Assignment 6

Weishi Wang, ID 260540022

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Problem 1. *(a) Let a be any integer. Prove that $a^n - an + n - 1$ is divisible by $(a-1)^2$ when $n \geq 2$.*

(b) We saw in class that

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

if we look at the following two sums, one might see a pattern emerging:

$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Prove the following, which generalizes the three summations above, for $n \geq 1$ where $m \geq 0$ is some fixed integer:

$\sum_{k=1}^{n} \frac{(k+m)!}{(k-1)!} = \frac{(n+m+1)!}{(n-1)!(m+2)}$

(c) You know about binary representations of integers, and I've asked you to prove things about integers in base 10. Now, show that every positive integer has a factorial representation. That is, prove that for every integer $n \geq 1$, we can write

$n = \sum_{i=1}^{k} c_i i!$

where the integer coefficients c_i satisfy $0 \leq c_i \leq i$ for each i.

Solution. (a) Proof by induction.
The base case is n = 2:
$a^2 - 2a + 2 - 1 = a^2 - 2a + 1 = (a-1)^2$

$(a-1)^2$ is divisible by (a-1).
The base case is satisfied.

Inductive step:
Assume that $a^n - an + n - 1$ is divisible by (a-1).
Then, check n+1:
$a^{n+1} - a(n+1) + n + 1 - 1$
$=a^{n+1} + a^2n - a^2n + (n-1)a - (n-1)a - a(n+1) + n$
$=a^{n+1} - a^2n + (n-1)a + a^2n - (n-1)a - a(n+1) + n$
$=a[a^n - 2n + (n-1)] + a^2n - (n-1)a - a(n+1) + n$
$=a[a^n - 2n + (n-1)] + a^2n - 2na + n$
$=a[a^n - 2n + (n-1)] + n(a-1)^2$
From the assumption, we know that $a[a^n - 2n + (n-1)]$ is divisible by (a-1), also we know that $n(a-1)^2$ is divisible by (a-1).
Therefore, $a[a^n - 2n + (n-1)] + n(a-1)^2$ is divisible by (a-1).
Thus the claim is true for n+1 case, and therefore true for all $n \geq 2$.

(b) Proof by induction.
The base case is n=1.
$\sum_{k=1}^{1} \frac{(k+m)!}{(k-1)!} = \frac{(1+m)!}{0!} = (1+m)!$
Also, when n=1, $\frac{(n+m+1)!}{(n-1)!(m+2)} = \frac{(m+2)!}{(0)!(m+2)} = (m+1)!$
They are equal, which proves the base case (n=1).

Inductive step:
Assume that $\sum_{k=1}^{n} \frac{(k+m)!}{(k-1)!} = \frac{(n+m+1)!}{(n-1)!(m+2)}$ is true.
Now, check n+1 case:
$\sum_{k=1}^{n+1} \frac{(k+m)!}{(k-1)!}$
$= \sum_{k=1}^{n} \frac{(k+m)!}{(k-1)!} + \frac{(n+m+1)!}{n!}$
by the inductive hypothesis of case n, we obtain:
$= \frac{(n+m+1)!}{(n-1)!(m+2)} + \frac{(n+m+1)!}{n!}$
$= \frac{(n+m+1)!(n+m+2)}{n!(m+2)}$
$= \frac{(n+m+2)!}{n!(m+2)}$
Which means that it holds for n+1 case.
Thus, the claim is true by induction.

(c) First, write down some of first few therms to find a pattern:
$0 = 0(0!)$
$1 = 0(0!) + 1(1!)$
$2 = 0(0!) + 0(1!) + 1(2!)$
$3 = 0(0!) + 1(1!) + 1(2!)$

$4 = 0(0!) + 0(1!) + 2(2!)$

$5 = 0(0!) + 1(1!) + 2(2!)$

$6 = 0(0!) + 0(1!) + 0(2!) + 1(3!)$

$7 = 0(0!) + 1(1!) + 0(2!) + 1(3!)$

something that we notice is that if we can increment by 1 each time, we can represent every integer.

Thus, we only need to prove that:

$0(0!) + 0(1!) + 0(2!) + ... + 0(n!) + (1)(n+1)! = 0(0!) + 1(1!) + 2(2!) + 3(3!) + ... + n(n!)$

Which implies a increment of 1.

We can prove this by induction.

Base case: n = 0,

$0(0!) = 0(0!) + 1(1!) - 1$

The base case satisfies the claim.

Inductive step:

Assume the claim is true, then we need to prove that it is true for n+1 case as well.

$0(0!) + 1(1!) + 2(2!) + 3(3!) + ... + n(n!) + (n+1)(n+1)!$

$= (n+1)! - 1 + (n+1)(n+1)!$

$= (n+2)(n+1)! - 1$

$= (n+2)! - 1$

Which proves that n+1 case is true.

Thus, by induction, the claim is true, and therefore, there is a factorial representation for all integers greater or equals to 0.

Problem 2. *(a) Prove that $f_1 - f_2 + f_3 + ... + (-1)^n f_{n+1} = (-1)^n f_n + 1$ for all $n \geq 1$.*

(b) Prove that $f_1 f_2 + f_2 f_3 + f_3 f_4 + ... + f_{2n-1} f_{2n} = f_{2n}^2$ for all $n \geq 1$.

Solution. (a) Proof by induction.

The base case is n = 1:

$f_1 - f_2 = 1 - 1 = 0$

$-f_1 + 1 = -1 + 1 = 0$

So, $f_1 - f_2 = -f_1 + 1$

which means that the claim holds for the base case.

Inductive step:

Assume that $f_1 - f_2 + f_3 + ... + (-1)^n f_{n+1} = (-1)^n f_n + 1$ is true for n.

Check n+1 case:
We can separate this into two cases:

1. If n is odd,
$f_1 - f_2 + f_3 + ... - f_{n+1} + f_{n+2}$
$= -f_n + 1 + f_{n+2}$
$= f_{n+1} + 1$
which is effectively $(-1)^{n+1} f_{n+1} + 1$ when n is odd.

2. If n is even,
$f_1 - f_2 + f_3 + ... + f_{n+1} - f_{n+2}$
$= f_n + 1 - f_{n+2}$
$= -f_{n+1} + 1$
which is again $(-1)^{n+1} f_{n+1} + 1$ when n is even.

Combining 2 cases, we conclude that the original claim is true for n+1 case.
Thus, the claim is true for all $n \geq 1$.

(b) Proof by induction again.
The base case is n=1,
$f_1 f_2 = 1 = f_2^2$
This proves that the claim holds for the base case.

Inductive step:
Assume that $f_1 f_2 + f_2 f_3 + f_3 f_4 + ... + f_{2n-1} f_{2n} = f_{2n}^2$ is true for n.
Check n+1 case:
$f_1 f_2 + f_2 f_3 + f_3 f_4 + ... + f_{2n-1} f_{2n} + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2}$
$= f_{2n}^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2}$
$= f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1} f_{2n+2}$
$= f_{2n} f_{2n+2} + f_{2n+1} f_{2n+2}$
$= (f_{2n} + f_{2n+1}) f_{2n+2}$
$= f_{2n+2} f_{2n+2}$
$= f_{2n+2}^2$

Therefore, the claim holds for n+1 case.
Thus, the claim is true for all $n \geq 1$.

Problem 3. *Recurrence relations. You've won a contest! You're going to win money! Your prize is determined as follows. You are given* \$40, *then asked to sit in a chair. At each minute mark of you being in the chair, your winnings are re-calculated as being 150 % of the amount you held during the previous minute* *but deducted from that is 25 % of the amount you held the minute before that (note that you held \$0 before the contest started). Whoever is holding the contest is no fool; it's not hard to see that there needs to be some cost to you sitting in the chair, or they'll go bankrupt! So, at each minute mark, you're going to lose \$6 for every minute you've been in the chair (after the first minute you'll lose \$6, after the second minute you'll lose an another \$12, after the third minute you'll lose another \$18, and so on). You can leave the chair any time you want, collect your winnings, and walk away.*

(c) Write a new recurrence relation that expresses the amount of money you win if you leave the chair after the m^{th} minute.

(d) Solve this recurrence relation to find an explicit function of m for your winnings after m minutes.

(e) How long should you stay in the chair to maximize your winnings? If you make any claims about the behaviour of the function after a given point, make sure you justify your answer (this can be done using basic calculus or by other means).

Solution. (c) Let a_m represent the money hold on m^{th} minutes.

Then, according to the rule, the following recurrence can be written:

$a_m = (1 + 50\%)a_{m-1} - (25\%)a_{m-2} - 6m$
$= 1.5a_{m-1} - 0.25a_{m-2} - 6m$

and $a_0 = 40, a_1 = 1.5(40) - 0 - 24 = 54$

(d) First, find the homogeneous solution to this recurrence.
Multiply the equation both sides by 4.
$4a_m = 6a_{m-1} - a_{m-2} - 24m$
Ignore the 24m to find the homogeneous solution:
$4a_m = 6a_{m-1} - a_{m-2}$
$4x^2 - 6x + 1 = 0$
$x = \frac{3+\sqrt{5}}{4}$ or $x = \frac{3-\sqrt{5}}{4}$
so the homogeneous solution is:
$a_m = C_1(\frac{3+\sqrt{5}}{4})^m + C_2(\frac{3-\sqrt{5}}{4})^m$

Now, find the particular solution:
Guess $p_m = am + b$
$4(am + b) = 6[a(m - 1)] - [a(m - 2) + b] - 24m$

$(a-24)m+(b-4a)=0$
The coefficients of m and constant must be 0, so:
a = 24 and b =96
So, $p_m = 24m + 96$
The solution is homogeneous + particular:
$a_m = C_1(\frac{3+\sqrt{5}}{4})^m + C_2(\frac{3-\sqrt{5}}{4})^m + (24m + 96)$

We can plug the initial conditions to find the coefficients C_1 and C_2.
$a_0 = C_1 + C_2 + 96 = 40$
$a_1 = C_1(\frac{3+\sqrt{5}}{4}) + C_2(x = \frac{3-\sqrt{5}}{4}) + (120) = 54$

Solve for C_1 and C_2:
$C_2 = \frac{48\sqrt{5}}{5} - 28$ and $C_1 = \frac{-48\sqrt{5}}{5} - 28$

Therefore, the solution to this recurrence is:
$a_m = (\frac{-48\sqrt{5}}{5} - 28)(\frac{3+\sqrt{5}}{4})^m + (\frac{48\sqrt{5}}{5} - 28)(\frac{3-\sqrt{5}}{4})^m + (24m + 96)$

(e) Find the derivative of a_m and make it equals to 0.
So, $\frac{da_m}{dm} = 0$
Solve the equation and the answer is approximately 2. Thus the value of m to maximize the price is m = 2.
To verify this, we can calculate the answer for m = 2.
$a_2 = 1.5(54) - 0.25(40) - 12 = 59$
Now calculate m = 3.
$a_2 = 1.5(59) - 0.25(54) - 18 = 57$
Which is smaller than a_2
Therefore, the value is maximized when m = 2.