Ted Johnson

1 (a) • The outcome of each persons test is a Bernoulli distributed random value, so the binomial distribution may be used to model the outcome of all tests.

• Let $n = 12$, the number of samples per occasion. Let $p$ be the probability that a test is positive. Then the number of positive cases per occasion is $X \sim \text{Binomial}(n, p)$

(b) $L(p) = P(X = x_1)\, P(X = x_2) = \prod_{i=1}^{2} P(X = x_i)$

But $P(X = x) = \binom{n}{x}(p)^x(1-p)^{n-x}$ where $n = 12$

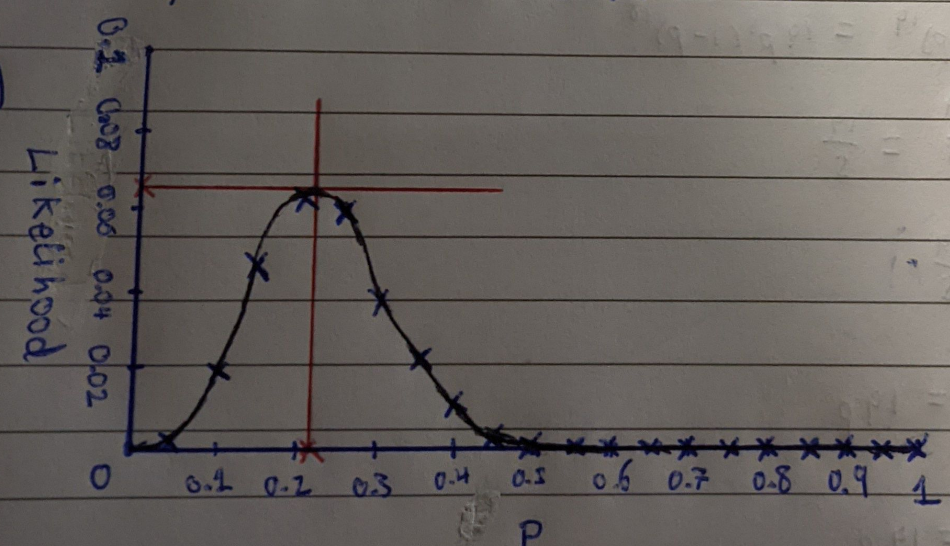
So $L(p) = \prod_{i=1}^{2} \binom{12}{x_i}(p)^{x_i}(1-p)^{12-x_i}$

Then $L(p) = \left(\prod_{i=1}^{2} \binom{12}{x_i}\right)\left((p)^s\right)\left((1-p)^{24-s}\right)$ where $s = \sum_{i=1}^{2} x_i$

From our data, we know $x_1 = 3$ and $x_2 = 2$

$\therefore L(p) = \left((220)(66)\right)\left((p)^5\right)\left((1-p)^{19}\right) = 14520(p)^5(1-p)^{19}$

(c)




From this sketch, it appears that $L(p)$ is maximised when $p \approx 0.21$, with a maximised value of $\sim 0.065$

(d) $\frac{dL}{dp} = \frac{d}{dp}\left(14520\,(p)^5(1-p)^{19}\right)$

$= 14520\,\frac{d}{dp}\left((p)^5(1-p)^{19}\right)$

$= 14520\left[\left(\frac{d}{dp}(p)^5\right)(1-p)^{19} + (p)^5\left(\frac{d}{dp}(1-p)^{19}\right)\right]$

$= 14520\left[5\,(p)^4(1-p)^{19} + 19\,(p)^5(1-p)^{18}\left(\frac{d}{dp}(1-x)\right)\right]$

$= 14520\left[5\,(p)^4(1-p)^{19} + 19\,(p)^5(1-p)^{18}(-1)\right]$

$= 14520\left(5(p)^4(1-p)^{19} - 19(p)^5(1-p)^{18}\right)$

To find the maximum value, we solve $\frac{dL}{dp} = 0$ :

$14520\left(5(p)^4(1-p)^{19} - 19(p)^5(1-p)^{18}\right) = 0$

$\Rightarrow 5p^4(1-p)^{19} - 19p^5(1-p)^{18} = 0$

$\Rightarrow 5p^4(1-p)^{19} = 19p^5(1-p)^{18}$

$\rightarrow \frac{p^4(1-p)^{19}}{p^5(1-p)^{18}} = \frac{19}{5}$

$\rightarrow \frac{1-p}{p} = \frac{19}{5}$

$\rightarrow 5(1-p) = 19p$

$\rightarrow 5 - 5p = 19p$

$\rightarrow 24p = 5$

$\rightarrow p = \frac{5}{24} \approx 0.2083$ so $14520\left(\frac{5}{24}\right)^5\left(1-\frac{5}{24}\right)^{19} = 0.0673$