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Astrophysical Fluids Numerical Project

Ted Johnson

1. OBLIQUE SHOCKS

In this work we will investigate oblique shocks – sharp discontinuities in the properties of a fluid caused by supersonic flow interacting with a barrier whose surface is not normal to the flow. Relevant quantities in this problem include the velocity of the pre-shock flow ($\boldsymbol{u_1}$), the velocity of the post-shock flow ($\boldsymbol{u_2}$), the angle between $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ (ψ), the angle between $\boldsymbol{u_1}$ and the normal vector to the shock ϕ, the pre-shock sound speed a_1, and the adiabatic index of the gas (γ).

In the case were the angle of the shock surface (ϕ) is not known in advance, we must deduce it from some other geometric conditions. Consider a shock caused by a wedge of known opening angle ψ. This setup is shown in Figure 1.

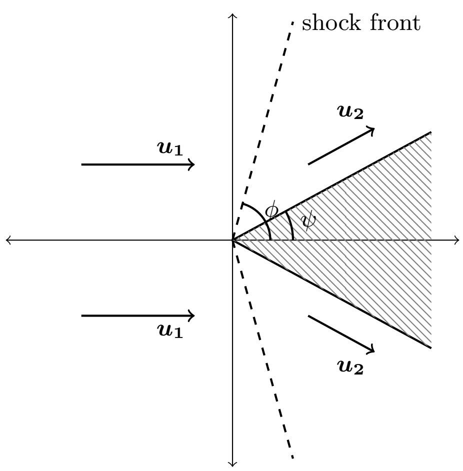

Figure 1. Oblique shock setup. A fluid traveling at supersonic velocity $\boldsymbol{u_1}$ meets a stationary barrier of opening angle ψ. A shock front of opening angle ϕ is formed, and the post-shock fluid flows parallel to the barrier's surface at a velocity $\boldsymbol{u_2}$. Other relevant quantities to the problem include the sound speed in the pre-shock medium (a_1) and the adiabatic index γ.

Shu (1992) gives two equations that relate these quantities:

$$u_1 \cos\phi = u_2 \cos\phi \cos\psi + u_2 \sin\phi \cos\psi \,, \tag{16.17}$$

$$\frac{u_2}{u_1}\cos\psi - \frac{u_2}{u_1}\cot\phi \sin\psi = \frac{2a_1^2}{(\gamma+1)u_1^2 \sin^2\phi} + \frac{\gamma-1}{\gamma+1} \,. \tag{16.18}$$

We can define the following dimensionless quantities to make these equations more manageable for our eventual numerical scheme:

$$\eta \equiv \frac{u_2}{u_1} \,, \tag{1}$$

$$M_1 \equiv \frac{u_1}{a_1} \,. \tag{2}$$

And equations (16.17) & (16.18) become

$$\cos\phi = \eta \cos\phi \cos\psi + \eta \sin\phi \sin\psi \,, \tag{3}$$

$$\eta \cos\psi - \eta \cot\phi \sin\psi = \frac{2}{(\gamma+1)M_1^2 \sin^2\phi} + \frac{\gamma-1}{\gamma+1} \,. \tag{4}$$

Following the work of Shu (1992, Ch. 16), we can rearrange equation (3) to find

$$\tan\phi = \frac{1-\eta\cos\psi}{\eta\sin\psi} \,, \tag{5}$$

$$\sin^2\phi = \frac{(1-\eta\cos\psi)^2}{(1-\eta\cos\psi)^2 + \eta^2\sin^2\psi} \,, \tag{6}$$

where the second equation is found via a trigonometric relation.

2. THE SHOCK POLAR

We can use equations (5) & (6) to remove ϕ from equation (4) and find that

$$\eta^2 \sin^2\psi = (1-\eta\cos\psi)^2 \frac{M_1^2 \eta \cos\psi - \alpha^2}{\alpha^2 + [2/(\gamma+1)]M_1^2 - M_1^2 \eta\cos\psi} \,, \tag{7}$$

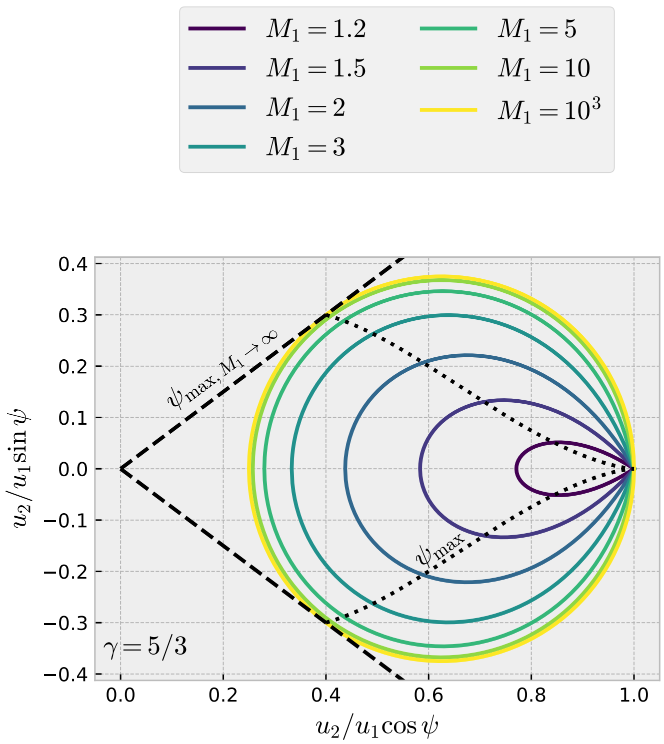

Figure 2. The shock polar for various Mach numbers. For a given ψ there are either two, one, or zero possible values of $\eta \equiv u_2/u_1$. When ψ is small, there are two solutions possible. The one closest to the origin is called a strong shock, as the fluid becomes subsonic after the shock. The second solution is called a weak shock, as the magnitudes of the discontinuities are less, and the post-shock flow remains supersonic. For any given M_1, there exists a value of $\psi = \psi_{\max}$ for which there is only one solution and the post-shock Mach number is 1 – this represented by the dashed line for the case $M_1 \to \infty$, and by the dotted line for the case of general M_1. When $\psi > \psi_{\max}$ there are no oblique shock solutions and a bow shock is formed.

where

$$\alpha^2 \equiv \frac{\gamma-1}{\gamma+1} M_1^2 + \frac{2}{\gamma+1} \tag{8}$$

is the dimensionless critical speed. This is equivalent to $\alpha^2 \equiv c_*^2/a_1^2$ in Shu's variables. Equation (7) is the *shock polar*, given by Shu in eq. (16.21). Solutions to this equation are shown in Figure 2. See the figure caption for a discussion on the number of solutions for a given ψ. The extrema of η are its two zeros: a maximum of 1, and a minimum of α^2/M_1^2.

2.1. *Finding ψ_{max}*

For any given Mach number it is possible to find the maximum allowed deflection angle $\psi_{\max}$ that still leads to an attached shock. It can be seen by inspection of Figure 2 that this angle satisfies

$$\frac{d(\eta \sin\psi_{\max})}{d(\eta \cos\psi_{\max})} = \tan\psi_{\max} \, . \tag{9}$$

The derivative on the left-hand side of equation (9) can be computed analytically from equation (7). However, it is much easier to compute the derivative numerically using the central difference method. If we let $x \equiv \eta\cos\psi$ and $y \equiv \eta \sin\psi$, then this problem reduces to finding the solution to

$$\frac{y(x+\delta) - y(x-\delta)}{2\delta} - \frac{y(x)}{x} = 0 \, , \tag{10}$$

for some $\delta \ll x$. For a solution $x = x_0$, we can then calculate

$$\psi_{\max} = \arctan y_0/x_0 \, . \tag{11}$$

Because there is only a single solution to equation (10) and x is bounded such that $\alpha^2/M_1^2 < x < 1$, we use the bisection method to find x_0. The dotted line in Figure 2 shows the position of (x_0, y_0) for general M_1. This line separates the strong and weak shock regimes.

3. FINDING ϕ & η

In general there are two unknowns in the oblique shock problem: η and ϕ. Equation (5) shows how ϕ can be easily computed once η is known. To compute η, however, we need to use a numerical solver. We use the bisection method to solve equation (7). We choose this solver again because of its simplicity and stability.

In order to find the value of η for both a strong and weak shock we run our solver twice over two different domains:

$$\alpha^2/M_1^2 < \eta_{\text{strong}} \cos\psi < \eta_{\psi_{\max}} \cos\psi_{\max} \, , \tag{12}$$

$$\eta_{\psi_{\max}} \cos\psi_{\max} < \eta_{\text{weak}} \cos\psi < 1 \, . \tag{13}$$

The bisection method allows this domain to be strictly enforced, while other methods (e.g. Newton-Raphson) are susceptible to large jumps which can lead to unexpectedly finding a solution other than that which is intended.

Now that η can be found for any (attached shock) setup in both the strong and weak regimes, we solve for ψ using equation (5). Figure 3 shows the results in both the strong and weak regimes using the parameterization $\phi = \phi(\psi; M_1, \gamma)$.

Strong and weak shock solutions for ϕ are shown in Figure 3.

4. DISCUSSION

4.1. *Conditions for an attached shock*

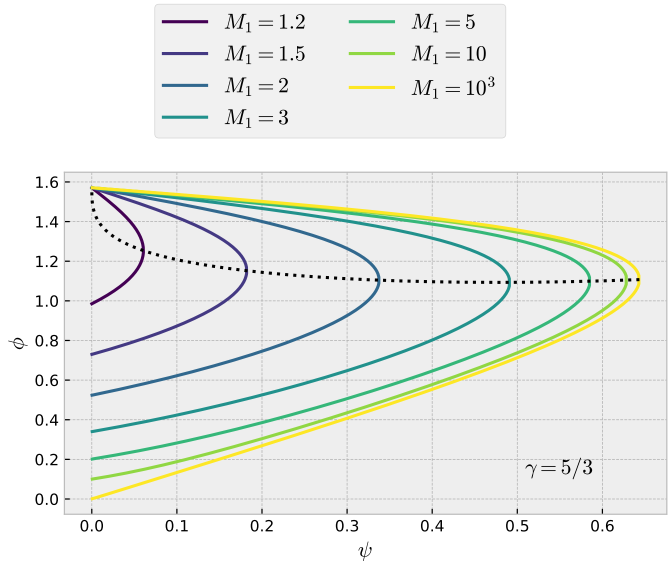

Figure 3. Shock angle solutions. The transition between strong (upper portion) and weak (lower portion) occurs and the ψ-maximum of each curve. This transition is marked by the dotted line.

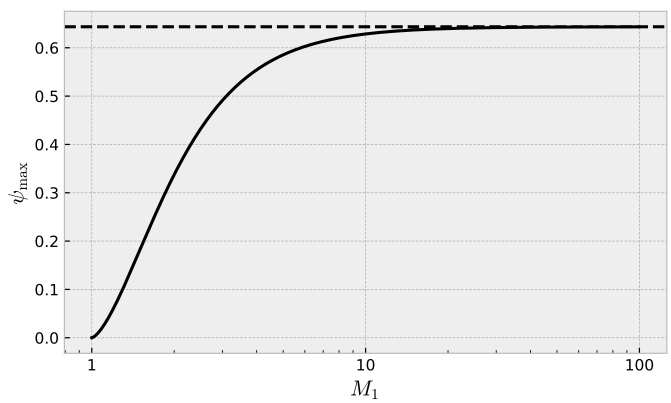

Figure 4. Maximum deflection angle as a function of Mach number for an attached shock. Barriers with opening angles greater than $\psi_{\max}$ will create a bow shock. The dashed line shows the limiting value as $M_1 \to \infty$. This figure assumes $\gamma = 5/3$.

It can be seen visually in Figure 3 there exists a maximum angle $\psi_{\max}$ for which we can calculate the angle of an attached shock. A barrier more blunt than $\psi_{\max}$ will create a bow shock. Figure 4 calculates this angle as a function of M_1. Qualitatively, this figure shows that a more blunt object requires a faster flow velocity (higher M_1) to maintain an attached shock than a more pointed object. While, formally, no attached shock can be created in sonic flow ($M_1 = 1$), in the limit that $\psi \to 0$ an attached shock can exist at very low Mach numbers ($M_1 \gtrapprox 1$).

On the other hand, there is an opening angle $\psi_{\max, M_1 \to \infty} = \arcsin(1/\gamma)$ for which a bow shock will form no matter how fast the flow velocity is.

4.2. *Choice between strong and weak shocks*

As discussed in the previous sections, when $\psi < \psi_{\max}$ the function $\phi(\psi)$ is double valued. While both solutions are mathematically valid, we can look to the limiting behavior of each branch to determine which is more likely to be observed in nature.

While looking at Figure 3, consider a barrier of infinitesimal opening angle and a flow of $M_1 \sim \infty$. The two valid solutions are $\phi \sim 0$ and $\phi \sim \pi/2$. In the case of a finite wedge, the choice $\phi \sim \pi/2$ is clearly not valid as it would create a shock front along the y-axis, and the whole of quadrants I & IV would be in the post-shock flow. Only the choice of the weak $\phi \sim 0$ solution allows for a finite post-shock area. If we then increase ψ and continue on the weak solution branch, then the only location where a transition would be allowed is at $\psi = \psi_{\max}$. It may be possible to cause the strong branch to take over at that point via artificial means, but there is no mathematical reason that the strong solution branch should be preferred. Shu (1992) notes that the strong shock solutions are irrelevant in the case of the idealized problem we have presented here.

REFERENCES

Shu, F. H. 1992, The physics of astrophysics. Volume II: Gas dynamics.