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COMP-551 : Applied Machine Learning	Given on: Sept 05, 08 am
Assignment #0	Due on : Sept 15, 10 pm

- The goal of this assignment is a self-assessment on fundamental problems from Probability Theory, Linear Algebra and Real Analysis. This assignment will NOT be graded.
- This is an individual assignment. Collaborations and discussions with others regarding the problems and solutions are strictly prohibited.
- You have to submit the `pdf` copy of the assignment on gradescope before the deadline. If you handwrite your solutions, you need to scan the pages, merge them to a single `pdf` file and submit.
- **Honour Code**: Please write out the following, digitally sign it or type your name under it and return it along with your assignment.
  By enrolling for COMP 551 Applied Machine Learning course, I agree that all the work submitted will be mine and original, and will not be plagiarized. Unless otherwise specifically stated by the instructor or TAs, I will not collaborate with anyone on my assignments or tests. I understand that any violation of this honor code will be strictly dealt with.

Probability Theory

1. (a) The Smiths have two children. At least one of them is a boy. What is the probability that both children are boys?

   (b) Suppose that 5% of men and 0.25% of women are color-blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male? Assume males and females to be in equal numbers.

2. Prasanna, a TA for COMP 551, will give the students a minimum mark of 1. The highest mark he would give, depending on his mood, will be based on this distribution function

$$F_Y(y) = P(Y \leq y) = 1 - \frac{1}{y^2}, 1 \leq y < \infty$$

   (a) Verify that $F_Y(y)$ is a cdf.

   (b) Find $f_Y(y)$, the pdf of Y.

   (c) Ali, however, gives a minimum mark of 0, and uses a different marking scale that is 1/10th of Prasanna's, and the highest mark becomes $Z = 10(Y - 1)$. Find $F_Z(z)$ and $f_Z(z)$.

3. Let X have the pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, 0 < x < \infty, \beta > 0$$

(a) Verify that f(x) is a pdf.

(b) Find E(X) and Var(X).

4. The random pair (X, Y) has the distribution:

	X=1	X=2	X=3
Y = 2	1/12	1/6	1/12
Y = 3	1/6	0	1/6
Y = 4	0	1/3	0

(a) Show that X and Y are dependent.

(b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

Linear Algebra

5. Let $\mathcal{Y} = \{y_1, y_2, y_3\}$ be a basis for R^3 where $y_1 = (1, 1, -1)$, $y_2 = (4, 1, 1)$ and $y_3 = (1, -1, 2)$. Let $\mathcal{W} = \{w_1, w_2\}$, $w_1 = (1, 1)$ and $w_2 = (2, 4)$ be a basis in R^2. Find the matrix w.r.t. $\mathcal{Y}$ and $\mathcal{W}$ for the linear transformation T defined by $Ty_1 = (4, 5)$, $Ty_2 = (1, 3)$ and $Ty_3 = (7, 1)$. Note that all coordinates are w.r.t. the standard basis.

6. Let E_1 be a set of eigenvectors with eigenvalue λ_1 and E_2 be a set of eigenvectors with eigenvalue $\lambda_2 \neq \lambda_1$.
Prove or disprove the following statement -
There exists a vector in E_2 which can be expressed as a linear combination of vectors in E_1.

7. You are given that the eigenvalues of a matrix A are 3, 2 and 2. What can you tell about the following statements?

   (a) A is invertible

   (b) A is diagonalizable

   Your answer can be "yes","no" or "depends". For each one, give arguments and/or examples to support your answer.

8. For a matrix A of size n × n you are told that all its n eigen vectors are independent. Let S denote the matrix whose columns are the n eigen vectors of A.

   (a) Is A invertible?

   (b) Is A diagonalizable?

   (c) If S invertible?

   (d) Is S diagonalizable?

   Your answer can be "yes","no" or "depends". For each one, give arguments and/or examples to support your answer.

Real Analysis

9. Compute the first order differentiation for the following functions. Show the steps and clearly mention the differentiation rule used to arrive at that step.
   (a) $\mathrm{y} = \mathrm{x}^4(\mathrm{sinx}^3 - \mathrm{cosx}^2)$. Find $\frac{dy}{dx}$.
   (b) $\mathrm{y} = 12\mathrm{x}^4 - 5\mathrm{x}^2 + 15$. Find $\frac{dy}{dx}$.
   (c) $\mathrm{y} = \log\ \mathrm{p},\ \ \mathrm{p} = \mathrm{x}^2$. Find $\frac{\partial\ y}{\partial x}$.

10. Compute the minima and maxima of the following function.
    (a) $f(x) = x^3 - 6x^2 + 9x + 15$
    (b) Now, use first and second order derivatives to identify which of the critical points are minima and maxima without directly evaluating $f(x)$, the value of the function at the critical points.

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Comp 551 : Assignment #0

Probability theory

1. (a) A : At least one is boy

B : Both are boys.

$$P(A) = 1 - \frac{1}{2}\cdot\frac{1}{2} = 1 - \frac{1}{4} = \frac{3}{4}.$$

$$P(A \cap B) = P(B) = \frac{1}{4}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \boxed{\frac{1}{3}} \quad \text{ans.}$$

(b) Assume there are x men and x women.

color-blind men : $5\% \cdot x$

color-blind women : $0.25\% \cdot x$

A : a person is color blind.

B : the person is a male.

$$P(A) = \frac{5\% \cdot x + 0.25\% \cdot x}{2x} = 0.02625$$

$$P(A \cap B) = \frac{5\% \cdot x}{2x} = 0.025$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.025}{0.02625} = \boxed{95.24\%}$$

2.

$$F_Y(y) = P(Y < y) = 1 - \frac{1}{y^2}, \quad 1 \le y < \infty$$

(a) ① $\lim_{y \to \infty} F_Y(y) = 1 - \frac{1}{\infty} = 1$

② $\lim_{y \to 1} F_Y(y) = 1 - \frac{1}{1} = 0$

Also,

③ $f = \frac{d F_Y(y)}{dy} = 2y^{-3}$

So, $2y^{-3}$ is greater than 0 when $1 \le y < \infty$

$F_Y(y)$ is non decreasing in $y \in [1, \infty)$

And in $1 \le y < \infty$,

there is no discontinuity, thus

④ $F_Y(y)$ is right-continuous.

Therefore $F_Y(y)$ is a cdf.

(b) $f_Y(y) = \dfrac{d F_Y(y)}{dy} = \dfrac{2}{y^3}$, $1 \le y < \infty$

(c) $$F_Z(z) = P(Z < z)$$
$$= P(10(Y-1) < z)$$
$$= P\left(Y < \frac{z}{10} + 1\right)$$
$$= F_Y\left(\frac{z}{10} + 1\right)$$
$$= \boxed{1 - \frac{1}{\left(\frac{z}{10} + 1\right)^2}}$$ ans.

$$f_Z(z) = F'_Z(z) = \frac{d}{dy} F_Y\left(\frac{z}{10} + 1\right)$$

$$= f_Y\left(\frac{z}{10} + 1\right) \cdot \frac{d}{dz}\left(\frac{z}{10} + 1\right)$$

$$= \frac{2}{\left(\frac{z}{10}+1\right)^3} \cdot \frac{1}{10}$$

$$= \boxed{\frac{1}{5\left(\frac{z}{10}+1\right)^3}} \quad \text{ans.}$$

3.

$$f(x) = \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-x^2/\beta^2}, \quad 0 < x < \infty, \quad \beta > 0$$

(a) $$\int_{-\infty}^{\infty} f(x)\, dx$$

$$= \int_0^{\infty} \frac{4}{\beta^3 \sqrt{\pi}} x^2 \cdot e^{-x^2/\beta^2}\, dx$$

$$= \frac{4}{\beta^3 \sqrt{\pi}} \int_0^{\infty} x^2 \cdot e^{-x^2/\beta^2}\, dx$$

$$= \frac{4}{\beta^3 \sqrt{\pi}} \frac{\sqrt{\pi}(\beta^2)^{\frac{3}{2}}}{2 \cdot 2} = \frac{4\sqrt{\pi} \cdot \beta^3}{\beta^3 \cdot \sqrt{\pi} \cdot 4}$$

$$= 1$$

Thus, $f(x)$ is a pdf. ans.

(b) $E(x) = \int x \cdot f(x)\, dx$

$$= \int_0^{\infty} \frac{4}{\beta^3 \sqrt{\pi}} x^3 \cdot e^{-x^2/\beta^2}\, dx.$$

let $u = x^2$. , $du = 2x\, dx$.

$$E(x) = \int_0^{\infty} \frac{4}{\beta^3\sqrt{\pi}} u \cdot x\, e^{-u/\beta^2}\, dx$$

$$= \int_0^{\infty} \frac{2}{\beta^3 \sqrt{\pi}} u \cdot e^{-u/\beta^2}\, du$$

let $w = u$ $\qquad dw = du$

$dv = e^{-u/\beta^2}\, du$ $\qquad v = -\beta^2 \cdot e^{-u/\beta^2}$

$$= \frac{2}{\beta^3\sqrt{\pi}} \left[-\beta^2 u \cdot e^{-u/\beta^2} \Big|_0^{\infty} + \int_0^{\infty} \beta^2 e^{-u/\beta^2}\, du \right]$$

$$= \frac{2}{\beta^3\sqrt{\pi}} \left[0 - 0 + \left(-\beta^4 e^{-u/\beta^2} \Big|_0^{\infty} \right) \right]$$

$$= \frac{2}{\beta^3\sqrt{\pi}} \left[0 - (-\beta^4) \right]$$

$$= \frac{2}{\beta^3 \sqrt{\pi}} \cdot \beta^4 = \frac{2\beta}{\sqrt{\pi}}$$ ans.

$$V(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_0^\infty x^2 \cdot f(x)\, dx$$

$$= \int_0^\infty \frac{4}{\beta^3 \sqrt{\pi}} x^4 \cdot e^{-x^2/\beta^2}\, dx.$$

$$= \frac{4}{\beta^3 \sqrt{\pi}} \int_0^\infty x^4 \cdot e^{-x^2/\beta^2}\, dx$$

$$= \frac{4}{\beta^3 \sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \frac{\sqrt{\pi}}{\left(\frac{1}{\beta^2}\right)^{5/2}}$$

$$= \frac{3}{2\beta^3 \sqrt{\pi}} \cdot \sqrt{\pi} \cdot \beta^5$$

$$= \frac{3\beta^2}{2}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \frac{3\beta^3}{2} - \frac{4\beta^2}{\pi}$$

$$= \frac{3\pi\beta^2 - 8\beta^2}{2\pi} = \frac{(3\pi - 8)\beta^2}{2\pi}$$ ans.

4. (a) $p(X=2) = \frac{1}{6} + \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$

$p(Y=4) = \frac{1}{3}$.

$p(X=2, Y=4) = \frac{1}{3}$.

$p(X=2) \cdot p(Y=4) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

So, $p(X=2, Y=4) \neq p(X=2)p(Y=4)$.

Therefore, X, Y are dependent.

(b) $p(U=1) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$.

$p(U=2) = \frac{1}{6} + \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$

$p(U=3) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} = \frac{1}{4}$.

$p(V=2) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}$

$p(V=3) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$p(V=4) = \frac{1}{3}$.

	$U=1$	$U=2$	$U=3$	
$V=2$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
$V=3$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
$V=4$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

ans.

linear Algebra.

5.

$$Ty_1 = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$a + b - c = 4$$
$$d + e - f = 5$$

$$Ty_2 = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$4a + b + c = 1$$
$$4d + e + f = 3$$

$$Ty_3 = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$a - b + 2c = 7$$
$$d - e + 2f = 1$$

So,

$$\begin{cases} 5a + 2b = 5 \\ 3a + b = 15 \end{cases} \Rightarrow \begin{cases} a = 25 \\ b = -60 \end{cases}$$

$$\Rightarrow \quad c = -39$$

$$\begin{cases} 2d + f = 6 \\ 5d + 3f = 4 \end{cases} \Rightarrow \begin{cases} d = 14 \\ f = -22 \end{cases}$$

$$e = -31$$

Therefore,

$$T = \begin{bmatrix} 25 & -60 & -39 \\ 14 & -31 & -22 \end{bmatrix}$$ ans.

6. let $\vec{v}_1$ be any vector in E_1,

$\vec{v}_2$ be any vector in E_2,

let $0 = a_1\vec{v}_1 + a_2\vec{v}_2$. $\circledast$

if $\vec{v}_1$ can be linear combination of $\vec{v}_2$,

then, there exist $a_1 \neq 0$ and $a_2 \neq 0$.

Otherewise, $a_1 = a_2 = 0$.

let $0 = \lambda_1 a_1 \vec{v}_1 + \lambda_2 a_2 \vec{v}_2$

but also, multiply $\circledast$ by λ_1,

$$0 = \lambda_1 a_1 \vec{v}_1 + \lambda_1 a_2 \vec{v}_2.$$

$$\Rightarrow \quad 0 = (\lambda_1 - \lambda_2) a_2 \vec{v}_2$$

but $\lambda_1 - \lambda_2 \neq 0$, and $\vec{v}_2$ is a eigenvector, can't be 0,

So, a_2 must be 0,

then a_1 must be 0 as well.

This concludes that $\vec{v}_2$ cannot be represented as linear combination of $\vec{v}_1$. $\blacksquare$

7.

(a) The matrix is invertible if and only if it does not has a zero eigenvalue.

Since the three eigenvalues are 3, 1 and 2, the matrix is invertible.

(b) We cannot tell if the matrix is diagnolizable since we don't know if it has n independent eigenvectors.
So, it depends.

8. (a) ~~It~~ depends,

$$BAB^{-1} = D.$$

D is the diagnal matrix
if D has 0 on its diagnal, then A is not invertible.

(b) It is diagonalizable since it has n independent eigenvectors

(c) S is invertible since each column is independent from another.

(d) It is diagonalizable since eigenvectors are independent.

Real Analysis.

9, (a) $y = x^4(\sin x^3 - \cos x^2)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4 \sin^3 x - x^4 \cdot \cos^2 x \right)$$

(product rule and chain rule)

$$= 4x^3 \cdot \sin^3 x + 3x^4 \cdot \sin^2 x \cdot \cos x$$
$$- [4x^3 \cdot \cos^2 x + 2x^4 \cdot \cos x \cdot (-\sin x))]$$

$$= 4x^3 \sin^3 x + 3x^4 \cdot \sin^2 x \cdot \cos x$$
$$- 4x^3 \cos^2 x + 2x^4 \cdot \cos x \cdot \sin x .$$ ans.

(b) $y = 12x^4 - 5x^2 + 15$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(12x^4 - 5x^2 + 15 \right)$$

$$= 48x^3 - 10x$$ ans.

(c) $y = \log p$, $p = x^2$.

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial p} \cdot \frac{\partial p}{\partial x}$$

$$= \frac{1}{p} \cdot 2x$$

$$= \frac{1}{x^2} \cdot 2x = \frac{2}{x}$$ ans.

10.

(a) $f(x) = x^3 - 6x^2 + 9x + 15$

$f'(x) = 3x^2 - 12x + 9 = 0$.

$x^2 - 4x + 3 = 0$

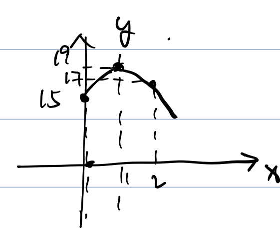
$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$.

$f(1) = 1 - 6 + 9 + 15 = 19$

$f(0) = 15$.

$f(2) = 8 - 24 + 18 + 15 = 17$.

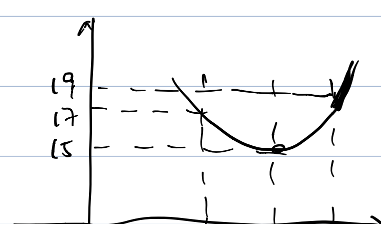
$\Rightarrow$ 

$\Rightarrow$ $f(1)$ is a maximum.

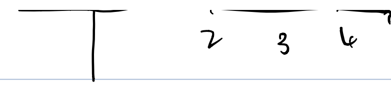
$f(3) = 27 - 54 + 27 + 15 = 15$

$f(2) = 17$

$f(4) = 64 - 96 + 36 + 15 = 19$



$\Rightarrow$ $f(3)$ is a minimum.

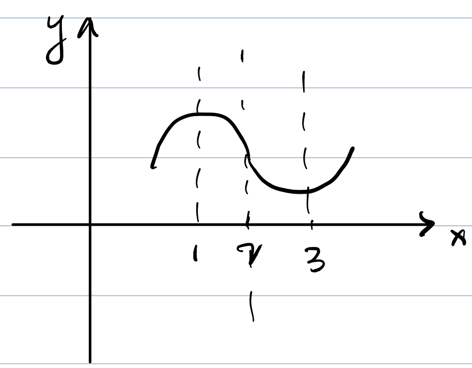


(b) $f''(x) = 6x - 12$

$f''(x) > 0$ when $x > 2$
$f''(x) < 0$ when $x < 2$.

So $f(x)$ is concave up when $x > 2$
and concave down when $x < 2$.

$\Rightarrow$



then $x=1$ is a maximum ans.
and $x = 3$ is a minimum. ans.

Weishi Wang.