Math 240: Discrete Structures I (W18) – Assignment 1

[12] 1. **Negation of predicates.** For each of the sentences below,

i. write it using symbolic logic notation, using the indicated predicates;
ii. find the negation of your sentence from (i) and simplify as much as possible; and
iii. re-write your answer from (ii) in English (with appropriate mathematical notation where applicable). Your answer should match the one from (ii) exactly; answers which differ will not receive full marks, even if logically equivalent.

(a) Something is rotten in the state of Denmark. [$D(x)$: x is in Denmark; $R(x)$: x is rotten]

Solution. Let x be a thing.

i. $\exists x\, D(x) \wedge R(x)$
ii. $\neg[\exists x\, D(x) \wedge R(x)] \equiv \forall x\, \neg D(x) \vee \neg R(x)$
iii. Everything is either not rotten or not in Denmark.

NOTE: The natural negation of the statement is "Nothing is rotten in the state of Denmark". Symbolized, this might written $\forall x D(x) \Rightarrow \neg R(x)$ ("For every thing in Denmark, that thing is not rotten."), or even $\forall x R(x) \Rightarrow \neg D(x)$ ("For every thing that is rotten, that thing is not in Denmark."). These are *equivalent* to the statement given in (iii) but are *not* how the statement is phrased.

(b) For every real number $\varepsilon > 0$ there is a positive real number N such that $x > N$ implies that $|f(x) - L| < \varepsilon$. [$B(x, y)$: $x > y$]

Solution. Let ε, N, x denote real numbers. Note that the symbolizations in (i) are not unique.

i. $\forall \varepsilon\, \exists N\, \forall x\ B(\varepsilon, 0) \wedge B(N, 0) \wedge B(x, N) \Rightarrow B(\varepsilon, |f(x) - L|)$, or
$\forall \varepsilon\, B(\varepsilon, 0)\Big(\exists N\, B(N, 0)\Big[\forall x\big(B(x, N) \Rightarrow B(\varepsilon, |f(x)-L|)\big)\Big]\Big)$, or other things in between.
ii. The first statement given in (i) is easier to negate, since all quantifiers are declared at the start of the sentence.

$$\begin{aligned}
&\neg\Big(\forall \varepsilon\, \exists N\, \forall x\ B(\varepsilon, 0) \wedge B(N, 0) \wedge B(x, N) \Rightarrow B(\varepsilon, |f(x) - L|)\Big) \\
\equiv&\neg\Big(\forall \varepsilon\, \exists N\, \forall x\ \neg\big[B(\varepsilon, 0) \wedge B(N, 0) \wedge B(x, N)\big] \vee B(\varepsilon, |f(x) - L|)\Big) \\
\equiv&\exists \varepsilon\, \forall N\, \exists x\ \big[B(\varepsilon, 0) \wedge B(N, 0) \wedge B(x, N)\big] \wedge \neg B(\varepsilon, |f(x) - L|)
\end{aligned}$$

iii. There exists a positive real number ε such that for all positive real numbers N there exists a real number $x > N$ for which $|f(x) - L| \geq \varepsilon$.

[10] 2. **Rules of inference.**

(a) Verify the transitivity inference rule by using a truth table.

Solution. In the table on the next page, the rows where $P \Rightarrow Q$ and $Q \Rightarrow R$ have been highlighted. Since $P \Rightarrow R$ is true in every one of those rows, the argument is valid.

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$P \Rightarrow R$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

(b) Verify the modus tollens inference rule by showing an appropriate statement is a tautology using only logical identities.

Solution. Show that $\big[P \Rightarrow Q\big] \wedge \neg Q \Rightarrow \neg P$ is a tautology.

$$\begin{aligned}
\big[P \Rightarrow Q\big] \wedge \neg Q \Rightarrow \neg P &\equiv \big[\neg P \vee Q\big] \wedge \neg Q \Rightarrow \neg P \\
&\equiv \big[\neg P \wedge \neg Q\big] \vee \big[Q \wedge \neg Q\big] \Rightarrow \neg P \\
&\equiv \big[\neg P \wedge \neg Q\big] \vee \mathbb{F} \Rightarrow \neg P \\
&\equiv \big[\neg P \wedge \neg Q\big] \Rightarrow \neg P \\
&\equiv \neg\big[\neg P \wedge \neg Q\big] \vee \neg P \\
&\equiv P \vee Q \vee \neg P \\
&\equiv \mathbb{T} \vee Q \\
&\equiv \mathbb{T}
\end{aligned}$$

(c) Use the rules of inference given on the handout to determine if the following argument is valid. Clearly state which rules you are using (you may symbolize if it is helpful).

If I study, then I will pass.
If I do not go to a movie, then I will study.
I did not pass.

Therefore, I went to a movie.

Solution. S: I study; P: I pass; M: I go to a movie. Using this symbolization, the argument becomes:

$$\begin{array}{r}
S \Rightarrow P \\
\neg M \Rightarrow S \\
\neg P \\
\hline
\therefore M
\end{array}$$

By transitivity, $\neg M \Rightarrow S$ and $S \Rightarrow P$ give $\neg M \Rightarrow P$. Then $\neg M \Rightarrow P$ and $\neg P$ give $\neg(\neg M)$ by modus tollens. Finally, double negation gives M, so the argument is valid.

[12] 3. **Set operations.** Let $A = \{n \in \mathbb{N} \,|\, n < 7\}$, $B = \{q \in \mathbb{Q} \,|\, |q-2| < 1\}$, $C = \{r \in \mathbb{R} \,|\, r^3 - r = 0\}$ and $D = \{1, 2, \{1, 2\}\}$. Find each of the following (recall that $\mathcal{P}(X)$ denotes the power set of X). Do not use "…" in your answer; give clear rules for set membership if you need them.

(a) $A \oplus C$
(b) $A \cap D$
(c) $C \cup D$
(d) $\{1, \{2\}\} \cup D$
(e) $B \setminus (A \oplus C)$
(f) $\{\emptyset\} \setminus \mathcal{P}(A)$
(g) $(A \setminus B) \setminus (C \setminus \overline{D})$
(h) $\mathcal{P}(D) \cap D$

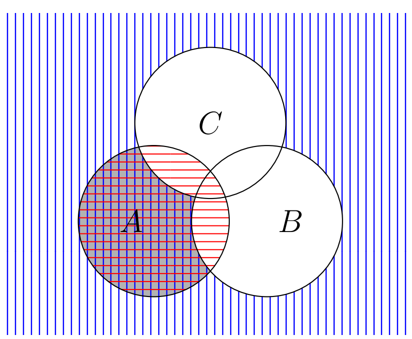
Solution. Note that $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{q \in \mathbb{Q} \,|\, 1 < q < 3\}$, and $C = \{-1, 0, 1\}$.

(a) $\{-1, 0, 2, 3, 4, 5, 6\}$
(b) $\{1, 2\}$
(c) $\{-1, 0, 1, 2, \{1, 2\}\}$
(d) $\{1, 2, \{2\}, \{1, 2\}\}$
(e) $B \setminus \{2\}$ or $\{q \in \mathbb{Q} \,|\, 0 < |q-2| < 1\}$
(f) $\emptyset$
(g) $\{3, 4, 5, 6\}$
(h) $\{1, 2\}$

[6] 4. **Venn diagrams.** Draw a Venn diagram for each of the following. If using LaTeX, you may include graphics for your diagrams (if hand-drawn, make sure they are very neat).

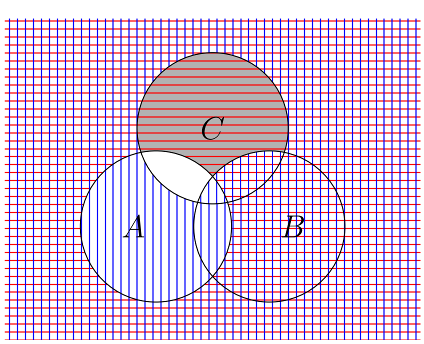
(a) $A \cap \overline{(B \cup C)}$

Solution. red = A, blue = $\overline{(B \cup C)}$, grey = $A \cap \overline{(B \cup C)}$



(b) $\overline{A} \setminus (B \cup \overline{C})$

Solution. red = $\overline{A}$, blue = $B \cup \overline{C}$, grey = $\overline{A} \setminus (B \cup \overline{C})$



(c) $(A \cup B) \oplus (C \setminus B)$

Solution. red = $A \cup B$, blue = $C \setminus B$, grey = $(A \cup B) \oplus (C \setminus B)$