi_2 : B' \rightarrow B$. Under what condition does the tensor product map $\phi_1 \otimes \phi_2 : A' \otimes B' \rightarrow A \otimes B$ define an algebra homomorphism $A' \otimes B' \rightarrow A \otimes B$? We claim it is precisely if the following holds:

$$(\phi_1 \otimes \phi_2) \circ \tau' = \tau \circ (\phi_2 \otimes \phi_1)\tag{17}$$

Recall $\phi_1 \otimes \phi_2$ is by definition an algebra homomorphism if $(\phi_1 \otimes \phi_2) \circ m_{X'} = m_X \circ [(\phi_1 \otimes \phi_2) \otimes (\phi_1 \otimes \phi_2)]$. Inserting the expanded expression for $m_{X'}$ into this we find:

$$\begin{aligned}(\phi_1 \otimes \phi_2) \circ (m_{A'} \otimes m_{B'}) \circ (\text{id}_{A'} \otimes \tau' \otimes \text{id}_{B'}) &= (\phi_1 \circ m_{A'}) \otimes (\phi_2 \circ m_{B'}) \circ (\text{id}_{A'} \otimes \tau' \otimes \text{id}_{B'}) \\ &= [(m_A \circ (\phi_1 \otimes \phi_2)) \otimes (m_B \circ (\phi_1 \otimes \phi_2))] \circ (\text{id}_{A'} \otimes \tau' \otimes \text{id}_{B'}) \\ &= (m_A \otimes m_B) \circ (\phi_1 \otimes ((\phi_1 \otimes \phi_2) \circ \tau') \otimes \phi_2)\end{aligned}$$

where in the second line we apply the fact ϕ_1, ϕ_2 are algebra homomorphisms. Using (17),

$$\begin{aligned} &= (m_A \otimes m_B) \circ (\phi_1 \otimes (\tau \circ (\phi_2 \otimes \phi_1))) \otimes \phi_2 \\ &= (m_A \otimes m_B) \circ (\text{id}_A \otimes \tau \otimes \text{id}_B) \circ [(\phi_1 \otimes \phi_2) \otimes (\phi_1 \otimes \phi_2)] \\ &= m_X \circ [(\phi_1 \otimes \phi_2) \otimes (\phi_1 \otimes \phi_2)] \end{aligned}$$

as required. Note that when ϕ_1, ϕ_2 are additionally both isomorphisms, then the tensor product map $\phi_1 \otimes \phi_2$ must be a linear isomorphism. When (17) is also satisfied we deduce $X \cong X'$ as algebras.

Now recall that we wished to construct an isomorphism $\phi : T(V') \rtimes G' \xrightarrow{\sim} (T(V) \rtimes G)_\mu$, and so far we have characterised the algebras in the domain and codomain of this map as algebra factorisations. Applying the above, if we can find algebra isomorphisms $\phi_1 : T(V') \rightarrow T(V)_\mu$, $\phi_2 : \mathbb{C}G' \rightarrow \mathbb{C}G_\mu$ such that the condition (17) holds, then we will have the desired isomorphism given by $\phi = \phi_1 \otimes \phi_2$.

We will construct ϕ_1, ϕ_2 via the graded isomorphism $\psi : S_{q'}(V') \rtimes G' \xrightarrow{\sim} (S_q(V) \rtimes G)_\mu$.

With ϕ_1, ϕ_2 in hand, and τ, τ' given in (16), (15) respectively, we check that (17) holds. Each side of (17) is a linear map $\mathbb{C}G' \otimes T(V') \rightarrow T(V) \otimes \mathbb{C}G$, and a general basis element of $\mathbb{C}G' \otimes T(V')$ is given by $\lambda = g' \otimes v'_1 \otimes \dots \otimes v'_k$ for some $g' \in G', v'_i \in V, k \in \mathbb{N}$. Applying both sides of (17) to λ we find:

$$\begin{aligned} (\phi_1 \otimes \phi_2) \circ \tau'(\lambda) &= \phi_1(g' \triangleright v'_1) \otimes \dots \otimes \phi_1(g' \triangleright v'_k) \otimes \phi_2(g') \\ \tau \circ (\phi_2 \otimes \phi_1)(\lambda) &= \sum_i (\mu_{(1)i} \triangleright \phi_2(g')) \triangleright [(\mu_{(2)i} \triangleright \phi_1(v'_1)) \otimes \dots \otimes (\mu_{(2)i} \triangleright \phi_1(v'_k))] \otimes (\mu_{(1)i} \triangleright \phi_2(g')) \end{aligned}$$

Checking these are equal reduces to checking:

$$\phi_1(g' \triangleright v') = \phi_2(g') \triangleright_\mu \phi_1(v') \quad \forall g' \in G', v' \in V' \quad (18)$$

Now $g' \triangleright v' \in V' \subset T(V')$, whilst $\phi_1(g' \triangleright v') \in V \subseteq T(V)_\mu$. Additionally V' embeds in $S_{q'}(V') \rtimes G'$, as does V into $(S_q(V) \rtimes G)_\mu$. We can therefore identify the subspaces of $T(V')$ and $S_{q'}(V') \rtimes G'$ corresponding to V' , and similarly for the subspaces of $T(V)_\mu$ and $(S_q(V) \rtimes G)_\mu$ corresponding to V . Restricted to the (identified) subspaces, we have $\phi_1|_{V'} = \psi|_{V'} : V' \rightarrow V$. A similar argument can be made to find $\phi_2|_{\mathbb{C}G'} = \psi|_{\mathbb{C}G'} : \mathbb{C}G' \rightarrow \mathbb{C}G_\mu$. Additionally the action of G' on V' is the same whether viewed inside $T(V') \rtimes G'$ or $S_{q'}(V') \rtimes G'$, and similarly for the twisted action \triangleright_μ of G on V . We can therefore say that (18) holds if and only if the following holds: $\psi(g' \triangleright v') = \psi(g') \triangleright_\mu \psi(v')$. This can be verified by using the fact that ψ is an algebra homomorphism:

$$\begin{aligned} [(\psi(g') \triangleright_\mu \psi(v')) \otimes 1_G] \cdot (1 \otimes \psi(g')) &= (\psi(g') \triangleright_\mu \psi(v')) \otimes \psi(g') \\ &= \psi(g') \cdot \psi(v') \\ &= \psi((g' \triangleright v') \otimes g') \\ &= [\psi(g' \triangleright v') \otimes 1_G] \cdot (1 \otimes \psi(g')) \end{aligned}$$

and additionally noting the ψ is an isomorphism, so we can multiply from the right by $\psi(g')^{-1}$ to get the desired result.

4.2.3. The quotient map $\hat{\phi}_{J'}$. Recall that above we showed $Q_{\alpha_\mu} = T(V') \rtimes G'/J'$, whilst $(Q_\alpha)_\mu = (T(V) \rtimes G)_\mu/J$. If ϕ satisfies $\phi(J') = J$, then the map $\hat{\phi}_{J'}$ (induced via the elementary fact above) will be an isomorphism $Q_{\alpha_\mu} \xrightarrow{\sim} (Q_\alpha)_\mu$, as required. So we finish by checking $\phi(J') = J$. Recall $J' := \langle v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i - \alpha'(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i) \rangle$, where $\alpha' := \psi^{-1} \circ \alpha_\mu \circ (\psi \otimes \psi)$ is the Hochschild 2-cocycle on $S_{q'}(V') \rtimes G'$ induced by α_μ and ψ . As we did for I' above, Applying the elementary fact which we stated at the start of

the proof to this map ϕ with the ideal $I' := \langle v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i \rangle$ we retrieve the map ψ , i.e. $\hat{\phi}_{I'} = \psi$. Indeed this holds if $\phi(I') = I$. Using the fact ϕ is an algebra morphism,

$$\phi(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i) = \mu^{-1} \triangleright (v_i \otimes v_j - q'_{ij} v_j \otimes v_i) \quad (19)$$

Now recalling that ψ is an algebra isomorphism sending $v'_i + I' \mapsto v_i + I$, it follows that

$$[m_\mu]((v_i + I) \otimes (v_j + I)) = q'_{ij} [m_\mu]((v_j + I) \otimes (v_i + I))$$

Hence $m_\mu(v_i \otimes v_j - q'_{ij} v_j \otimes v_i) = m(\mu^{-1} \triangleright v_i \otimes v_j - q'_{ij} v_j \otimes v_i) \in I$. Now since the action of H is degree-preserving, $\mu^{-1} \triangleright v_i \otimes v_j - q'_{ij} v_j \otimes v_i \in V \otimes V$, and m in this case is just the product on $T(V)$, so

$$m(\mu^{-1} \triangleright v_i \otimes v_j - q'_{ij} v_j \otimes v_i) = \mu^{-1} \triangleright v_i \otimes v_j - q'_{ij} v_j \otimes v_i \in I \cap (V \otimes V) \quad (20)$$

A general element of I is of the form $\sum_{k,l} r_{kl}(v_k \otimes v_l - q_{kl} v_l \otimes v_k) s_{kl}$ for $r_{kl}, s_{kl} \in T(V) \rtimes G$, so elements in $I \cap (V \otimes V)$ take the form $\sum_{k,l} \lambda_{kl}(v_k \otimes v_l - q_{kl} v_l \otimes v_k)$ for $\lambda_{kl} \in \mathbb{C}$. Therefore by (20) we find:

$$\mu^{-1} \triangleright v_i \otimes v_j - q'_{ij} v_j \otimes v_i = \sum_{k,l} \lambda_{kl}(v_k \otimes v_l - q_{kl} v_l \otimes v_k) \quad (21)$$

for some $\lambda_{kl} \in \mathbb{C}$. By (19) we then see $\phi(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i) \in I$, so ϕ maps generators of I' to elements of I and we deduce $\phi(I') \subset I$. By a similar argument using ϕ^{-1} we get $\phi(I') \supset I$, and so $\phi(I') = I$.

we show that ϕ maps the generators of J' to J . Indeed the first part of each generator of J' is of the same form as the generators in I' , and we showed above that $\phi(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i) = \sum_{k,l} \lambda_{kl}(v_k \otimes v_l - q_{kl} v_l \otimes v_k)$ for some $\lambda_{kl} \in \mathbb{C}$. Now note that $\alpha'(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i) \in \mathbb{C}G'$, and $\phi = \psi$ when both are restricted to $\mathbb{C}G'$. Using the definition of α' ,

$$\begin{aligned} \phi(\alpha'(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i)) &= \alpha_\mu \circ (\psi \otimes \psi)(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i) \\ &= \alpha_\mu(v_i \otimes v_j - q'_{ij} v_j \otimes v_i) \\ &= \alpha(\mu^{-1} \triangleright v_i \otimes v_j - q'_{ij} v_j \otimes v_i) \\ &= \sum_{k,l} \lambda_{kl} \alpha(v_k \otimes v_l - q_{kl} v_l \otimes v_k) \end{aligned}$$

where moving to the final line applies (21). We need to be careful though since the expression $v_i \otimes v_j - q'_{ij} v_j \otimes v_i$ appearing inside the α represents something slightly different to the $v_i \otimes v_j - q'_{ij} v_j \otimes v_i$ appearing in (21). This was discussed in the proof of Proposition 4.3 (see just below (12)). **However we find that (21) also holds for this second meaning of the expression $v_i \otimes v_j - q'_{ij} v_j \otimes v_i$.** Therefore

$$\phi(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i - \alpha'(v'_i \otimes v'_j - q'_{ij} v'_j \otimes v'_i)) = \sum_{k,l} \lambda_{kl}(v_k \otimes v_l - q_{kl} v_l \otimes v_k - \alpha(v_k \otimes v_l - q_{kl} v_l \otimes v_k)) \in J$$

and so $\phi(J') \subset J$. **Similarly one shows $\phi(J') \supset J$,** and the result follows. \square

5 Examples

In the following we give several examples showing how twisting results between skew group algebras can lift to quantum Drinfeld Hecke algebras. In the first example we take the twisting result of the preprint [], and show that it arises from this lifting process. The groups involved in this example include the Coxeter groups of type B , but not the Coxeter groups of type A . In the second example we use the lifting process to show

that a rational Cherednik algebra of the Coxeter group of type A_3 is related by via a new twist to a braided Cherednik algebra.

Example 5.1. Our motivating example of Claims 2/3 in work will be the pair of algebras $A = S(V \oplus V^*) \rtimes G$ and $B = S_{-1}(V \oplus V^*) \rtimes \mu(G)$. Note it remains to check that G and $\mu(G)$ act by algebra automorphisms on $\Lambda(V \oplus V^*)$ and $\bigwedge_{-1}(V \oplus V^*)$. In fact in [6] Sections 6 and 7 Witherspoon does not actually check this, and surely without this it isn't guaranteed her constant cocycles generate QDHA's...?

Example 5.2.

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