Math 240: Discrete Structures I (W18) – Assignment 3

Solutions must typed or very neatly written and uploaded to MyCourses no later than **6 pm** on **Saturday, February 10, 2018**. Up to 4 bonus marks will be awarded for solutions typeset in LaTeX; both the .tex file and .pdf file must be uploaded.

You may use theorems proven or stated in class, but you must state the theorem you are using.

[15] 1. **Proofs with sets.** Let A, B, C be arbitrary sets. For each of the following statements, either prove it is true (without a Venn diagram) or give a counterexample to show that it is false.

(a) $(A \setminus B) \setminus C = A \setminus (B \cup C)$

Solution. TRUE. We first show $(A \setminus B) \setminus C \subseteq A \setminus (B \cup C)$. If $x \in (A \setminus B) \setminus C$, then $x \in A \setminus B$ and $x \notin C$. This mean that $x \in A$, $x \notin B$ and $x \notin C$, and if $x \notin B$ and $x \notin C$ then $x \notin B \cup C$. Thus $x \in A \setminus (B \cup C)$, and so $(A \setminus B) \setminus C \subseteq A \setminus (B \cup C)$.

We now show $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$. If $y \in A \setminus (B \cup C)$, then $y \in A$ and $y \notin B \cup C$. This means that $y \notin B$ and $y \notin C$, and so $y \in A \setminus B$ and $y \notin C$, or $y \in (A \setminus B) \setminus C$. Thus $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$, and so $(A \setminus B) \setminus C = A \setminus (B \cup C)$

(b) $\Big(A \oplus B = A \oplus C \text{ and } A \cap B = A \cap C\Big) \Rightarrow B = C$

Solution. TRUE. We suppose that $A \oplus B = A \oplus C$ and $A \cap B = A \cap C$, and show that $B \subseteq C$ and $C \subseteq B$. Let $x \in B$. If $x \in A$, then $x \in A \cap B = A \cap C$, and so $x \in C$. If $x \notin A$, then $x \in A \oplus B = A \oplus C$, and so $x \in C$ (since $x \notin A$). In either case, $x \in B \Rightarrow x \in C$, and so $B \subseteq C$.

Let $y \in C$. If $y \in A$, then $y \in A \cap C = A \cap B$, and so $y \in B$. If $x \notin A$, then $y \in A \oplus C = A \oplus B$, and so $y \in B$. In either case, $y \in C \Rightarrow y \in B$, and so $C \subseteq B$. Therefore $B = C$.

NOTE: You were given unnecessary extra information! In fact, one can prove that $A \oplus B = A \oplus C \Rightarrow B = C$. Suppose $x \in B$. If $x \in A$, then $x \notin A \oplus B = A \oplus C$. But, if $x \notin A \oplus C$ and $x \in A$, then $x \in C$. On the other hand, if $x \notin A$, then $x \in A \oplus B = A \oplus C$. But, if $x \in A \oplus C$ and $x \notin A$, then $x \in C$. Thus $B \subseteq C$. The proof that $C \subseteq B$ is symmetric.

(c) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

Solution. FALSE. If $A = \{1, 2\}, B = \{3\}, C = \{4\}, D = \{5\}$, then

$$\begin{aligned}(A \cup B) \times (C \cup D) &= \{1, 2, 3\} \times \{4, 5\} \\ &= \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\} \\ (A \times C) \cup (B \times D) &= (\{(1, 4), (2, 4)\}) \cup (\{(3, 5)\}) \\ &= \{(1, 4), (2, 4), (3, 5)\}.\end{aligned}$$

2. **Relations.**

[11] (a) Determine whether or not each relation is reflexive, symmetric, antisymmetric, and/or transitive. For each property, if the relation has that property, prove it. If it doesn't have that property, give a counterexample. State if the relation is a total order, partial order but not a total order, or neither; justify your answer.

i. $\mathcal{R} = \{(X, Y) \in (\mathcal{P}(A))^2 \,|\, X \cap Y \neq \emptyset\}$ where A is some arbitrary set

Solution.

Reflexive: **No.** $\emptyset \subseteq A$ for any set A, but since $\emptyset \cap \emptyset = \emptyset$, $(\emptyset, \emptyset) \notin \mathcal{R}$.

Symmetric: **Yes.** By associativity, $X \cap Y = Y \cap X$. Thus $(X, Y) \in \mathcal{R} \Rightarrow X \cap Y \neq \emptyset \Rightarrow Y \cap X \neq \emptyset \Rightarrow (Y, X) \in \mathcal{R}$.

Antisymmetric: **No.** If $A = \{1, 2\}$, then $(\{1\}, A), (A, \{1\}) \in \mathcal{R}$, but $\{1\} \neq A$.

Transitive: **No.** If $A = \{1, 2\}$, then $(\{1\}, A), (A, \{2\}) \in \mathcal{R}$, but $(\{1\}, \{2\}) \notin A$.

Since A is not reflexive, antisymmetric, and transitive, A is not a partial order (or a total order).

ii. $\mathcal{R} = \{(a, b) \in \mathbb{N}^2 \,|\, a \text{ divides } b\}$ (a divides b means that there is some integer k such that $b = ka$)

Solution.

Reflexive: **Yes.** For every $a \in \mathbb{N}$, a divides $a \Rightarrow (a, a) \in \mathcal{R}$.

Symmetric: **No.** 2 divides 4 but 4 does not divide 2, so $(2, 4) \in \mathcal{R}$ and $(4, 2) \notin \mathcal{R}$.

Antisymmetric: **Yes.** If $(a, b), (b, a) \in \mathcal{R}$, then there are $k, l \in \mathbb{Z}$ such that $b = ka$ and $a = lb \Rightarrow b = k(lb) \Rightarrow kl = 1 \Rightarrow k = l = 1$ ($k = l = -1$ is ruled out since a and b are positive) $\Rightarrow a = b$.

Transitive: **Yes.** If $(a, b), (b, c) \in \mathcal{R}$, then there are $k, l \in \mathbb{Z}$ such that $b = ka$ and $c = lb \Rightarrow c = l(ka) = (lk)a$. Since $kl \in \mathbb{Z}$, $(a, c) \in \mathcal{R}$.

Since $\mathcal{R}$ is reflexive, antisymmetric, and transitive, it is a partial order. However, it is not a total order because there exist elements of $\mathbb{N}$ which cannot be compared; e.g. $(3, 5), (5, 3) \notin \mathcal{R}$.

[6] (b) For $a, b \in \mathbb{R} \setminus \{0\}$, define $a \sim b$ iff $\frac{a}{b} \in \mathbb{Q}$. Prove that $\sim$ defines an equivalence relation on $\mathbb{R} \setminus \{0\}$. Show that $\left[\frac{9-\sqrt{5}}{1-\sqrt{5}}\right] = \left[\frac{2}{3-6\sqrt{5}}\right]$.

Solution.

Reflexive: **Yes.** For every $a \in \mathbb{R} \setminus \{0\}$, $\frac{a}{a} = 1 \in \mathbb{Q} \Rightarrow (a, a) \in \mathcal{R}$.

Symmetric: **Yes.** $(a, b) \in \mathcal{R} \Rightarrow \frac{a}{b} \in \mathbb{Q} \Rightarrow \frac{b}{a} \in \mathbb{Q} \Rightarrow (b, a) \in \mathcal{R}$.

Transitive: **Yes.** If $(a, b), (b, c) \in \mathcal{R}$, then $\frac{a}{b}, \frac{b}{c}, \in \mathbb{Q}$. Note that we may not assume that $\frac{a}{b}, \frac{b}{c}$ are the rational representations; in other words, a, b, c need not be integers. However, there exist $m, n, p, q \in \mathbb{Z}$ such that $\frac{a}{b} = \frac{m}{n}, \frac{b}{c} = \frac{p}{q}$. Then $\left(\frac{a}{b}\right)\left(\frac{b}{c}\right) = \frac{mp}{nq} \Rightarrow \frac{a}{c} = \frac{mp}{nq}$ since $b \neq 0$. Thus $(a, c) \in \mathcal{R}$.

We proved in class that $[a] = [b] \Leftrightarrow a \sim b$ for any equivalence relation $\sim$. Thus, it suffices to show that $\frac{9-\sqrt{5}}{1-\sqrt{5}} \sim \frac{2}{3-6\sqrt{5}}$:

$$\begin{aligned}
\frac{9-\sqrt{5}}{1-\sqrt{5}} \div \frac{2}{3-6\sqrt{5}} &= \frac{9-\sqrt{5}}{1-\sqrt{5}} \times \frac{3-6\sqrt{5}}{2} \\
&= \frac{(9-\sqrt{5})(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} \times \frac{3-6\sqrt{5}}{2} \\
&= \frac{4+8\sqrt{5}}{-4} \times \frac{3-6\sqrt{5}}{2} \\
&= -\frac{3}{2}(1+2\sqrt{5})(1-2\sqrt{5}) \\
&= -\frac{3}{2}(1-20) = \frac{57}{2} \in \mathbb{Q}
\end{aligned}$$

Since $\frac{9-\sqrt{5}}{1-\sqrt{5}} \div \frac{2}{3-6\sqrt{5}} \in \mathbb{Q}$, $\left(\frac{9-\sqrt{5}}{1-\sqrt{5}}, \frac{2}{3-6\sqrt{5}}\right) \in \mathcal{R}$ as required.

[8] 3. **Proof techniques.** Prove the following statements using the method of your choice (direct proof, proof of the contrapositive, proof by contradiction).

(a) Let $a, b \in \mathbb{R}$. If $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$, then $a \pm b \notin \mathbb{Q}$.

Solution. Proof by contradiction. Suppose that $a \pm b \in \mathbb{Q}$; let $a = \frac{m}{n}$, $a \pm b = \frac{p}{q}$ $(m, n, p, q \in \mathbb{Z})$. Then,

$$\begin{aligned}
\pm b &= (a \pm b) - a \\
&= \frac{p}{q} - \frac{m}{n} \\
&= \frac{pn - mq}{qn} \in \mathbb{Q},
\end{aligned}$$

contradicting $b \notin \mathbb{Q}$. Thus $a \pm b \notin \mathbb{Q}$.

(b) If the average of 4 distinct integers is 10, then at least one of the integers is greater than 11.

Solution. Proof by contradiction. If all 4 numbers are at most 11, then the greatest possible sum is $11 + 10 + 9 + 8 = 38$, whose average is $\frac{38}{4} < 10$, a contradiction. Thus, at least one number must be greater than 11.