Math 240: Discrete Structures I (W18) – Assignment 1

1. Use a truth table to determine if each statement is a tautology, contradiction, or contingency.

[3] (a) $(P \vee Q) \Rightarrow \neg P$

Solution. *Contingency*

P	Q	$P \vee Q$	$\neg P$	$P \vee Q \Rightarrow \neg P$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	T

[4] (b) $(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R) \Rightarrow (P \Leftrightarrow R)$

Solution. *Tautology*

P	Q	R	$P \Leftrightarrow Q$	$Q \Leftrightarrow R$	$(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R)$	$P \Leftrightarrow R$	$(P \Leftrightarrow Q) \wedge (Q \Leftrightarrow R) \Rightarrow (P \Leftrightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	F	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

[4] (c) $\Big[(P \oplus Q) \oplus \neg Q\Big] \Leftrightarrow P$, where $P \oplus Q$ is defined by the following truth table:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Solution. *Contradiction*

P	Q	$P \oplus Q$	$\neg Q$	$(P \oplus Q) \oplus \neg Q$	$[(P \oplus Q) \oplus \neg Q] \Leftrightarrow P$
T	T	F	F	F	F
T	F	T	T	F	F
F	T	T	F	T	F
F	F	F	T	T	F

2. Verify the following statements using only identities (see the list posted on MyCourses). Show all of your work and name the identity or identities used in each step.

[4] (a) $\Big[(P \Rightarrow Q) \wedge P\Big] \Rightarrow Q$ is a tautology

Solution.
$$\begin{aligned}
\Big[(P \Rightarrow Q) \wedge P\Big] \Rightarrow Q &\equiv \Big[(\neg P \vee Q) \wedge P\Big] \Rightarrow Q && \textit{(conditional)}\\
&\equiv \Big[(\neg P \wedge P) \vee (Q \wedge P)\Big] \Rightarrow Q && \textit{(distributive)}\\
&\equiv \Big[\mathbb{F} \vee (Q \wedge P)\Big] \Rightarrow Q && \textit{(complement)}\\
&\equiv (Q \wedge P) \Rightarrow Q && \textit{(identity)}\\
&\equiv \neg(Q \wedge P) \vee Q && \textit{(conditional)}\\
&\equiv (\neg Q \vee \neg P) \vee Q && \textit{(DeMorgan)}\\
&\equiv (\neg Q \vee Q) \vee \neg P && \textit{(associative \& commutative)}\\
&\equiv \mathbb{T} \vee \neg P && \textit{(complement)}\\
&\equiv \mathbb{T} && \textit{(domination)}
\end{aligned}$$

[4] (b) $\neg(P \wedge Q) \wedge (Q \Rightarrow P) \equiv \neg Q$

Solution.
$$\begin{aligned}
\neg(P \wedge Q) \wedge (Q \Rightarrow P) &\equiv (\neg P \vee \neg Q) \wedge (\neg Q \vee P) && \textit{(conditional)}\\
&\equiv \Big[(\neg P \vee \neg Q) \wedge \neg Q\Big] \vee \Big[(\neg P \vee \neg Q) \wedge P\Big] && \textit{(distributive)}\\
&\equiv \neg Q \vee \Big[(\neg P \wedge P) \vee (\neg Q \wedge P)\Big] && \textit{(absorption)}\\
&\equiv \neg Q \vee \Big[\mathbb{F} \vee (\neg Q \wedge P)\Big] && \textit{(complement)}\\
&\equiv \neg Q \vee (\neg Q \wedge P) && \textit{(identity)}\\
&\equiv \neg Q && \textit{(absorption)}
\end{aligned}$$

[4] (c) $\neg\Big[(P \vee Q) \vee \big[(Q \vee \neg R) \wedge (P \vee R)\big]\Big] \equiv \neg P \wedge \neg Q$

Solution.
$$\begin{aligned}
\neg\Big[(P \vee Q) \vee \big[(Q \vee \neg R) \wedge (P \vee R)\big]\Big] &\equiv \neg\Big[\big[(P \vee Q) \vee (Q \vee \neg R)\big] \wedge \big[(P \vee Q) \vee (P \vee R)\big]\Big] && \textit{(distributive)}\\
&\equiv \neg\Big[\big[P \vee Q \vee Q \vee \neg R\big] \wedge \big[P \vee Q \vee P \vee R\big]\Big] && \textit{(associative)}\\
&\equiv \neg\Big[\big[P \vee Q \vee \neg R\big] \wedge \big[P \vee Q \vee R\big]\Big] && \textit{(idempotent \& associative)}\\
&\equiv \neg\Big[\big[(P \vee Q) \vee \neg R\big] \wedge \big[(P \vee Q) \vee R\big]\Big] && \textit{(associative)}\\
&\equiv \neg\Big[(P \vee Q) \vee (\neg R \wedge R)\Big] && \textit{(distributive)}\\
&\equiv \neg\Big[(P \vee Q) \vee \mathbb{F}\Big] && \textit{(complement)}\\
&\equiv \neg(P \vee Q) && \textit{(identity)}\\
&\equiv \neg P \wedge \neg Q && \textit{(DeMorgan)}
\end{aligned}$$

3. Of the following conditional and biconditional statements, which are true and which are false? Briefly justify your answers.

[2] (a) π is an integer if and only if $\sqrt{e+3}$ is a vowel.

Solution. *This statement is* $F \Leftrightarrow F$ *and is thus true.*

[2] (b) $0 > 1$ whenever $2 + 2 = 4$.

Solution. *This statement is* $(2 + 2 = 4) \Rightarrow (0 > 1)$*, which is* $T \Rightarrow F$*, which is false.*

[3] (c) If (a) implies (b), then pigs cannot fly.

Solution. *Since (a) is true and (b) is false, we have* $(T \Rightarrow F) \Rightarrow T$*, or* $F \Rightarrow T$*, which is true.*

4. Symbolize the following English sentences in logic, using the abbreviation scheme provided.

[2] (a) "Thunder only happens when it's raining."
T : thunder happens; R : it's raining

Solution. *"T only if R" gives* $T \Rightarrow R$

[4] (b) "For every positive integer n there is a prime number that is bigger than n but at most $2n$."
$I(x)$: x is a positive integer; $P(x)$: x is a prime number; $B(x, y)$: x is bigger than y.

Solution. $\forall n\, \exists p\, \big(I(n) \Rightarrow (P(p) \wedge B(p, n) \wedge \neg B(p, 2n))\big)$ *OR* $\forall n\, \big(I(n) \Rightarrow\ \exists p(P(p) \wedge B(p, n) \wedge \neg B(p, 2n))\big)$
NOTE: $\neg B(p, 2n)$ *is not the same as* $B(2n, p)$*. The former says* p *is NOT bigger than* $2n$*, or* $p \leq 2n$*. The latter say* $2n$ *is bigger than* p*, or* $p < 2n$*.*

[4] (c) "Goldbach's Conjecture is true if every even integer greater than 2 can be written as the sum of two primes."
G : Goldbach's Conjecture is true; $E(x)$: x is an even integer; $T(x)$: x is greater than 2; $P(x)$: x is the sum of two primes.

Solution. $\forall x \left[\big(E(x) \wedge T(x)\big) \Rightarrow P(x)\right] \Rightarrow G$