Assignment 1

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at here. I have also completed the Online Tutorial on avoiding plagiarism 'Ready Steady Write', located here.

Exercise 1

Please carry out the following proof in propositional logic following the proof format in tutorial 1.
Hypotheses: $P \Rightarrow (Q \Leftrightarrow \neg R),\ P \vee \neg S,\ R \Rightarrow S,\ \neg Q \Rightarrow \neg R$
Conclusion: $\neg R$

Solution

These are the provided hypotheses:

- $P \Rightarrow (Q \Leftrightarrow \neg R)$ (a)
- $P \vee \neg S$ (b)
- $R \Rightarrow S$ (c)
- $\neg Q \Rightarrow \neg R$ (d)

First, let's used tautology #21 to construct some implications:

	$P \vee \neg S$	(Using (b))
$\rightarrow$	$\neg S \vee P$	(#32: Law of commutativity)
$\rightarrow$	$S \Rightarrow P$	(#21)
$\rightarrow$	$\neg P \Rightarrow \neg S$	(#24: Law of contraposition)
•	$\neg P \Rightarrow \neg S$	(1)
	$R \Rightarrow S$	(Using (c))
$\rightarrow$	$\neg S \Rightarrow \neg R$	(#24: Law of contraposition)
•	$\neg S \Rightarrow \neg R$	(2)
	$P \Rightarrow (Q \Leftrightarrow \neg R)$	(Using (a))
$\rightarrow$	$\neg(Q \Leftrightarrow \neg R) \Rightarrow \neg P$	(#21)
•	$\neg(Q \Leftrightarrow \neg R) \Rightarrow \neg P$	(3)

Now we use tautology #14 to prune the unnecessary variables:

$$
\begin{array}{lll}
 & \neg P \Leftrightarrow \neg R & \text{(Using Biconditional Rule with (1) and (2))} \\
\rightarrow & \neg(Q \Leftrightarrow \neg R) \Rightarrow \neg R & \text{(Substituting into (3))} \\
\bullet & \neg(Q \Leftrightarrow \neg R) \Rightarrow \neg R & (4)
\end{array}
$$

Next, we are going to reorganise (4):

$$
\begin{array}{lll}
 & \neg(Q \Leftrightarrow \neg R) \Rightarrow \neg R & \text{(Using (4))} \\
\rightarrow & \neg((Q \wedge \neg R) \vee (\neg Q \wedge \neg\neg R)) \Rightarrow \neg R & \text{(\#23)} \\
\rightarrow & \neg(Q \wedge \neg R) \wedge \neg(\neg Q \wedge \neg\neg R) \Rightarrow \neg R & \text{(\#19: De Morgan's Law)} \\
\rightarrow & (\neg Q \vee \neg\neg R) \wedge (\neg\neg Q \vee \neg\neg\neg R) \Rightarrow \neg R & \text{(\#18: De Morgan's Law)} \\
\rightarrow & (\neg Q \vee R) \wedge (Q \vee \neg R) \Rightarrow \neg R & \text{(\#3: Law of Double Negation)} \\
\rightarrow & \neg Q \vee R \Rightarrow Q \vee \neg R \Rightarrow \neg R & \text{(\#27)} \\
\bullet & \neg Q \vee R \Rightarrow Q \vee \neg R \Rightarrow \neg R & (5)
\end{array}
$$

Finally, we assert $\neg R$ by using *Modus Ponens*:

$$
\begin{array}{lll}
 & \neg Q \Rightarrow \neg R & \text{(Using (d))} \\
\rightarrow & Q \vee \neg R & \text{(\#21)} \\
\rightarrow & Q \vee \neg R \Rightarrow \neg R & \text{(\#10: Modus Ponens with (5))} \\
\rightarrow & \neg R & \text{(\#10: Modus Ponens with (5))} \\
 & \neg R &
\end{array}
$$

$\blacksquare$

Exercise 2

Prove the following statement: If n is any integer, then $n^2 - 3n$ must be even.

Solution

For $n^2 - 3n$ to be even, the expression must take the form $2k$ for any integer n. Here, we will prove this is true if n is either even or odd.

In the case that n is an even number, $2k$ can be substituted for n:

$$\begin{aligned} & n^2 - 3n \\ \rightarrow\ & (2k)^2 - 3(2k) \\ \rightarrow\ & 4k^2 - 6k \\ \rightarrow\ & 2(2k^2 - 3k) \end{aligned}$$

We have proven $n^2 - 3n$ is even when n is even, as the expression $2(2k^2 - 3k)$ takes the form $2k$.

In the case that n is an odd number, $2k + 1$ can be substituted for n:

$$\begin{aligned} & n^2 - 3n \\ \rightarrow\ & (2k+1)^2 - 3(k+1) \\ \rightarrow\ & 4k^2 + 1 - 6k - 3 \\ \rightarrow\ & 4k^2 - 6k - 2 \\ \rightarrow\ & 2(2k^2 - 3k - 1) \end{aligned}$$

We have proven $n^2 - 3n$ is even when n is odd, as the expression $2(2k^2 - 3k - 1)$ takes the form $2k$.

$\therefore$ $n^2 - 3n$ is even for any integer n.

$\blacksquare$

Exercise 3

Prove via inclusion in both directions that for any three sets A, B and C:

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$

Solution

To prove via double inclusion, we must prove both $A \cap (B \setminus C)$ contains $(A \cap B) \setminus (A \cap C)$ and $(A \cap B) \setminus (A \cap C)$ contains $A \cap (B \setminus C)$, or, more formally, prove both of these:

$$(A \cap B) \setminus (A \cap C) \subseteq A \cap (B \setminus C) \qquad \text{(a)}$$
$$A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C) \qquad \text{(b)}$$

To prove (a), take $\forall x \in (A \cap B) \setminus (A \cap C)$:

$$\begin{aligned}
& x \in (A \cap B) \setminus (A \cap C) & \\
\rightarrow\ & x \in A \cap B \cap (A^c \cup C^c) & \text{(De Morgan's law)} \\
\rightarrow\ & x \in ((A \cap B) \cap A^c) \cup ((A \cap B) \cap C^c) & \text{(Law of distributivity)} \\
\rightarrow\ & x \in (A \cap A^c \cap B) \cup (A \cap B \cap C^c) & \text{(Law of associativity)}
\end{aligned}$$

As $A \cap A^c \cap B$ is always false, we can exclude the left-hand side:

$$\begin{aligned}
\rightarrow\ & x \in A \cap B \cap C^c & \text{(Excluding false left-side of union)} \\
\rightarrow\ & x \in A \cap (B \setminus C) & \text{(Applying definition of set subtraction)} \\
& x \in A \cap (B \setminus C) & \text{Here (a) is proven}
\end{aligned}$$

To prove (b), take $\forall x \in A \cap (B \setminus C)$:

$$\begin{aligned}
& x \in A \cap (B \setminus C) & \\
\rightarrow\ & x \in A \cap B \cap C^c & \text{(Definition of set subtraction)} \\
\rightarrow\ & x \in A \cap B \cap (A^c \cup C^c) & (A \cup A^c \text{ is an empty set}) \\
\rightarrow\ & x \in A \cap B \cap (A \cap C)^c & \text{(De Morgan's law)} \\
\rightarrow\ & x \in (A \cap B) \setminus (A \cap C) & \text{(Applying definition of set subtraction)} \\
& x \in (A \cap B) \setminus (A \cap C) & \text{Here (b) is proven}
\end{aligned}$$

With both (a) and (b) proven, it holds that the equivalence is true. ∎

Exercise 4

Let $\mathbb{N} \times \mathbb{N}$ be the Cartesian product of the set of natural numbers with itself consisting of all ordered pairs $(x1, x2)$, such that $x1 \in \mathbb{N}$ and $x2 \in \mathbb{N}$. We define a relation on its power set $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ as follows:

$$\forall A, B \in \mathcal{P}(\mathbb{N}\times\mathbb{N}),\ A \sim B \text{ iff } (A \setminus B) \cup (B \setminus A) = C \text{ and } C \text{ is a finite set.}$$

Determine whether or not $\sim$ is an equivalence relation and justify your answer by checking each of the three properties in the definition of an equivalence relation.

Solution

To determine if $\sim$ is an equivalence relation, we must check the three properties of an equivalence relation: *Reflexivity*, *symmetry* and *transitivity*.

To check *reflexivity*, we must verify $\forall A \in \mathcal{P}(\mathbb{N} \times \mathbb{N}),\ A \sim A = C$, where C is a finite set:

$$\begin{aligned}
& (A \setminus A) \cup (A \setminus A) = C \\
\rightarrow\ & \emptyset \cup \emptyset = C \\
\rightarrow\ & \emptyset = C
\end{aligned}$$

An empty set contains 0 elements, thus C is finite. So the relation is *reflexive*.

To check *symmetry*, we now verify $\forall A, B \in \mathcal{P}(\mathbb{N} \times \mathbb{N}),\ A \sim B \Rightarrow (A \setminus B) \cup (B \setminus A) = C$, where C is a finite set:

$$\begin{aligned}
& (A \setminus B) \cup (B \setminus A) = C & \\
\rightarrow\ & (A \setminus B) \vee (B \setminus A) = C & \text{(Definition of a union)} \\
\rightarrow\ & (B \setminus A) \vee (A \setminus B) = C & \text{(\#32: Law of commutativity)} \\
\rightarrow\ & (B \setminus A) \cup (A \setminus B) = C & \text{(Applying definition of a union)}
\end{aligned}$$

Here $B \Rightarrow A$ and C is still finite. Therefore the relation is *symmetric*.

To check *transitivity*, we finally verify $\forall A, B, C \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$, $A \sim B$ and $B \sim C$ such that:

$A \sim B \Rightarrow (A \setminus B) \cup (B \setminus A) = D$ where D is a finite set,
$B \sim C \Rightarrow (B \setminus C) \cup (C \setminus B) = E$ where E is a finite set,
To show $A \sim C$, which is $(A \setminus C) \cup (C \setminus A) = F$ where F is a finite set:

$$
\begin{array}{llr}
 & (A \setminus B) \cup (B \setminus A) & \\
\to & (A \cap B^c) \cup (B \cap A^c) & \text{(Definition of set subtraction)} \\
\to & (A \Rightarrow B)^c \cup (B \Rightarrow A)^c & \text{(\#20)} \\
\to & ((A \Rightarrow B) \cap (B \Rightarrow A))^c & \text{(\#19: De Morgan's Law)} \\
\to & (A \Leftrightarrow B)^c & \text{(\#22)} \\
\bullet & (A \Leftrightarrow B)^c & 1
\end{array}
$$

$$
\begin{array}{llr}
 & (B \setminus C) \cup (C \setminus B) & \\
\to & (B \cap C^c) \cup (C \cap B^c) & \text{(Definition of set subtraction)} \\
\to & (B \Rightarrow C)^c \cup (C \Rightarrow B)^c & \text{(\#20)} \\
\to & ((B \Rightarrow C) \cap (C \Rightarrow B))^c & \text{(\#19: De Morgan's Law)} \\
\to & (B \Leftrightarrow C)^c & \text{(\#22)} \\
\bullet & (B \Leftrightarrow C)^c & 2
\end{array}
$$

$$
\begin{array}{llr}
 & (A \Leftrightarrow B)^c \cup (B \Leftrightarrow C)^c & \text{(\#6: Addition with 1 and 2)} \\
\to & ((A \Leftrightarrow B) \cap (B \Leftrightarrow C))^c & \text{(\#18: De Morgan's Law)} \\
\to & (A \Leftrightarrow C)^c & \text{(\#17)} \\
\to & ((A \Rightarrow C) \cap (C \Rightarrow A))^c & \text{(\#22)} \\
\to & (A \Rightarrow C)^c \cup (C \Rightarrow A)^c & \text{(\#19: De Morgan's Law)} \\
\to & (A \cap C^c) \cup (C \cap A^c) & \text{(\#20)} \\
\to & (A \setminus C) \cup (C \setminus A) & \text{(Definition of set subtraction)}
\end{array}
$$

Here we have proven F is a subset of finite sets D and E. Thus F is a finite set and *transitivity* of $\sim$ is proven.

As the three properties of equivalence have been validated, we can conclude that relation $\sim$ is an equivalence relation.

$\blacksquare$