Arithmetic and Combinatorics Part 2

Training problems for M1 2018 term 2

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1. I have have 5 objects. I want to choose 3 of them. Draw all the different ways that this can be done. How many are there?

2. I have have 5 objects. I want to choose 2 of them. Draw all the different ways that this can be done. How many ways are there?

3. I have four fruits: apple, banana, strawberry and peach. I want to choose three of them to make a milkshake. Write down all the different ways of doing this. Order doesn't matter.

4. What does $\binom{n}{k}$ mean? Explain it.

5. Figure these out.

(a) $\binom{1}{0}$. (b) $\binom{5}{5}$. (c) $\binom{5}{0}$. (d) $\binom{n}{0}$. (e) $\binom{n}{n}$.

6. Figure these out.

(a) $\binom{1}{2}$. (b) $\binom{0}{1}$. (c) $\binom{5}{6}$. (d) $\binom{n}{n+1}$. (e) $\binom{2}{-1}$.

7. Draw Pascal's triangle, circle these elements and label them:

$$\binom{5}{2} \quad \binom{3}{3} \quad \binom{1}{0} \quad \binom{7}{6} \quad \binom{4}{2} \quad \binom{2}{2} \quad \binom{6}{0} \quad \binom{0}{0}.$$

8. What is the sum of row $n = 5$ of Pascal's triangle?

9. What is the sum of row $n = 12$ of Pascal's triangle? Do it *without* using the numbers of row $n = 12$.

10. Draw Pascal's triangle up to row $n = 8$ and circle the central Pascal numbers $\binom{2m}{m}$.

11. Figure out $\binom{8}{4}$ by summing the squares of the elements in row $n = 4$.

12. Figure out $\binom{14}{7}$. by summing the squares of the elements in row $n = 7$.

13. What is a set? Explain it. What are the rules for sets?

14. What is a subset? Explain it.

15. Write down all the subsets of $\{a, b, c\}$. How many are there?

16. Write down all the subsets of $\{1, 2, 3, 4\}$. How many are there?

17. Let $S = \{a, b, c, d, e, f, g\}$. How many size-3 subsets does S have?

18. Let $|S| = 10$ and let $A \subseteq S$ with $|A| = 5$. How many such A are there?

19. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many subsets does S have?

20. Let $|S| = 14$ and $A \subseteq S$. How many such A are there?

21. Let $S = \{a, b, c, d, e, f, g, h, i, j\}$. Let $A \subseteq S$ such that $|A|$ is an even number. How many such A are there? Use Pascal numbers and Pascal's triangle.

22. Let $S = \{a, b, c, d, e, f, g, h, i, j\}$. Let A be an odd-sized subset of S. How many such A are there? Use Pascal numbers and Pascal's triangle to solve this.

23. This is the menu at the Italian restaurant.

Main dishes	*Side dishes*	*Desserts*
Fetuccine alfredo	Porcini mushroom bruschetta	Hazelnut tartufo
Pepperoni pizza	Grilled polenta	Cannoli
Three-cheese lasagna	Spinach ricotta gnocci	White chocolate panna cotta
	Radicchio with lemon	Sfogliatelle
	Stuffed artichokes	

I want to get one main dish, one side dish, *and* one dessert. How many ways can I do that? Write the definitions for what you are doing. Be clear. Write the principle. Do the computation.

24. At the Italian restaurant in problem **23**, I want to get either one main dish *or* one side dish *or* one dessert. How many ways can I do that? Show definitions, principle, calculation. Do a proper job, don't just write the answer.

25. At the Italian restaurant in **23**, I want to get either only a main dish or a side dish and a dessert. How many ways can I do that? Show definitions, principle, calculation.

26. At the Italian restaurant in **23**, I want to get either both a main dish and a dessert or both a side dish and a dessert. How many ways can I do that? Definitions. Principle. Calculation.

27. The Boring Book Library has nothing but the most boring books on the dullest topics.

Topic	*Number of books*
Dishwashing	7
K-Pop	6
Mops and brooms	5
British tophats	4

How many ways can I choose 2 diswashing books, 2 K-Pop books, 2 books about mops and brooms, and 2 books about British tophats? Show your definitions, combinatorics principle, and calculation.

28. From the library in problem **27**, I want to choose either 4 dishwashing books or 3 K-pop books *or* 2 books about mops or 1 tophat book. How many ways can I do this? Definition—principle—calculation.

29. From the Boring Book Library, choose either three dishwashing books and two mops books, *or* two K-Pop books and three British tophat books. How many ways can you do that? Show definition—principle—calculation.

30. From the Boring Book Library, choose 3 Dishwashing books or 2 British tophat books *and* 4 K-Pop books and 3 books on mops and brooms. Definitions. Principle. Calculation.

31. We have 7 girls and 5 boys. How many ways can we make a team by choosing 3 girls and 3 boys? Definitions, principle, calculation.

32. We have 7 girls and 5 boys. We want a small team of only 3 people: either all girls or all boys. How many ways can I do this? Definitions, principle, calculation.

33. How many 3-digit numbers are there? Show definitions, principle, calculation.

34. How many 3-digit numbers can you make using only *even* digits? Show your definitions, combinatorics principle, and your calculations.

35. How many 3-digit numbers can you make using only *odd* digits? Definitions. Principle. Calculation. Be clear. Explain what you are doing. Don't just write an answer.

36. What is a permutation? Explain it.

37. Write down all different permutations of the letters EFG. How many are there?

38. Write down all the different permutations of the digits 1234. How many are there.

39. Prove that the number of different ways to arrange k objects in order is $k!$.

40. I have 6 books and I want to arrange them in order on a bookself. How many different ways can I do it?

41. I have 6 books and I want to choose 3 to arrange on my bookshelf in order. How many ways can I do this?

42. I have 8 students. I want to choose 3 of them and give them prizes: 1st, 2nd and 3rd place. Does order matter? How many ways can I do this? Do it in two steps and show definitions, principle, calculation.

43. We have n students. We want to choose k of them and arrange them in order. How many ways can we do this? Show your definitions, what principle you use, and the calculation of the final answer.

44. We have 5 girls and 5 boys in our class. I want to choose 2 girls and one boy and give them prizes: 1st place, 2nd place, 3rd place. Does order matter? How many ways can I do this? Definitions (be clear). Principle (what combinatorics principle are you using?) Calculation (get the final answer).

45. We have 5 girls and 5 boys. I want to choose 3 girls and 3 boys to make a team of six. Then I want to choose three in the team to be president, secretary and messenger. Is order important? How many ways can I do this? Show definitions, principle, calculation.

46. We have 6 girls and 5 boys. I want to give three prizes to the girls (1st, 2nd, 3rd) *and* three prizes to the boys (1st, 2nd, 3rd). How many ways can I do this? Definitions. Principle. Calculation.

47. We have 6 consonants *mnpqrs*, 5 vowels *aeiou* and 5 digits 12345. I want to make passwords by choosing two of each, a total of 6 symbols. Of course order matters when you make passwords. For example: 2*aqur*5 and 5*ruqa*2 are two different passwords. How many such passwords can I make? Definitions. Principle. Calculation.

48. (A) I have n objects. I select k of them in some special order. This one first, then that one, then another one, and so on.

(B) I have n objects. I select k of them without order, but then I arrange them in some special order later. I put one first, then another one second, and so on.

Is there a difference between (A) and (B)? Think about it.

49. How many ways can you make a combination of n objects taken k at a time?

50. How many ways can you make a permutation of n objects taken k at a time?

51. When do you have the smallest non-zero number of combinations of n objects taken k at a time?

52. When do you have the largest number of permutations of n objects taken k at a time?

53. Prove that the number of permutations of n objects taken k at a time is

$$n(n-1)\dots(n-k+1).$$

Use drawings of balls with numbers on them and boxes that are in order. Make sure you clearly explain why the last factor is $(n-k+1)$.

54. Prove that

$$\frac{n!}{(n-k)!} = n(n-1)\dots(n-k+1).$$

55. Calculate $\binom{25}{4}$ using the formulas we proved in class.

56. Calculate $\binom{25}{21}$ using formulas we proved in class.

57. Calculate the central Pascal number $\binom{20}{10}$ in two ways:

(a) By summing the squares of the elements in row $n = 10$ of Pascal's triangle.

(b) By using the formulas we proved in class.

58. Find the number of permutations of 10 objects taken 4 at a time.

59. Find the number of combinations of 10 objects taken 4 at a time.

60. Find the number of permutations of 12 objects taken 9 at a time.

61. Find the number of combinations of 12 objects taken 9 at a time.

62. I have 7 different fruits. I want to throw 4 of them in the blender to make a milkshake. Is this a permutation or a combination? How many ways can I do it?

63. W have 7 different students. We want to give 4 of them different prizes: \$100, \$50, \$20 and \$10. Is this a permutation or a combination? How many ways can I do it?

64. I have these characters: `a`, `b`, `c`, `d`, `e`, `f`, `g`, `h`, `i`. I want to make passwords by choosing 5 different characters from what I have. Is order important? Is this a permutation or a combination? Figure out how many different passwords I can make.

65. I have two groups of characters. Group A: `A`, `B`, `C`, `D`, `E`, `F`. Group B: `1`, `2`, `3`, `4`, `5`. I want to make license plates of the form `XXX-YYY` where `XXX` are three different characters chosen from group A, and `YYY` are three different characters chosen from group B. How many different license plates can I make? Use definitions, principles, etc.

66. At the Blender Company, we have to put a product code on every blender we make. The code looks like this `XX-YYY-ZZ` where `XX` are two different characters chosen from group A, `YYY` are three different characters from group B, and `ZZ` are two different characters from group C.

Group A:	`a, b, c, d, e, f, g`
Group B:	`1, 2, 3, 4, 5, 6, 7, 8`
Group C:	`u, v, w, x, y, z`

Does order matter? How many blender codes can I make? Use definitions and principles and give a clear explanation of how you got the answer.

67. Calculate the number of possible winning combinations for the 6/49 lottery. You have to choose 6 numbers out of 49 correctly. Order doesn't matter.

68. In the 5/32 lottery, you have to choose 5 numbers from one to 32. On Friday the lotto machine gives the winning numbers on TV. If your numbers are the same as the machine's numbers then you win. Order doesn't matter. How many possibilities are there?

69. I want pack my books into boxes. I have 18 books and 3 boxes. I want to put 6 books in each box. How many ways can I do this? Use your imagination. Does the order of the books inside the boxes make a difference in this problem? Figure out the number of ways I can do this. Use definitions and principles and give a clear explanation.

70. I have 12 different objects and 4 different numbered boxes. I want to choose an object and put it into a box. One object per box. Find the number of ways to do this. Draw pictures and give a complete explanation using definitions and principles of combinatorics.

71. I have 12 different objects and 4 different numbered boxes. I want to put three objects into each box. Find the number of ways to do this. Draw pictures and give a complete explanation using definitions and principles of combinatorics.

72. I have 12 different objects and 3 different numbered boxes. I want to put four objects into each box. Find the number of ways to do this. Draw pictures and give a complete explanation using definitions and principles of combinatorics.

73. We have 9 people: Anne, Jim, Bob, Rick, William, Kenny, Pete, Harry and Joe. We want to hide them in four rooms: bedroom, kitchen, living room, and bathroom. Three go in the bedroom, two go in the kitchen, two go in the living room and two go in the bathroom.

(a) Draw an example of a correct arrangement of people in the rooms.

(b) Think about the order of the three people that you put in the bedroom. Is the order important?

(c) Calculate the number of ways this can happen. Always use definitions and principles.

74. We have 9 people: Rick, Ken, Jim, Bob, James, Anne, Willie, Zoe and Frank. We have four different rooms: the garage, the attic, the kitchen and the workshop. We want to put two people in each room.

(a) Make some drawings and explain why you think the order is or is not important inside a room.
(b) What should we use to calculate the number of ways to fill a room? Permutations or combinations?
(c) Figure out the number of ways to fill all rooms. Give definitions, combinatorics principles. Don't just write an answer.

75. Ten people go to a fast food shop. Five of them get hamburgers. Three get chickenburgers. One gets a fishburger. Does order matter between three people who get the same kind of burger? Does order matter between three people who each get different kinds of burgers? How many ways can this happen? Always use definitions and principles when you answer a question like this. It's also good if you make drawings.

76. Suppose we have different objects like numbered balls, and we have numbered boxes. We put one ball into each box. This is called a permutation. What if we put any number of the balls into each box? Like maybe two balls in box 1, none in box 2, one in box 3, and so on. We can call this a *super-permutation.*

(a) When does a super-permutation become a permutation?
(b) When does a super-permutation become a combination?

77. We have 9 people and three rooms: the bedroom, kitchen and attic. In each room we have 4 chairs: red, green, yellow and blue. I want to put two people in each room and arrange them in the chairs. Does order matter inside the rooms? How many ways can I do this? Use definitions, principles.

78. We have 9 people and three rooms. In each room there are three chairs: red, green and yellow. We want to put three people in each room and arrange them in the chairs. How many ways can we do this? Use definitions, principles.

79. We have 8 people: Ken, Bob, Larry, Tim, James, Joe, Dave and Rick. We have three rooms: the bedroom, livingroom, and workshop. In the bedroom there are four chairs: yellow, red, blue and green. In the livingroom there are two chairs: yellow and red. In the workshop there are also two chairs: yellow and red. I want to arrange 4 people in the bedroom chairs, 2 people in the livingroom chairs and 2 people in the workshop chairs. How many ways can I do that? Definitions, principles, etc.

80. Think about the answers you got for problems **78** and **79**. Why do you get a simple answer like $n!$?

81. We have 8 people. We have two rooms. In each room there are five differently colored chairs: red, blue, yellow, black, green. We want to put three people in each room and arrange them sitting in the chairs. How many ways can we do this? Definitions, principles.

82. What does *distinguishable* mean?

83. What does *indistinguishable* mean?

84. Where do we see the effects of indestinguishable objects in nature?

85. Who were some famous scientists who worked on the idea of distinguishable and indistinguishable objects in physics?

86. What are the indistinguishable objects (symbols) of `okefenokee`? Maybe you get an extra point if you can tell me what Okefenokee is.

87. We have three red balls, two black balls and five white balls in a row. Draw some permutations of these and show which objects are indistinguishable.

88. Which of these are distinguishable and which are indistinguishable?

`Janet Elroy Ed Jackie Elroy Willie Janet Elroy`

89. We have six people. Three of them are named Jim and three of them are named Larry. What are the distinguishable objects? What are the indistinguishable objects?

90. Make a list of all the permutations of `cat`. How many are there?

91. make a list of all the *different* permutations of `crab`. How many are there?

92. Make a list of all the different permutations of `mama`. How many are there?

93. Make a list of all the different permutations of `naana`. How many are there?

94. Make a list of all the different permutations of `tarat`. How many are there?

95. What are the indistinguishable symbols in `ongopongo`? Find the number of different permutations. Draw some pictures of positions of symbols and boxes for different kinds of symbols. Use the super-permutation idea. Write definitions and combinatorics principle.

96. Find the number of different permutations of `AKUAKU`. Draw balls and boxes and use the super-permutation idea (any number of balls in each box.) Give definitions, combinatorics principle. What is Aku-Aku? Find out.

97. How many permutations of `SASSAFRAS` are there? Draw balls and boxes. Use the super-permutation idea. Give definitions, principles, etc. What is sassafras?

98. We have these objects: $aa\cdots abb\cdots b$. There are k_1 of the a objects and k_2 of the b objects.

(a) How many objects are there in all?
(b) Find a beautiful (multinomial) formula for the number of different permutaions using algebra and the formula $\binom{m}{r} = \dfrac{m!}{r!(m-r)!}$.
(c) Use logical, combinatorial thinking to get a beautiful formula for the number of permutations. Explain how you get this formula.

99. We have these objects: $aa\cdots abb\cdots bcc\cdots c$. There are k_1 of the a objects, k_2 of the b objects and k_3 of the c objects.

(a) How many objects are there in total?
(b) Find a multinomial formula for the number of permutations using algebra.
(c) Find a beautiful (multinomial) formula for number of permutations using logical combinatorial thinking. Explain how you get it.

100. We have these objects:

$$aa\cdots abb\cdots bcc\cdots cc\cdots cdd\cdots d\,.$$

There are k_1 of the a objects, k_2 of the b objects, k_3 of the c objects and k_4 of the d objects.
(a) How many objects are there in all?
(b) Use algebra to find a beautiful formula for the number of permutations.
(c) Use logical combinatorial thinking to get the same formula. Explain how you get it.

101. We have r kinds of indistiguishable objects. There are k_1 of the first kind, k_2 of the second kind, ... and k_r of the rth kind
(a) What is the total number of objects?
(b) Use logical combinatorial thinking to find a multinomial formula for the number of different permutations of these objects.

102. Use the formula $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ and write $\binom{n}{k}$ as a multinomial. This problem shows that all Pascal numbers are just simple multinomials.

103. Find the number of different permutations of `MISSISSIPPI`. Use the multinomial formula. What are the indistinguishable symbols? What is Mississippi? Where is it?

104. What are the indistinguishable symbols in `ABRACADABRA`? Find the number of permutations using multinomial formula. What does abracadabra mean?

105. We have three red balls, three green balls and four white balls arranged in a row. Draw some examples of permutations of these balls. What are the indistinguishable objects? Use the multinomal formula to find the number of permutations of these balls.

106. Find the number of permutations of `ICELAND` by using the multinomial formula. Notice that you can think of each symbol as being repeated *once*. This is an interesting problem because it shows that even when all objects are distinguishable, you can still calculate their permutations by the multinomial formula.

107. What are the coefficients and and degrees of these terms?

(a) $-\frac{1}{2}x^3y^2$. (b) $5xyz^2$. (c) $-2x^5z^2$. (d) $7y^{12}$. (e) $\frac{4}{3}x^4y^3$.

108. Expand by polynomial multiplication.

(a) $(x+y)^2$. (b) $(x+y)^4$. (c) $(x+y+z)^2$. (d) $(x+y+z)^3$.

109. Expand and write the coefficients using $\binom{n}{k}$.

$$\begin{array}{l|c}
(x+y)^0 & \binom{0}{0} \\
(x+y)^1 & \binom{1}{0}x + \binom{1}{1}y \\
(x+y)^2 & \\
(x+y)^3 & \\
(x+y)^4 & \\
(x+y)^5 &
\end{array}$$

110. Use the ideas in problem **109** to quickly expand $(x+y)^8$ without multiplying anything.