Problem Solving 2019

Training problems for M1, M2 and M3

Ted Szylowiec
tedszy@gmail.com

1. Write down the elements of these sequences.

(a) A sequence of consecutive integers beginning with 10.
(b) A sequence of consecutive multiples of 7 beginning at 21.
(c) A sequence of consecutive prime numbers beginning with 11.
(d) A sequence of consecutive odd numbers beginning with 13.
(e) A sequence of consecutive square numbers beginning with 25.

2. Give a non-consecutive example of each of the sequences in problem **1**.

3. Describe these sequences using words.

(a) 15, 16, 17, ...
(b) 77, 79, 81, ...
(c) 12, 14, 16, ...
(d) 21, 28, 35, ...

4. Describe these sequences in words.

(a) 1, 9, 25, 49, ...
(b) 64, 144, 196, ...
(c) 32, 128, 512, ...
(d) 64, 256, 1024, ...

5. Count the elements in these sequences.

(a) 11, 13, ... 23.
(b) 18, 20, ... 32.
(c) 35, 40, ... 75.
(d) $16, 25, 36 \ldots 121$.

6. Consider the sequence of consecutive integers a, $a+1$, $\ldots$, $b-1$, b. Prove that the number of elements in this sequence is $b-a+1$.

7. Count the number of elements in these sequences. Use the result of problem **6**.

(a) 12, 13, ... 77.
(b) 87, 88, ... 152.
(c) -14, -13, ... 17, 18.
(d) -199, -198, ... 98, 99.

8. Count the elements in these sequences. Use the counting formula of problem **6**. Explain how you got your answer.

(a) 8, 10, ... 192.
(b) 77, 79, ... 151.
(c) 55, 60, ... 500.
(d) 85, 102, ... 748.
(e) -100, -69, ... 682.
(f) 25, 36, 49, ... 8100.

9. Consider the sequence of consecutive even numbers $p, \ldots q$, Find a counting formula for the number of elements in this sequence.

10. Let $m, \ldots\ n$, be a sequence of consecutive odd numbers. Find a formula for the number of elements in this sequence.

11. Let $x, \ldots y$ be a sequence of consecutive multiples of h. Find a formula that counts the elements of this sequence.

12. Let $x, \ldots y$ be a sequence of consecutive numbers that have remainder r when divided by h. Prove that the number of elements in this sequence is

$$\frac{y - x + h}{h}.$$

13. Find a counting formula for consecutive square numbers.

14. How many perfect square integers are there in between 10000 and 100000?

15. Count these sequences:

(a) 35, 42, 49, ... 427.
(b) 484, 529, ... 14400.

16. How many three-digit numbers are there? How many four-digit numbers are there?

17. How many *even* three-digit numbers are there?

18. How many *odd* 4-digit numbers are there?

19. How many 3-digit multiples of 7 are there?

20. How many 4-digit multiples of 5 are there?

21. How many three-digit numbers are both multiples of 5 and multiples of 7?

22. Let A and B be sets. Explain why $n(A) + n(B) - n(A \cap B)$ is the total number of elements. Why do we have to subtract $n(A \cap B)$? Use drawings.

23. How many three-digit numbers are either multiples of 5 or multiples of 7. Explain all of your thinking clearly.

24. How many three-digit numbers are multiples of 2 and also multiples of 3 and also multiples of 7? Explain every step of your thinking.

25. Let A, B and C be sets. Find a formula that counts the total number of elements in all of them. Explain why it is true. Use drawings. Give examples.

26. How many three-digit numbers are either multiples of 2 or multiples of 3 or multiples of 7? Explain every step.

27. Find the altitude of an equilateral triangle if the length of one side is a.

28. Find the area of an equilateral triangle if the length of one side is a.

29. Consider an equilateral triangle ABC. Choose a point O anywhere inside ABC. Draw perpendicular lines from O to the sides of ABC. Prove that the sum of the lengths of these perpendiculars is equal to the altitude of ABC.

30. What heppens when you choose O to be right in the center of the equilateral triangle? Given that a side of the triangle is a, what is the length of each perpendicular line, given that the length of one side of the triangle is a?

31. What happens when O is exacly on the midpoint of one side of the equilateral triangle? What are the lengths of the perpendiculars? You are given a, the length of one side of the equilateral triangle.

32. What happens when O is chosen to be on one of the vertices of the equilateral triangle? What are the lengths of the perpendiculars? The length of one side of the triangle is a.

33. Suppose O is on the midpoint of one side of the equilateral triangle. Let P and Q be the points where the perpendiculars from O meet the other sides. Find the length of PQ.

34. Express the area of a trapezoid in terms of arithmetic mean.

35. Let $a = 9$ and $b = 16$. Find the arithmetic mean, geometric mean, harmonic mean and root-mean-square of a and b. Is it true that

$$9 < \mathrm{HM}(9,16) < \mathrm{GM}(9,16) < \mathrm{AM}(9,16) < \mathrm{RMS}(9,16) < 16\,?$$

36. Let a and b be the lengths of the parallel sides of a trapezoid and let h be the height. Prove that area of the trapezoid is the arithmetic mean of a and b multiplied by h.

37. Solve for x:

(a) $(a+b)\left(\dfrac{1}{x} + \dfrac{1}{x+b}\right) = 2.$ (b) $(a+b)\left(\dfrac{1}{x+a} + \dfrac{1}{x}\right) = 2.$

38. Solve for x:

(a) $(a+b)\left(\dfrac{1}{a} + \dfrac{1}{x+b}\right) = 2.$ (b) $(a+b)\left(\dfrac{1}{x+a} + \dfrac{1}{b}\right) = 2.$

39. Solve for x:

(a) $(a+b)\left(\dfrac{1}{ax} + \dfrac{1}{bx}\right) = 2.$ (b) $(a+b)\left(\dfrac{1}{x+a} + \dfrac{1}{x+b}\right) = 2.$

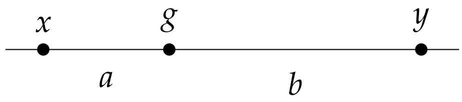
40. Let $ABCD$ be a trapezoid and let AB and CD be the parallel sides. Draw EF parallel to AB and CD such that it bisects the area of $ABCD$. Prove that the length of EF is the root-mean-square of the lengths of the parallel sides AB and CD.

41. In problem **40**, let a, b and x be the lengths of AB, CD and EF. Show that $a+b$ is equal to the harmonic mean of $x+a$ and $x+b$.

42. Draw x and y on the number line such that $x < y$ and let p be the harmonic mean of x and y:

Prove that for harmonic mean, the ratio a/b is equal to x/y.

43. Draw x and y on the number line such that $x < y$ and let g be the geometric mean of x and y:



Prove that for geometric mean, the ratio a/b is equal to $\sqrt{x/y}$.

44. Draw lines AB and $A'B'$ with these proportions:

(a) $AB : A'B' = 3 : 2$.
(b) $A'B' : AB = 3 : 2$.
(c) $AB : 3 = A'B' : 2$.
(d) $3 : AB = 2 : A'B'$.
(e) $AB : 3 = 2 : A'B'$.
(f) $3 : AB = A'B' : 2$.

45. Draw rectangles with these side ratios:

(a) $1 : 3$. (b) $5 : 2$. (c) $2 : 3$. (d) $\sqrt{5} : 2$. (e) $\sqrt{2} : \sqrt{3}$.

46. Sketch (freehand) two similar triangles. Label the vertices, sides and angles using A, A', a, a', α, α' etc. Write down the six fundamental relationships between the sides of the similar triangles.

47. Let ABC be a right triangle with right angle at vertex C. Drop an altitude line CD from C to the hypotenuse AB. Let a and b be the lengths of the legs of the triangle and let h be the length of the altitude line. Prove the following:

(a) $h = \dfrac{ab}{\sqrt{a^2 + b^2}}$.
(b) $2h^2$ is the harmonic mean of a^2 and b^2.
(c) h is the geometric mean of AD and DB.

48. Consider a right triangle. The lengths of the legs are a and b. The length of the altitude through the right vertex is h. Develop an analogy between the squares of a, b, h and resistors connected in parallel.

49. Use the classical definition of Golden ratio ϕ:

$$\frac{b}{a} = \frac{a+b}{b}$$

to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

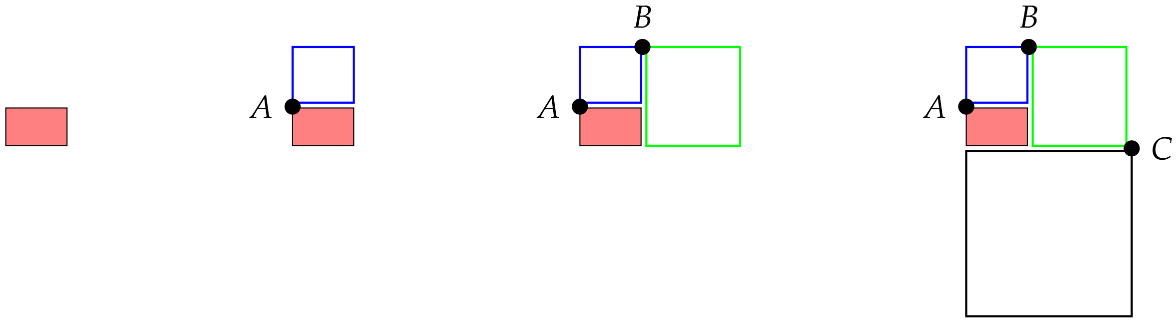
50. Use $\phi = \dfrac{1+\sqrt{5}}{2}$ to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

51. Make a table of the first 20 Fibonacci numbers. They begin like this: $F(1) = 1$, $F(2) = 1$.

52. Find a simple formula for ϕ^n using Fibonacci numbers.

53. The Kepler triangle. Is it possible to construct a right triangle with sides 1, x and x^2? Find x and sketch the Kepler triangle.

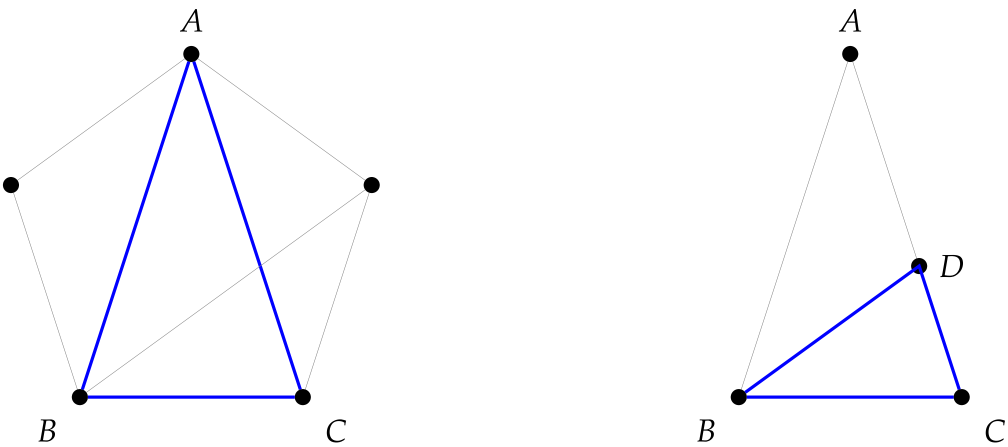
54. Construct a golden spiral. Start with a small golden rectangle (shown in red). Build more golden rectangles by adding squares. The blue square, green square, black square, etc. Work in a clockwise direction.



Build as many squares as you can. Sketch the spiral through the points A, B, C, etc.

55. Build the golden spiral like in problem **54**, but this time go in a counterclockwise direction.

56. The Descartes spiral. Triangle ABC is made from the side and diagonals of a perfect pentagon.



ABC is a *golden triangle* because $AB/BC = (AB + BC)/AB$. In other words, AB/BC is ϕ. If we cut the golden triangle ABC at point D, we get another golden triangle: DBC. You can make smaller and smaller golden triangles this way. Construct the Descartes spiral by joining A, B, C, D, etc. with a smooth curve. In a similar way, you can also make a Descartes spiral by constructing bigger and bigger golden triangles.

57. What are the interior angles of the golden triangle?