Creative math and beautiful problems

Do them! Show your friends! Become a genius the easy way!

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Competition problems are often breathtakingly clever or beautiful or both. It's good to see beauty in mathematics. By doing beautiful and interesting competition problems you will learn mysterious things that no teacher can directly teach. Not everything can be explained on the board. Some things can only be learned by experience and training.

Each section starts with an olympiad problem. Olympiad problems are hard, but they can be broken down into steps that are easy to understand. Each step is numbered. After you finish a step, color the number with a highlighter. Work on each step until they are all completely clear. Then put the pieces together and get a clear understanding of the olympiad problem.

A new dimension has been added here: computer programming. We will use Racket to study and analyze problems in order to get a deeper feeling for what is going on. There's nothing quite like seeing practical calculations unfolding before your eyes. Programming gives that experience. I have chosen Racket for this book because it's incredibly easy to set up and learn, and incredibly powerful as a language for expressing ideas. If you prefer Python or your own favorite programming implementation, go ahead and use it! The Racket code in the text can serve as a model for you to do the same in any language.

You can take pictures of your work and send it to me by Messenger, LINE or email.

1

Let $F(x) = \dfrac{2}{4^x + 2}$. Evaluate the sum

$$F\left(\frac{1}{2001}\right) + F\left(\frac{2}{2001}\right) + \cdots + F\left(\frac{1999}{2001}\right) + F\left(\frac{2000}{2001}\right).$$

This was from a Korean math competition.

1. This, like most competition problems, seems impossibly hard. It's hard because you must combine elementary ideas in very clever ways. The ideas, the building blocks of the solution, are not hard. This problem is a good example of how several mathematical ideas are combined in beautiful and non-obvious ways to give a simple solution.

2. Take a piece of paper, draw the x–y axes, and plot the curve $y = \dfrac{2}{4^x + 2}$. You don't need a calculator for this. Just examine how the function behaves for important values of x. For example, when $x \to \infty$ (x goes to infinity), $y \to 0$ (y goes to zero). Sometimes you have to use your intuition to guess what values are important. Find where y goes when...

(a) $x \to -\infty$.
(b) $x \to 0$.

(c) $x \to 1/2$
(d) $x \to 1$.
Now it's easy to sketch the curve $y = F(x)$. Do it.

3. *Reflect* and *shift* operations allow you to easily sketch new functions from old ones that you know already. Let's look at reflect first.

The operation $x \to -x$ reflects the curve over the y-axis. Think of the y-axis as a mirror. For example, $y = x^3$ is the cubic curve and if you change x to $-x$ you get the mirror image of the cubic curve (where the y-axis is the mirror.)

For each of these, sketch the curve and then sketch the curve with $x \to -x$:
(a) $y = x^3, \quad -\infty < x < \infty$.
(b) $y = x, \quad -\infty < x < \infty$.
(c) $y = -x, \quad -\infty < x < \infty$.
(d) $y = 1/x, \quad -\infty < x < \infty$.
(e) $y = 2^x, \quad -\infty < x < \infty$.
(f) $y = x^2, \quad -\infty < x < \infty$.
Explain the last one.

4. Plot $y = F(-x)$ by doing a a reflection operation $x \to -x$ on $y = F(x) = \dfrac{2}{4^x + 2}$. Sketch the result. You don't need to plot any points or to do any thinking beyond using your imagination to reflect $y = F(x)$ over the y-axis.

5. Can you think of an operation on a function $f(x)$ that will reflect the curve $y = f(x)$ over the x-axis?

6. The other graph operation we need for this problem is *shift*. Given some function $f(x)$, the operation $x \to x + 1$ shifts the curve $y = f(x)$ one step to the left. So, for example, since we know what $y = x^2$ looks like, it's easy to plot $y = (x + 1)^2$. Just shift everything to the left by 1. No further thinking is necessary.

Plot these curves by doing a left shift operation on curves that you already know. Each one should take only a few seconds of thinking.
(a) $y = (x + 1)^3$. Start with $y = x^3$.
(b) $y = x + 2$. Start with $y = x$.
(c) $y = -(x + 2)$. Start with $y = -x$.
(d) $y = 1/(x + 1)$.
(e) $y = 2^{x+1}$.
(f) $y = (x + 3/2)^2$.

7. Likewise the transformation $x \to x - 1$ will shift the curve $y = f(x)$ one step *to the right*. Now you can plot $y = (x - 1)^2$ instantly, without thinking for more than a second. Just take the curve for $y = x^2$ and shift everything to the right by 1.

Plot these curves by doing right shift operations on curves that you already know.
(a) $y = (x - 1)^3$.
(b) $y = x - 5$.
(c) $y = -(x - 1)$. Start with $y = -x$.
(d) $y = 1/(x - 2)$.
(e) $y = 2^{x-1}$.
(f) $y = (x - 3/2)^2$.

8. Plot $y = F(x+1)$ and $y = F(x-1)$. Start with the curve $y = F(x)$ which you already did. Use your imagination to do the shifts. Don't do any thinking beyond that.

9. Can you think of operations that shift a curve up and down, rather than left and right?

10. Reflect and shift operations can be combined. You can start with a curve, do a shift, and then do a reflect. For example, starting with $y = f(x)$, we can instantly plot $y = f(-x+1)$ by first doing a left shift $x \to x+1$ and then a reflection $x \to -x$. Notice that it's not the same if we do the operations the other way around! Starting with the curve $y = f(x)$, if we first do a reflect $x \to -x$ and then a left shift $x \to x+1$, we get $f(-(x+1))$ or $f(-x-1)$, which is not the same.

For the given curve, do the operations in order and plot the result. Don't calculate anything. Use your imagination only.

(a) $y = x^3$, shift $x \to x-1$, then reflect $x \to -x$.
(b) $y = x^3$, $x \to -x$, then $x \to x-1$.
(c) $y = 2^x$, $x \to x+1$, then $x \to -x$.
(d) $y = 2^x$, $x \to -x$, $x \to x+1$.
(e) $y = x^2$, $x \to x-2$, $x \to -x$.
(f) $y = x^2$, $x \to -x$, $x \to x-2$.

Explain what is going on in the last two.

11. Plot the given function by doing two operations on a simpler function that you already know how to sketch.

(a) $y = (-(x-2))^3$. Start with $y = x^3$. First do $x \to -x$, then do $x \to x-2$..
(b) $y = (1-x)^3$. Write as $y = (-(x-1))^3$ and do similar to above.
(c) $y = -(x-1)$. Start with $y = x$.
(d) $y = 1/(-x+2)$. Start with $y = 1/x$.
(e) $y = 2^{-(x-1)}$. Start with $y = 2^x$.

12. Draw the x–y axes. Plot the curve $y = F(x)$ again. Now, on top of that, plot $y = F(1-x)$ by doing operations. $1-x$ is the same as $-x+1$, so it is clear that you have to do a left shift by 1 first, followed by a reflection (not the other way around!) Or, if you like, $1-x = -(x-1)$, so you can do a reflection followed by a right shift by 1. If you do it right, both curves will go through the point $(\frac{1}{2}, \frac{1}{2})$.

13. Look at the curves $y = F(x)$ and $y = F(1-x)$. Notice that when $F(x)$ is close to 1, $F(1-x)$ is close to 0. And when $F(1-x)$ is close to 1, $F(x)$ is close to 0. Also, when $F(x)$ is 1/2, $F(1-x)$ is also 1/2. Thinking about all this, we can make take a guess, based on the shape of these curves, that $F(x) + F(1-x)$ is always 1. If this guess is true, then it should be possible to prove that

$$F(x) + F(1-x) = 1.$$

Go ahead and prove this by algebra! This result is the key that is needed to solve the original competition problem.

14. Consider this sum:

$$1 + 2 + \cdots + 50 + 51 + \cdots + 99 + 100$$

You want to find the result. You don't have to sum the terms in the order that they are given. You can rearrange them and sum them in some other way. This is another key idea in competition problems.

Take the first term, 1, and the last term, 100. Add them. You get 101. Now take the second term, and the second-last term, and add them: $2 + 99 = 101$. Continuing this way we see there are 50 pairs of numbers that can be added to form 101. Thus the sum is $50 \times 101 = 5050$.

Use the same idea to find the following sums. Note that when there is an odd number of terms, there will be a term in the middle that is left over, and cannot be paired with another term.

(a) $1 + 2 + \cdots + 49 + 50$.
(b) $1 + 2 + \cdots + 50 + \cdots + 100 + 101$.
(c) $25 + 26 + \cdots + 80$;
(d) $30 + 31 + \cdots + 70$;
(e) $1 + 3 + 5 + \cdots + 95 + 97 + 99$.

15. And now you are ready to solve the competition problem! Use the idea of pairing first and last terms, second with second last term, etc., together with the relationship $F(x) + F(1 - x) = 1$. The rest is easy!

16. For extra enlightenment, write a computer program that does the sum. Use any programming language that you like. Here is a Racket version:

```
#lang racket

(define (f x)
  (/ 2
     (+ (expt 4 x) 2)))

(define (do-sum)
  (let loop ((k 1) (result 0))
    (if (> k 2000)
        result
        (loop (+ k 1)
              (+ result (f (/ k 2001)))))))
```

Now test it in Dr Racket:

```
> (do-sum)
999.9999999999984
```

Almost 1000. Close enough. Machine calculations are often not perfect. Download and install the Racket programming language system from `racket-lang.org` and try it yourself.

2

Given angles A, B and C of a triangle, find $\phi = \angle BAM$ where M is the midpoint of side BC. From a Russian problem book.