SME M2
2/10   2/9   2/8
Practice Problems

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Part of the challenge of problem-solving is figuring out exactly what the question wants you to do. So you have to read the question carefully.

When solving problems, use sensible symbols, like m for mass or p for pressure or h for height.

Use simplifications, assumptions and approximations as long as these do not affect the essence of the answer. It takes experience to know when to make assumptions and simplifications. You can only get this kind of experience from doing problems.

Pay careful attention to units: meters, kilograms etc. Never write just 3.5 if you mean a mass of 3.5 kg. Likewise for all other quantities that have units.

If you do all these problems, you will have no trouble at all on class tests, midterm or final test.

1 Planet versus atmosphere

1. Venus is almost the same size as the Earth, but its atmosphere is much heavier. Venus has the following physical characteristics:

Diameter	$12,104$ km
mass	5×10^{21} tons
gravitational acceleration	$8.9\,\mathrm{m/s^2}$
atmospheric pressure	$9,200,000$ Pa

How many times heavier is Venus compared to its atmosphere? Calculate the ratio $M_{\text{planet}}/M_{\text{atm}}$.

2. Mars is much smaller than the Earth and its atmosphere is much thinner. Here is the data for Mars.

Diameter	$6,780$ km
mass	6.4×10^{20} tons
gravitational acceleration	$3.7\,\mathrm{m/s^2}$
atmospheric pressure	640 Pa

Calculate the planet-atmosphere ratio $M_{\text{planet}}/M_{\text{atm}}$ for Mars.

3. Titan is a moon of Saturn. It's an unusual because it has a pretty thick atmosphere even though the gravity is low. We used to think that Titan was the largest moon in the Solar System, but now we know that Ganymede is a bit bigger.

Diameter	5150 km
mass	1.35×10^{20} tons
gravitational acceleration	$1.35\,\mathrm{m/s^2}$
atmospheric pressure	147,000 Pa

Calculate $M_{\text{planet}}/M_{\text{atm}}$ for Titan.

4. Arrange the $M_{\text{planet}}/M_{\text{atm}}$ ratios for Earth, Venus, Mars, and Titan in a table, from smallest to largest. Where does the Earth fit in?

2 Coffee cans

5. Two cylindrical paint cans are different sizes but geometrically similar. Their dimensions are proportional. The bigger paint can has a height h_1 while the smaller one has height h_2. Prove the following surface area and volume proportionalities:

$$\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2, \quad \frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3. \tag{1}$$

Be careful to consider the top and bottom of the cans when calculating surface area!

6. Consider two rectangular boxes of proportional shape. The bigger one has a height h_1 and the smaller one has height h_2. These boxes have general dimensions of length, width, height, and all three may be different. Show that the area-volume relationships (1) are true for these boxes.

7. Repeat the same problem but for two cube-shaped boxes of height h_1 and h_2. Naturally they are proportional because all cubes are geometrically similar. Prove the relationships (1).

8. Consider two cone-shaped geometrically proportional objects of height h_1 and h_2. Prove that the same area-volume relationships (1) apply. You will have to look up the formulas for surface area and volume of a cone. Also, don't forget about the bottom of the cone when calculating surface area.

9. Finally do the same exercise but with two spheres of height h_1 and h_2. Naturally all spheres are proportional to each other; they are all geometrically similar. Prove (1).

10. Two cans of coffee are similar geometrically proportional The first can has a height of 13 cm and total mass of 3.2 kg. The other can has a height of 11 cm and a total mass of 2 kg. Find the mass of coffee in both cans, and find the mass of the empty cans.

11. As above, two geometrically similar cans of coffee. One is 14 cm high and the other is 11 cm high. The total masses of the cans are 2.5 kg and 1.25 kg respectively. Find the mass of coffee in each can and the mass of each empty can.

12. In M2/10 class I gave a coffee-can problem, but it turned out to have no reasonable solution. That leads to a very interesting question: *what conditions are necessary for there to be a solution to the coffee can problem?* This question has a beautiful and intriguing answer.

Suppose two geometrically similar cans have heights h_1 and h_2 with total masses m_1 and m_2. Suppose x_1 and x_2 are the masses of coffee in each can, and y_1, y_2 are the masses of the cans themselves.

We want a reasonable solution: all the numbers x_1, x_2, y_1, y_2 must be positive.

Let $\phi = h_1/h_2$ and $\mu = m_1/m_2$, where h_1 and m_1 is the height and mass of the bigger can. Show that if there is a solution, the following conditions are necessary:

$$\begin{aligned} \phi \neq 1, \quad \phi \neq 0 \\ \phi^2 < \mu < \phi^3. \end{aligned} \tag{2}$$

13. We have two geometrically similar cans of sugar. The first can is 14 cm high and weighs 3 kg. The other can is 10 cm high and weighs 1 kg. Is it possible to find the mass of the sugar in each can? Use the conditions in (2).

14. We have the typical coffee can problem, as above. First can has height 14 cm and total mass 2 kg. Second can has height 12 cm and mass 1.3 kg. Show that the coffee cans satisfy the necessary conditions (2) for a reasonable solution.

3 3D thinking

Solid geometry problems are difficult because it's hard to imagine what is going on in three dimensions. Use the construction that we learned in class: a simple piece of paper folded at 90°. Make your drawings on this paper. It will help you visualize the problem. Be careful about multiple solutions: *always check to see if more than one solution is possible.*

15. Two right triangles having legs a and b share the leg b in common. The triangles are in planes perpendicular to each other. What is the distance between the vertices opposite the common leg? Is there more than one solution? Give symbolic answers first, then give answers for the specific case of $a = 3$, $b = 4$.

16. Two right triangles with legs of length 5 and 12 share a common hypotenuse. The triangles are in planes perpendicular to each other. Find the distance between the vertices opposite the hypotenuse.

17. Two isoceles triangles with legs $1/\sqrt{2}$ and $1/\sqrt{2}$ share a hypotenuse in common. The two triangles are in perpendicular planes. What is the distance between the vertices? How many solutions does this problem have?

18. Two triangles with legs $1/2$ and $\sqrt{3}/2$ share a hypotenuse. The two triangles are in perpendicular planes. Find the distance between the vertices of the triangles.

19. Right triangles A and B are in perpendicular planes and share a common hypotenuse. Triangle A has legs $1/2$ and $\sqrt{3}/2$. Triangle B has legs $1/\sqrt{2}$ and $1/\sqrt{2}$. Make a drawing of these triangles on folded paper. Find the distance between the vertices of the triangles.

4 Finance problems

20. Jim starts with some money. In the first year, he spends \$100 but increases the remainder by $1/4$. In the second year he also spends \$100 but then increases the remainder by $1/3$. In the third year, Jim again spends \$100 but then increases the remainder by $1/2$. At the end of the third year, Jim has twice as much money than what he started with. How much did Jim start with, and how much did he end up with?

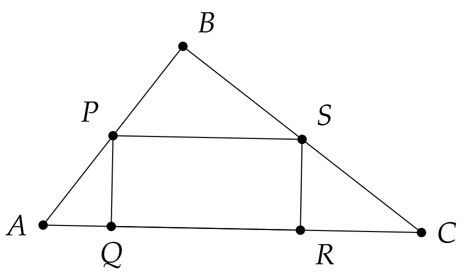
21. We have been using the word *remainder* to mean the money remaining in the bank. In finance problems this is also called the *balance*. Bob has some money in the bank. In the first year, he spent \$100 but then increased the balance by 1/5. In the second year he spent \$200 and then increased the balance by 1/4. In the third year he spent \$300 and then increased the remaining balance by 1/3. Finally in the fourth year he spent \$400 and increased the balance by 1/2. At the end, he had twice as much money as what he started with. How much money did Bob start with? How much did he end up with?

22. Mr. Somchai is not a very good businessman, as you will see. He has a taxi. He began with some money in the bank. In the first year he spent \$100 on gas, and then increased the balance by 1/2. In the second year he spent \$300 on gas and then increased the balance by only 1/4. Finally in the third year he spent a lot of money on gas, \$600, and then increased his balance by 1/3. Unfortunately at the end of all this, Mr. Somchai had only half of what he started with. How much money did he start with, and how much did he end up with?

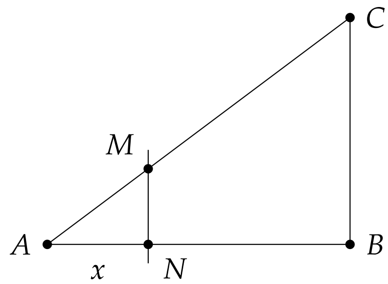
23. Five brothers have \$1024 all together. If you increase the first brother's money by \$4, decrease the second's by \$4, triple the third brother's money, halve the fourth's and leave the fifth's as it is, then they would all be equal. How much money does each brother actually have?

5 Area and perimiter

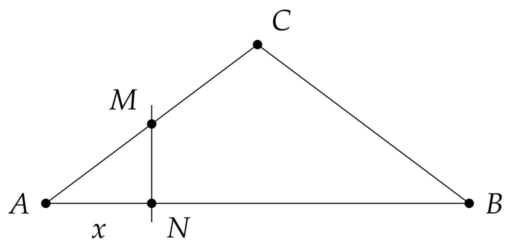
24. Rectangle $PQRS$ is inscribed in triangle ACB. Let $\overline{AB} = 3$ and $\overline{BC} = 4$. Find the height h and base b of the triangle. If $y = \overline{SR} = 1$, what is the area and perimiter of rectangle $PQRS$?



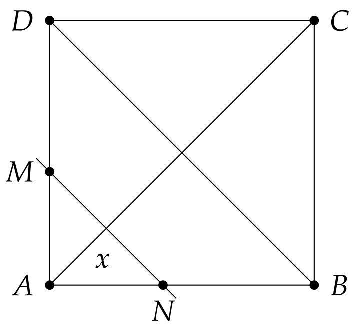
25. The line MN cuts away a piece ANM of the right triangle ABC. Find the area $A(x)$ and the perimiter $P(x)$ of the part that is cut away. $\overline{AB} = 4$ and $\overline{BC} = 3$.



26. Line MN cuts away a part ANM of triangle ABC. What is the area $A(x)$ and perimiter $P(x)$ of the part that is cut away? Let $\overline{AB} = 8$, $\overline{AC} = \overline{CB} = 5$. Note that this problem has two solutions, one when x is small (on the left side of the altitude), and one when x is big (on the right side of the altitude.)



27. Let $ABCD$ be a square with side length 3 (we did side length 2 in class.) Line MN is parallel to diagonal DB and cuts away a part ANM of the square. What is the area of the part that is cut away? Note that there are two solutions, one when x is small (below the center of AC) and another one when x is large (above the center of AC.)



What is the area cut away when $x = 1$? What is the area cut away when $x = 3$?

28. Same diagram as above, except the square has side length 2. We want to find the *perimiter* of the part that is cut away. Again there will be two solutions depending on how big x is.

6 Logic grid problems

All our logic problems will be 3×4 problems. This means each problem has three grids and each grid is 4×4.

Draw the three grids carefully. It's a good habit to write little notes explaining why you think that something is or is not true. Give complete answer matching all the items in the problem.

29. Four friends went to a restaurant. They each had two dishes: a main dish and a dessert. Figure out who ate which main dish and which dessert.

The four friends are Arnold, Kathy, Jim and Billy. The main dishes are cheese pizza, lasagna, Swedish meatballs and meatloaf. The four desserts are creme brulee, pecan pie, cheesecake and apple pie.

1. The person who had the apple pie (whose main dish was not expensive), the person who had the creme brulee, and Jim all arrived at different times.
2. Arnold didn't have the cheese pizza and neither did Kathy.
3. The person who had the creme brulee did not have lasagna and did not have Swedish meatballs.
4. Kathy did not have the meatloaf. Kathy's main dish was very expensive.
5. Jim, who did not have the lasagna, didn't have cheesecake.
6. The person who had the pecan pie did not have the cheese pizza.
7. Billy and the person who had the creme brulee sat next to each other.