M2 Training Problems

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1 Functions, identity, inverses and plots

1. Let $f(x) = 2x + 1$. Find...
   (a) Find $f(f(x))$.
   (b) Find $f(f(f(x)))$.
   (c) Find $f(f(f(f(x))))$.

2. Let $f(x) = 3x^2 + 1$ and $g(x) = 2x - 3$.
   (a) Find $f(g(x))$.
   (b) Find $g(f(x))$.
Are they the same?

3. Let $f(x) = ax + b$.
   (a) Find $f(f(x))$.
   (b) Find $f(f(f(x)))$.

4. Let $f(x) = ax + b$ and $g(x) = cx + d$.
   (a) Find $f(g(x))$.
   (b) Find $g(f(x))$.
Are they the same?

5. Sketch $y = x$ and $y = -x$. Put them on the same axes. Label everything.

6. Sketch $y = 2x$ and $y = -2x$. Put them on the same axes.

7. Sketch these lines on the same axes.

$$y = \frac{x}{2}, \quad y = -\frac{x}{2}.$$

8. Make an exact plot of $y = 3x + 2$ by finding the x-intercept and y-intercept.

9. Make an exact plot of

$$y = -\frac{x}{2} - 1.$$

10. If $f(x)$ and $g(x)$ are linear, show that
   (a) $f(g(x))$ is linear.
   (b) $g(f(x))$ is linear.

11. Let $f(x) = 3x + 2$. Find $f^{-1}(x)$. Do it two ways:

(a) By $f(f^{-1}(x)) = I(x)$.
(b) And by $f^{-1}(f(x)) = I(x)$.

12. Let $f(x) = ax + b$. Find $f^{-1}(x)$. Do it two ways:
(a) By $f(f^{-1}(x)) = I(x)$.
(b) And by $f^{-1}(f(x)) = I(x)$.

13. Let $f(x) = 2x + 1$. Find $f^{-1}(x)$. Make exact plots of f and f^{-1}. Also draw I.

14. Let $f(x) = -2x + 3$. Find $f^{-1}(x)$. Make exact plots of f and f^{-1}. Also draw I.

15. Consider the function
$$f(x) = -\frac{x}{2} + 3.$$
Find $f^{-1}(x)$. Make exact plots of f and f^{-1}. Also draw I.

16. Sketch the curve $y = x^2$. Use the unit square idea.

17. Let $f(x) = x^2$. Sketch f, I and f^{-1} on the same axes.

18. Let $f(x) = x^2 + 1$. Sketch f, I and f^{-1} on the same axes.

19. Are there functions that are inverses of themselves? Does there exist any functions with the property $f(x) = f^{-1}(x)$? In other words, f is its own inverse.
(a) Find one such self-inverse function f.
(b) Try to find more, as many as you can.

2 Introducing logarithms

20. Draw the fastest-growing function f that you can imagine. Draw $I(x)$ and use it to find the inverse f^{-1}.

21. Draw the slowest-growing function f that you can imagine. Draw the identity $I(x)$ and use it to find the inverse f^{-1}.

22. Given f, tell me about the inverse f^{-1}. Does it grow fast, slowly, very fast etc.?
(a) f is a fast-growing function.
(b) f does not grow at all.
(c) f is a slow-growing function.
(d) f is a very slow-growing function.
(e) f is a very fast-growing function.

23. Fill in this table about the behavior of $f(x) = 2^x$ for different values of x.

x	$f(x)$
$x = 0$	$f = 1$
$x > 0$	
$x < 0$	
$x > 1$	
$x \to \infty$	
$x \to -\infty$	

24. Plot 2^x, 3^x and 5^x all on the same axes.

25. Consider the function $f(n) = \left(1 + \frac{1}{n}\right)^n$. Use a calculator. Fill in this table

x	$f(x)$
1	2
2	
5	
10	
100	
1000	

Notice how $f(n)$ keeps increasing as n gets bigger. But also notice how $f(n)$ does not increase to infinity, but approaces the magic number *e from below*.

26. Now consider the slightly different function $g(n) = \left(1 + \frac{1}{n}\right)^{n+1}$. Use a calculator. Fill in this table

x	$f(x)$
1	2
2	
5	
10	
100	
1000	

Notice how $g(n)$ keeps decreasing as n gets bigger. But also notice how $g(n)$ does not decrease to minus infinity, but approaces the magic number *e from above*.

27. Plot 2^x, e^x and 10^x all on the same axes.

28. Plot $f(x) = 2^x$ and the identity line $I(x)$. Use the identity line to draw the inverse $f^{-1}(x) = \log_2 x$.

29. Plot $f(x) = 3^x$, $I(x)$ and $f^{-1}(x) = \log_3 x$ all on the same axes.

30. Fill in this table about the behavior of $g(x) = \log_2 x$ for different values of x.

x	$g(x)$
$x = 1$	$g = 0$
$x > 1$	
$x < 1$	
$x = 2$	
$x \to \infty$	
$x \to 0$	

31. Does 2^x ever touch the x-axis? Does $\log_2 x$ ever touch the y-axis?

32. Fill in the table.

x	3^x
1	
2	
3	
4	
5	

x	$\log_3 x$
1	
9	
27	
243	
59049	

33. Fill in the table.

x	10^x
1	
2	
3	
4	
5	

x	$\log_{10} x$
1	
10	
1000	
$100,000$	
$10,000,000$	

34. Plot $\log_2 x$, $\log_3 x$ and $\log_5 x$ all on the same axes. Label all the important points.

35. Plot $\log_2 x$, $\log_e x$ and $\log_{10} x$ all on the same axes. Label all the important points.

36. The formulas relating f, f^{-1} and I establish the two most important properties of logarithms and exponentials. Use $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ and tell me what these formulas imply:

(a) $f(f^{-1}(x)) = I(x)$.
(b) $f^{-1}(f(x)) = I(x)$.

3 Properties of $\log_a$

37. Figure out $a^{\log_a a^x}$.

38. Figure out $\log_a a^{\log_a a^x}$.

39. Begin with the well-known property of exponential functions $(a^x)^p = a^{xp}$ and prove the following property of logarithms:

$$\log_a u^p = p \log_a u.$$

Notice how log changes powers to multiplications.

40. Begin with something we all know: $a^x a^y = a^{x+y}$ and prove the following property of logarithms:

$$\log_a uv = \log_a u + \log_a v.$$

Notice how log changes multiplication into addition.

41. Using an idea similar to the one in problem **40**, prove that

$$\log_a uvw = \log_a u + \log_a v + \log_a w.$$

42. Use the results of problems **39** and **40** to prove this:

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

Notice how log changes division into subtraction.

43. Use **39**, **40** and **42** to figure these out.

(a) $\log_2 8 \times 32 \times 64$.
(b) $\log_5 25 \times 125 \times 625$.
(c) $\log_2 2^7 8^5 16^3$.
(d) $\log_3 \sqrt{27}\sqrt[3]{81}$.
(e) $\log_5 \dfrac{\sqrt{125}}{\sqrt[3]{625}}$.
(f) $\log_e \dfrac{\sqrt{e}}{e^3}\sqrt[3]{e}$.

44. Use plots to explain why $0 < x < 1$ when $\log x$ is negative.

45. Use plots to explain why $\log x$ is positive when $x > 1$.

46. Let $x = a/b$. Use the property in **42** to show that $\log x$ is negative when $0 < x < 1$.

47. Let $x = a/b$. Use the property in **42** to show that $\log x$ is positive when $x > 1$.

48. Positive or negative? Explain why.

(a) $\log_a \dfrac{2}{3}$.

(b) $\log_a \dfrac{5}{2}$.

(c) $\log_a 0.8$.

(d) $\log_a 1.8$

(e) $\log_a \dfrac{e}{\pi}$.

(f) $\log_a \dfrac{\pi}{e}$.