

Problem Solving 2019

Training problems for M1, M2 and M3

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- Count the number of elements in these sequences.
 - 12, 13, ... 77.
 - 87, 88, ... 152.
 - 14, -13, ... 17, 18.
 - 199, -198, ... 98, 99.
- Consider the sequence $a, a + 1, \dots, b - 1, b$. Prove that the number of elements in this sequence is $b - a + 1$.
- How many three-digit numbers are there? How many four-digit numbers are there?
- How many *even* three-digit numbers are there?
- How many *odd* 4-digit numbers are there?
- How many 3-digit multiples of 7 are there?
- How many 4-digit multiples of 5 are there?
- Find the altitude of an equilateral triangle if the length of one side is a .
- Find the area of an equilateral triangle if the length of one side is a .
- Consider an equilateral triangle ABC . Choose a point O anywhere inside ABC . Draw perpendicular lines from O to the sides of ABC . Prove that the sum of the lengths of these perpendiculars is equal to the altitude of ABC .
- What happens when you choose O to be right in the center of the equilateral triangle? Given that a side of the triangle is a , what is the length of each perpendicular line, given that the length of one side of the triangle is a ?
- What happens when O is exactly on the midpoint of one side of the equilateral triangle? What are the lengths of the perpendiculars? You are given a , the length of one side of the equilateral triangle.
- What happens when O is chosen to be on one of the vertices of the equilateral triangle? What are the lengths of the perpendiculars? The length of one side of the triangle is a .
- Suppose O is on the midpoint of one side of the equilateral triangle. Let P and Q be the points where the perpendiculars from O meet the other sides. Find the length of PQ .
- Express the area of a trapezoid in terms of arithmetic mean.

16. Let $a = 9$ and $b = 16$. Find the arithmetic mean, geometric mean, harmonic mean and root-mean-square of a and b . Is it true that

$$9 < \text{HM}(9, 16) < \text{GM}(9, 16) < \text{AM}(9, 16) < \text{RMS}(9, 16) < 16?$$

17. Let a and b be the lengths of the parallel sides of a trapezoid and let h be the height. Prove that area of the trapezoid is the arithmetic mean of a and b multiplied by h .

18. Solve for x :

$$(a) \quad (a + b) \left(\frac{1}{x} + \frac{1}{x + b} \right) = 2.$$

$$(b) \quad (a + b) \left(\frac{1}{x + a} + \frac{1}{x} \right) = 2.$$

19. Solve for x :

$$(a) \quad (a + b) \left(\frac{1}{a} + \frac{1}{x + b} \right) = 2.$$

$$(b) \quad (a + b) \left(\frac{1}{x + a} + \frac{1}{b} \right) = 2.$$

20. Solve for x :

$$(a) \quad (a + b) \left(\frac{1}{ax} + \frac{1}{bx} \right) = 2.$$

$$(b) \quad (a + b) \left(\frac{1}{x + a} + \frac{1}{x + b} \right) = 2.$$

21. Let $ABCD$ be a trapezoid and let AB and CD be the parallel sides. Draw EF parallel to AB and CD such that it bisects the area of $ABCD$. Prove that the length of EF is the root-mean-square of the lengths of the parallel sides AB and CD .

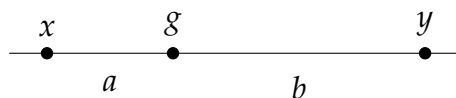
22. In problem 21, let a , b and x be the lengths of AB , CD and EF . Show that $a + b$ is equal to the harmonic mean of $x + a$ and $x + b$.

23. Draw x and y on the number line such that $x < y$ and let p be the harmonic mean of x and y :



Prove that for harmonic mean, the ratio a/b is equal to x/y .

24. Draw x and y on the number line such that $x < y$ and let g be the geometric mean of x and y :



Prove that for geometric mean, the ratio a/b is equal to $\sqrt{x/y}$.

25. Draw lines AB and $A'B'$ with these proportions:

$$(a) \quad AB : A'B' = 3 : 2.$$

$$(c) \quad AB : 3 = A'B' : 2.$$

$$(e) \quad AB : 3 = 2 : A'B'.$$

$$(b) \quad A'B' : AB = 3 : 2.$$

$$(d) \quad 3 : AB = 2 : A'B'.$$

$$(f) \quad 3 : AB = A'B' : 2.$$

26. Draw rectangles with these side ratios:

- (a) $1 : 3$. (b) $5 : 2$. (c) $2 : 3$. (d) $\sqrt{5} : 2$. (e) $\sqrt{2} : \sqrt{3}$.

27. Sketch (freehand) two similar triangles. Label the vertices, sides and angles using $A, A', a, a', \alpha, \alpha'$ etc. Write down the six fundamental relationships between the sides of the similar triangles.

28. Let ABC be a right triangle with right angle at vertex C . Drop an altitude line CD from C to the hypotenuse AB . Let a and b be the lengths of the legs of the triangle and let h be the length of the altitude line. Prove the following:

- (a) $h = \frac{ab}{\sqrt{a^2 + b^2}}$.
 (b) $2h^2$ is the harmonic mean of a^2 and b^2 .
 (c) h is the geometric mean of AD and DB .

29. Consider a right triangle. The lengths of the legs are a and b . The length of the altitude through the right vertex is h . Develop an analogy between the squares of a, b, h and resistors connected in parallel.

30. Use the classical definition of Golden ratio ϕ :

$$\frac{b}{a} = \frac{a+b}{b}$$

to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

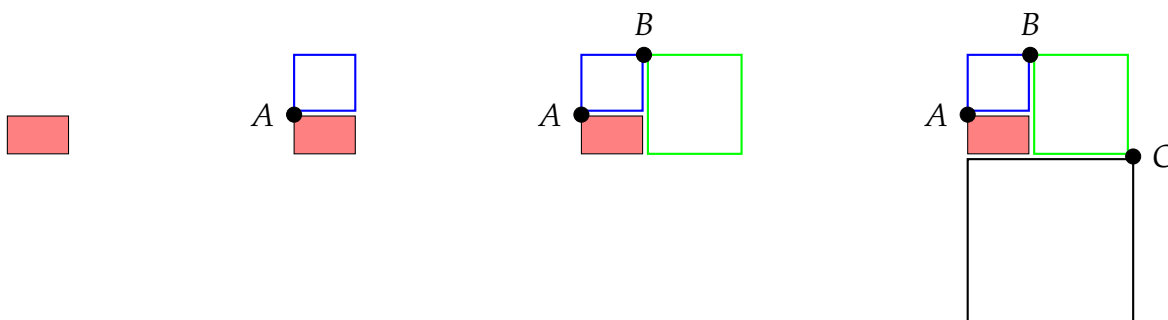
31. Use $\phi = \frac{1 + \sqrt{5}}{2}$ to prove that $\phi^2 = \phi + 1$ and $1/\phi = \phi - 1$.

32. Make a table of the first 20 Fibonacci numbers. They begin like this: $F(1) = 1, F(2) = 1$.

33. Find a simple formula for ϕ^n using Fibonacci numbers.

34. The Kepler triangle. Is it possible to construct a right triangle with sides 1, x and x^2 ? Find x and sketch the Kepler triangle.

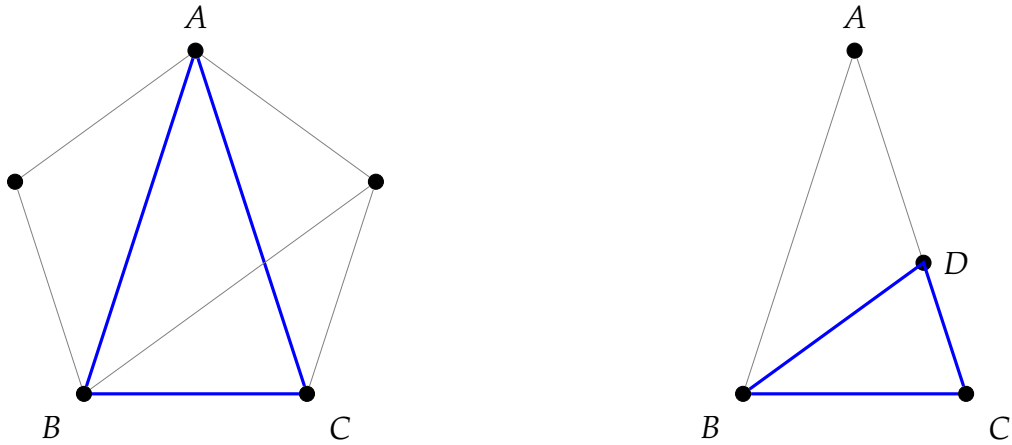
35. Construct a golden spiral. Start with a small golden rectangle (shown in red). Build more golden rectangles by adding squares. The blue square, green square, black square, etc. Work in a clockwise direction.



Build as many squares as you can. Sketch the spiral through the points A, B, C , etc.

36. Build the golden spiral like in problem 35, but this time go in a counterclockwise direction.

37. The Descartes spiral. Triangle ABC is made from the side and diagonals of a perfect pentagon.



ABC is a *golden triangle* because $AB/BC = (AB + BC)/AB$. In other words, AB/BC is ϕ . If we cut the golden triangle ABC at point D , we get another golden triangle: DBC . You can make smaller and smaller golden triangles this way. Construct the Descartes spiral by joining A, B, C, D , etc. with a smooth curve. In a similar way, you can also make a Descartes spiral by constructing bigger and bigger golden triangles.

38. What are the interior angles of the golden triangle?