1. Studiați convergența (natura) următoarelor integrale improprii:

a) $\int_1^{\infty} \frac{1}{x^4+1}\,dx$

Sol.:

Fie $f, g : [1,+\infty) \to [0,+\infty)$, $f(x) = \frac{1}{x^4+1}$, $g(x) = \frac{1}{x^4}$

Avem $f(x) \le g(x)$, $(\forall)\, x \in [1,+\infty)$

$$\int_1^{\infty} g(x)\,dx = \lim_{d \to \infty} \int_1^d \frac{1}{x^4}\,dx = \lim_{d \to +\infty} \left(\frac{x^{-4+1}}{-4+1} \Big|_1^d \right)$$

$$= \lim_{d \to +\infty} \left(-\frac{1}{3d^3} + \frac{1}{3} \right) = \frac{1}{3} \in \mathbb{R}$$

Deci $\int_1^{\infty} g(x)\,dx$ este convergentă.

Cf. Criteriului de comparație cu inegalități, avem că $\int_1^{\infty} f(x)\,dx$ este convergentă $\square$

b) $\int_2^{\infty} \frac{1}{x^4-1}\,dx$

Sol.:

Fie $f, g : [2,+\infty) \to (0,+\infty)$, $f(x) = \frac{1}{x^4-1}$, $g(x) = \frac{1}{x^4}$

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x^4}{x^4-1} = 1 \in (0,+\infty)$$

Cf. Criteriului de comparație cu limită avem că

$$\int_2^{\infty} f(x)\,dx \sim \int_2^{\infty} g(x)\,dx$$

$$\int_2^{\infty} g(x)\,dx = \lim_{d\to\infty} \int_2^{d} \frac{1}{x^4}\,dx = \lim_{d\to+\infty} \left(\frac{x^{-4+1}}{-4+1} \Big|_2^d \right)$$

$$= \lim_{d\to+\infty} \left(-\frac{1}{3d^3} + \frac{1}{3\cdot 8} \right) = \frac{1}{24} \in \mathbb{R}$$

Deci, $\int_2^{\infty} g(x)\,dx$ este convergentă

Așadar, $\int_2^{\infty} f(x)\,dx$ este convergentă $\square$

c) $\int_1^{\infty} \sin \frac{1}{x^9}\,dx$

Sol.: $x \in [1,+\infty) \Rightarrow \frac{1}{x^9} \in (0,1] \subset (0,\frac{\pi}{2}) \Rightarrow$

$\sin \frac{1}{x^9} > 0 \ \left((\forall)\ x \in [1,+\infty) \right) \Rightarrow f(x) > 0$

$((\forall)\ x \in [1,+\infty))$

$$\begin{array}{ccc} x & \longmapsto & \frac{1}{x^9} \\ \cap & & \cap \\ [1,+\infty) & & (0,1] \end{array}$$ strict descrescătoare

$$\begin{array}{ccc} x & \longmapsto & \sin x \\ \cap & & \cap \\ (0,\frac{\pi}{2}) & & (0,1) \end{array}$$ strict crescătoare

Deci f este strict descrescătoare.

Cf. Criteriului integral al lui Cauchy, avem

că $\int_1^\infty f(x)\,dx \sim \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \sin\frac{1}{n^9} \sim$

$\sim \sum_{n=1}^{\infty} \frac{1}{n^9}$ convergentă (serie armonică generalizată, $\alpha = 9$)

$$\lim_{n\to+\infty} \frac{\sin\frac{1}{n^9}}{\frac{1}{n^9}} = 1$$

2. Folosind funcțiile Γ și B, determinați:

a) $\int_0^\infty e^{-x^2}\,dx =$

Sol.:

s.v. $x^2 = t \Rightarrow x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}}\,dt$

$x = 0 \Rightarrow t = 0$

$x \to +\infty \Rightarrow t \to +\infty$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$$= \int_0^\infty e^{-t}\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt =$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1} e^{-t}\,dt = \frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \quad \square$$

a) $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

c) $\int_0^{\infty} \sqrt{x}\, e^{-x^3}\,dx = \int_0^{\infty} \left(t^{\frac{1}{3}}\right)^{\frac{1}{2}} e^{-t}\, \frac{1}{3} t^{-\frac{2}{3}}\,dt =$

S.V. $x^3 = t \Rightarrow x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} t^{-\frac{2}{3}}$

$x = 0 \Rightarrow t = 0$

$x \to +\infty \Rightarrow t \to +\infty$

$= \frac{1}{3} \int_0^{\infty} t^{\frac{1}{6}-\frac{2}{3}} e^{-t}\,dt = \frac{1}{3} \int_0^{\infty} t^{-\frac{1}{2}} e^{-t}\,dt =$

$= \frac{1}{3} \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t}\,dt = \frac{1}{3}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3} \quad \square$

d) $\int_0^{2} \frac{x^2}{\sqrt{2-x}}\,dx = \int_0^{2} \frac{x^2}{\sqrt{2}\sqrt{1-\frac{x}{2}}}\,dx$

$= \int_0^{1} \frac{4t^2}{\sqrt{2}\sqrt{1-t}} \cdot 2\,dt = \frac{8}{\sqrt{2}} \int_0^{1} t^2 (1-t)^{-\frac{1}{2}}\,dt =$

S.V. $\frac{x}{2} = t \Rightarrow x = 2t$

$dx = 2\,dt$

$x = 0 \Rightarrow t = 0$

$x \to 2 \Rightarrow t \to 1$

$= \frac{8}{\sqrt{2}} \int_0^{1} t^{3-1} (1-t)^{\frac{1}{2}-1}\,dt = \frac{8}{\sqrt{2}}\, B\left(3, \frac{1}{2}\right)$

$B\left(3, \frac{1}{2}\right) = \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)} = \frac{2! \cdot \sqrt{\pi}}{\Gamma\left(3+\frac{1}{2}\right)}$

$$\Gamma\left(3+\frac{1}{2}\right)=\Gamma\left(1+\underbrace{2+\frac{1}{2}}\right)=\left(2+\frac{1}{2}\right)\cdot\Gamma\left(2+\frac{1}{2}\right)=\frac{5}{2}\,\Gamma\left(1+\underbrace{1+\frac{1}{2}}\right)$$

$$=\frac{5}{2}\left(1+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\;\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\cdot\sqrt{\pi}$$

$$B\left(3,\frac{1}{2}\right)=\frac{2\cancel{\sqrt{\pi}}}{\frac{15}{8}\cancel{\sqrt{\pi}}}=\frac{16}{15}$$

$$\text{Deci } \int_0^2\frac{x^2}{\sqrt{2-x}}\,dx=\frac{8}{\sqrt{2}}\cdot\frac{16}{15}=\frac{8\cdot16}{15\sqrt{2}}^{\sqrt{2})}=\frac{8\cdot16\cdot\sqrt{2}}{15\cdot2}=\frac{64\sqrt{2}}{15}$$

$\square$

e) $\displaystyle\int_0^{\frac{\pi}{2}}(\sin t)^{\frac{5}{2}}(\cos t)^{\frac{3}{2}}\,dt$

$$B(x,y)=2\int_0^{\frac{\pi}{2}}(\sin t)^{2x-1}(\cos t)^{2y-1}\,dt,\ (\forall)\ x,y\in(0,+\infty)$$

$$2x-1=\frac{5}{2}\Leftrightarrow x=\frac{7}{4}$$

$$2y-1=\frac{3}{2}\Leftrightarrow x=\frac{5}{4}$$

$$\int_0^{\frac{\pi}{2}}(\sin t)^{\frac{5}{2}}(\cos t)^{\frac{3}{2}}\,dt=\int_0^{\frac{\pi}{2}}(\sin t)^{2\cdot\frac{7}{4}-1}(\cos t)^{2\cdot\frac{5}{2}-1}\,dt$$

$$=\frac{1}{2}B\left(\frac{7}{4},\frac{5}{4}\right)$$

$$B\left(\frac{7}{4},\frac{5}{4}\right)=\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}+\frac{5}{4}\right)}=\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)}=$$

$$=\frac{\Gamma\left(\frac{7}{4}\right)\,\Gamma\left(\frac{5}{4}\right)}{2!}$$

$$\Gamma\left(\frac{7}{4}\right)=\Gamma\left(1+\frac{3}{4}\right)=\frac{3}{4}\,\Gamma\left(\frac{3}{4}\right)$$

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(1+\frac{1}{4}\right) = \frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)$$

$$\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right) = \frac{3}{16}\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{3}{16}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) =$$

$$= \frac{3}{16}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{3}{16}\cdot\frac{\pi}{\frac{\sqrt{2}}{2}} = \frac{3}{\cancel{16}\,8}\cdot\frac{\cancel{2}\pi}{\sqrt{2}} = \frac{3\pi}{8\sqrt{2}}$$

$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x}$, $(\forall)\ x\in(0,1)$

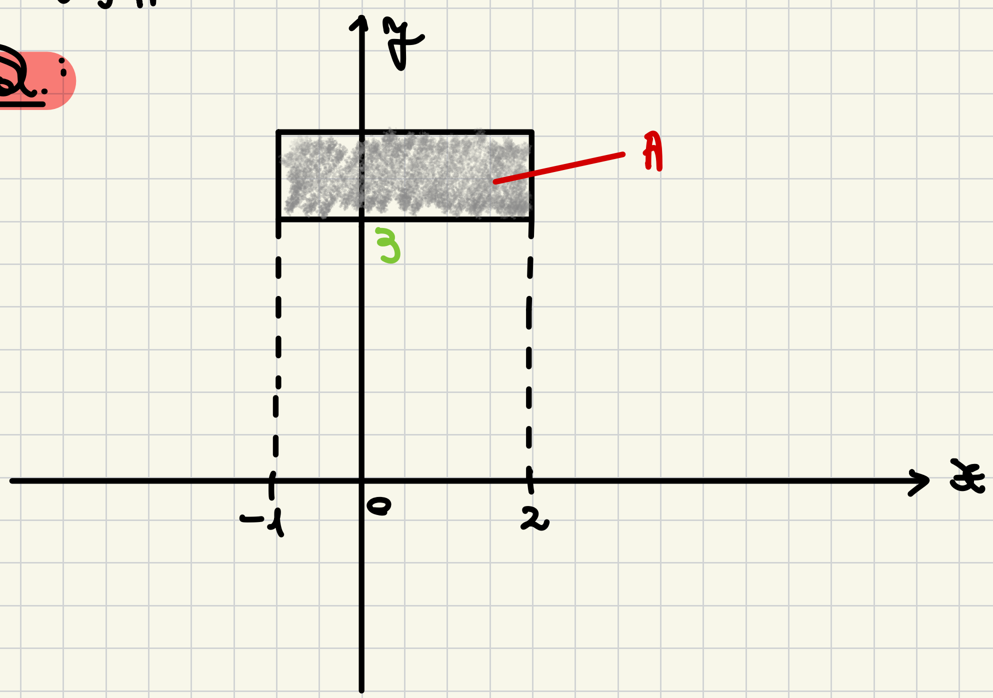
$$B\left(\frac{7}{4},\frac{5}{4}\right) = \frac{\frac{3\pi}{8\sqrt{2}}}{2} = \frac{3\pi}{16\sqrt{2}}$$

$$\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}} (\cos t)^{\frac{3}{2}}\,dt = \frac{1}{2}\cdot\frac{3\pi}{16\sqrt{2}} = \frac{3\pi}{32\sqrt{2}} \quad \square$$

3. Determinați:

a) $\iint_A x\,dx\,dy$, unde $A=[-1,2]\times[3,4]$

Sol.:



$A=[-1,2]\times[3,4] \Rightarrow A\in \mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f: A\to\mathbb{R}$, $f(x,y)=x$

f continuă

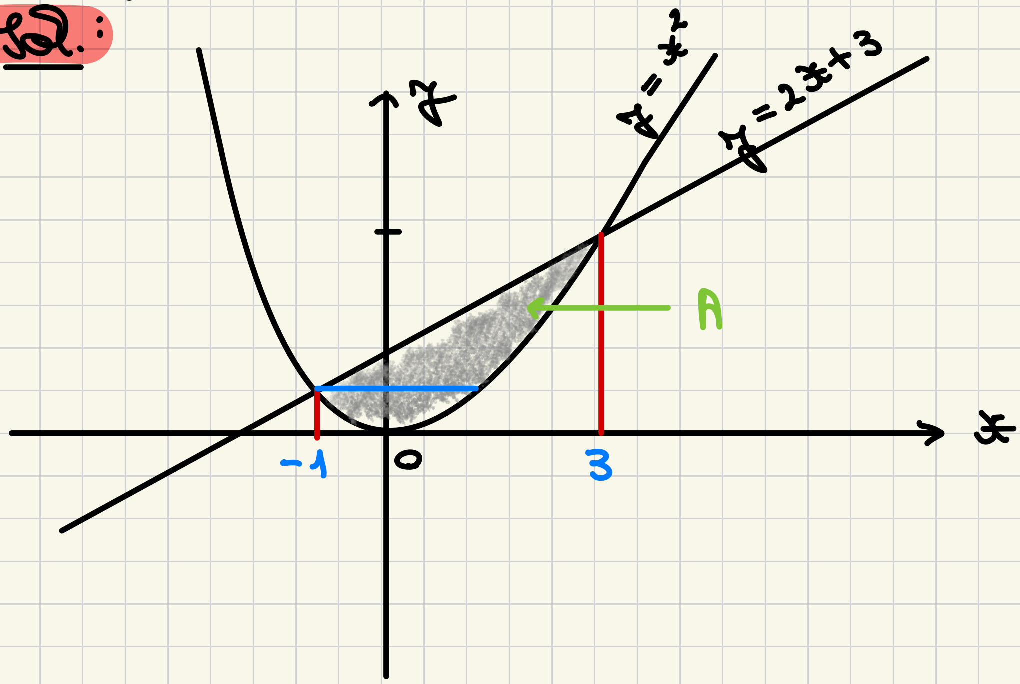
$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{2}\left(\int_3^4 x\,dy\right)dx =$$

$$= \int_{-1}^{2} \left(xy \Big|_{y=3}^{y=4} \right) dx = \int_{-1}^{2} x(4-3)\, dx = \frac{x^2}{2} \Bigg|_{x=-1}^{x=2} =$$

$$= \frac{3}{2} \square$$

2) $\iint_A x \, dx\, dy$, unde A este mulțimea plană mărginită de $y = x^2$ și $y = 2x+3$

Sol.:



Determinăm punctele de intersecție dintre $y = x^2$ și $y = 2x+3$.

$$\begin{cases} y = x^2 \\ y = 2x+3 \end{cases} \Rightarrow x^2 = 2x+3 \Rightarrow x^2 - 2x - 3 = 0$$

$$\Delta = 4+12 = 16 \Rightarrow \sqrt{\Delta} = 4$$

$$x_1 = \frac{2+4}{2} = 3 \Rightarrow y_1 = 9$$

$$x_2 = \frac{2-4}{2} = -1 \Rightarrow y_2 = 1$$

$$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1, 3],\ x^2 \le y \le 2x+3\}$$

Fie $\alpha, \beta : [-1,3] \to \mathbb{R}$, $\alpha(x) = x^2$; $\beta(x) = 2x+3$

α, β cont.

$A \in \mathcal{J}(\mathbb{R}^2)$ şi A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = x$

f continuă

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{3}\left(\int_{x^2}^{2x+3} x\,dy\right)dx =$$

$$= \int_{-1}^{3}\left(xy \Big|_{y=x^2}^{y=2x+3} \right) dx = \int_{-1}^{3} x(2x+3-x^2)\,dx =$$

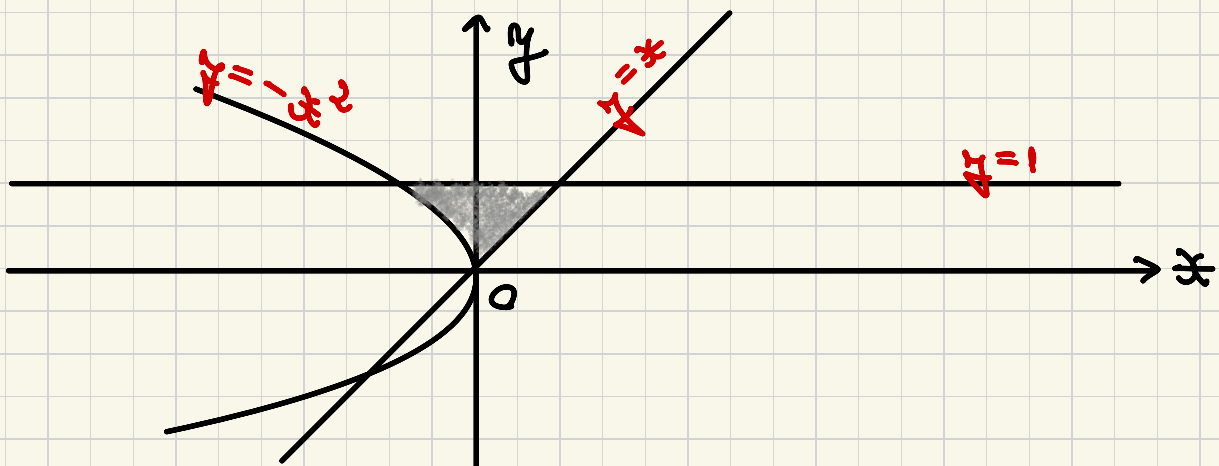
$$= \int_{-1}^{3}(2x^2+3x-x^3)\,dx = 2\,\frac{x^3}{3}\Big|_{x=-1}^{x=3} + 3\,\frac{x^2}{2}\Big|_{x=-1}^{x=3}$$

$$- \frac{x^4}{4}\Big|_{x=-1}^{x=3} = \frac{56}{3} + 12 - 20 = \frac{56}{3} - 8 = \frac{56-24}{3} = \frac{32}{3} \quad \square$$

c) $\iint_A x\,dx\,dy$, unde A este mulţimea plană mărginită de $y = -x^2 - x + 2$ şi $y = x - 1$

d) $\iint_A y\,dx\,dy$, unde A este mulţimea plană limitată de $x = -y^2$, $y = x$ şi $y = 1$

Sol:



$A = \{(x,y) \in \mathbb{R}^2 \mid y \in [0,1] ; -y^2 \leq x \leq y\}$

Fie $\varphi, \psi : [0,1] \to \mathbb{R}$, $\varphi(y) = -y^2$, $\psi(y) = y$

φ, ψ continue

$A \in \mathcal{J}(\mathbb{R}^2)$, A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = y$

f continuă

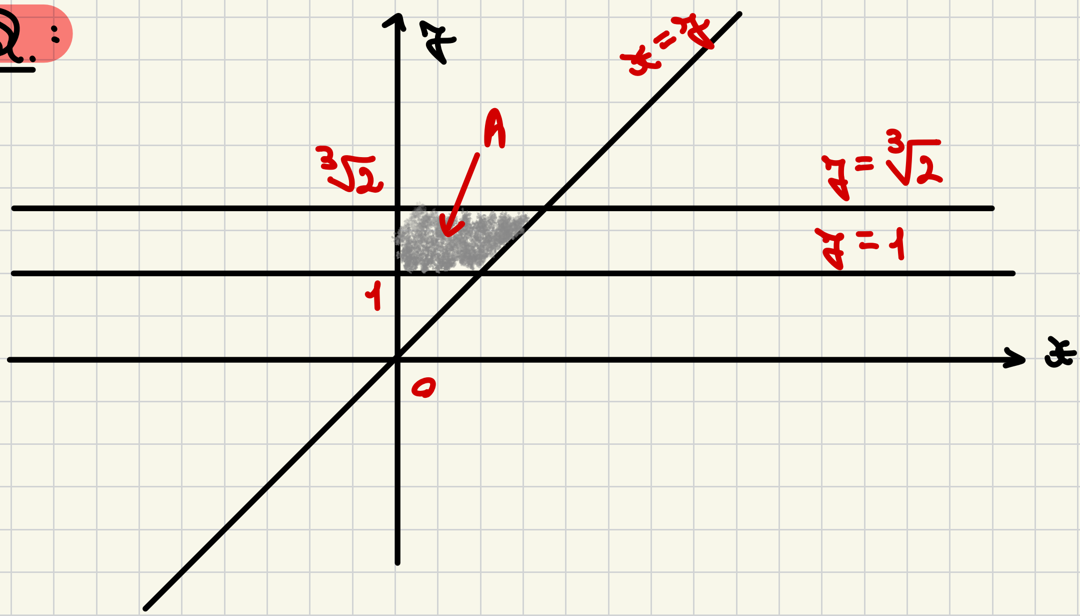
$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-y^2}^{y} y \, dx \right) dy =$$

$$= \int_0^1 xy \Big|_{x=-y^2}^{x=y} dy = \int_0^1 y(y+y^2)\,dy = \int_0^1 y^3 + y^2\,dy$$

$$= \left(\frac{y^4}{4} + \frac{y^3}{3} \right) \Big|_{y=0}^{y=1} = \frac{1}{4} + \frac{1}{3} = \frac{3+4}{12} = \frac{7}{12} \quad \square$$

e) $\iint_A y \, dx\,dy$, unde A este mulțimea plană mărginită de $x=0$, $y=1$, $y=\sqrt[3]{2}$, $x=y$

Sol.:



$A = \{(x,y) \in \mathbb{R}^2 \mid y \in [1, \sqrt[3]{2}], 0 \le x \le y\}$

Fie $\varphi, \psi : A \to \mathbb{R}$, $\varphi(y) = 0$, $\psi(y) = y$

φ, ψ continue, $A \in \mathcal{J}(\mathbb{R})$ și A compactă

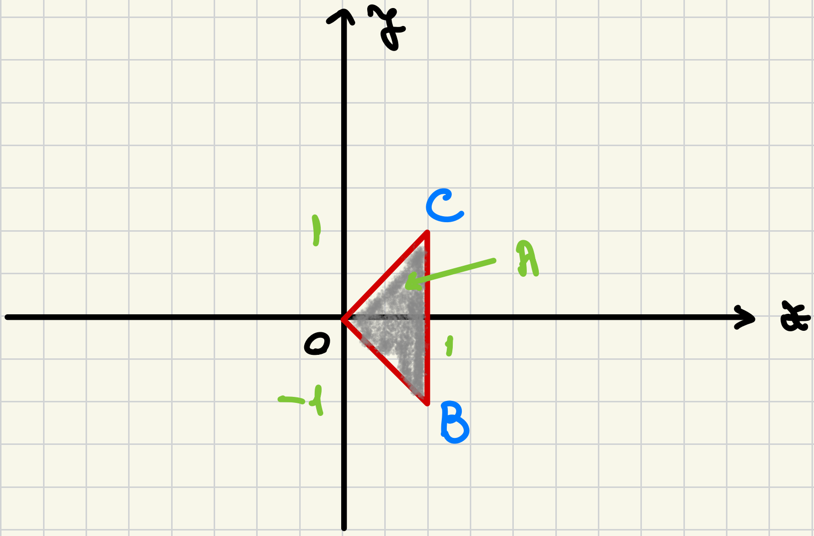
Fie $f : A \to \mathbb{R}$, $f(x,y) = y$, f continuă

$$\iint_A f(x,y)\,dx\,dy = \int_1^{\sqrt[3]{2}} \left(\int_0^y y \, dx \right) dy =$$

$$= \int_1^{\sqrt[3]{2}} \left(yx \Big|_{x=0}^{x=y} \right) dy = \int_1^{\sqrt[3]{2}} y^2 \, dy = \frac{y^3}{3} \Big|_{y=1}^{y=\sqrt[3]{2}} =$$

$$= \frac{(\sqrt[3]{2})^3}{3} - \frac{1}{3} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \ \square$$

f) $\iint_A x \, dx\,dy$, unde A este mulțimea plană mărginită de triunghiul OBC, $O(0,0)$, $B(1,-1)$, $C(1,1)$



$$OB: \frac{y-y_O}{y_B-y_O} = \frac{x-x_O}{x_B-x_O} \Leftrightarrow OB: \frac{y}{-1} = \frac{x}{1} \Leftrightarrow OB: y=-x$$

$$OC: \frac{y-y_O}{y_C-y_O} = \frac{x-x_O}{x_C-x_O} \Leftrightarrow OC: \frac{y}{1} = \frac{x}{1} \Leftrightarrow OC: y = x$$

$$BC: \frac{y-y_B}{y_C-y_B} = \frac{x-x_B}{x_C-x_B} \Leftrightarrow BC: \frac{y+1}{1+1} = \frac{x-1}{1-1}$$

$$\Leftrightarrow BC: x-1=0 \Leftrightarrow BC: x=1$$

$$A = \{(x,y)\in\mathbb{R}^2 \mid x\in[0,1],\ -x \le y \le x\}$$

Fie $\alpha,\beta:[0,1]\to\mathbb{R}$, $\alpha(x)=-x$, $\beta(x)=x$, α,β cont.

$A \in J(\mathbb{R}^2)$, A compactă

$$\iint_A f(x,y) = \int_0^1\left(\int_{-x}^{x} x\,dy\right)dx = \int_0^1 yx \Big|_{y=-x}^{y=x} dx$$

$$= \int_0^1 2x^2\,dx = 2\,\frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{2}{3} \quad \square$$