The problem is $\frac{1}{2}\|\underbrace{x}_{\in R^{M\times N\times 3}}-\underbrace{y}_{}\|_2^2+\lambda\|\underbrace{D}_{D:\ R^{M\times N\times 3}\to R^{M\times N\times 3\times 2}}x\|_{S_1,1}$

Therefore, there is a matrix $A\in R^{3\times 2}$ associated with each pixel. We show below that the SVD of $A\in R^{3\times 2}$ has a closed-form solution.

Let $\underset{3\times 2}{A}=\underset{3\times 3}{U}\underset{3\times 2}{\Sigma}\underset{2\times 2}{V^T}$, then V is orthogonal so can be parameterised by $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

We also have $A^TA=V\Sigma^2V^T$

Denote A by $\begin{bmatrix}| & |\\ a_1 & a_2\\ | & |\end{bmatrix}$ then $A^TA=\begin{bmatrix}\langle a_1,a_1\rangle & \langle a_1,a_2\rangle\\ \langle a_2,a_1\rangle & \langle a_2,a_2\rangle\end{bmatrix}$

$$V\Sigma^2V^T=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

σ_1 and σ_2 are the two non-zero singular values of A

$$=\begin{bmatrix}\sigma_1^2\cos\theta & -\sigma_2^2\sin\theta\\ \sigma_1^2\cos\theta & \sigma_2^2\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\overbrace{\sigma_1^2\cos^2\theta+\sigma_2^2\sin^2\theta}^{S_{11}} & \overbrace{(\sigma_1^2-\sigma_2^2)\sin\theta\cos\theta}^{S_{12}}\\ \overbrace{(\sigma_1^2-\sigma_2^2)\sin\theta\cos\theta}^{S_{21}} & \overbrace{\sigma_1^2\sin^2\theta+\sigma_2^2\cos^2\theta}^{S_{22}}\end{bmatrix}$$

Therefore,
$$\begin{aligned}\theta&=\frac{1}{2}\operatorname{atan2}(\sin 2\theta,\ \cos 2\theta)\\ &=\frac{1}{2}\operatorname{atan2}\left((\sigma_1^2-\sigma_2^2)\sin 2\theta,\ (\sigma_1^2-\sigma_2^2)\cos 2\theta\right)\\ &=\frac{1}{2}\operatorname{atan2}(2S_{21},\ S_{11}-S_{22})\\ &=\frac{1}{2}\operatorname{atan2}(2\langle a_1,a_2\rangle,\ \langle a_1,a_1\rangle-\langle a_2,a_2\rangle)\end{aligned}$$

The nuclear norm of A : $\|A\|_{S_1} = \sigma_1 + \sigma_2$

We also have : $\underset{3\times2\ 2\times2}{AV} = \underset{3\times3\ 3\times2}{U\Sigma}$

$$\text{Left} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} | & | \\ \cos\theta a_1 + \sin\theta a_2 & -\sin\theta a_1 + \cos\theta a_2 \\ | & | \end{bmatrix}$$

$$\text{Right} = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} | & | \\ \sigma_1 u_1 & \sigma_2 u_2 \\ | & | \end{bmatrix}$$

So $\|A\|_{S_1}$ is just the sum of the norm of the two columns of $U\Sigma = AV$ and the latter can be efficiently calculated.

This also allows for efficient calculation of the projections.
In the dual update, we deal with the S_∞ norm, which amount to

$$\hat{\sigma} = \min\{\sigma, \lambda\}$$

We can rescale the norm of the two columns of $\begin{bmatrix} | & | \\ \sigma_1 u_1 & \sigma_2 u_2 \\ | & | \end{bmatrix}$ to $\begin{bmatrix} | & | \\ \hat{\sigma}_1 u_1 & \hat{\sigma}_2 u_2 \\ | & | \end{bmatrix}$

Then
$$\hat{A} = \begin{bmatrix} | & | \\ \hat{\sigma}_1 u_1 & \hat{\sigma}_2 u_2 \\ | & | \end{bmatrix} V^T$$

$$= \begin{bmatrix} | & | \\ \hat{\sigma}_1 u_1 & \hat{\sigma}_2 u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ \cos\theta \hat{\sigma}_1 u_1 - \sin\theta \hat{\sigma}_2 u_2 & \sin\theta \hat{\sigma}_1 u_1 + \cos\theta \hat{\sigma}_2 u_2 \\ | & | \end{bmatrix}$$