Assignment 5

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Exercise 1

Let $A = \mathbb{N}\times\mathbb{Z}\times\mathbb{Q}\times\mathbb{C}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution

Note that $\mathbb{R} \subseteq \mathbb{C} \subseteq A$. As we have proven in lectures, $\mathbb{R}$ is uncountably infinite. $\therefore$ $\mathbb{C}$ has an uncountably infinite subset, so $\mathbb{C}$ itself is uncountably infinite. From this it is clear A also has an uncountably infinite subset, so A itself is uncountably infinite.

Exercise 2

Let A be the set of points in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy the equation $r^2 = (\sin(\theta) - 1)^2$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution

To convert the Cartesian coordinates (x, y) to Polar coordinates (r, θ), we can use the following equations:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\frac{y}{x}$$

We can substitute r and simplify as such:

$$r^2 = (\sin\theta - 1)^2$$
$$\rightarrow (\sqrt{x^2 + y^2})^2 = (\sin\theta - 1)^2$$
$$\rightarrow x^2 + y^2 = (\sin\theta - 1)^2$$

Notice the values of $\sin\theta$ always equal $\frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

$$x^2 + y^2 = (\sin\theta - 1)^2$$
$$\rightarrow x^2 + y^2 = (\frac{\pi}{2} + 2\pi n - 1)^2$$

From this, $A = \{\ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = (\frac{\pi}{2} + 2\pi n - 1)^2, n \in \mathbb{Z}\}$.

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A is uncountably infinite.

Exercise 3

Let $A = \{\ (x, y) \in \mathbb{C}^2 \mid x^6 - 3x^2 + 1 = 0\ \}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution

A consists of all roots of the polynomial $x^6 - 3x^2 + 1$ paired with $\forall y \in \mathbb{C}$. The polynomial has a degree of 6, so according to the Fundamental Theorem of Algebra, this polynomial must have exactly 6 roots over $\mathbb{C}$. Let these set of roots be $(x_1, x_2, \ldots, x_6)$.
As A is the set of roots to $x^6 - 3x^2 + 1$ paired with all values of $\mathbb{C}$, we can redefine A as $\{\ x_n \times \mathbb{C}\ \}_{n=1\ldots6}$. It is now clear that A is the Cartesian product of a finite set and $\mathbb{C}$. As we proved in Exercise 1, the set of $\mathbb{C}$ is uncountably infinite. Therefore A is the Cartesian product of a finite set and an uncountably infinite set. From the definition of the Cartesian product, A itself must be an uncountably infinite set.

Exercise 4

Let $A = \{ \ (x, y) \in \mathbb{R} \times \mathbb{R}^+ \mid 1 + xy = 0 \ \} \ \cap \ \{ \ (x, y) \in \mathbb{R}^2 \mid \frac{(x-7)^2}{25} + \frac{(y+4)^2}{9} = 1 \ \}$

$\mathbb{R}^+$ stands for all positive real numbers. Consider $\mathcal{P}(A)$, the power set of A. Is $\mathcal{P}(A)$ finite, countably infinite or uncountably infinite? Justify your answer.

Solution

Let $B = \{ \ (x, y) \in \mathbb{R} \times \mathbb{R}^+ \mid 1 + xy = 0 \ \}$ and $C = \{ \ (x, y) \in \mathbb{R}^2 \mid \frac{(x-7)^2}{25} + \frac{(y+4)^2}{9} = 1 \ \}$. Notice that $A = B \cap C$. In other words, A consists of all the elements common to both B and C.
The domain of B is limited to $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ while the domain of C is limited to $(x, y) \in \mathbb{R}^2$. Clearly, the intersection A must therefore be confined to the more restricted domain $(x, y) \in \mathbb{R} \times \mathbb{R}^+$. Therefore, A consists of all common elements from sets B and C within the domain $(x, y) \in \mathbb{R} \times \mathbb{R}^+$.
Consider the elements within C that can be selected from the domain of A. Notice that this domain only permits positive real values for y. Due to this, we can show that there exists no value within $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ such that $\frac{(x-7)^2}{25} + \frac{(y+4)^2}{9} = 1$. We can accomplish this by replacing $\frac{(y+4)^2}{9}$ with a smaller value. As y must be a positive real number, letting $y = 0$ results in the smaller value $\frac{16}{9}$. Let us assume there exists a valid value for x.

$$\begin{aligned} &\frac{(x-7)^2}{25} + \frac{16}{9} < 1 \\ \rightarrow &\frac{(x-7)^2}{25} < -\frac{7}{9} \\ \rightarrow &(x-7)^2 < -\frac{175}{9} \end{aligned}$$

Here, we have met a contradiction. $x \in \mathbb{R}$, so it is impossible for any value for $(x-7)^2$ to be less than a negitive number. As such, C cannot contain any elements within the domain $(x, y) \in \mathbb{R} \times \mathbb{R}^+$. Clearly, the intersection of this set with a set restricted to this domain must contain exactly zero elements.
$\therefore A = B \cap C$ results in an empty set.
Futhermore, $\mathcal{P}(A)$, the power set of A, is the power set of the empty set. As such, $\mathcal{P}(A)$ only consists of the empty set. Therefore, $\mathcal{P}(A)$ is a finite set.

Exercise 5

Let A consist of all 2×2 matrices with entries in the real numbers $\mathbb{R}$ and determinant equal to 1. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution

A consists of matrices in the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
We can express $A \sim B$ where $B = \{\ (a, b, c, d) \in \mathbb{R}^4 \mid ad - bc = 1\ \}$, as each matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in A is mapped to (a, b, c, d) in B.
Consider the subset of B that is $C = B \cap (\mathbb{R} \times \{0\} \times \{0\} \times \mathbb{R})$. Notice we have simply created $C = \{\ (a, 0, 0, d) \in \mathbb{R}^4 \mid ad = 1\ \}$ as $b = 0$ and $c = 0$.

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A is uncountably infinite.

Exercise 6

Let $A = \{\ (x, y, z) \in \mathbb{R}^3 \mid 3x - y + 2z = 0 \text{ and } x + 2y + 3z = 0\ \}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution

Consider the subset of A that is $B = A \cap ((0, 1) \times (0, 1) \times \mathbb{R})$

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As B is uncountably infinite, we have proven A has an uncountably infinite subset. Therefore A is uncountably infinite.

Exercise 7

Let $A = \{0, 1\}$. Is the language $[\ (0 \cup \epsilon)^* \circ (1 \cup \epsilon)\ ]\ \cap\ (A \circ A)^*$ finite, countably infinite, or uncountably infinite? Justify your answer.

Solution

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L can be enumerated.

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L is countably infinite.

Exercise 8

Let A be a countably infinite alphabet. Is A^* finite, countably infinite or uncountably infinite? Justify your answer.

Solution

Recall that $A^* = \bigcup_{j=1}^{\infty} A^j$ where A^j is the sequence of all words of length j in the alphabet A. Note that the set of all words of a fixed length in an countably infinite alphabet is itself countably infinite, so A^j is a sequence of countably infinite sets. $\therefore$ A^* is the union of a sequence of countably infinite sets. As proven in lectures, it holds that A^* itself is then countably infinite.

Exercise 9

Let $A = \{0, 1, 2, 3, 4, 5\}$. Let the language L consist of all even length strings containing at least three odd letters. Is L finite, countably infinite or uncountably infinite? Justify your answer.

Solution

L can be expressed as the regular expression $(A \circ A)^*\ \cap\ (A^* \circ \{1, 3, 5\} \circ A^* \circ \{1, 3, 5\} \circ A^* \circ \{1, 3, 5\} \circ A^*)$ so it must be a regular language.

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L is countably infinite.

Exercise 10

Does there exist a sequence $\{x1, x2, x3, \cdots\}$ of languages over a finite alphabet A such that xi is not a regular language $\forall i \geq 1$? Justify your answer.

Solution

As proven in lectures, there are an uncountably infinite number of languages over the finite alphabet A.

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