SME M2 Training Problems

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Some problems may be harder than usual (or more advanced.) They are marked with a dot •. Problems that are possibly even harder (and even more advanced) are marked with two dots ••. All problems are worth doing! Try to do all of them. Most don't take very long. Some problems include answers.

1. Show that $n^4 + 10n^3 + 27n^2 + 10n + 1$ is a perfect square for all n. Do it by writing it in the form $(an^2 + bn + c)^2$ and finding the coefficients a, b, c.

2. Show that $n^4 - 14n^3 + 51n^2 - 14n + 1$ is a perfect square.

3. Show that $n^4 + 14n^3 + 51n^2 + 14n + 1$ is a perfect square.

4. • Show that $n^6 + 2n^5 + 5n^4 + 6n^3 + 6n^2 + 4n + 1$ is a perfect square.

5. • Show that $n^6 + 10n^5 + 27n^4 + 12n^3 + 11n^2 + 2n + 1$ is a perfect square.

6. • Show that $n^6 + 4n^5 + 10n^4 + 14n^3 + 13n^2 + 6n + 1$ is a perfect square.

7. Consider positive integers in steps of five: n, $n + 5$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi + 625$ is always a perfect square.

8. Consider positive integers in steps of 10: n, $n + 10$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi + 10000$ is always a perfect square.

9. • Even numbers have the form $2n$. Let ϕ be the product of four consecutive even numbers. Show that $\phi + 16$ is always a perfect square.

10. • Odd numbers have the form $2n + 1$. Let ϕ be the product of four consecutive odd numbers. Prove that $\phi + 16$ is always a perfect square.

11. • Examine the patterns in problems **7**, **8**, **9** and **10**. Can you combine all these results into one mathematical statement?

12. • Prove the answer you gave to problem **11**.

13. Prove that 7 always divides $n^7 - n$ for any integer n.

14. Prove that it is not true that 9 always divides $n^9 - n$. Use modular arithmetic to find an n such that $9 \nmid n^9 - n$.

15. Use modular arithmetic to prove that $11 \mid n^{11} - n$ for all integers n.

16. Show that 10 does not always divide $n^{10} - n$ and give an example of a number n such that $10 \nmid n^{10} - n$.