

# M2 Training Problems

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## 1 Functions, identity, inverses and plots

1. Let  $f(x) = 2x + 1$ . Find...

- (a) Find  $f(f(x))$ .
- (b) Find  $f(f(f(x)))$ .
- (c) Find  $f(f(f(f(x))))$ .

2. Let  $f(x) = 3x^2 + 1$  and  $g(x) = 2x - 3$ .

- (a) Find  $f(g(x))$ .
- (b) Find  $g(f(x))$ .

Are they the same?

3. Let  $f(x) = ax + b$ .

- (a) Find  $f(f(x))$ .
- (b) Find  $f(f(f(x)))$ .

4. Let  $f(x) = ax + b$  and  $g(x) = cx + d$ .

- (a) Find  $f(g(x))$ .
- (b) Find  $g(f(x))$ .

Are they the same?

5. Sketch  $y = x$  and  $y = -x$ . Put them on the same axes. Label everything.

6. Sketch  $y = 2x$  and  $y = -2x$ . Put them on the same axes.

7. Sketch these lines on the same axes.

$$y = \frac{x}{2}, \quad y = -\frac{x}{2}.$$

8. Make an exact plot of  $y = 3x + 2$  by finding the  $x$ -intercept and  $y$ -intercept.

9. Make an exact plot of

$$y = -\frac{x}{2} - 1.$$

10. If  $f(x)$  and  $g(x)$  are linear, show that

- (a)  $f(g(x))$  is linear.
- (b)  $g(f(x))$  is linear.

11. Let  $f(x) = 3x + 2$ . Find  $f^{-1}(x)$ . Do it two ways:

- (a) By  $f(f^{-1}(x)) = I(x)$ .
- (b) And by  $f^{-1}(f(x)) = I(x)$ .

12. Let  $f(x) = ax + b$ . Find  $f^{-1}(x)$ . Do it two ways:

- (a) By  $f(f^{-1}(x)) = I(x)$ .
- (b) And by  $f^{-1}(f(x)) = I(x)$ .

13. Let  $f(x) = 2x + 1$ . Find  $f^{-1}(x)$ . Make exact plots of  $f$  and  $f^{-1}$ . Also draw  $I$ .

14. Let  $f(x) = -2x + 3$ . Find  $f^{-1}(x)$ . Make exact plots of  $f$  and  $f^{-1}$ . Also draw  $I$ .

15. Consider the function

$$f(x) = -\frac{x}{2} + 3.$$

Find  $f^{-1}(x)$ . Make exact plots of  $f$  and  $f^{-1}$ . Also draw  $I$ .

16. Sketch the curve  $y = x^2$ . Use the unit square idea.

17. Let  $f(x) = x^2$ . Sketch  $f$ ,  $I$  and  $f^{-1}$  on the same axes.

18. Let  $f(x) = x^2 + 1$ . Sketch  $f$ ,  $I$  and  $f^{-1}$  on the same axes.

19. Are there functions that are inverses of themselves? Does there exist any functions with the property  $f(x) = f^{-1}(x)$ ? In other words,  $f$  is its own inverse.

- (a) Find one such self-inverse function  $f$ .
- (b) Try to find more, as many as you can.

## 2 Introducing logarithms

20. Draw the fastest-growing function  $f$  that you can imagine. Draw  $I(x)$  and use it to find the inverse  $f^{-1}$ .

21. Draw the slowest-growing function  $f$  that you can imagine. Draw the identity  $I(x)$  and use it to find the inverse  $f^{-1}$ .

22. Given  $f$ , tell me about the inverse  $f^{-1}$ . Does it grow fast, slowly, very fast etc.?

- (a)  $f$  is a fast-growing function.
- (b)  $f$  does not grow at all.
- (c)  $f$  is a slow-growing function.
- (d)  $f$  is a very slow-growing function.
- (e)  $f$  is a very fast-growing function.

23. Fill in this table about the behavior of  $f(x) = 2^x$  for different values of  $x$ .

$x$	$f(x)$
$x = 0$	$f = 1$
$x > 0$	
$x < 0$	
$x > 1$	
$x \rightarrow \infty$	
$x \rightarrow -\infty$	

24. Plot  $2^x$ ,  $3^x$  and  $5^x$  all on the same axes.

25. Consider the function  $f(n) = \left(1 + \frac{1}{n}\right)^n$ . Use a calculator. Fill in this table

$x$	$f(x)$
1	2
2	
5	
10	
100	
1000	

Notice how  $f(n)$  keeps increasing as  $n$  gets bigger. But also notice how  $f(n)$  does not increase to infinity, but approaches the magic number  $e$  from below.

26. Now consider the slightly different function  $g(n) = \left(1 + \frac{1}{n}\right)^{n+1}$ . Use a calculator. Fill in this table

$x$	$f(x)$
1	2
2	
5	
10	
100	
1000	

Notice how  $g(n)$  keeps decreasing as  $n$  gets bigger. But also notice how  $g(n)$  does not decrease to minus infinity, but approaches the magic number  $e$  from above.

27. Plot  $2^x$ ,  $e^x$  and  $10^x$  all on the same axes.

28. Plot  $f(x) = 2^x$  and the identity line  $I(x)$ . Use the identity line to draw the inverse  $f^{-1}(x) = \log_2 x$ .

29. Plot  $f(x) = 3^x$ ,  $I(x)$  and  $f^{-1}(x) = \log_3 x$  all on the same axes.

30. Fill in this table about the behavior of  $g(x) = \log_2 x$  for different values of  $x$ .

$x$	$g(x)$
$x = 1$	$g = 0$
$x > 1$	
$x < 1$	
$x = 2$	
$x \rightarrow \infty$	
$x \rightarrow 0$	

31. Does  $2^x$  ever touch the  $x$ -axis? Does  $\log_2 x$  ever touch the  $y$ -axis?

32. Fill in the table.

$x$	$3^x$	$x$	$\log_3 x$
1		1	
2		9	
3		27	
4		243	
5		59049	

33. Fill in the table.

$x$	$10^x$	$x$	$\log_{10} x$
1		1	
2		10	
3		1000	
4		100,000	
5		10,000,000	

34. Plot  $\log_2 x$ ,  $\log_3 x$  and  $\log_5 x$  all on the same axes. Label all the important points.

35. Plot  $\log_2 x$ ,  $\log_e x$  and  $\log_{10} x$  all on the same axes. Label all the important points.

36. The formulas relating  $f$ ,  $f^{-1}$  and  $I$  establish the two most important properties of logarithms and exponentials. Use  $f(x) = a^x$  and  $f^{-1}(x) = \log_a x$  and tell me what these formulas imply:

(a)  $f(f^{-1}(x)) = I(x)$ .

(b)  $f^{-1}(f(x)) = I(x)$ .

### 3 Properties of $\log_a$

37. Figure out  $a^{\log_a a^x}$ .

38. Figure out  $\log_a a^{\log_a a^x}$ .

39. Begin with the well-known property of exponential functions  $(a^x)^p = a^{xp}$  and prove the following property of logarithms:

$$\log_a u^p = p \log_a u.$$

Notice how log changes powers to multiplications.

40. Begin with something we all know:  $a^x a^y = a^{x+y}$  and prove the following property of logarithms:

$$\log_a uv = \log_a u + \log_a v.$$

Notice how log changes multiplication into addition.

41. Using an idea similar to the one in problem 40, prove that

$$\log_a uvw = \log_a u + \log_a v + \log_a w.$$

42. Use the results of problems 39 and 40 to prove this:

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

Notice how log changes division into subtraction.

43. Use 39, 40 and 42 to figure these out.

(a)  $\log_2 8 \times 32 \times 64$ .

(b)  $\log_5 25 \times 125 \times 625$ .

(c)  $\log_2 2^7 8^5 16^3$ .

(d)  $\log_3 \sqrt{27} \sqrt[3]{81}$ .

(e)  $\log_5 \frac{\sqrt{125}}{\sqrt[3]{625}}$ .

(f)  $\log_e \frac{\sqrt{e}}{e^3} \sqrt[3]{e}$ .

44. Use plots to explain why  $0 < x < 1$  when  $\log x$  is negative.

45. Use plots to explain why  $\log x$  is positive when  $x > 1$ .

46. Let  $x = a/b$ . Use the formula in 42 to show that  $\log x$  is negative when  $0 < x < 1$ .

47. Let  $x = a/b$ . Use the formula in 42 to show that  $\log x$  is positive when  $x > 1$ .

48. Positive or negative? Use log, algebra and plots to explain why. Assume the logarithm base is  $a > 1$  as usual.

(a)  $\log_a \frac{2}{3}$

(b)  $\log_a \frac{5}{2}$

(c)  $\log_a 0.8$

(d)  $\log_a 1.8$

(e)  $\log_a \frac{e}{\pi}$

(f)  $\log_a \frac{\pi}{e}$

49. Positive or negative? Explain why.

(a)  $\log \frac{\pi}{\sqrt{10}}$

(b)  $\log \frac{\sqrt{10}}{\pi}$

(c)  $\log \frac{3e}{2\pi}$

(d)  $\log \frac{2\pi}{3e}$

50. Bigger or smaller than 1? Explain using logarithms, algebra and plots.

(a)  $\left(\frac{3}{2}\right)^{2/3}$

(b)  $\left(\frac{2}{3}\right)^{-3/2}$

(c)  $\left(\frac{\pi}{5}\right)^{\sqrt{5/2}}$

(d)  $\left(\frac{1}{\sqrt{3}}\right)^{-1/\sqrt{2}}$

(e)  $(\sqrt{2})^{-\sqrt{2}}$

(f)  $\left(\frac{1}{\pi}\right)^{1/e}$

(g)  $(\sqrt{e})^{-\sqrt{\pi}}$

51. Given the inequality, determine which is bigger:  $x$  or  $y$ . Prove it using logarithms.

(a)  $(0.5)^x > (0.5)^y$ .

(b)  $(1.5)^x < (1.5)^y$ .

(c)  $\left(\frac{3}{2}\right)^x > \left(\frac{3}{2}\right)^y$ .

52. Which is bigger:  $x$  or  $y$ ?

(a)  $\left(\frac{2}{3}\right)^x < \left(\frac{2}{3}\right)^y$ .

(b)  $\left(\frac{e}{\pi}\right)^x > \left(\frac{e}{\pi}\right)^y$ .

(c)  $\left(\frac{\pi}{e}\right)^x < \left(\frac{\pi}{e}\right)^y$ .

53. Which is bigger? Explain using logarithms, plots, algebra etc.

(a)  $2^{70}$  or  $7^{20}$ ?

(b)  $5^{30}$  or  $3^{50}$ ?

(c)  $2^{50}$  or  $5^{20}$ ?

54. Simplify.

(a)  $\frac{\log_2 81}{\log_2 27}$

(b)  $81^{\log_3 2}$

(c)  $64^{\log_4 3}$

(d)  $e^{\log_e (\log_e e^2)}$

(e)  $\left(\frac{1}{100}\right)^{\log_{10} 2}$

55. Prove that

$$\log_a b = \frac{1}{\log_b a}.$$

56. Simplify.

(a)  $a^{1/\log_b a^2}$

(b)  $a^{1/\log_{b^2} a}$

(c)  $a^{1/\log_{b^6} a^3}$

(d)  $3^{1/\log_5 3}$

(e)  $3^{1/\log_5 3^2}$

(f)  $2^{1/\log_{27} 8}$

57. Simplify and explain what values of  $a$  and  $b$  are possible.

(a)  $\log(ab) - \log |a|$ .

(b)  $\log(ab) - \log |b|$ .

(c)  $\log(ab) - \log |a| - \log |b|$ .

58. Prove that

$$\log_{a^2} x = \frac{1}{2} \log_a x.$$

59. Prove that

$$\log_{a^p} x = \frac{1}{p} \log_a x.$$

60. Simplify.  $\log_a b^2 + \log_{a^2} b^4 + \cdots + \log_{a^n} b^{2n}$ .

61. Simplify.  $(\log_a b)(\log_b c)(\log_c a)$ .
62. Simplify.  $(\log_a b)(\log_b c)(\log_c d)(\log_d a)$ .
63. Simplify.  $(\log_a b)(\log_{b^2} c^2)(\log_{c^3} a^3)$ .
64. Simplify.  $(\log_a b)(\log_{b^2} c^2)(\log_{c^3} d^3)(\log_{d^4} a^4)$ .
65. Simplify.  $(\log_a b^4)(\log_{b^2} c^3)(\log_{c^3} d^2)(\log_{d^4} a)$ .
66. Simplify.  $\frac{\log_8 5}{\log_4 25}$ .
67. Simplify.  $(\log_8 5)(\log_{125} 4)$ .
68. Simplify.  $(\log_7 2)(\log_3 5)(\log_5 7)(\log_2 3)$ .
69. Simplify.  $(\log_5 3)(\log_4 5)(\log_{125} 8)$ .
70. Simplify  $(\log_5 2)(\log_{27} 125)(\log_2 3)$ .
71. Use  $\sqrt{\log_a b} \sqrt{\log_a b} = \log_a b$  and prove that

$$a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}.$$

For what values of  $a$  and  $b$  is this true?

72. Prove the famous change-of-base formula:

$$\log_a x = (\log_a b) \log_b x.$$

4  $\log_a$  with  $0 < a < 1$ . What happens?