Problem Solving 2019

Training problems for M1, M2 and M3

Ted Szylowiec
tedszy@gmail.com

1. Count the elements in these sequences.

(a) 11, 13, . . . 23.
(b) 18, 20, . . . 32.
(c) 35, 40, . . . 75.
(d) $16, 25, 36 \ldots 121$.

2. Count the number of elements in these sequences.

(a) 12, 13, . . . 77.
(b) 87, 88, . . . 152.
(c) -14, -13, . . . 17, 18.
(d) -199, -198, . . . 98, 99.

3. Consider the sequence of consecutive integers a, $a+1$, . . . , $b-1$, b. Prove that the number of elements in this sequence is $b-a+1$.

4. Consider the sequence of consecutive even numbers $p, \ldots q$, $p < q$. Find a counting formula for the number of elements in this sequence.

5. Let $m, \ldots n$, $m < n$ be a sequence of consecutive odd numbers. Find a formula for the number of elements in this sequence.

6. Let $x, \ldots y$ be a sequence where every pair of consecutive elements are different by a step size of h. I.e., the sequence looks like this:

$$x,\ x+h,\ \ldots\ y-h,\ y.$$

Find a formula that counts the elements of this sequence. Can you now explain why the formulas in problems **4** and **5** are the same?

7. Use counting formulas to count the elements in these sequences.

(a) 118, 20, . . . 592.
(b) 277, 279, . . . 1151.
(c) 155, 60, . . . 810.
(d) 85, 102, . . . 748.
(e) -100, -69, . . . 682.

8. Find a counting formula for consecutive square numbers and use it to find the number of squares in between 1000 and 10000.

9. How many three-digit numbers are there? How many four-digit numbers are there?

10. How many *even* three-digit numbers are there?

11. How many *odd* 4-digit numbers are there?

12. How many 3-digit multiples of 7 are there?

13. How many 4-digit multiples of 5 are there?

14. Find the altitude of an equilateral triangle if the length of one side is a.

15. Find the area of an equilateral triangle if the length of one side is a.

16. Consider an equilateral triangle ABC. Choose a point O anywhere inside ABC. Draw perpendicular lines from O to the sides of ABC. Prove that the sum of the lengths of these perpendiculars is equal to the altitude of ABC.

17. What heppens when you choose O to be right in the center of the equilateral triangle? Given that a side of the triangle is a, what is the length of each perpendicular line, given that the length of one side of the triangle is a?

18. What happens when O is exacly on the midpoint of one side of the equilateral triangle? What are the lengths of the perpendiculars? You are given a, the length of one side of the equilateral triangle.

19. What happens when O is chosen to be on one of the vertices of the equilateral triangle? What are the lengths of the perpendiculars? The length of one side of the triangle is a.

20. Suppose O is on the midpoint of one side of the equilateral triangle. Let P and Q be the points where the perpendiculars from O meet the other sides. Find the length of PQ.

21. Express the area of a trapezoid in terms of arithmetic mean.

22. Let $a = 9$ and $b = 16$. Find the arithmetic mean, geometric mean, harmonic mean and root-mean-square of a and b. Is it true that

$$9 < \mathrm{HM}(9,16) < \mathrm{GM}(9,16) < \mathrm{AM}(9,16) < \mathrm{RMS}(9,16) < 16\,?$$

23. Let a and b be the lengths of the parallel sides of a trapezoid and let h be the height. Prove that area of the trapezoid is the arithmetic mean of a and b multiplied by h.

24. Solve for x:

(a) $(a+b)\left(\dfrac{1}{x} + \dfrac{1}{x+b}\right) = 2.$ (b) $(a+b)\left(\dfrac{1}{x+a} + \dfrac{1}{x}\right) = 2.$

25. Solve for x:

(a) $(a+b)\left(\dfrac{1}{a} + \dfrac{1}{x+b}\right) = 2.$ (b) $(a+b)\left(\dfrac{1}{x+a} + \dfrac{1}{b}\right) = 2.$

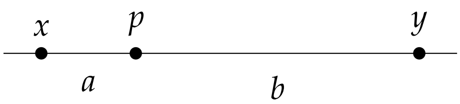
26. Solve for x:

(a) $(a+b)\left(\dfrac{1}{ax} + \dfrac{1}{bx}\right) = 2.$ (b) $(a+b)\left(\dfrac{1}{x+a} + \dfrac{1}{x+b}\right) = 2.$

27. Let $ABCD$ be a trapezoid and let AB and CD be the parallel sides. Draw EF parallel to AB and CD such that it bisects the area of $ABCD$. Prove that the length of EF is the root-mean-square of the lengths of the parallel sides AB and CD.

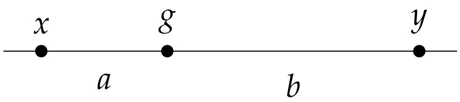
28. In problem **27**, let a, b and x be the lengths of AB, CD and EF. Show that $a+b$ is equal to the harmonic mean of $x+a$ and $x+b$.

29. Draw x and y on the number line such that $x<y$ and let p be the harmonic mean of x and y:



Prove that for harmonic mean, the ratio a/b is equal to x/y.

30. Draw x and y on the number line such that $x<y$ and let g be the geometric mean of x and y:



Prove that for geometric mean, the ratio a/b is equal to $\sqrt{x/y}$.

31. Draw lines AB and $A'B'$ with these proportions:

(a) $AB : A'B' = 3:2$.
(b) $A'B' : AB = 3:2$.
(c) $AB : 3 = A'B' : 2$.
(d) $3 : AB = 2 : A'B'$.
(e) $AB : 3 = 2 : A'B'$.
(f) $3 : AB = A'B' : 2$.

32. Draw rectangles with these side ratios:

(a) $1:3$.
(b) $5:2$.
(c) $2:3$.
(d) $\sqrt{5}:2$.
(e) $\sqrt{2}:\sqrt{3}$.

33. Sketch (freehand) two similar triangles. Label the vertices, sides and angles using A, A', a, a', α, α' etc. Write down the six fundamental relationships between the sides of the similar triangles.

34. Let ABC be a right triangle with right angle at vertex C. Drop an altitude line CD from C to the hypotenuse AB. Let a and b be the lengths of the legs of the triangle and let h be the length of the altitude line. Prove the following:

(a) $h = \dfrac{ab}{\sqrt{a^2+b^2}}$.
(b) $2h^2$ is the harmonic mean of a^2 and b^2.
(c) h is the geometric mean of AD and DB.

35. Consider a right triangle. The lengths of the legs are a and b. The length of the altitude through the right vertex is h. Develop an analogy between the squares of a, b, h and resistors connected in parallel.

36. Use the classical definition of Golden ratio ϕ:

$$\frac{b}{a} = \frac{a+b}{b}$$

to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

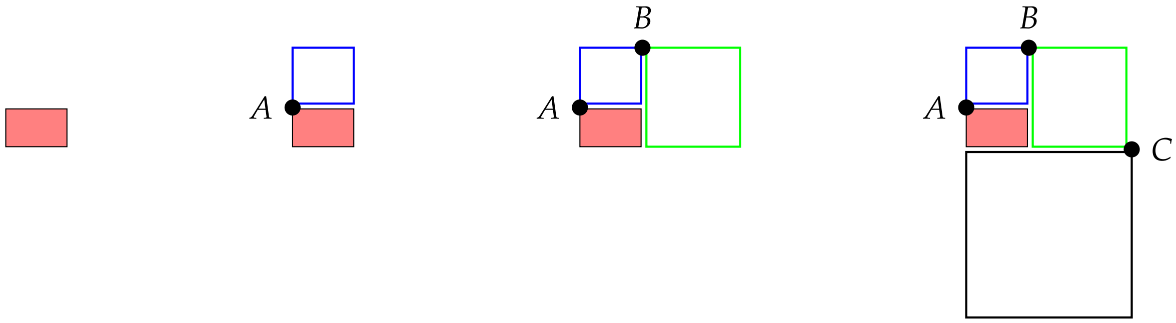
37. Use $\phi = \dfrac{1+\sqrt{5}}{2}$ to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

38. Make a table of the first 20 Fibonacci numbers. They begin like this: $F(1) = 1$, $F(2) = 1$.

39. Find a simple formula for ϕ^n using Fibonacci numbers.

40. The Kepler triangle. Is it possible to construct a right triangle with sides 1, x and x^2? Find x and sketch the Kepler triangle.

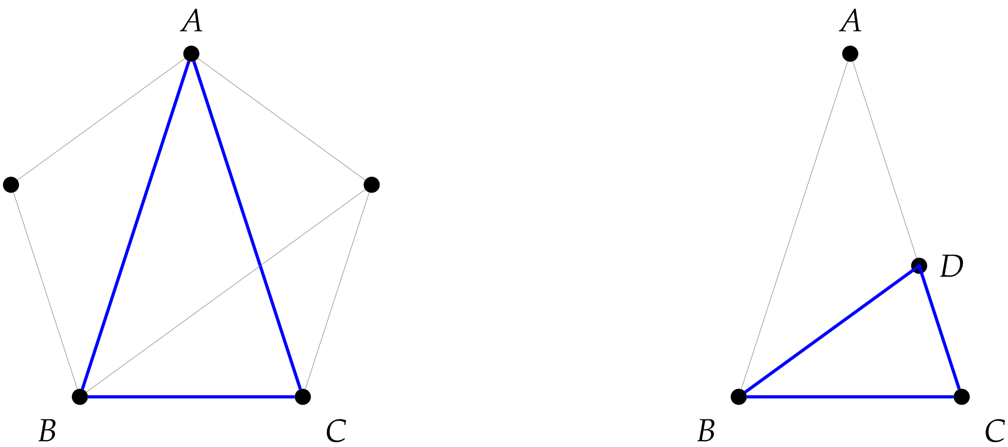
41. Construct a golden spiral. Start with a small golden rectangle (shown in red). Build more golden rectangles by adding squares. The blue square, green square, black square, etc. Work in a clockwise direction.



Build as many squares as you can. Sketch the spiral through the points A, B, C, etc.

42. Build the golden spiral like in problem **41**, but this time go in a counterclockwise direction.

43. The Descartes spiral. Triangle ABC is made from the side and diagonals of a perfect pentagon.



ABC is a *golden triangle* because $AB/BC = (AB + BC)/AB$. In other words, AB/BC is ϕ. If we cut the golden triangle ABC at point D, we get another golden triangle: DBC. You can make smaller and smaller golden triangles this way. Construct the Descartes spiral by joining A, B, C, D, etc. with a smooth curve. In a similar way, you can also make a Descartes spiral by constructing bigger and bigger golden triangles.

44. What are the interior angles of the golden triangle?