SME M2 Training Problems

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Some problems may be harder than usual (or more advanced.) They are marked with a dot $\bullet$. Problems that are possibly even harder (and even more advanced) are marked with two dots $\bullet\bullet$. All problems are worth doing! Try to do all of them. Most don't take very long. Some problems include answers.

1. Show that $n^4 + 10n^3 + 27n^2 + 10n + 1$ is a perfect square for all n. Do it by writing it in the form $(an^2 + bn + c)^2$ and finding the coefficients a, b, c.

2. Show that $n^4 - 14n^3 + 51n^2 - 14n + 1$ is a perfect square.

3. Show that $n^4 + 14n^3 + 51n^2 + 14n + 1$ is a perfect square.

4. $\bullet$ Show that $n^6 + 2n^5 + 5n^4 + 6n^3 + 6n^2 + 4n + 1$ is a perfect square.

5. $\bullet$ Show that $n^6 + 10n^5 + 27n^4 + 12n^3 + 11n^2 + 2n + 1$ is a perfect square.

6. $\bullet$ Show that $n^6 + 4n^5 + 10n^4 + 14n^3 + 13n^2 + 6n + 1$ is a perfect square.

7. Consider positive integers in steps of five: n, $n+5$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi + 625$ is always a perfect square.

8. Consider positive integers in steps of 10: n, $n+10$, etc. Let ϕ be the product of four such consecutive numbers. Prove that $\phi + 10000$ is always a perfect square.

9. $\bullet$ Even numbers have the form $2n$. Let ϕ be the product of four consecutive even numbers. Show that $\phi + 16$ is always a perfect square.

10. $\bullet$ Odd numbers have the form $2n + 1$. Let ϕ be the product of four consecutive odd numbers. Prove that $\phi + 16$ is always a perfect square.

11. $\bullet$ Examine the patterns in problems **7**, **8**, **9** and **10**. Can you combine all these results into one mathematical statement?

12. $\bullet$ Prove the answer you gave to problem **11**.

13. Prove that 7 always divides $n^7 - n$ for any integer n.

14. Prove that it is not true that 9 always divides $n^9 - n$. Use modular arithmetic to find an n such that $9 \nmid n^9 - n$.

15. Use modular arithmetic to prove that $11 \mid n^{11} - n$ for all integers n.

16. Show that 10 does not always divide $n^{10} - n$ and give an example of a number n such that $10 \nmid n^{10} - n$.

17. Show that 7 does not divide $7n^3 + 3n^2 + 3n + 5$. Use modular arithmetic.

18. Prove that for all values of n, $121n^3 + 77n^2 + 66n + 11$ is never square.

19. $\bullet$ Prove that the sum of four consecutive fourth powers can never be square.