

# Permutations and Groups

Training problems for M2 2018 term 2

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## 1 Permutations


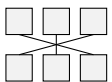
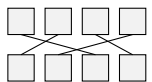
1. What is a permutation? Explain it.
2. Is it a permutation or not? Explain why.  
(a)  $abcd \rightarrow aabc$ .      (b)  $abcd \rightarrow cadb$ .      (c)  $cbda \rightarrow aebdc$ .      (d)  $dcba \rightarrow bca$ .
3. What is a transposition?
4. What is a regular permutation? Regular permutations are also called *derangements*.
5. What is the identity permutation?
6. Write down all the different permutations of  $uv$ .
7. Write down all the different permutations of  $abc$ .
8. Write down all the different permutations of  $wxyz$ .
9. I have five boxes colored red, green, blue, yellow, and orange. I have five balls colored red, green, blue, yellow and orange. How many different ways can I arrange the balls into the boxes, with one ball in each box?
10. I want to arrange 10 different people in a row. How many ways can I do this?
11. Prove that the number permutations of  $m$  objects is  $m!$ .
12. Prove that the number of permutation machines having  $m$  boxes per row is  $m!$ .
13. How many elements are in...  
(a)  $S_2$ ?      (b)  $S_3$ ?      (c)  $S_4$ ?      (d)  $S_5$ ?      (e)  $S_7$ ?
14. What is the difference between a permutation symbol and a permutation machine?
15. Write the permutation symbol that does the given permutaion.  
(a)  $abc \rightarrow bac$ .      (b)  $bac \rightarrow abc$ .      (c)  $abcd \rightarrow badc$ .      (d)  $badc \rightarrow abcd$ .
16. Draw the permutation machine that does the given permutaion.

- (a)  $abc \rightarrow cab$ .      (b)  $cab \rightarrow abc$ .      (c)  $abcd \rightarrow dcba$ .      (d)  $dcba \rightarrow abcd$ .

17. Change from permutation symbol to permutation machine.

- (a)  $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$       (b)  $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$       (c)  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$       (d)  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$

18. Change from permutation machine to permutation symbol.

- (a)       (b)       (c)       (d) .

19. Apply the permutation symbol to the objects. What is the result?

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} abc.$$

20. Put the objects into the permutation machine. What is the result?

$$abc \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix}.$$

21. Apply the permutations to the objects. What happens?

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} abc.$$

22. Put the objects into the permutation machines. What happens?

$$abc \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix}.$$

23. Apply the permutation symbols to the objects.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} abcd.$$

24. Put the objects into the permutation machines. What do you get?

$$abcd \begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} \begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} \begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix}$$

25. Multiply permutation symbols.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

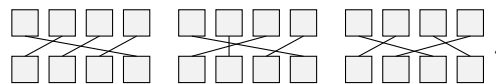
26. Multiply permutation symbols.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

27. Multiply permutation machines.

$$\begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix} \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix}.$$

28. Multiply permutation machines.



## 2 $S_3$ and $S_4$

29. Fill in this table for the elements of  $S_3$ .

machine	symbol	symbol	machine
		$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	
		$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	
		$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	

30. Write down all the permutation symbols for  $S_3$  and examine the size of the derangements (how many elements are changed). Then fill in this table:

Size of derangement	Number of elements that do it
0	
1	
2	
3	

31. Write down all the permutation symbols for  $S_4$ . Examine them and fill in this table (like you did in problem 30):

Size of derangement	Number of elements that do it
0	
1	
2	
3	
4	

32. Can you find an organized way to write down all the regular permutations (derangements) of  $S_5$ ? It's a big project. There should be 44 of them.

33. Use these standard definitions for  $S_3$  permutation symbols...

$$\begin{aligned}
 e &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & t_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} & t_2 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \\
 t_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} & s_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & s_2 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}
 \end{aligned}$$

...to fill in this mini  $S_3$  multiplication table:

	$e$	$s_1$	$s_2$
$e$			
$s_1$			$s_1s_2$
$s_2$			

The entry  $s_1s_2$  tells you how to combine the symbols. Take  $s_1$  from the leftmost column, and then put  $s_2$  from the top row.

**34.** Use the standard  $S_3$  definitions from problem 33 to construct the full  $S_3$  multiplication table:

	$e$	$t_1$	$t_2$	$t_3$	$s_1$	$s_2$
$e$						
$t_1$						$t_1s_2$
$t_2$						
$t_3$						
$s_1$						
$s_2$						

The entry  $t_1s_2$  tells you how to combine the symbol from the leftmost column ( $t_1$ ), with the symbol from the top row ( $s_2$ ).

**35.** Examine the table in problem 34. Notice that no row has two of the same elements. Also notice that no column has two of the same elements. You can use these facts to fill in the table faster. I was able to get 9 free table entries this way, where I did not have to do any multiplication of permutation symbols. Can you do it in such a way as to get more than 9 free ones?

**36.** Define the symbols  $e$  and  $t$  and use them to construct multiplication tables for  $S_2$  and  $S_1$ . How many elements do  $S_2$  and  $S_1$  have?

**37.** Look at the multiplication tables for  $S_1$ ,  $S_2$  and  $S_3$ . What permutation symbols behave like the identity in  $S_1$ ,  $S_2$  and  $S_3$ ?

**38.** What permutation symbols behave like the identity in  $S_5$ ? In  $S_6$ ?

**39.** Is  $S_2$  inside  $S_3$ ? Explain how.

**40.** Is  $S_3$  inside  $S_4$ ? Explain how.

**41.** Consider these elements of  $S_4$ :

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Make a multiplication table with  $e$ ,  $a$ ,  $b$ ,  $c$ . Is the table perfect (each row contains each symbol exactly once and each column contains each symbol exactly once)?

42. Consider these elements of  $S_4$ :

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

Make a multiplication table with  $e, p, q, r$ . Is the table perfect?

### 3 Inverse

43. Use the  $S_3$  multiplication table in problem 34 to find the inverses of  $e, t_1, t_2, t_3, s_1$  and  $s_2$ . Do it two different ways:

- (a) using  $x \cdot x^{-1} = e$ . (b) using  $x^{-1} \cdot x = e$ .

44. Find the inverses of  $e, t_1, t_2, t_3, s_1$  and  $s_2$  *without* using the  $S_3$  multiplication table. Do it two different ways:

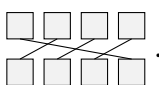
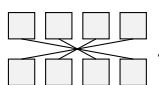
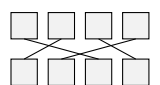
- (a) using  $x \cdot x^{-1} = e$ . (b) using  $x^{-1} \cdot x = e$ .

45. Find the inverses of these  $S_4$  permutation symbols.

- (a)  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ . (b)  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ . (c)  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ .

Do it two different ways: using  $x \cdot x^{-1} = e$  and then using  $x^{-1} \cdot x = e$ .

46. Find the inverses of these  $S_4$  permutation machines.

- (a) . (b) . (c) .

Do it two different ways: using  $x \cdot x^{-1} = e$  and then using  $x^{-1} \cdot x = e$ . Remember that machines multiply to the right.

47. Find the inverses of these  $S_5$  symbols and machines. Do it two different ways: using  $x \cdot x^{-1} = e$  and  $x^{-1} \cdot x = e$ .

- (a)  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$ . (b) .

48. Study the patterns in the  $S_3$  multiplication table of problem 34. Is it possible for an element to have two different inverses? Prove that if  $x$  is an element of  $S_n$  then  $x$  cannot have two different inverses.

49. Prove that the inverse of  $abc$  is  $c^{-1}b^{-1}a^{-1}$ . Hint: use  $xx^{-1} = e$  and  $x^{-1}x = e$ .

### 4 Symmetries

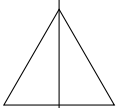
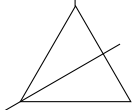
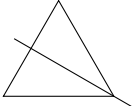
50. What is an isometry?

51. Write down the three different kinds of isometries.

52. What are symmetries?

53. What kind of thing has translational symmetries? Draw some examples.

54. What did the Ancient Greeks think about beauty and symmetry?
55. According to the Ancient Greeks, what is the most beautiful geometric shape? Why did they think so?
56. Find all symmetries of a scalene triangle. Make a multiplication table. (It's not very big.)
57. An isosceles triangle has two symmetries. Find them. Use Roman letters  $a, b, \dots$  for rotational symmetries and Greek letters  $\alpha, \beta, \dots$  for reflection symmetries. Make a multiplication table. Which symmetry behaves like the identity?
58. Find all symmetries of an equilateral triangle. How many are there? Which symmetry behaves like the identity?
59. An equilateral triangle has six symmetries. Three rotational symmetries and three reflection symmetries. We use Roman and Greek letters to give them names:

Symbol	Symmetry
$a$	$0^\circ$ rotation.
$b$	$120^\circ$ rotation.
$c$	$240^\circ$ rotation.
$\alpha$	
$\beta$	
$\gamma$	

Construct a multiplication table for the symmetries of the equilateral triangle:

	$a$	$b$	$c$	$\alpha$	$\beta$	$\gamma$
$a$						
$b$						
$c$						
$\alpha$						
$\beta$						
$\gamma$						

Remember: symmetries are combined from right to left, just like permutation symbols.

60. Compare  $S_1$  to the symmetries of a scalene triangle. Are the multiplication tables similar?

61. Compare  $S_2$  to the symmetries of an isosceles triangle. Are the multiplication tables similar?
62. Compare  $S_3$  to the symmetries of an equilateral triangle. What can you say about the multiplication tables of these two things?

## 5 Groups

63. Write down the symmetries of an isosceles triangle and construct the multiplication table. Prove that they form a group by showing that there is an identity element, that the multiplication table is closed and all symmetries have inverses.
64. Prove that  $S_2$  is a group by showing the three group properties: identity, closure and inverse.
65. Let  $K = \{a, b, c, d\}$  with the following multiplication table:

	$a$	$b$	$c$	$d$
$a$	$a$	$b$	$c$	$d$
$b$	$b$	$a$	$d$	$c$
$c$	$c$	$d$	$a$	$b$
$d$	$d$	$c$	$b$	$a$

Prove that  $K$  is a group.

66. Let  $L = \{x, y, z, w\}$  with the following multiplication table:

	$x$	$y$	$z$	$w$
$x$	$z$	$w$	$x$	$y$
$y$	$w$	$z$	$y$	$x$
$z$	$x$	$y$	$z$	$w$
$w$	$y$	$x$	$w$	$z$

Prove that  $L$  is a group.

67. Let  $G = \{e, t_1, t_2, t_3, s_1, s_2\}$  be the symmetries of an equilateral triangle, as in 59. Construct the multiplication table for  $G$  and prove that  $G$  is a group. Show identity, closure, inverse.
68. Let  $S_3 = \{e, t_1, t_2, t_3, s_1, s_2\}$  according to the standard definitions that we used in problem 33. Show that  $S_3$  is a group by showing that it has the properties of identity, closure and inverse.
69. Let  $G$  be a group. Prove that the multiplication table for  $G$  has the following magical property: no row has more than one of the same element.
70. Let  $G$  be a group. Prove that the multiplication table for  $G$  has another magical property: no column has more than one of the same element.

71. Let  $G$  be a group and let  $g \in G$ . Prove that if  $g^2 = g$  then  $g$  must be the identity element. You can try using contradiction. Assume  $g$  isn't the identity, and try to get a contradiction.
72. Show that  $g^{|S_3|} = e$  for all elements  $g \in S_3$ .
73. Find the order of every element in  $S_3$ .
74. Construct an order-3 group  $G$ . Find the order of each element in  $G$ . Verify that if  $g \in G$  then  $g^{|G|}$  is the identity. Find all generators (if any). Is this group cyclic?
75. Let  $G = \{a, b, c, d\}$  with the following Cayley table:

	$a$	$b$	$c$	$d$
$a$	$a$	$b$	$c$	$d$
$b$	$b$	$d$	$a$	$c$
$c$	$c$	$a$	$d$	$b$
$d$	$d$	$c$	$b$	$a$

Find the order of each element in  $G$ . Verify that  $g^{|G|}$  is the identity for all  $g$  in  $G$ . Find all generators of  $G$ . Is  $G$  a cyclic group?

76. Construct a group  $G$  of order 5. Find the order each element  $g \in G$  and show that  $g^{|G|}$ . Find all generators of  $G$ . Is  $G$  cyclic?
77. Find all generators of  $S_3$  (if any). Is  $S_3$  a cyclic group?
78. Construct a group of order 7. Prove that it is a group. Find the orders of all the elements. Find all generators. Is the group cyclic?
79. Is it possible to construct two groups of order 3 that are not isomorphic? In other words, can we construct two different Cayley tables with the same symbols  $a, b, c$  in the same order, which cannot be matched by an isomorphism map?

	$a$	$b$	$c$
$a$			
$b$			
$c$			

	$a$	$b$	$c$
$a$			
$b$			
$c$			

Try it.