SME M2 Training Problems

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Some problems may be harder than usual (or more advanced.) They are marked with a dot •. Problems that are possibly even harder (and even more advanced) are marked with two dots ••. All problems are worth doing! Try to do all of them. Most don't take very long.

1. Translate this into an algebraic expression: *the sum of the cubes of six consecutive integers.*

2. Even integers have the form $2n$. Translate this into an algebraic expression: *the sum of the squares of five consecutive even integers.*

3. Odd integers have the form $2n+1$. Translate this into an algebraic expression: *the sum of the fourth powers of three consecutive odd integers.*

4. Show that $n^4 + 10n^3 + 27n^2 + 10n + 1$ is a perfect square for all n. Do it by writing it in the form $(an^2 + bn + c)^2$ and finding the coefficients a, b, c.

5. Show that $n^4 - 14n^3 + 51n^2 - 14n + 1$ is a perfect square.

6. Show that $n^4 + 14n^3 + 51n^2 + 14n + 1$ is a perfect square.

7. • Show that $n^6 + 2n^5 + 5n^4 + 6n^3 + 6n^2 + 4n + 1$ is a perfect square.

8. • Show that $n^6 + 10n^5 + 27n^4 + 12n^3 + 11n^2 + 2n + 1$ is a perfect square.

9. • Show that $n^6 + 4n^5 + 10n^4 + 14n^3 + 13n^2 + 6n + 1$ is a perfect square.

10. Consider positive integers in steps of five: n, $n+5$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi + 625$ is always a perfect square.

11. Consider positive integers in steps of 10: n, $n+10$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi + 10000$ is always a perfect square.

12. • Even numbers have the form $2n$. Let ϕ be the product of four consecutive even numbers. Show that $\phi + 16$ is always a perfect square.

13. • Odd numbers have the form $2n+1$. Let ϕ be the product of four consecutive odd numbers. Prove that $\phi + 16$ is always a perfect square.

14. • Examine the patterns in problems **10**, **11**, **12** and **13**. Can you combine all these results into one mathematical statement?

15. • Prove the answer you gave to problem **14**.

16. Draw $17 \mod 4$ using a dot drawing.

17. If $x \equiv 2 \mod 5$ and $y \equiv 3 \mod 5$ then show that 5 divides $x + y$ by using dot drawings.

18. Find the last digit of 3^{20}. Use modular arithmetic. Don't use a calculator.

19. What is the last digit of 7^{101}? Use modular arithmetic.

20. Use negative residues to find $36^5 - 6^{50} + 48^{100} \mod 7$. Hint: use $48 = 6 \times 8$.

21. Use modular arithmetic to show that 33 divides $2^{50} - 1$. Use negative residues.

22. Use negative residues to prove that 10 divides $3^{20} - 1$.

23. Prove that 15 divides $2^{100} - 1$.

24. Prove that 6 divides the product of three consecutive integers.

25. Prove that 10 divides the product of five consecutive integers.

26. Prove that 35 divides the product of seven consecutive integers.

27. Prove that 7 always divides $n^7 - n$ for any integer n.

28. Prove that it is not true that 9 always divides $n^9 - n$. Use modular arithmetic to find an n such that $9 \nmid n^9 - n$.

29. Use modular arithmetic to prove that $11 \mid n^{11} - n$ for all integers n.

30. Show that 10 does not always divide $n^{10} - n$ and give an example of a number n such that $10 \nmid n^{10} - n$.

31. Is it true that 12 always divides $n^{12} - n$? Use modular arithmetic to find a counterexample.

32. Show that 7 does not divide $7n^3 + 3n^2 + 3n + 5$. Use modular arithmetic.

33. Prove that for all values of n, $121n^3 + 77n^2 + 66n + 11$ is never square.

34. • Prove that the sum of four consecutive fourth powers can never be square.

35. • Prove that the sum of the squares of five consecutive odd numbers is never square.

36. •• Prove that the sum of the fourth powers of four consecutive odd numbers is never square.

37. • • • Open question. Consider every kth integer. Let ϕ be the sum of the squares of five such consecutive integers. Investigate different values of k. Can you prove that it is never square for all k?

38. • • • Open question. Consider again every kth integer. Let ϕ be the sum of the fourth powers of four such consecutive integers. Is ϕ ever square? Can you prove that it can or cannot be square?

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39. Sum the first n odd integers beginning with 1. Use Euler's reverse method.

40. Sum the first n even integers beginning with 2. Use Euler's reverse method.

41. Use dot diagrams mod 2 to show how the odd numbers fit together to form a square having n^2 dots.

42. Use dot diagrams mod 2 to show how the even numbers fit together to form a square without a diagonal, having $n^2 - n$ dots.

43. Use the reverse method to sum all the integers from 1 to n^2.