

Creative Math and Beautiful Problems

Do them! Show your friends! Become a genius the easy way!

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Why competition problems?

Competition problems are often breathtakingly clever or beautiful or both. It's good to see beauty in mathematics. By doing beautiful and interesting competition problems you will learn mysterious things that no teacher teach you directly. Not everything can be explained on the board. Some things can only be learned by experience and training.

There is no better way to develop your mathematical abilities than by working on competition problems. You don't even have to compete in competitions to get the benefits!

I chose interesting problems for this book. Problems that I found personally compelling and enlightening. It is my hope that the world of mathematics education moves away from uninspired repetition, drudgery and rote, into a world of beautiful, entertaining and illuminating competitive and creative mathematics.

Computer programming

A new dimension has been added here: computer programming. We will use Racket to study and analyze problems to get a deeper feeling for what is going on. There's nothing like seeing practical calculations unfolding before your eyes. Programming gives that experience. You can play with otherwise intractable calculation, experiment, modify, see what happens, explore possibilities that you would not have time for if you were confined to paper and pencil.

For us, computer science principles are more important than a particular choice of programming language. But whatever language we choose to work in, it must be accessible to students that are new to programming. I have chosen the Racket programming system for this book because it's very easy to set up and learn, and incredibly powerful for expressing ideas. Racket is a descendant of the Scheme, which is the language used for the famous textbook *Structure and Interpretation of Computer Programs*. If you are new to Racket, there are resources on my website to help you get started. In my experience, Racket has proven to be a great tool for exploring ideas in computer science and mathematics.

If you prefer your own favorite programming language implementation, go ahead and use it! The Racket code in the text can serve as a model for you to do the same in any language.

Geometric constructions

Get familiar with the techniques of ruler-compass construction: bisection of segments and angles, construction of polygons, perpendiculars, parallel lines, etc. Use ruler and compass wherever you can. Ruler and compass constructions lead to intuitive understanding of geometry. Never miss an opportunity to translate a competition question into a ruler and compass construction.

How to use the book

Each section starts with an Olympiad problem or some interesting competition-level problem. Olympiad problems are hard, but in this book they are broken down into steps that are easy to understand. Each step is numbered. Work on each step until they are all completely clear. Then put the pieces together and get a clear understanding of the olympiad problem.

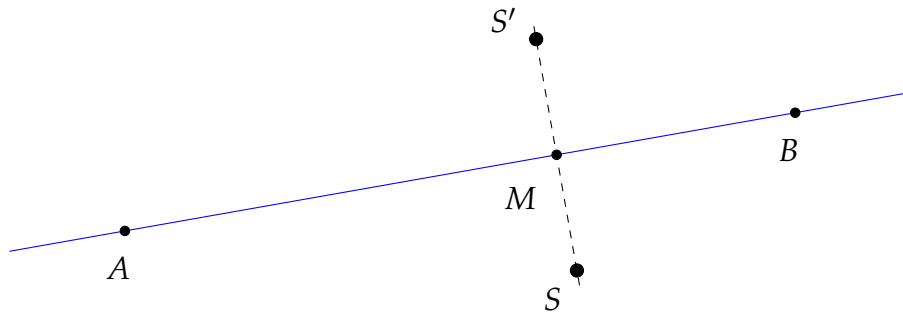
When you see Racket code, try it out in DrRacket. It's not necessary to understand every detail the first time you encountered a piece of code. Try it out. The details will become clear with practice.

Part 1

Problem 1. Show that two reflections is equivalent to a rotation. Also show that a rotation through angle 2ϕ around point A can be obtained by reflections over any two lines on A separated by angle ϕ .

1. A remarkable idea, which no doubt will be surprising to any who have not seen it before! It's difficult to imagine exactly how these two reflections combine to create a rotation, and why this should be so in the first place. Therefore it pays to examine this in detail from different points of view. For maximum enlightenment, we will look at it from three points of view: classical geometry, vectors and complex numbers.

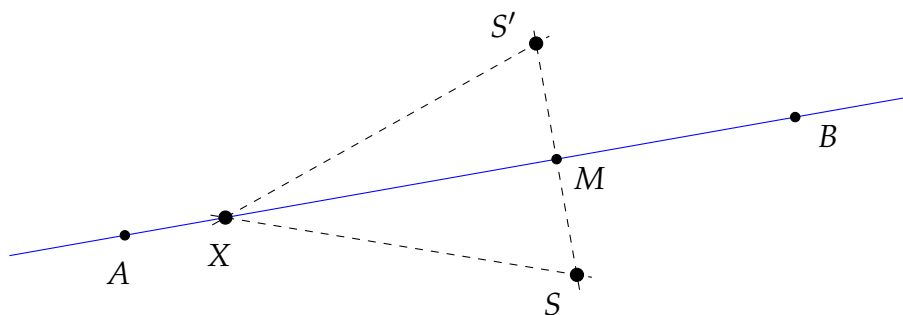
2. What is a reflection? Line AB is the mirror-line. It will do the reflection.



Point S is reflected to point S' with the following properties: SS' is perpendicular to AB and M is the midpoint of SS' . In other words, $\angle AMS$ is a right angle and $SM = MS'$.

3. Begin with mirror-line AB and a point S as in 2. With a ruler and compass, construct the reflected point S' . You'll have to construct a line through S perpendicular to AB , then mark off equal segments, etc.

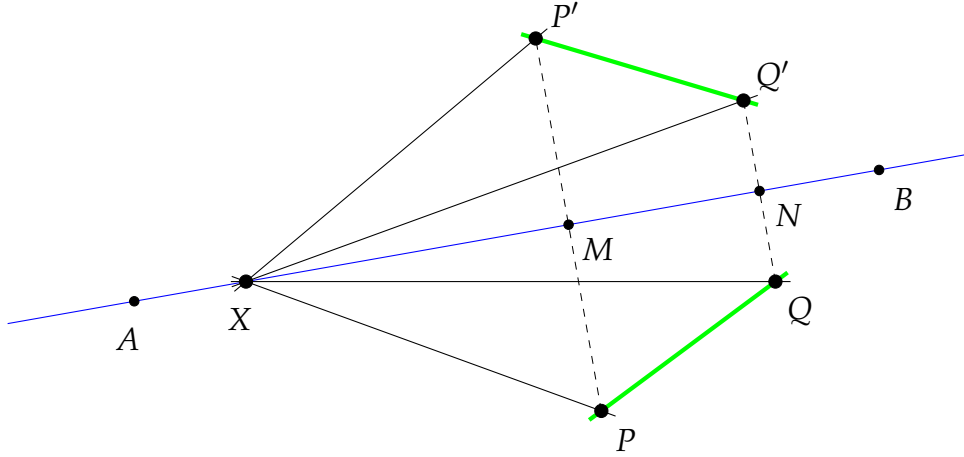
4. Let X be any point on the mirror-line AB . Prove that the distances from S and S' to point X are the same. Also prove that $\angle MXS = \angle MXS'$.



This is easily demonstrated by similar triangles. Consider $\triangle XMS$ and $\triangle XMS'$. By the properties of reflections 2, both $\angle XMS$ and $\angle XMS'$ are right angles and $SM = MS'$. But also, $\triangle XMS$ and $\triangle XMS'$ share side XM in common. Therefore by *side-angle-side* (SAS), the two triangles are congruent! Matching sides and matching angles of these triangles must be equal. Hence we have $XS = XS'$ and $\angle MXS = \angle MXS'$.

5. Show that reflections preserve distance between points.

Consider a segment PQ and its reflected version $P'Q'$. By choosing a point X somewhere on the mirror-line AB , we can create similar (in this case congruent) triangles.



By the properties of reflections, 2, $\angle NXQ = \angle NXQ'$ and $\angle MXP = \angle MXP'$. Therefore it must be that $\angle PXQ = \angle P'XQ'$. By what was proved in 4, $XP = XP'$ and $XQ = XQ'$. Therefore by *side-angle-side* (SAS), the two triangles $\triangle PXQ$ and $\triangle P'XQ'$ must be congruent. Therefore the segments PQ and $P'Q'$ have equal length. Reflection preserves distance between points!

6. Use 4 and the cosine law to form an argument that reflections preserve distance.

7. Now we will show the same thing again, that reflections preserve distance, but using vectors. This is a lot more work than using similar triangles, but it's interesting and worth doing.

Refer to the figure in 5 and give some shorter names to important angles:

$$\angle MXP = \alpha, \quad \angle NXQ = \beta, \quad \angle QXP = \alpha - \beta = \phi.$$

Fix X to be the origin of all vectors in the figure. The vectors \vec{M} and \vec{N} can be found by midpoints:

$$\vec{M} = \frac{\vec{P} + \vec{P'}}{2}, \quad \vec{N} = \frac{\vec{Q} + \vec{Q'}}{2}.$$

From these we obtain expressions for $\vec{P'}$ and $\vec{Q'}$:

$$\vec{P'} = 2\vec{M} - \vec{P}, \quad \vec{Q'} = 2\vec{N} - \vec{Q}. \quad (1)$$

Why did we do this? Because if you examine the figure in 5, vector \vec{M} is the perpendicular projection of vector \vec{P} onto the mirror-line. Likewise for \vec{N} and \vec{Q} . Therefore we can eliminate \vec{M} and \vec{N} by expressing them in terms of \vec{P} and \vec{Q} . The idea is to express the reflected points in terms of the original points with a bit of added trigonometry.

Let \hat{u} be the unit vector in the direction of the mirror-line AB . Since \vec{M} and \vec{N} are projections of \vec{P} and \vec{Q} onto the mirror line, we have:

$$\vec{M} = \hat{u}p \cos \alpha, \quad \vec{N} = \hat{u}q \cos \beta$$

where p is the length of vector \vec{P} and q is the length of vector \vec{Q} . Substituting into (1) gives

$$\vec{P'} = 2\hat{u}p \cos \alpha - \vec{P}, \quad \vec{Q'} = 2\hat{u}q \cos \beta - \vec{Q}. \quad (2)$$

We will do some fancy vector calculations to show that the length of segment $P'Q'$ is the same as the length of PQ . The length of the segment $P'Q'$ is the norm $\|\vec{Q}' - \vec{P}'\|$ of the vector $\vec{Q}' - \vec{P}'$. Using the expressions (2) for P' and Q' ,

$$\begin{aligned}\vec{Q}' - \vec{P}' &= \vec{P} - \vec{Q} + 2\hat{u}(q \cos \beta - p \cos \alpha) \\ &= \vec{P} - \vec{Q} + 2\hat{u}A.\end{aligned}$$

Here A is short for $q \cos \beta - p \cos \alpha$. Using $\|\vec{V}\|^2 = \vec{V} \cdot \vec{V}$, and the fact that dot product is distributive,

$$\begin{aligned}\|\vec{Q}' - \vec{P}'\|^2 &= (\vec{Q}' - \vec{P}') \cdot (\vec{Q}' - \vec{P}') \\ &= (\vec{P} - \vec{Q} + 2\hat{u}A) \cdot (\vec{P} - \vec{Q} + 2\hat{u}A) \\ &= \vec{P} \cdot \vec{P} + \vec{Q} \cdot \vec{Q} - 2\vec{P} \cdot \vec{Q} + 4A\hat{u} \cdot \vec{P} - 4A\hat{u} \cdot \vec{Q} + 4\hat{u} \cdot \hat{u}A^2.\end{aligned}$$

By the definition of dot product, $\vec{P} \cdot \vec{P} = \|\vec{P}\|^2 = p^2$ and $\vec{P} \cdot \vec{Q} = pq \cos \phi$ where ϕ is the angle between \vec{P} and \vec{Q} . Also note that $\hat{u} \cdot \hat{u} = 1$ because \hat{u} is a unit vector. So, combining all these, we have

$$\begin{aligned}\|\vec{Q}' - \vec{P}'\|^2 &= p^2 + q^2 - 2pq \cos \phi + 4Ap \cos \alpha - 4Aq \cos \beta + 4A^2 \\ &= p^2 + q^2 - 2pq \cos \phi + 4A(p \cos \alpha - q \cos \beta + A)\end{aligned}$$

The term $p^2 + q^2 - 2pq \cos \phi$ looks familiar. It's precisely the cosine law expression for the square of the length of segment PQ , i.e., $\|\vec{Q} - \vec{P}\|^2$. It had better be the case, then, that the term $4A(p \cos \alpha - q \cos \beta + A)$ vanishes. Let's see:

$$\begin{aligned}4A(p \cos \alpha - q \cos \beta + A) &= 4A(p \cos \alpha - q \cos \beta + q \cos \beta - p \cos \alpha) \\ &= 0.\end{aligned}$$

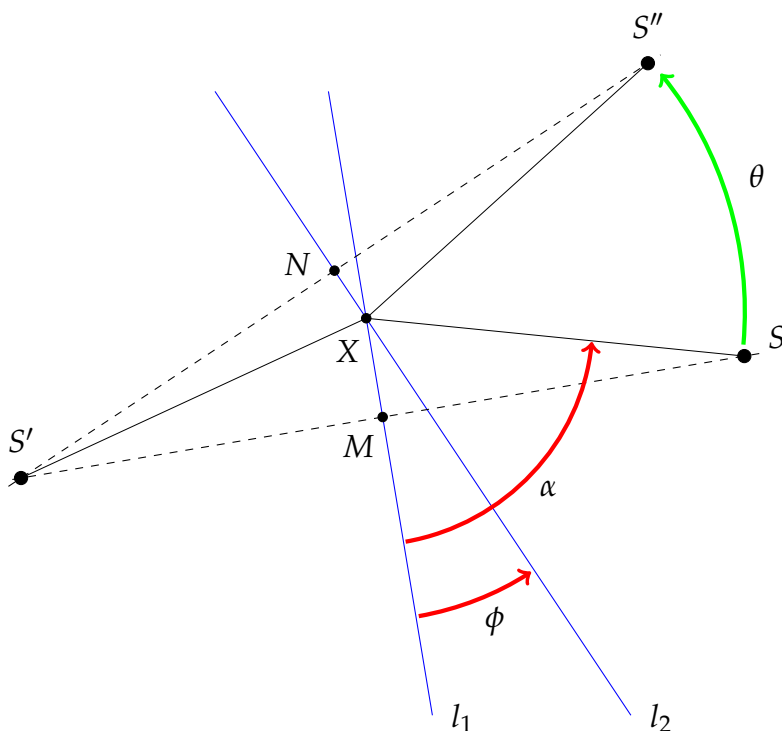
It does vanish! Therefore

$$\|\vec{Q}' - \vec{P}'\|^2 = \|\vec{Q} - \vec{P}\|^2$$

and $P'Q' = PQ$! That was a lot of work to demonstrate something fairly obvious from similar triangles. But nevertheless it's interesting how the cosine law appears and the rest of the complicated expression vanishes.

8. Two reflections is a rotation.

9. Two reflections make a rotation. What rotation? How does that work exactly? We will show that if two mirror-lines are separated by angle ϕ , then no matter how they are oriented, the result of reflecting point S over the two mirrors will be S rotated by 2ϕ around the intersection of the mirror-lines.



Lines l_1 and l_2 are the mirror-lines. They are separated by angle ϕ . The orientation of these mirrors is determined by α . Point S will undergo the two reflections. We already know that two reflections make a rotation. But what rotation? In other words: what is angle θ ?

Point S is first reflected over l_1 giving S' . According to the results of 4, $\angle SXM = \angle S'XM$, with both of these being α . Likewise S' is reflected over l_2 giving S'' , and $\angle S'XN = \angle S''XN$. But by basic geometry both of these are equal to $180 - (\alpha + \phi)$. Going around X full circle and adding all the angles we get:

$$\alpha + \alpha + 180 - (\alpha + \phi) + 180 - (\alpha + \phi) + \theta = 360.$$

Simplifying, $\theta = 2\phi$. Notice the amazing effect: the problem does not depend on α . The mirror-lines can be oriented with any α , and the resulting rotation of S to S'' will always be $\theta = 2\phi$!

10. Explain why $\angle S'XN = 180 - (\alpha + \phi)$ and $\angle S''XN = 180 - (\alpha + \phi)$.

11. Prove that $\angle SS'S'' = \phi$.

12. Complex numbers application.

13. Ruler-compass exercise: rotate 360 degrees around triangle angles, using compass copy technique.

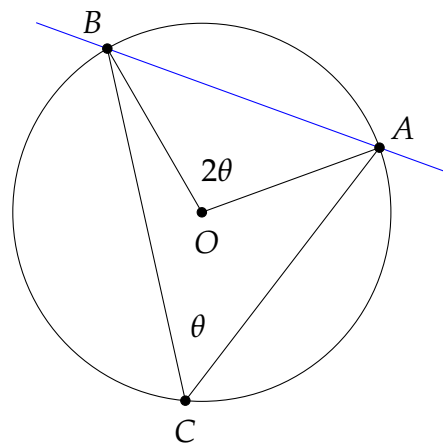
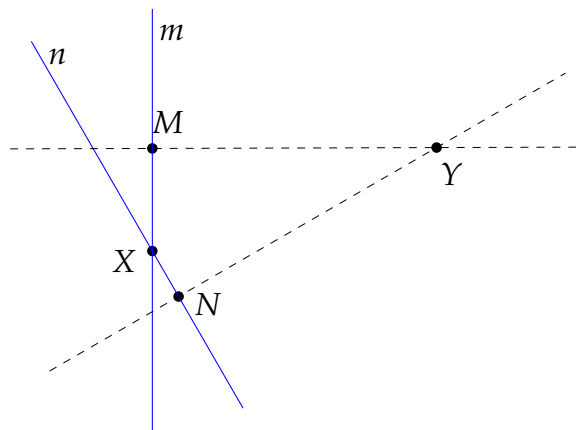
14. Racket applications: rotations around origin, angles between lines, tan formula.

15. All that was very interesting and illuminating, especially the complex numbers stuff. One cannot help but wonder if there are simple principles or some essential idea by which the phenomenon can be intuitively grasped—something that would make all this obvious.

Rephrasing the problem from a physics picture (mirrors, light, reflected images, mechanical rotation) to a pure geometry picture (lines, circles, theorems) does exactly that.

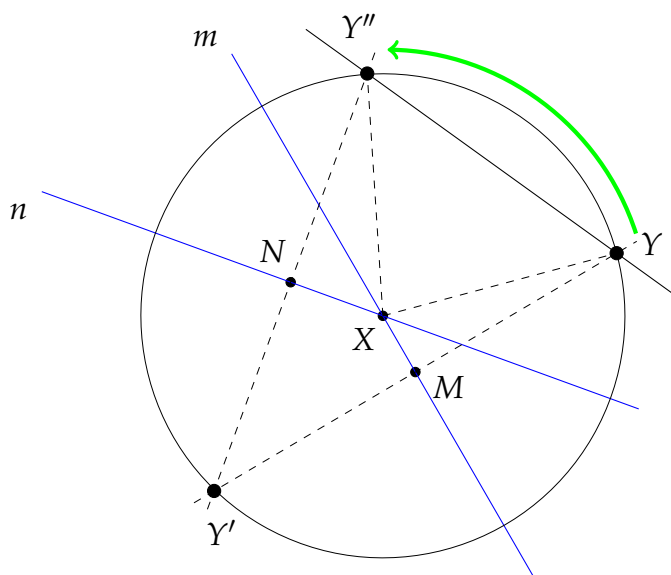
The relationship between reflections and rotation becomes instantly obvious. So obvious that we might laugh at all the work we did. Nevertheless it's a good thing we did all that work, otherwise we would not appreciate the simplicity and beauty of what comes next.

16. Prove these two classic geometry theorems.



On the left: if lines m and n are separated by angle ϕ , and MY , NY are perpendicular to m , n , then $\angle MYN$ is also ϕ . On the right: AB is a chord line and O is the center of the circle. C is a point anywhere on the circumference, but on the same side of AB as center O . If $\angle ACB = \theta$, then $\angle AOB$ will be 2θ .

17. Putting these ideas together,



Problem 2. Let $F(x) = \frac{2}{4^x + 2}$. Evaluate the sum

$$F\left(\frac{1}{2001}\right) + F\left(\frac{2}{2001}\right) + \cdots + F\left(\frac{1999}{2001}\right) + F\left(\frac{2000}{2001}\right). \quad (3)$$

This was from a Korean math competition.

18. This, like most competition problems, seems impossibly hard. It's hard because you must combine elementary ideas in very clever ways. The ideas, the building blocks of the solution, are not hard. This problem is a good example of how several mathematical ideas are combined in beautiful and non-obvious ways to give a simple solution.

19. Take a piece of paper, draw the x - y axes, and plot the curve $y = \frac{2}{4^x + 2}$. You don't need a calculator for this. Just examine how the function behaves for important values of x . For example, when $x \rightarrow \infty$ (x goes to infinity), $y \rightarrow 0$ (y goes to zero). Sometimes you have to use your intuition to guess what values are important. Find where y goes when...

- (a) $x \rightarrow -\infty$.
- (b) $x \rightarrow 0$.
- (c) $x \rightarrow 1/2$
- (d) $x \rightarrow 1$.

Now it's easy to sketch the curve $y = F(x)$. Do it.

20. *Reflect* and *shift* operations allow you to easily sketch new functions from old ones that you know already. Let's look at reflect first.

The operation $x \rightarrow -x$ reflects the curve over the y -axis. Think of the y -axis as a mirror. For example, $y = x^3$ is the cubic curve and if you change x to $-x$ you get the mirror image of the cubic curve (where the y -axis is the mirror.)

For each of these, sketch the curve and then sketch the curve with $x \rightarrow -x$:

- (a) $y = x^3$, $-\infty < x < \infty$.
- (b) $y = x$, $-\infty < x < \infty$.
- (c) $y = -x$, $-\infty < x < \infty$.
- (d) $y = 1/x$, $-\infty < x < \infty$.
- (e) $y = 2^x$, $-\infty < x < \infty$.
- (f) $y = x^2$, $-\infty < x < \infty$.

Explain the last one.

21. Plot $y = F(-x)$ by doing a reflection operation $x \rightarrow -x$ on $y = F(x) = \frac{2}{4^x + 2}$. Sketch the result. You don't need to plot any points or to do any thinking beyond using your imagination to reflect $y = F(x)$ over the y -axis.

22. Can you think of an operation on a function $f(x)$ that will reflect the curve $y = f(x)$ over the x -axis?

23. The other graph operation we need for this problem is *shift*. Given some function $f(x)$, the operation $x \rightarrow x + 1$ shifts the curve $y = f(x)$ one step to the left. So, for example, since we know what $y = x^2$ looks like, it's easy to plot $y = (x + 1)^2$. Just shift everything to the left by 1. No further thinking is necessary.

Plot these curves by doing a left shift operation on curves that you already know. Each one should take only a few seconds of thinking.

- (a) $y = (x + 1)^3$. Start with $y = x^3$.
- (b) $y = x + 2$. Start with $y = x$.
- (c) $y = -(x + 2)$. Start with $y = -x$.
- (d) $y = 1/(x + 1)$.
- (e) $y = 2^{x+1}$.
- (f) $y = (x + 3/2)^2$.

24. Likewise the transformation $x \rightarrow x - 1$ will shift the curve $y = f(x)$ one step *to the right*. Now you can plot $y = (x - 1)^2$ instantly, without thinking for more than a second. Just take the curve for $y = x^2$ and shift everything to the right by 1.

Plot these curves by doing right shift operations on curves that you already know.

- (a) $y = (x - 1)^3$.
- (b) $y = x - 5$.
- (c) $y = -(x - 1)$. Start with $y = -x$.
- (d) $y = 1/(x - 2)$.
- (e) $y = 2^{x-1}$.
- (f) $y = (x - 3/2)^2$.

25. Plot $y = F(x + 1)$ and $y = F(x - 1)$. Start with the curve $y = F(x)$ which you already did. Use your imagination to do the shifts. Don't do any thinking beyond that.

26. Can you think of operations that shift a curve up and down, rather than left and right?

27. Reflect and shift operations can be combined. You can start with a curve, do a shift, and then do a reflect. For example, starting with $y = f(x)$, we can instantly plot $y = f(-x + 1)$ by first doing a left shift $x \rightarrow x + 1$ and then a reflection $x \rightarrow -x$. Notice that it's not the same if we do the operations the other way around! Starting with the curve $y = f(x)$, if we first do a reflect $x \rightarrow -x$ and then a left shift $x \rightarrow x + 1$, we get $f(-(x + 1))$ or $f(-x - 1)$, which is not the same.

For the given curve, do the operations in order and plot the result. Don't calculate anything. Use your imagination only.

- (a) $y = x^3$, shift $x \rightarrow x - 1$, then reflect $x \rightarrow -x$.
- (b) $y = x^3$, $x \rightarrow -x$, then $x \rightarrow x - 1$.
- (c) $y = 2^x$, $x \rightarrow x + 1$, then $x \rightarrow -x$.
- (d) $y = 2^x$, $x \rightarrow -x$, $x \rightarrow x + 1$.
- (e) $y = x^2$, $x \rightarrow x - 2$, $x \rightarrow -x$.
- (f) $y = x^2$, $x \rightarrow -x$, $x \rightarrow x - 2$.

Explain what is going on in the last two.

28. Plot the given function by doing two operations on a simpler function that you already know how to sketch.

- (a) $y = (-(x - 2))^3$. Start with $y = x^3$. First do $x \rightarrow -x$, then do $x \rightarrow x - 2$.
- (b) $y = (1 - x)^3$. Write as $y = (-(x - 1))^3$ and do similar to above.
- (c) $y = -(x - 1)$. Start with $y = x$.
- (d) $y = 1/(-x + 2)$. Start with $y = 1/x$.
- (e) $y = 2^{-(x-1)}$. Start with $y = 2^x$.

29. Draw the x - y axes. Plot the curve $y = F(x)$ again. Now, on top of that, plot $y = F(1 - x)$ by doing operations. $1 - x$ is the same as $-x + 1$, so it is clear that you have to do a left shift by 1 first, followed by a reflection (not the other way around!) Or, if you like, $1 - x = -(x - 1)$, so you can do a reflection followed by a right shift by 1. If you do it right, both curves will go through the point $(\frac{1}{2}, \frac{1}{2})$.

30. Look at the curves $y = F(x)$ and $y = F(1 - x)$. Notice that when $F(x)$ is close to 1, $F(1 - x)$ is close to 0. And when $F(1 - x)$ is close to 1, $F(x)$ is close to 0. Also, when $F(x)$ is $1/2$, $F(1 - x)$ is also $1/2$. Thinking about all this, we can make take a guess, based on the

shape of these curves, that $F(x) + F(1 - x)$ is always 1. If this guess is true, then it should be possible to prove that

$$F(x) + F(1 - x) = 1.$$

Go ahead and prove this by algebra! This result is the key that is needed to solve the original competition problem.

31. Consider this sum:

$$1 + 2 + \cdots + 50 + 51 + \cdots + 99 + 100$$

You want to find the result. You don't have to sum the terms in the order that they are given. You can rearrange them and sum them in some other way. This is another key idea in competition problems.

Take the first term, 1, and the last term, 100. Add them. You get 101. Now take the second term, and the second-last term, and add them: $2 + 99 = 101$. Continuing this way we see there are 50 pairs of numbers that can be added to form 101. Thus the sum is $50 \times 101 = 5050$.

Use the same idea to find the following sums. Note that when there is an odd number of terms, there will be a term in the middle that is left over, and cannot be paired with another term.

- (a) $1 + 2 + \cdots + 49 + 50$.
- (b) $1 + 2 + \cdots + 50 + \cdots + 100 + 101$.
- (c) $25 + 26 + \cdots + 80$;
- (d) $30 + 31 + \cdots + 70$;
- (e) $1 + 3 + 5 + \cdots + 95 + 97 + 99$.

32. And now you are ready to solve the competition problem! Use the idea of pairing first and last terms, second with second last term, etc., together with the relationship $F(x) + F(1 - x) = 1$. The rest is easy!

33. I know what you're thinking. *Why did I have to go through all that plotting stuff when I could have just guessed that this is one of those problems where I have to add terms in some peculiar order, and go from there?* Good question. The reason is that the plotting exercises give depth to your understanding of this problem and others like it. And, most of all, the way it gels together is beautiful.

34. So then, do it by guessing that the leftmost terms added to the rightmost terms give some special value. Return to (3), take the leftmost term and add it with the rightmost:

$$\frac{2}{4^{\frac{1}{2001}} + 2} + \frac{2}{4^{\frac{2000}{2001}} + 2}.$$

Do the algebra and make sure you get 1. Now try the *second* leftmost term, and the *second* rightmost:

$$\frac{2}{4^{\frac{2}{2001}} + 2} + \frac{2}{4^{\frac{1999}{2001}} + 2}$$

and verify that this is 1. So you feel justified in guessing that these partial sums are all 1, which then leads you the idea of proving that it must always be so by showing $F(x) + F(1 - x) = 1$ and so on.

35. Evaluate the sum

$$F\left(\frac{641}{6201}\right) + F\left(\frac{642}{6201}\right) + \cdots + F\left(\frac{5559}{6201}\right) + F\left(\frac{5560}{6201}\right).$$

36. Let's write Racket programs that compute the sum (3) by brute force. We will use named-let and recursion, which is an elegant technique: `let loop...` creates a function named `loop` with variables `k` and `result` initialized to 1 and 0. Then `loop` is called again and again, incrementing `k` and accumulating the sum into `result`. Finally `loop` returns `result` when `k` goes above 2000.

```
#lang racket

(define (f x)
  (/ 2
     (+ (expt 4 x) 2)))

;; Using named-let and recursion.
(define (do-sum)
  (let loop ((k 1) (result 0))
    (if (> k 2000)
        result
        (loop (+ k 1)
              (+ result (f (/ k 2001)))))))
```

Test it in DrRacket:

```
> (do-sum)
999.9999999999984
```

Almost 1000. Close enough. Unless we use only integers and rational numbers (here we are using fraction exponents, which yield real numbers), machine calculations are not going to be perfect.

It's interesting to compare this recursive style to other styles of expressing the same computation. Here is the same thing but using a for-loop and accumulator. The accumulator `result` is initialized to 0, then `for/fold` iterates through the range 1 to 2000, the steps are 'folded' together into `result` according to the expression in the body of the for-loop.

```
;; Using for-loop with accumulator.
(define (do-sum-2)
  (for/fold ((result 0))
            ((k (in-range 1 2001)))
    (+ result (f (/ k 2001)))))
```

Now, using lists and higher-order functions (`map`, `foldl` etc.) First, `range` creates a big list of integers from 1 to 2000. These integers are turned into fractions by mapping. Then the function $F(x)$ is mapped on to each fraction. Finally the whole thing is folded together by addition, with initial value 0.

```
;; Using lists and higher-order functions.
(define (do-sum-3)
  (foldl + 0 (map f (map (lambda (x)
                          (/ x 2001))
                        (range 1 2001))))))
```

37. Try `do-sum-2` and `do-sum-3` in DrRacket. Make sure they give correct answers.

38. Write a Racket program to verify your answer for 35.

39. Challenge. Consider the series

$$3 + 4 \left(\frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \frac{1}{8 \times 9 \times 10} + \cdots \right).$$

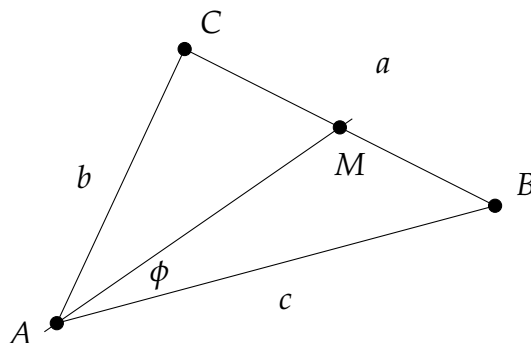
Write Racket functions that compute N terms of this series. Do it in three different styles: by named-let and recursion, by `for/fold` and then by higher-order functions. You should start off the `result` with real initial value 0.0 rather than integer 0, to force Racket to compute with real numbers rather than fractions (which become huge as N becomes large). Compute this sum for $N = 5$, $N = 10$, $N = 100$ and so on. What do you think this series converges to as $N \rightarrow \infty$? A standard trick for handling the alternating $+$ and $-$ signs is to use -1 raised to an appropriate power.

Problem 3. Given angles A , B and C of a triangle, find $\phi = \angle BAM$ where M is the midpoint of side BC . From a Russian problem book.

40. Notice that we are only given the angles of the triangle and nothing else. We do not know the lengths of the sides. Therefore our solution must express ϕ in terms of these given angles. We may involve the sides or other quantities while we are working toward the solution, but the final answer must be in terms of A , B and C only. It's critical to read between the lines of a problems this way.

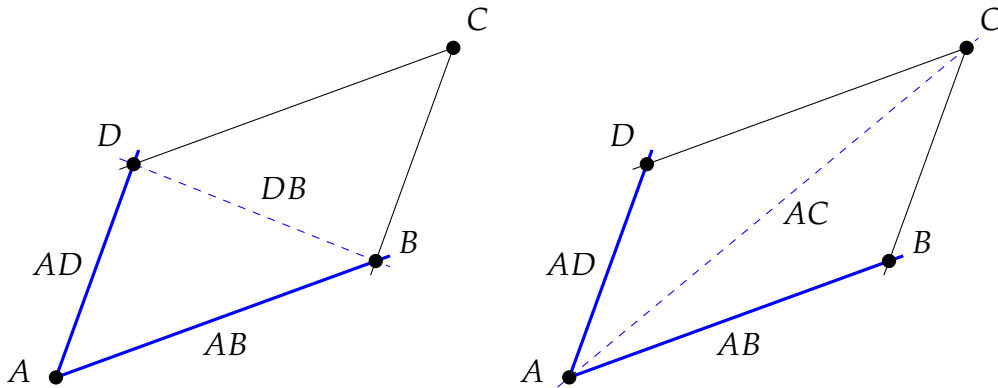
41. The problem can be solved by applying the cosine law, the sine law and some ideas about parallelograms and similar triangles. Reflect on this during the study of this problem and others like it in this book. Parallelogram theorems and similar triangles appear universally in all that we do concerning geometry. The cosine and sine laws are themselves expressions of ideas about parallelograms and similar triangles, as we will see.

42. Here is a figure corresponding to the problem, using standard conventions for labeling the parts of a triangle: side opposite to angle A is called a , etc.



We have drawn the median line AM and the angle ϕ that is to be found. Draw this figure carefully yourself and construct the midpoint M and the median line AM by ruler and compass.

43. While most high-school textbooks present the cosine law as a tool for solving triangles, we will take a different (and much better) approach. The cosine laws are statements about parallelograms. We can say *laws* rather than *law* because there are two of them. It is best to learn both. Consider a parallelogram $ABCD$. The cosine laws tell you the lengths of the diagonals in terms of some data about sides and angles.



There are two diagonals: DB and AC , and so there are two forms of the cosine law telling you the lengths of these diagonals. Given angle A and the sides AD and AB adjacent to it, we have for diagonal DB :

$$DB^2 = AB^2 + AD^2 - 2 AB AD \cos A$$

and we also have (this is the part nobody talks about) for diagonal AC :

$$AC^2 = AB^2 + AD^2 + 2 AB AD \cos A$$

44. Vectors provide a beautiful demonstration of the cosine laws in a unified manner which you are unlikely to ever forget. Expressing lengths in terms of dot product of vectors, we have, for the first figure of **43**:

$$\begin{aligned} DB^2 &= |\vec{AB} - \vec{AD}|^2 = (\vec{AB} - \vec{AD}) \cdot (\vec{AB} - \vec{AD}) \\ &= \vec{AB} \cdot \vec{AB} + \vec{AD} \cdot \vec{AD} - 2\vec{AB} \cdot \vec{AD} \\ &= AB^2 + AD^2 - 2 AB AD \cos A. \end{aligned}$$

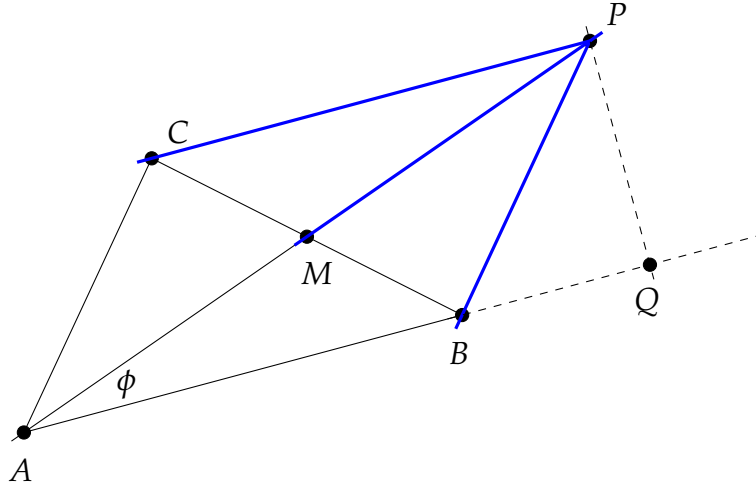
For the second figure,

$$\begin{aligned} AC^2 &= |\vec{AB} + \vec{AD}|^2 = (\vec{AB} + \vec{AD}) \cdot (\vec{AB} + \vec{AD}) \\ &= \vec{AB} \cdot \vec{AB} + \vec{AD} \cdot \vec{AD} + 2\vec{AB} \cdot \vec{AD} \\ &= AB^2 + AD^2 + 2 AB AD \cos A. \end{aligned}$$

And there you have it! Cosine laws, rolled into one beautiful show, if you happen to know just a little bit about vectors. You see? It's not really about triangles. It's about parallelograms.

45. Using the cosine laws, prove that the sum of the squares of the diagonals of a parallelogram is twice the sum of the squares of the sides.

46. A parallelogram can be constructed from triangle ABC in 42 by adding some extra segments (in blue). We have added CP parallel to AB and BP parallel to AC , thus completing parallelogram $ABPC$ out of the original triangle. Why do this? So that we can apply cosine laws!



Also notice that we have finished the diagonal AP by drawing MP , extended AB to AQ and dropped a perpendicular PQ on it from P . We can find $\cos \phi$ by trigonometry:

$$\cos \phi = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB + BQ}{AP}. \quad (4)$$

Since $ABPC$ is a parallelogram, $\angle PBQ$ is the same as $\angle A$ and $AC = BP$. The segment BQ can be found by trigonometry:

$$BQ = AC \cos A.$$

Plugging this into (4) and using the cosine law to find diagonal AP , we have:

$$\cos \phi = \frac{AB + AC \cos A}{\sqrt{AB^2 + AC^2 + 2 AB AC \cos A}}.$$

Or, using the convenient labels a, b, c for the side lengths, as in 42:

$$\cos \phi = \frac{c + b \cos A}{\sqrt{c^2 + b^2 + 2bc \cos A}}. \quad (5)$$

Let's pause here. This isn't finished because we are not given any information about the sides of triangle ABC and therefore b and c should not appear in the final formula for $\cos \phi$. Nevertheless this is progress and we are almost there.

Problem 4. Consider a triangle where the medians to sides a and b are perpendicular. Construct an example of this triangle by ruler and compass. Then construct an example where such a triangle *does not exist*! Finally, determine the conditions on a and b for such a triangle to exist. Adapted from a Russian collection. One of my favorite problems.

47. This problem contains such a wealth of wonderful ideas that every student on Earth should work through it at least once!