# Problem Solving 2019

Training problems for M1, M2 and M3

Ted Szylowiec
tedszy@gmail.com

**1.** Describe these sequences using words.

(a) 15, 16, 17, ...
(b) 77, 79, 81, ...
(c) 12, 14, 16, ...
(d) 21, 28, 35, ...

**2.** Describe these sequences in words.

(a) 1, 9, 25, 49, ...
(b) 64, 144, 196, ...
(c) 32, 128, 512, ...
(d) 64, 256, 1024, ...

**3.** Count the elements in these sequences.

(a) 11, 13, ... 23.
(b) 18, 20, ... 32.
(c) 35, 40, ... 75.
(d) $16, 25, 36 \ldots 121$.

**4.** Consider the sequence of consecutive integers $a$, $a+1$, $\ldots$, $b-1$, $b$. Prove that the number of elements in this sequence is $b-a+1$.

**5.** Count the number of elements in these sequences. Use the result of problem **4**.

(a) 12, 13, ... 77.
(b) 87, 88, ... 152.
(c) $-14$, $-13$, ... 17, 18.
(d) $-199$, $-198$, ... 98, 99.

**6.** Count the elements in these sequences. Use the counting formula of problem **4**. Explain how you got your answer.

(a) 8, 10, ... 192.
(b) 77, 79, ... 151.
(c) 55, 60, ... 500.
(d) 85, 102, ... 748.
(e) $-100$, $-69$, ... 682.
(f) 25, 36, 49, ... 8100.

**7.** Consider the sequence of consecutive even numbers $p, \ldots q$, Find a counting formula for the number of elements in this sequence.

**8.** Let $m, \dots n$, be a sequence of consecutive odd numbers. Find a formula for the number of elements in this sequence.

**9.** Let $x, \dots y$ be a sequence where every pair of consecutive elements are different by a step size of $h$. I.e., the sequence looks like this:

$$x,\ x+h,\ x+2h,\ \dots\ y.$$

Find a formula that counts the elements of this sequence. Can you now explain why the formulas in problems **7** and **8** are the same?

**10.** Find a counting formula for consecutive square numbers. Use it to find the number of squares in between 1000 and 10000.

**11.** How many three-digit numbers are there? How many four-digit numbers are there?

**12.** How many *even* three-digit numbers are there?

**13.** How many *odd* 4-digit numbers are there?

**14.** How many 3-digit multiples of 7 are there?

**15.** How many 4-digit multiples of 5 are there?

**16.** Find the altitude of an equilateral triangle if the length of one side is $a$.

**17.** Find the area of an equilateral triangle if the length of one side is $a$.

**18.** Consider an equilateral triangle $ABC$. Choose a point $O$ anywhere inside $ABC$. Draw perpendicular lines from $O$ to the sides of $ABC$. Prove that the sum of the lengths of these perpendiculars is equal to the altitude of $ABC$.

**19.** What heppens when you choose $O$ to be right in the center of the equilateral triangle? Given that a side of the triangle is $a$, what is the length of each perpendicular line, given that the length of one side of the triangle is $a$?

**20.** What happens when $O$ is exacly on the midpoint of one side of the equilateral triangle? What are the lengths of the perpendiculars? You are given $a$, the length of one side of the equilateral triangle.

**21.** What happens when $O$ is chosen to be on one of the vertices of the equilateral triangle? What are the lengths of the perpendiculars? The length of one side of the triangle is $a$.

**22.** Suppose $O$ is on the midpoint of one side of the equilateral triangle. Let $P$ and $Q$ be the points where the perpendiculars from $O$ meet the other sides. Find the length of $PQ$.

**23.** Express the area of a trapezoid in terms of arithmetic mean.

**24.** Let $a = 9$ and $b = 16$. Find the arithmetic mean, geometric mean, harmonic mean and root-mean-square of $a$ and $b$. Is it true that

$$9 < \mathrm{HM}(9, 16) < \mathrm{GM}(9, 16) < \mathrm{AM}(9, 16) < \mathrm{RMS}(9, 16) < 16\,?$$

**25.** Let $a$ and $b$ be the lengths of the parallel sides of a trapezoid and let $h$ be the height. Prove that area of the trapezoid is the arithmetic mean of $a$ and $b$ multiplied by $h$.

**26.** Solve for $x$:

(a) $(a+b)\left(\dfrac{1}{x}+\dfrac{1}{x+b}\right)=2.$ (b) $(a+b)\left(\dfrac{1}{x+a}+\dfrac{1}{x}\right)=2.$

**27.** Solve for $x$:

(a) $(a+b)\left(\dfrac{1}{a}+\dfrac{1}{x+b}\right)=2.$ (b) $(a+b)\left(\dfrac{1}{x+a}+\dfrac{1}{b}\right)=2.$

**28.** Solve for $x$:

(a) $(a+b)\left(\dfrac{1}{ax}+\dfrac{1}{bx}\right)=2.$ (b) $(a+b)\left(\dfrac{1}{x+a}+\dfrac{1}{x+b}\right)=2.$

**29.** Let $ABCD$ be a trapezoid and let $AB$ and $CD$ be the parallel sides. Draw $EF$ parallel to $AB$ and $CD$ such that it bisects the area of $ABCD$. Prove that the length of $EF$ is the root-mean-square of the lengths of the parallel sides $AB$ and $CD$.

**30.** In problem **29**, let $a$, $b$ and $x$ be the lengths of $AB$, $CD$ and $EF$. Show that $a+b$ is equal to the harmonic mean of $x+a$ and $x+b$.

**31.** Draw $x$ and $y$ on the number line such that $x<y$ and let $p$ be the harmonic mean of $x$ and $y$:


Prove that for harmonic mean, the ratio $a/b$ is equal to $x/y$.

**32.** Draw $x$ and $y$ on the number line such that $x<y$ and let $g$ be the geometric mean of $x$ and $y$:


Prove that for geometric mean, the ratio $a/b$ is equal to $\sqrt{x/y}$.

**33.** Draw lines $AB$ and $A'B'$ with these proportions:

(a) $AB : A'B' = 3:2.$ (c) $AB : 3 = A'B' : 2.$ (e) $AB : 3 = 2 : A'B'.$
(b) $A'B' : AB = 3:2.$ (d) $3 : AB = 2 : A'B'.$ (f) $3 : AB = A'B' : 2.$

**34.** Draw rectangles with these side ratios:

(a) $1:3.$ (b) $5:2.$ (c) $2:3.$ (d) $\sqrt{5}:2.$ (e) $\sqrt{2}:\sqrt{3}.$

**35.** Sketch (freehand) two similar triangles. Label the vertices, sides and angles using $A$, $A'$, $a$, $a'$, $\alpha$, $\alpha'$ etc. Write down the six fundamental relationships between the sides of the similar triangles.

**36.** Let $ABC$ be a right triangle with right angle at vertex $C$. Drop an altitude line $CD$ from $C$ to the hypotenuse $AB$. Let $a$ and $b$ be the lengths of the legs of the triangle and let $h$ be the length of the altitude line. Prove the following:

(a) $h=\dfrac{ab}{\sqrt{a^2+b^2}}.$

(b) $2h^2$ is the harmonic mean of $a^2$ and $b^2$.
(c) $h$ is the geometric mean of $AD$ and $DB$.

**37.** Consider a right triangle. The lengths of the legs are $a$ and $b$. The length of the altitude through the right vertex is $h$. Develop an analogy between the squares of $a$, $b$, $h$ and resistors connected in parallel.

**38.** Use the classical definition of Golden ratio $\phi$:

$$\frac{b}{a} = \frac{a+b}{b}$$

to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

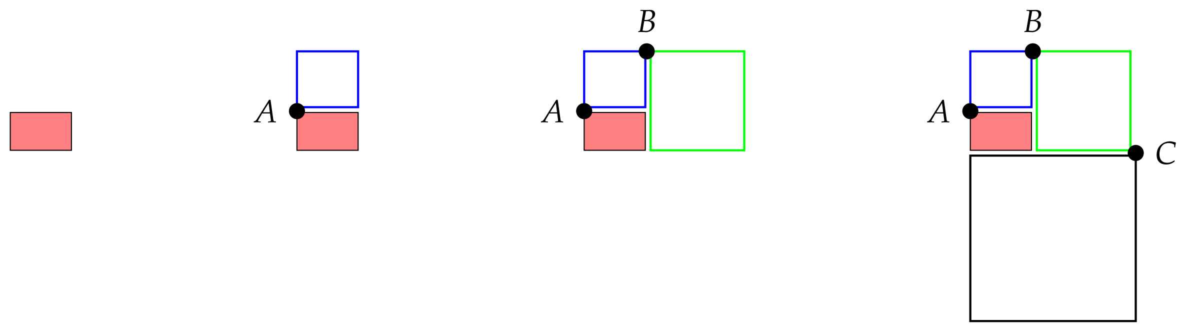
**39.** Use $\phi = \dfrac{1+\sqrt{5}}{2}$ to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

**40.** Make a table of the first 20 Fibonacci numbers. They begin like this: $F(1) = 1$, $F(2) = 1$.

**41.** Find a simple formula for $\phi^n$ using Fibonacci numbers.

**42.** The Kepler triangle. Is it possible to construct a right triangle with sides 1, $x$ and $x^2$? Find $x$ and sketch the Kepler triangle.

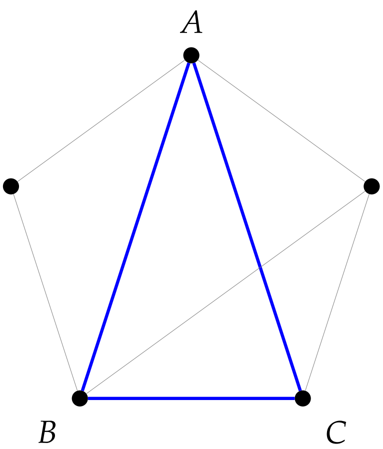
**43.** Construct a golden spiral. Start with a small golden rectangle (shown in red). Build more golden rectangles by adding squares. The blue square, green square, black square, etc. Work in a clockwise direction.

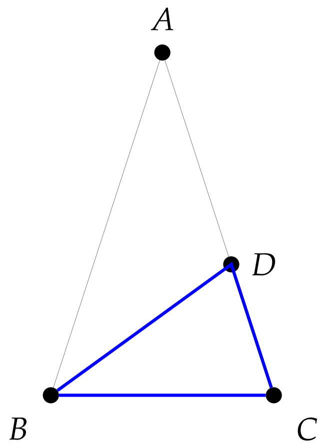



Build as many squares as you can. Sketch the spiral through the points $A$, $B$, $C$, etc.

**44.** Build the golden spiral like in problem **43**, but this time go in a counterclockwise direction.

**45.** The Descartes spiral. Triangle $ABC$ is made from the side and diagonals of a perfect pentagon.






$ABC$ is a *golden triangle* because $AB/BC = (AB + BC)/AB$. In other words, $AB/BC$ is $\phi$. If we cut the golden triangle $ABC$ at point $D$, we get another golden triangle: $DBC$. You can make smaller and smaller golden triangles this way. Construct the Descartes spiral by joining $A$, $B$, $C$, $D$, etc. with a smooth curve. In a similar way, you can also make a Descartes spiral by constructing bigger and bigger golden triangles.

**46.** What are the interior angles of the golden triangle?