Arithmetic and Combinatorics Part 2

Training problems for M1 2018 term 2

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1. I have have 5 objects. I want to choose 3 of them. Draw all the different ways that this can be done. How many are there?

2. I have have 5 objects. I want to choose 2 of them. Draw all the different ways that this can be done. How many ways are there?

3. I have four fruits: apple, banana, strawberry and peach. I want to choose three of them to make a milkshake. Write down all the different ways of doing this. Order doesn't matter.

4. What does $\binom{n}{k}$ mean? Explain it.

5. Figure these out.

(a) $\binom{1}{0}$. (b) $\binom{5}{5}$. (c) $\binom{5}{0}$. (d) $\binom{n}{0}$. (e) $\binom{n}{n}$.

6. Figure these out.

(a) $\binom{1}{2}$. (b) $\binom{0}{1}$. (c) $\binom{5}{6}$. (d) $\binom{n}{n+1}$. (e) $\binom{2}{-1}$.

7. Draw Pascal's triangle, circle these elements and label them:

$$\binom{5}{2} \quad \binom{3}{3} \quad \binom{1}{0} \quad \binom{7}{6} \quad \binom{4}{2} \quad \binom{2}{2} \quad \binom{6}{0} \quad \binom{0}{0}.$$

8. What is the sum of row $n = 5$ of Pascal's triangle?

9. What is the sum of row $n = 12$ of Pascal's triangle? Do it *without* using the numbers of row $n = 12$.

10. Draw Pascal's triangle up to row $n = 8$ and circle the central Pascal numbers $\binom{2m}{m}$.

11. Figure out $\binom{8}{4}$ by summing the squares of the elements in row $n = 4$.

12. Figure out $\binom{14}{7}$. by summing the squares of the elements in row $n = 7$.

13. What is a set? Explain it. What are the rules for sets?

14. What is a subset? Explain it.

15. Write down all the subsets of $\{a, b, c\}$. How many are there?

16. Write down all the subsets of $\{1, 2, 3, 4\}$. How many are there?

17. Let $S = \{a, b, c, d, e, f, g\}$. How many size-3 subsets does S have?

18. Let $|S| = 10$ and let $A \subseteq S$ with $|A| = 5$. How many such A are there?

19. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many subsets does S have?

20. Let $|S| = 14$ and $A \subseteq S$. How many such A are there?

21. Let $S = \{a, b, c, d, e, f, g, h, i, j\}$. Let $A \subseteq S$ such that $|A|$ is an even number. How many such A are there? Use Pascal numbers and Pascal's triangle.

22. Let $S = \{a, b, c, d, e, f, g, h, i, j\}$. Let A be an odd-sized subset of S. How many such A are there? Use Pascal numbers and Pascal's triangle to solve this.

23. This is the menu at the Italian restaurant.

Main dishes	*Side dishes*	*Desserts*
Fetuccine alfredo	Porcini mushroom bruschetta	Hazelnut tartufo
Pepperoni pizza	Grilled polenta	Cannoli
Three-cheese lasagna	Spinach ricotta gnocci	White chocolate panna cotta
	Radicchio with lemon	Sfogliatelle
	Stuffed artichokes	

I want to get one main dish, one side dish, *and* one dessert. How many ways can I do that? Write the definitions for what you are doing. Be clear. Write the principle. Do the computation.

24. At the Italian restaurant in problem **23**, I want to get either one main dish *or* one side dish *or* one dessert. How many ways can I do that? Show definitions, principle, calculation. Do a proper job, don't just write the answer.

25. At the Italian restaurant in **23**, I want to get either only a main dish or a side dish and a dessert. How many ways can I do that? Show definitions, principle, calculation.

26. At the Italian restaurant in **23**, I want to get either both a main dish and a dessert or both a side dish and a dessert. How many ways can I do that? Definitions. Principle. Calculation.

27. The Boring Book Library has nothing but the most boring books on the dullest topics.

Topic	*Number of books*
Dishwashing	7
K-Pop	6
Mops and brooms	5
British tophats	4

How many ways can I choose 2 diswashing books, 2 K-Pop books, 2 books about mops and brooms, and 2 books about British tophats? Show your definitions, combinatorics principle, and calculation.

28. From the library in problem **27**, I want to choose either 4 dishwashing books or 3 K-pop books *or* 2 books about mops or 1 tophat book. How many ways can I do this? Definition—principle—calculation.

29. From the Boring Book Library, choose either three dishwashing books and two mops books, *or* two K-Pop books and three British tophat books. How many ways can you do that? Show definition—principle—calculation.

30. From the Boring Book Library, choose 3 Dishwashing books or 2 British tophat books *and* 4 K-Pop books and 3 books on mops and brooms. Definitions. Principle. Calculation.

31. We have 7 girls and 5 boys. How many ways can we make a team by choosing 3 girls and 3 boys? Definitions, principle, calculation.

32. We have 7 girls and 5 boys. We want a small team of only 3 people: either all girls or all boys. How many ways can I do this? Definitions, principle, calculation.

33. How many 3-digit numbers are there? Show definitions, principle, calculation.

34. How many 3-digit numbers can you make using only *even* digits? Show your definitions, combinatorics principle, and your calculations.

35. How many 3-digit numbers can you make using only *odd* digits? Definitions. Principle. Calculation. Be clear. Explain what you are doing. Don't just write an answer.

36. What is a permutation? Explain it.

37. Write down all different permutations of the letters EFG. How many are there?

38. Write down all the different permutations of the digits 1234. How many are there.

39. Prove that the number of different ways to arrange k objects in order is $k!$.

40. I have 6 books and I want to arrange them in order on a bookself. How many different ways can I do it?

41. I have 6 books and I want to choose 3 to arrange on my bookshelf in order. How many ways can I do this?

42. I have 8 students. I want to choose 3 of them and give them prizes: 1st, 2nd and 3rd place. Does order matter? How many ways can I do this? Do it in two steps and show definitions, principle, calculation.

43. We have n students. We want to choose k of them and arrange them in order. How many ways can we do this? Show your definitions, what principle you use, and the calculation of the final answer.

44. We have 5 girls and 5 boys in our class. I want to choose 2 girls and one boy and give them prizes: 1st place, 2nd place, 3rd place. Does order matter? How many ways can I do this? Definitions (be clear). Principle (what combinatorics principle are you using?) Calculation (get the final answer).

45. We have 5 girls and 5 boys. I want to choose 3 girls and 3 boys to make a team of six. Then I want to choose three in the team to be president, secretary and messenger. Is order important? How many ways can I do this? Show definitions, principle, calculation.

46. We have 6 girls and 5 boys. I want to give three prizes to the girls (1st, 2nd, 3rd) *and* three prizes to the boys (1st, 2nd, 3rd). How many ways can I do this? Definitions. Principle. Calculation.

47. We have 6 consonants *mnpqrs*, 5 vowels *aeiou* and 5 digits 12345. I want to make passwords by choosing two of each, a total of 6 symbols. Of course order matters when you make passwords. For example: 2*aqur*5 and 5*ruqa*2 are two different passwords. How many such passwords can I make? Definitions. Principle. Calculation.

48. (A) I have n objects. I select k of them in some special order. This one first, then that one, then another one, and so on.

(B) I have n objects. I select k of them without order, but then I arrange them in some special order later. I put one first, then aonther one second, and so on.

Is there a difference between (A) and (B)? Think about it.

49. How many ways can you make a combination of n objects taken k at a time?

50. How many ways can you make a permutation of n objects taken k at a time?

51. When do you have the smallest non-zero number of combinations of n objects taken k at a time?

52. When do you have the largest number of permutations of n objects taken k at a time?

53. Prove that the number of permutations of n objects taken k at a time is

$$n(n-1)\dots(n-k+1).$$

Use drawings of balls with numbers on them and boxes that are in order. Make sure you clearly explain why the last factor is $(n-k+1)$.

54. Prove that

$$\frac{n!}{(n-k)!} = n(n-1)\dots(n-k+1).$$

55. We have 9 people: Anne, Jim, Bob, Rick, William, Kenny, Pete, Harry and Joe. We want to hide them in four rooms: bedroom, kitchen, living room, and bathroom. Three go in the bedroom, two go in the kitchen, two go in the living room and two go in the bathroom.

(a) Draw an example of a correct arrangement of people in the rooms.
(b) Think about the order of the three people that you put in the bedroom. Is the order important?
(c) Calculate the number of ways this can happen. Always use definitions and principles.

56. Ten people go to a fast food shop. Five of them get hamburgers. Three get chickenburgers. One gets a fishburger. Does order matter between three people who get the same kind of burger? Does order matter between three people who each get different kinds of burgers? How many ways can this happen? Always use definitions and principles when you answer a question like this. It's also good if you make drawings.