Arithmetic and Combinatorics

Training problems for M1 2018 term 1

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1 Calculating prodigies

1. *"He who refuses to do arithmetic is doomed to talk nonsense."* Which famous computer scientist said that? You can use the internet to find out.

2. Which famous American calculating prodigy became an astronomer when he grew up?

3. This prodigy was from Germany. He could multiply 100-digit numbers in his head. Who was he?

4. Write down a few interesting things about the life of Jedediah Buxton. Where was he from? When was he born? How did he die? What did he do?

5. What do the amazing powers of calculating prodigies prove about the human mind? Tell me some opinions.

2 Euclidean division

6. Who was Euclid and where was he from? When did he live?

7. Use the terms *dividend*, *divisor*, *remainder* and *quotient* to label the parts of the expression

$$29 = 4 \times 6 + 5.$$

Choose the divisor carefully. Is it 4 or is it 5? Check that $0 \leqslant r < d$. Is it true? Is this a correct expression of Euclidean division?

8. Check that $0 \leqslant r < d$. Is it true?

$$111 = 9 \times 11 + 12$$

Is this a correct expression of Euclidean division?

9. Label the parts of these expressions with the terms *dividend*, *divisor*, *remainder* and *quotient*.

(a) $101/39$.
(b) $m = q \times d + r,\ 0 \leqslant r < d$.
(c) $59 = 5 \times 11 + 4$.
(d) $a = b \times c + d,\ 0 \leqslant d < c,\ d > b$.
(e) $r \div s$.
(f) $39/101$.

10. Which number is the divisor and which one is the quotient?

(a) $42 = 11 \times 3 + 9$.
(b) $99 = 7 \times 13 + 8$.
(c) $69 = 10 \times 6 + 9$.
(d) $23 = 4 \times 5 + 3$.

11. Do by Euclidean division. Label all the parts of your expressions. Give a clear answer in terms of two numbers.

(a) 99/91.
(b) 919/7.
(c) 1001/651.
(d) 1/19.
(e) 19/1.
(f) 17/17.
(g) 0/15.
(h) 15/0.

12. These matrices are filled with consecutive integers $0, 1, 2, \ldots$ Figure out what goes into the boxes.

(a) 7×7 matrix.
(b) 10×10 matrix.
(c) $n \times n$ matrix.

$$\begin{bmatrix} \Box & \cdots & \Box \\ \vdots & & \vdots \\ \Box & \cdots & \Box \end{bmatrix} \qquad \begin{bmatrix} \Box & \cdots & \Box \\ \vdots & & \vdots \\ \Box & \cdots & \Box \end{bmatrix} \qquad \begin{bmatrix} \Box & \cdots & \Box \\ \vdots & & \vdots \\ \Box & \cdots & \Box \end{bmatrix}$$

13. Each matrix is $n \times n$ and is filled with consecutive integers $0, 1, 2, \ldots$ Figure out what goes into the boxes.

(a) $\begin{bmatrix} \Box & \cdots & \Box \\ \Box & \cdots & \Box \\ \vdots & & \vdots \\ \Box & \cdots & \Box \\ \Box & \cdots & \Box \end{bmatrix}$
(b) $\begin{bmatrix} \Box & \Box & \cdots & \Box & \Box \\ \vdots & & & & \vdots \\ \Box & \Box & \cdots & \Box & \Box \end{bmatrix}$

14. For a 6×6 matrix filled with consecutive integers $0, 1, 2, \ldots$, find out what goes into the boxes.

(a) $\begin{bmatrix} \Box & & & & \\ & \Box & & & \\ & & \ddots & & \\ & & & \Box & \\ & & & & \Box \end{bmatrix}$
(b) $\begin{bmatrix} & & & & \Box \\ & & & \Box & \\ & & \iddots & & \\ & \Box & & & \\ \Box & & & & \end{bmatrix}$

15. Now, these are $n \times n$ square matrices filled with consecutive integers $0, 1, 2, \ldots$ Figure out what goes into the boxes.

(a) $\begin{bmatrix} \Box & & & & \\ & \Box & & & \\ & & \ddots & & \\ & & & \Box & \\ & & & & \Box \end{bmatrix}$
(b) $\begin{bmatrix} & & & & \Box \\ & & & \Box & \\ & & \iddots & & \\ & \Box & & & \\ \Box & & & & \end{bmatrix}$

16. Fill in the boxes for a 100×100 matrix of consecutive integers $0, 1, 2, \ldots$

$$\begin{bmatrix} \square & \square & \cdots & \cdots & \square & \square \\ \square & \square & & & \square & \square \\ \vdots & & \ddots & \iddots & & \vdots \\ \vdots & & \iddots & \ddots & & \vdots \\ \square & \square & & & \square & \square \\ \square & \square & \cdots & \cdots & \square & \square \end{bmatrix}$$

17. Consider a 30×30 matrix filled with consecutive integers starting at 0. Where is 611? Give the row and column. Use Euclidean division.

18. Consider a 99×99 matrix of consecutive integers starting at zero. Find the row and column position of 3333.

19. Given an 80×80 matrix of consective numbers beginning at 0, find the number is at the given position:

(a) row 25, column 68. (b) row 68, column 25.

20. In a 5000×5000 matrix of consective integers $0, 1, 2, \ldots,$ what number is at row 991, column 599?

3 Numbers in base-d

21. What is an algorithm? Explain it in your own words. Give some examples.

22. Make base-10 Euclidean division tables for these numbers.

(a) 3351. (b) 4096. (c) 12801.

23. Change these numbers into base-2 by making $d = 2$ Euclidean division tables.

(a) 331. (b) 409. (c) 1280.

24. Make Euclidean division tables with dividend $m = 3721$, using these divisors:

(a) $d = 10$. (b) $d = 4$. (c) $d = 2$.

25. Use Euclidean division tables to change 757 into these bases:

(a) base-5. (b) base-3. (c) base-7.

26. There are two ways we can handle digits bigger than 9. One way is to use alphabetical symbols like A, B, etc. Another way is to use Taylor digits like (10), (11). Sometimes it's better to use alphabetical symbols. Other times it is better to use Taylor digits. Write down some reasons why.

27. How many different digits are there in base-d? Write them down.

28. Write down all the digits of base-20. Use Taylor's idea for digits bigger than 9.

29. How many different digits are there in base-16? Write them down...

(a) using capital letters. (b) using Taylor digits.

30. Change these base-16 numbers from alphabetical symbols to Taylor digits.

(a) FE199A6. (b) 123ABCD12. (c) D00E00F0C. (d) 99FF11BB0A.

31. Change these base-16 Taylor digit numbers into numbers using alphabetical symbols.

(a) (15)(14)(13)0. (b) 53(10)10(10)5. (c) (11)111(11)(11). (d) (14)(12)4(15).

32. Count the digits.

(a) 3DCF918B. (c) (123)(456)789. (e) (89)(71)2(14)(11). (g) 785(21)871.
(b) 348(11)1(13). (d) FFA000A1. (f) (123321). (h) 11(11)1(111).

33. Change 39101 into base-20. Use Taylor digits.

34. Change 569112 into base-100. Use Taylor digits.

35. Change 569112 into base-1000. Use Taylor digits.

36. Now that you see the idea for base-100, 1000, etc, it is very easy to fill in this table without doing any calculations at all...

base-10	891283481785
base-100	
base-1000	
base-10000	
base-1000000	

37. This is Pascal's triangle. You can make it as big as you like.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Change these numbers into base-9 by using Euclidean division tables. Think about the patterns you see in the digits.

(a) 1. (b) 10. (c) 100. (d) 1000.

38. Change 10000 into base-9. Use Pascal's triangle to guess the answer. Check your answer by Euclidean division table. Is your guess right?

39. Change these numbers into base-99 by Euclidean division tables. Think about the patterns you see.

(a) 1. (b) 100. (c) 10000.

40. Change these numbers into base-99 by using Pascal's triangle and Taylor digits.

(a) 1000000. (b) 100000000. (c) 10000000000.

41. Change these numbers into base-999 by Euclidean division. Think about the patterns.

(a) 1. (b) 1000. (c) 1000000.

42. Change these numbers into base-999 by Pascal's triangle and Taylor digits.

(a) 10^9. (b) 10^{12}. (c) 10^{15}. (d) 10^{24}.

43. Change 10^{24} into base-9999 by using Pascal's triangle and Taylor digits.

44. Change 10^{48} into base-999999 using Pascal's triangle and Taylor digits.

4 Computer science arithmetic

45. Why do we use base-2, 4, 8, 16, etc. in the design of computers? Give two reasons.

46. Make a counting table from 0 to 15 in base-2.

base-10	base-2
0	
1	
2	
3	
4	
5	
6	
7	

base-10	base-2
8	
9	
10	
11	
12	
13	
14	
15	

47. Make a counting table from 0 to 15 in base-4.

base-10	base-4
0	
1	
2	
3	
4	
5	
6	
7	

base-10	base-4
8	
9	
10	
11	
12	
13	
14	
15	

48. Fill in this table. One digit in base-4, 8, 16, etc., is equivalent to how many base-2 digits?

kind of digit	equivalent digits in base-2
one base-4 digit	
one base-8 digit	
one base-16 digit	
one base-32 digit	
one base-64 digit	
one base-256 digit	

49. Fill in this table. One digit in base-16, 64, etc., is equivalent to how many base-4 digits?

kind of digit	equivalent digits in base-4
one base-16 digit	
one base-64 digit	
one base-256 digit	

50. Fill in this table. One digit in base-64 or base-512, etc., is equivalent to how many base-8 digits?

kind of digit	equivalent digits in base-8
one base-64 digit	
one base-512 digit	

5 Power bases

51. What is combinatorics? Explain. Give an example of a combinatorial problem or question.

52. Start with the base-10 number 53122. If we start from the *right* and make groups of two digits, we get 5(31)(22). If we start from the *left* and make groups of two digits, we get (53)(12)2.

(a) Change 5(31)(22) from base-100 into base-10.
(b) Change (53)(12)2 from base-100 into base-10.
(c) Are they the same?

53. Start with the base-10 number 24189281.

(a) Start from the right and make groups of three digits. Next, change it from base-1000 into base-10
(b) Start from the left and make groups of three digits. Then change it from base-1000 into base-10.
(c) Are they the same?

54. Fill in the table of powers

n	2^n	7^n	9^n	11^n
1				
2				
3				
4				
5				
6				
7				
8				

55. Write these numbers as powers.

(a) 2048 as power of 2.
(b) $161,051$ as power of 11.
(c) $40,353,607$ as power of 7.
(d) $4,782,969$ as power of 3.

56. At the chinese restaurant they have, in column A: chicken in black bean sauce, prawns with chili, sweet and sour pineapple pork and squid with pepper. In column B they have chow mein, fried rice with cashews, mushrooms in oyster sauce, and egg foo yong. In column C they have spring rolls, green tea ice cream, honeydew melon and monkey-brain jello.

Draw the chinese menu organized into columns. Write down the combinatorics principle and figure out how many different dinners are possible if we choose one from A, one from B and one from C.

57. Jim, Bill, Bob and Fred are on green team. Alice, Ken and Tim are on blue team. Percy, Oliver, Arnold, Lenny and Hank are on yellow team. Yaz, Richard and Zelda are on team red. How many ways can I choose one person from yellow, one person from blue, one person from green and one person from red? Write down the combinatorics principle and figure it out.

58. At the library they have 8 Russian books, 5 Japanese books, 6 English books and 4 Mongolian books.

(a) How many ways can you choose one Russian book, and one Japanese, and one English and one Mongolian book?
(b) How many ways can you choose one Russian book or one Japanese or one English or one Mongolian book?

59. If I have five columns A, B, C, D, E in my menu, and each column has 10 dishes, how many different dinners can I make by choosing one from each column? Write the combinatorics principle and figure it out.

60. Fill in the blanks in the table of equvalent digits.

base-1000	base-10
(51)	
	005
(188)	
	001
7	
	559

61. Change from base-100 to base-10.
(a) (79)(31)(18)(77).
(b) (79)(31)(18)77.
(c) (79)(31)1877.
(d) (79)311877.
(e) 79311877.

62.

Change from base-10 to base-100.
(a) 111222333444555.
(b) 55544433322211.

63. One base-1,000,000 digit is equivalent to how many digits in base-1000?

64. One digit of base-1,000,000 is equivalent to how many digits of base-100?

65. One digit of base-1,000,000 is equivalent to how many digits in base-10?

66. One base-10,000 digit is equivalent to how many digits of base-100? And now:
(a) Change (59)(17)(40)(22)(19)(91) from base-100 to base-10,000.
(b) Change (9300)(3421)(1245) from base-10,000 to base-100.

67. Change (314)(21)9(6731) from base-10,000 to base-100.

68. Change 2(75)6(13)1(99)4(58) from base-100 to base-10,000.

69. One digit of base-81 is equivalent to how many digits of base-3? Illustrate this with the chinese menu drawing. Draw the menu columns and show that there are 81 ways to choose.

70. One digit of base-27 is equivalent to how many digits of base-3? Draw the chinese menu picture for this.

71. One digit of base-625 is equivalent to how many digits of base-5? Draw the chinese menu boxes.

72. One digit of base-625 is equivalent to how many digits of base-25? Draw the chinese menu picture.

73. How can I easily change base-81 into base-27 by two steps?

74. Make a counting table in base-3 and base-9.

base-10	base-3	base-9
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		

base-10	base-3	base-9
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		
26		
27		

75. Change this number from base-27 to base-3.

$$(11)(22)(10)(20)_{27}.$$

76. Change this number from base-3 to base-27.

$$212012102212_3.$$

77. Change this from base-81 to base-9.

$$5(15)9(30)3(60)_{81}.$$

78. Change from base-9 to base-81.

$$617158233149_9.$$

79. Change from base-625 to base-25:

$$(111)(222)(333)_{625}.$$

80. Change from base-625 to base-125 in two steps:

$$(150)(250)(350)(550)_{625}.$$

6 $d-1$ in base-d

81. Prove that if q divides a and q divides b, then it is also true that q divides $a+b$.

82. Let N be a four-digit number with digits $ABCD$. Prove that if 9 divides $A+B+C+D$ then 9 divides N.

83. Let N be a five-digit number with digits $ABCDE$. Prove that if 9 divides $A+B+C+D+E$ then 9 divides N.

84. Let N be a six-digit number with digits $ABCDEF$. Prove that if 9 divides $A+B+C+D+E+F$ then 9 divides N.

85. Show that 19 divides $N = 6023$ by changing N to base-20 and summing the digits.

86. Does 19 divide $N = 14649$? Change N to base-20 and sum the digits to find out.

87. Does 29 divide $N = 51359$? Change to base-30, sum the digits and find out.

88. Determine if these numbers are divisible by 99 by changing to base-100 and summing digits

(a) 12308100948. (b) 4236233503392. (c) 168823196161.

7 GCD and Euclidean algorithm

d is the greatest common divisor of a and b if both of these properties are true:

(1) d is a common divisor of a and b.
(2) All common divisors of a and b also divide d.

It helps to think of it like this: *all common divisors are inside the GCD.*

89. The GCD of 195 and 45 is 15. Find all the common divisors of these numbers and show that properties (1) and (2) are both true.

90. Let $a = 84$ and $b = 72$.
(a) Find all divisors of a.
(b) Find all divisors of b.
(c) Find all common divisors of a and b.
(d) Which of these common divisors is the GCD?
(e) Show that all common divisors divide the GCD.

91. Let $a = 312$ and $b = 168$.
(a) Find all divisors of a.
(b) Find all divisors of b.
(c) Find all common divisors of a and b.
(d) Which of these common divisors is the GCD?
(e) Show that all common divisors divide the GCD.

92. Let $a = 192$ and $b = 112$.
(a) Find all divisors of a.
(b) Find all divisors of b.
(c) Find all common divisors of a and b.
(d) Which of these common divisors is the GCD?
(e) Show that all common divisors divide the GCD.

93. Find the greatest common divisor of 224 and 96 by factoring into primes.

94. Find the GCD of 945 and 675 by factoring into primes.

95. Find $(5390, 1925)$ by factoring into primes.

96. Find $(6720, 7260)$ by factoring into primes.

97. What does it mean when we say that two numbers are *coprime*?

98. Who was Euclid? Were was he from? When did he live?

99. What is an algorithm? Give some examples.

100. Compare these side-by-side.
(a) Euclidean algorithm for GCD of 999 and 21.
(b) Euclidean table for changing 999 into base-21.

101. Find the GCD of 6642 and 4838. Use the Euclidean algorithm.

102. Find the GCD of 7931 and 6699. Use the Euclidean algorithm.

103. Find $(100899, 329966)$. Use the Euclidean algorithm.

104. Show that 8891 and 1177 are coprime. Use the Euclidean algorithm.

105. Show that 6649 and 4813 are coprime. Use the Euclidean algorithm.

8 Fibonacci numbers

Fibonacci numbers have the most incredible properties and relationships. Especially GCD properties.

106. Who was Fibonacci? Where was he from? When did he live?

107. Make a table of Fibonacci numbers from $F(1)$ to $F(30)$.

108. Verify (check) that

$$(F(n+1), F(n)) = 1$$

for these values of n:

(a) $n = 8$. (b) $n = 10$. (c) $n = 12$.

Use the Euclidean algorithm. What is special about the quotient column? What is special about the remainder column?

109. Construct Pascal's triangle carefully up to row $n = 8$. Show how the numbers in diagonals of Pascal's triangle can be added to give Fibonacci numbers.

110. The sum of the squares of two Fibonacci numbers is also a Fibonacci number: $F(n)^2 + F(n+1)^2 = F(2n+1)$. Use your Fibonacci number table and verify this for various values of n:

(a) $n = 5$. (b) $n = 8$. (c) $n = 12$.

111. Who was Giovanni Cassini and where was he from?

112. Cassini discovered this amazing property of Fibonacci numbers:

$$F(n)^2 - F(n-1)F(n+1) = \pm 1.$$

Check this for various values of n. Is it true for these n?

(a) $n = 10$. (b) $n = 15$. (c) $n = 20$.

113. Here is another amazing property of Fibonacci numbers. The sum of the first n Fibonnaci numbers is a Fibonacci number minus one:

$$F(1) + F(2) + \cdots + F(n) = F(n+2) - 1.$$

Use your Fibonacci number table and verify this property for these values of n:

(a) $n = 5$. (b) $n = 9$. (c) $n = 15$.

114. Even more amazing, the sum of the *squares* of the first n Fibonacci numbers is the product of two other Fibonacci numbers!

$$F(1)^2 + F(2)^2 + \cdots + F(n)^2 = F(n)F(n+1).$$

Verify this for these values of n:

(a) $n = 5$. (b) $n = 10$. (c) $n = 15$.

115. Use a calculator and calculate the ratio

$$\frac{F(n+1)}{F(n)}$$

for $n = 5$, $n = 15$ and $n = 30$. Notice that, as n gets bigger, the ratio gets closer to a magic number. This number is called the Golden Ratio.

116. Probably the most amazing of all Fibonacci properties is called the *Lucas property*: The GCD of two Fibonacci numbers is another Fibonacci number! Like so:

$$(F(m), F(n)) = F\left((m, n)\right).$$

Verify the Lucas property for these values of m and n:

(a) $m = 18$, $n = 12$. (b) $m = 28$, $n = 14$. (c) $m = 30$, $n = 18$.

9 Continued fractions

117. What is the Golden Ratio? Who discovered it? Where can we find it?

118. Expand this into a continued fraction by changing all numerators to 1:

$$\frac{355}{109}.$$

119. Expand into a continued fraction by using the Euclidean algorithm:

$$\frac{345}{254}.$$

120. Expand into a continued fraction by using the Euclidean algorithm: $\frac{44}{149}$.

121. Change from continued fraction into a normal fraction:

$$1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7}}}}}}$$

122. Change into a normal fraction:

$$\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1}}}}}}$$

123. Change into a normal fraction:

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}}}}}$$

124. Choose every other Fibonacci number: $F(n)$ and $F(n+2)$. Make fractions with them:

$$\frac{F(n+2)}{F(n)}.$$

Take some values of n and expand into continued fractions using the Euclidean algorithm:

(a) $n = 10$. (b) $n = 15$. (c) $n = 23$. (d) $n = 28$.