Assignment 2

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Exercise 1

Let $A = \mathbb{R}^n = \{(x_1,\ \dots,\ x_n) \mid x_i \in \mathbb{R},\ 1 \leq i \leq n\}$.
For $x, y \in A,\ x = (x_1,\ \dots,\ x_n)$ and $y = (y_1,\ \dots,\ y_n)$, xQy if and only if $\forall i,\ 1 \leq i \leq n,\ x_i = y_i$ or $\exists i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j \forall j,\ j < i$. Determine:

(i) Whether or not the relation Q is *reflexive*;

(ii) Whether or not the relation Q is *symmetric*;

(iii) Whether or not the relation Q is *anti-symmetric*;

(iv) Whether or not the relation Q is *transitive*;

(v) Whether or not the relation Q is an *equivalence relation*;

(vi) Whether or not the relation Q is a *partial order*.

Solution

There are two conditions in relation Q we will label (a) and (b):

(a) $\forall i,\ 1 \leq i \leq n,\ x_i = y_i$ or

(b) $\exists i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j \forall j,\ j < i$

If either conditions are valid for a case, the relation holds for that case. We must prove each relevant case for each property to show Q has that property.

(i) Relation Q is reflexive iff $\forall a \in A,\ aQa$.
In condition (a), $a_i = a_i$ always holds, so relation Q must be reflexive.

(ii) Relation Q is symmetric iff $\forall a, b \in A,\ aQb \Rightarrow bQa$.
In the case that $a = b$, condition (a) always holds.
However, when $a \neq b$, condition (b) is not symmetric:
$a_i < b_i \wedge a_j = b_j$ does not imply $b_i < a_i \wedge a_j = b_j$ as $a_i < b_i \not\equiv b_i < a_i$
$\therefore$ Relation Q is *not* symmetric.

(iii) Relation Q is anti-symmetric iff $\forall a, b \in A,\ aQb \land bQa \Rightarrow a = b$.
In the case that (a) holds, then $a = b$ must hold.
In a case where (a) does not hold, (b) will never hold as $a_i < b_i$ and $b_i < a_i$ cannot both be true, thus $a \neq b$.
$\therefore aQb \land bQa \Rightarrow a = b$, so relation Q is anti-symmetric.

(iv) Relation Q is transitive iff $\forall a, b, c \in A,\ aQb \land bQc \Rightarrow aQc$.
There are four cases for the conditions in $aQb \land bQc$ to hold:

1. (a) and (a) held: $a_i = b_i$ and $b_i = c_i$
2. (a) and (b) held: $a_i = b_i$ and $b_i < c_i \land b_j = c_j$
3. (b) and (a) held: $a_i < b_i \land a_j = b_j$ and $b_i = c_i$
4. (b) and (b) held: $a_i < b_i \land a_j = b_j$ and $b_i < c_i \land b_j = c_j$

For $aQb \land bQc \Rightarrow aQc$ to be valid, each case must be able to prove aQc.

1. If $\forall i,\ a_i = b_i$ and $\forall i,\ b_i = c_i$, then it follows that $a = b = c$.
   $\therefore aQc$ must hold according to condition (a).
2. If $\forall i,\ a_i = b_i$ and $\exists i,\ b_i < c_i \land b_j = c_j\ \forall j$, then we can substitute b_i for a_i in $b_i < c_i \land b_j = c_j$, that is $a_i < c_i \land a_j = c_j$.
   This exactly fulfills the condition (b) for aQc. $\therefore aQc$ holds.
3. If $\exists i,\ a_i < b_i \land a_j = b_j\ \forall j$ and $\forall i,\ b_i = c_i$, then we can substitute c_i for b_i in $a_i < b_i \land a_j = b_j$, that is $a_i < c_i \land a_j = c_j$.
   This, again, exactly fulfills the condition (b) for aQc. $\therefore aQc$ holds.
4. If $\exists i,\ a_i < b_i \land a_j = b_j\ \forall j$ and $\exists i,\ b_i < c_i \land b_j = c_j\ \forall j$, then it follows that $a_i < b_i < c_i$ and $a_j = b_j = c_j$.
   Then it is clear that $a_i < c_i$ and $a_j = c_j$, which is simply the condition (b) for aQc. $\therefore aQc$ holds.

As aQc holds for all four of the possible cases, we can conclude relation Q is transitive.

(v) Relation Q is *not* an equivalence relation as it does not exhibit *symmetry* as proven above.

(vi) Relation Q is a partial order as it exhibits *reflexivity*, *anti-symmetry* and *transitivity* as proven above.

Exercise 2

Use mathematical induction to prove that for all $n \geq 7$, $n! > 3^n$.

Solution

Base case: Prove true for $n = 7$

$n! > 3^n$
$7! > 3^7$
$5040 > 2187$ $\therefore$ true for $n = 7$

Inductive step: Assume true for $n = k$, prove true for $n = k + 1$.

$(k + 1)! > 3^{(k+1)}$
$(k + 1)(k!) > (3^k)(3^1)$

Because we assumed $n = k$ (i.e. $k! > 3^k$) is true, if we substitute 3^k for $k!$ in the LHS, then the resulting expression must have a smaller value, that is:

$(k + 1)(k!) > (k + 1)(3^k)$

Now we can take this smaller expression and see if it is *still* greater than the RHS of our original statement:

$(k + 1)(3^k) > (3^k)(3^1)$
$(k + 1) > 3$ ...dividing by 3^k as k is always positive
$k > 2$ $\therefore$ true for $n = k + 1$ as $n \geq 7$

As our smaller expression has been proven to still be greater than the RHS, it follows that the initial LHS (which is even greater) must also be greater than the RHS, thus proving $n! > 3^n$ is true for $n = k + 1$.

As $n! > 3^n$ is true for $n = 7$ and $n = k + 1$, it follows that $n! > 3^n$ must be true for $n \geq 7$.

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Exercise 3

Part (a)

Let $\{C_n\}_{n=1,2,\ldots} = \{C_1,\ C_2,\ \ldots\}$ be a sequence of sets satisfying that $C_n \subseteq C_{n+1}$ $\forall n \geq 1$. Prove by mathematical induction that $C_m \subseteq C_n$ whenever $m < n$.

Solution

Base case: Prove true for $n = m + 1$.

$C_m \subseteq C_n$
$C_m \subseteq C_{(m+1)}$ ...substituting $m + 1$ for n
$C_m \subseteq C_{m+1}$ $\therefore$ true for $n = m + 1$ as $C_n \subseteq C_{n+1}$

Inductive step: Assume true for $n = k$, prove true for $n = k + 1$.

$C_m \subseteq C_n$
$C_m \subseteq C_{(k)}$ ...our assumption $n = k$ it true
$C_m \subseteq C_{k+1}$ C_m is a subset of C_k

Then, if we substitute m for k in our base case, we get:

$C_k \subseteq C_{k+1}$ C_k is a subset of C_{k+1}

As C_m is a subset of C_k and C_k is a subset of C_{k+1}, we can apply the transitivity of $\subseteq$:

$C_m \subseteq C_k \subseteq C_{k+1}$
$C_m \subseteq C_{k+1}$ $\therefore$ true for $n = k + 1$

As $C_m \subseteq C_n$ is true for $n = m + 1$ and $n = k + 1$, it follows that $C_m \subseteq C_n$ must be true for all $n < m$.

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Part (b)

Recall that the graph of a function $f : A \to B$ is given by

$$\Gamma(f) = \{(x, y) \mid x \in A \text{ and } y = f(x)\} \subseteq A \times B$$

Let $Funct(A, B)$ the set of all functions $f : \tilde{A} \to \tilde{B}$ such that $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$. We define a relation on $Funct(A, B)$ as follows:

$$\forall f, g \in Funct(A, B)\ f \subseteq g \text{ iff } \Gamma(f) \subseteq \Gamma(g)$$

Prove that this relation is a partial order on $Funct(A, B)$.

Solution

For relation $\subseteq$ to be a partial order, it must have the properties *reflexivity*, *anti-symmetry* and *transitivity*.

(i) $\subseteq$ is reflexive iff $\forall f \in Funct(A, B),\ f \subseteq f$.
That is, $\Gamma(f) \subseteq \Gamma(f)$, which clearly holds as $\Gamma(f) = \Gamma(f)$.

(ii) $\subseteq$ is anti-symmetric iff $\forall f, g \in Funct(A, B),\ f \subseteq g \wedge g \subseteq f \Rightarrow f = g$.
That is, $\Gamma(f) \subseteq \Gamma(g) \wedge \Gamma(g) \subseteq \Gamma(f)$ implies $\Gamma(f) = \Gamma(g)$. For $\Gamma(f)$ to be a subset of $\Gamma(g)$ while $\Gamma(g)$ is also a subset of $\Gamma(f)$, it must hold that $\Gamma(f) = \Gamma(g)$. $\therefore f = g$.

(iii) $\subseteq$ is transitive iff $\forall f, g, h \in Funct(A, B),\ f \subseteq g \wedge g \subseteq h \Rightarrow f \subseteq h$.
That is, $\Gamma(f) \subseteq \Gamma(g) \wedge \Gamma(g) \subseteq \Gamma(h)$ implies $\Gamma(f) \subseteq \Gamma(h)$. If $\Gamma(f)$ is a subset of $\Gamma(g)$ and $\Gamma(g)$ is a subset of $\Gamma(h)$, then it follows that $\Gamma(f)$ is a subset of $\Gamma(h)$. $\therefore f \subseteq h$.

$\therefore$ As relation $\subseteq$ exhibits all three of these properties, it is a partial order.

Part (c)

Let $\{f_n\}_{n=1,2,\ldots} = \{f_1,\ f_2,\ \ldots\}$ be a sequence of functions in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$. Since functions are in one-to-one correspondence with their graphs, we identify $\bigcup_{n \in \mathbb{N}} f_n$ with $\bigcup_{n \in \mathbb{N}} \Gamma(f_n)$. Using part (a), prove that $\bigcup_{n \in \mathbb{N}} f_n$ is a function and $\bigcup_{n \in \mathbb{N}} f_n \in Funct(A, B)$.

Solution

To prove that $\bigcup\limits_{n\in\mathbb{N}} f_n$ is a function, we must show that each element $x \in A$ corresponds with exactly one element $y \in B$ for $\Gamma(\bigcup\limits_{n\in\mathbb{N}} f_n)$. By assuming $\bigcup\limits_{n\in\mathbb{N}} f_n$ is not a function, that is, assume there exists an element x which maps to both y_1 and y_2, then we should be able to prove via a contradiction.

Our assumption, where (x, y) $s.t$ $y = f_i(x)$, entails that $(x, y_1), (x, y_2) \in \Gamma(\bigcup\limits_{n\in\mathbb{N}} f_n)$. If that is the case, then $(x, y_1) \in \Gamma(f_{i1})$ and $(x, y_2) \in \Gamma(f_{i2})$, where $f_{i1} : A_1 \to B_1$ and $f_{i2} : A_2 \to B_2$, with $A_1 \subseteq A_2 \subseteq A$ and $B_1 \subseteq B_2 \subseteq B$ as $f_n \subseteq f_{n+1}$.

For $\bigcup\limits_{n\in\mathbb{N}} f_n$ to not be a function, it must hold that $y_1 \neq y_2$ as then x would have two different outputs. Let's consider the three cases for $i1$ and i_2:

(i) If $i_1 = i_2$, that is, $f_{i1} = f_{i2}$, then it must be that $y_1 = y_2$ as inputting x to the equal functions f_{i1} and f_{i2} must be mapped to a single corresponding output.

(ii) If $i_1 < i_2$, then $f_{i1} \subseteq f_{i2}$ and $B_1 \subseteq B_2$. $\therefore y_1 \in B_2$, so the input x into functions f_{i1} and f_{i2} have equal outputs, thus $y_1 = y_2$.

(iii) Similarly, if $i_1 > i_2$, then $f_{i2} \subseteq f_{i1}$ and $B_2 \subseteq B_1$. $\therefore y_2 \in B_1$, so the input x into functions f_{i1} and f_{i2} have equal outputs, thus $y_1 = y_2$.

As we have seen for all three cases, we have proven $y_1 = y_2$, which contradicts our initial assumption. $\therefore \bigcup\limits_{n\in\mathbb{N}} f_n$ must be a function.

To show that $\bigcup\limits_{n\in\mathbb{N}} f_n \in Funct(A, B)$, we must prove that it is a function with domain $\tilde{A}$ and codomain $\tilde{B}$ where $\tilde{A} \in A$ and $\tilde{B} \in B$.

The domain of $\bigcup\limits_{n\in\mathbb{N}} f_n$ is $d_1 \cup d_2 \cup \ldots \cup d_n$ where d_i is the domain of f_i. The union of these sets is still within A as each set d_i is a subset of $\tilde{A}$, therefore the domain of $\bigcup\limits_{n\in\mathbb{N}} f_n \in A$.

Similarly, the sub-domain of $\bigcup_{n\in\mathbb{N}} f_n$ is $s_1 \cup s_2 \cup \ldots \cup s_n$ where s_i is the sub-domain of f_i. The union of these sets is still within B as each set s_i is a subset of $\tilde{B}$, therefore the sub-domain of $\bigcup_{n\in\mathbb{N}} f_n \in B$.

As the domain and codomain of the function $\bigcup_{n\in\mathbb{N}} f_n$ are subsets of A and B, it is shown that $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A, B)$.

Part (d)

For every $f \in Funct(A, B)$, let $Dom(f)$ be the domain of f, namely if $f : \tilde{A} \to \tilde{B}$ with $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$, $Dom(f) = \tilde{A}$. Prove that $Dom(\bigcup_{n\in\mathbb{N}} f_n) = \bigcup_{n\in\mathbb{N}} Dom(f_n)$ for every sequence of the functions $\{f_n\}_{n=1,2,\ldots} = \{f_1,\ f_2,\ \ldots\}$ in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$.

Solution

To prove $Dom(\bigcup_{n\in\mathbb{N}} f_n) = \bigcup_{n\in\mathbb{N}} Dom(f_n)$ via double inclusion, we will show $Dom(\bigcup_{n\in\mathbb{N}} f_n) \subseteq \bigcup_{n\in\mathbb{N}} Dom(f_n)$ and $Dom(\bigcup_{n\in\mathbb{N}} f_n) \supseteq \bigcup_{n\in\mathbb{N}} Dom(f_n)$.

- $Dom(\bigcup_{n\in\mathbb{N}} f_n) \subseteq \bigcup_{n\in\mathbb{N}} Dom(f_n)$
  As we have seen, the domain of $\bigcup_{n\in\mathbb{N}} f_n$, called A_1, is a subset of A.

  $\bigcup_{n\in\mathbb{N}} Dom(f_n)$ would be $\tilde{A}_1 \cup \tilde{A}_2 \cup \ldots \cup \tilde{A}_n$, called A_2, which is also a subset of A.

  ...

- $Dom(\bigcup_{n\in\mathbb{N}} f_n) \supseteq \bigcup_{n\in\mathbb{N}} Dom(f_n)$
  ...

By proving inclusion in both directions, it holds that $Dom(\bigcup_{n\in\mathbb{N}} f_n) = \bigcup_{n\in\mathbb{N}} Dom(f_n)$.

Exercise 4

Let $\mathbb{R}[x]$ be the set of all polynomials in variable x with coefficients $\mathbb{R}$. In other words,

$$\mathbb{R}[x] = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \mid n \in \mathbb{N} \text{ and } a_0, \ \ldots, \ a_n \in \mathbb{R}\}$$

Part a

Give three examples of $\mathbb{R}[x]$.

Solution

- $5x^3 + 0.3x^2 - 97x + 0.2$
- 4
- x^8

Part b

Prove that $(\mathbb{R}[x], +)$, $\mathbb{R}[x]$ with addition as the operator, is a semi-group.

Solution

A semigroup is a set with an associative binary operator applied to it. Here, we must prove that the addition of $\mathbb{R}[x]$ is both a binary operations and associative.

Addition of $\mathbb{R}[x]$ is a binary operation as $\forall a, b \in \mathbb{R}[x], \ a + b \in \mathbb{R}[x]$ because the sum of any two real numbers is itself a real number.
Addition of $\mathbb{R}[x]$ is also associative as $(a+b)+c = a+(b+c)$ is valid because addition of any two real numbers is associative.

$\therefore$ $(\mathbb{R}[x], +)$, $\mathbb{R}[x]$ is a semi-group.

Part c

Is $(\mathbb{R}[x], +)$ a monoid? Justify your answer.

Solution

For the semi-group $(\mathbb{R}[x], +)$ to be a monoid, the set $\mathbb{R}[x]$ must contain the identity element e to $(\mathbb{R}[x], +)$. The identify of addition with $\mathbb{R}[x]$ is 0, as the sum of any real number a and 0 is a. Additionally, 0 is an element of $\mathbb{R}[x]$, thus $(\mathbb{R}[x], +)$ is a monoid, where $e = 0$.

Part d

Does $(\mathbb{R}[x], +)$ have invertable elements? If so, which of its elements are invertable? Justify your answer.

Solution

The inverse element a^{-1} of any element a in $(\mathbb{R}[x], +)$ is such that $a + a^{-1} = e$. It follows that $a^{-1} = e - a$. In the case of $(\mathbb{R}[x], +)$, we have found $e = 0$. Thus, any element $a \in \mathbb{R}[x]$ has an inverse element $a^{-1} = -a$.