1\. a) Arătați că $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$.

b) Studiați convergența seriei $\sum\limits_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right) x^n$, $x>0$.

Sol.:

a) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} \overset{\left(\frac{0}{0}\right)}{\underset{L'H}{=}} \lim\limits_{x\to 0} \dfrac{\sin x}{2x} = \dfrac{1}{2}$ 1

b) Fie $x_n = \left(1-\cos\frac{1}{n}\right) x^n$, $(\forall)\, n\in\mathbb{N}^*$ și

$y_n = \dfrac{x^n}{n^2}$, $(\forall)\, n\in\mathbb{N}^*$

Folosim criteriul de comparație cu limită:

$$\lim_{n\to+\infty} \frac{x_n}{y_n} = \lim_{n\to+\infty} \frac{\left(1-\cos\frac{1}{n}\right)\cancel{x^n}}{\frac{\cancel{x^n}}{n^2}} = \lim_{n\to+\infty} \frac{1-\cos\frac{1}{n}}{\left(\frac{1}{n}\right)^2} \overset{a)}{=}$$

$= \dfrac{1}{2} \in (0,+\infty)$

Deci $\sum\limits_{n=1}^{\infty} x_n \sim \sum\limits_{n=1}^{\infty} y_n$.

Studiem convergența $\sum\limits_{n=1}^{\infty} y_n$.

Aplicăm Criteriul raportului:

$$\lim_{n\to+\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to+\infty} \frac{\frac{x^{\cancel{n+1}}}{(n+1)^2}}{\frac{\cancel{x^n}}{n^2}} = \lim_{n\to+\infty} x\cdot\frac{n^2}{(n+1)^2} = x$$

1) Dacă $x<1$ (i.e. $x\in(0,1)$), atunci $\sum\limits_{n=1}^{\infty} y_n$ este conv.

2) Dacă $x>1$ (i.e. $x\in(1,+\infty)$), atunci $\sum\limits_{n=1}^{\infty} y_n$ este div.

3) Dacă $x=1$, atunci criteriul nu decide.

Fie $x=1$.

$$y_n = \frac{1^n}{n^2} = \frac{1}{n^2}, \forall n\in\mathbb{N}^*$$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$, convergentă (serie armonică generalizată, $\alpha=2$) □

2. Studiați convergența (natura) seriilor:

a) $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$.

Sol.: Fie $x_n = \frac{1}{2^n+3^n}$ și $y_n = \frac{1}{3^n}$, $\forall n\in\mathbb{N}^*$

$x_n < y_n$, $\forall n\in\mathbb{N}^*$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$, convergentă (serie geometrică, $q=\frac{1}{3}$)

Conform Criteriului de comparație cu inegalități, $\sum_{n=1}^{\infty} x_n$ este convergentă. □

b) $\sum_{n=1}^{\infty} \frac{7\cdot 13\cdot 19\cdot\ldots\cdot(6n+1)}{8\cdot 13\cdot 18\cdot\ldots\cdot(5n+3)}\cdot x^n$, $x>0$

Sol.: Aplicăm Criteriul raportului.

$$\lim_{n\to+\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to+\infty}\frac{\dfrac{\cancel{7\cdot13\cdot19\cdot\ldots\cdot(6n+1)}(6(n+1)+1)}{\cancel{8\cdot13\cdot18\cdot\ldots\cdot}(5n+3)(5(n+1)+3)}\cdot x^{n+\cancel{1}}}{\dfrac{\cancel{7\cdot13\cdot19\cdot\ldots\cdot(6n+1)}}{\cancel{8\cdot13\cdot18\cdot\ldots\cdot(5n+3)}}\cdot\cancel{x^n}}$$

$$=\lim_{n\to+\infty}\frac{6n+7}{5n+8}\cdot x=\frac{6}{5}x$$

1) Dacă $\frac{6}{5}x<1$ (i.e. $x\in\left(0,\frac{5}{6}\right)$), atunci $\sum_{n=1}^{\infty}x_n$ conv.

2) Dacă $\frac{6}{5}x>1$ (i.e. $x\in\left(\frac{5}{6},+\infty\right)$), atunci $\sum_{n=1}^{\infty}x_n$ div.

3) Dacă $x=\frac{5}{6}$, criteriul nu decide.

Aplicăm Criteriul Raabe-Duhamel.

Fie $x=\frac{5}{6}$.

$$x_n=\frac{7\cdot13\cdot19\cdot\ldots\cdot(6n+1)}{8\cdot13\cdot18\cdot\ldots\cdot(5n+3)}\cdot x^n,\ \forall\, n\in\mathbb{N}^*$$

$$\lim_{n\to+\infty}n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to+\infty}n\left(\frac{5n+8}{6n+7}\cdot\frac{6}{5}-1\right)=$$

$$=\lim_{n\to+\infty}n\left(\frac{30n+48}{30n+35}-1\right)=\lim_{n\to\infty}n\cdot\frac{\cancel{30n}+48-\cancel{30n}-35}{30n+35}=$$

$$=\lim_{n\to+\infty}n\cdot\frac{13}{30n+35}=\frac{13}{30}<1$$

Deci $\sum_{n=1}^{\infty}x_n$ este divergentă. $\square$

c) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Sol.:

Fie $x_n = \frac{1}{n\ln n}$, $\forall n\in\mathbb{N}, n\geq 2$

$(x_n)_{n\geq 2}$ este strict descrescător

Aplicăm Criteriul condensării.

Deci $\sum_{n=2}^{\infty} x_n \sim \sum_{n=2}^{\infty} 2^n \cdot x_{2^n}$.

$$\sum_{n=2}^{\infty} 2^n \cdot x_{2^n} = \sum_{n=2}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \ln 2^n} = \sum_{n=2}^{\infty} \frac{1}{n\ln 2}$$

$= \sum_{n=2}^{\infty} \left(\frac{1}{\ln 2}\right) \cdot \frac{1}{n}$ divergentă, deoarece $\sum_{n=2}^{\infty} \frac{1}{n}$ este divergentă (serie armonică generalizată, $\alpha = 1$)

Așadar, $\sum_{n=1}^{\infty} x_n$ este divergentă $\square$

d) $\sum_{n=1}^{\infty} \frac{a^n + n}{3^n + n^3}$, $a > 0$.

Sol.:

Fie $x_n = \frac{a^n + n}{3^n + n^3}$, $\forall n\in\mathbb{N}^*$

$$x_n = \frac{a^n}{3^n + n^3} + \frac{n}{3^n + n^3}$$

Fie $y_n = \frac{a^n}{3^n + n^3}$, $\forall n\in\mathbb{N}^*$

$$z_n = \frac{n}{3^n + n^3}, \forall n\in\mathbb{N}^*$$

Fie $t_n = \frac{n}{n^3}$, $\forall n \in \mathbb{N}^*$

$z_n \le t_n$, $\forall n \in \mathbb{N}^*$

$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \frac{\cancel{n}}{n^{\cancel{3}\,2}} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, convergentă (serie armonică generalizată, $\alpha = 2$)

Conform Criteriului de comparație cu inegalități, avem $\sum_{n=1}^{\infty} z_n$ convergentă

Deci $\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} z_n$

Fie $u_n = \frac{2^n}{3^n}$, $\forall n \in \mathbb{N}^*$

$$\lim_{n\to+\infty} \frac{z_n}{u_n} = \lim_{n\to+\infty} \frac{\frac{\cancel{2^n}}{3^n+n^3}}{\frac{\cancel{2^n}}{3^n}} = \lim_{n\to+\infty} \frac{3^n}{3^n+n^3} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{3^n}}{\cancel{3^n}\left(1+\frac{n^3}{3^n}\right)} = \lim_{n\to+\infty} \frac{1}{1+\frac{n^3}{3^n}}$$

• $$\lim_{n\to+\infty} \frac{\frac{(n+1)^3}{3^{\cancel{n}+1}}}{\frac{n^3}{\cancel{3^n}}} = \lim_{n\to+\infty} \frac{(n+1)^3}{n^3} \cdot \frac{1}{3} = \frac{1}{3} < 1$$

Conform Criteriului Raportului pentru șiruri cu termeni strict pozitivi, avem că $\lim_{n\to+\infty} \frac{n^3}{3^n} = 0$

Deci $\lim_{n\to+\infty} \frac{z_n}{u_n} = 1 \in (0, +\infty)$

Conform Criteriului de comparație cu limită, avem că $\sum\limits_{n=1}^{\infty} y_n \sim \sum\limits_{n=1}^{\infty} u_n = \sum\limits_{n=1}^{\infty} \left(\frac{a}{3}\right)^n$

$= \begin{cases} \text{convergentă, dacă } \frac{a}{3} \in (-1,1) \\ \text{divergentă, dacă } \frac{a}{3} \in \mathbb{R} \setminus (-1,1) \end{cases}$

(serie geometrică, $q = \frac{a}{3}$)

Dar $a>0$, deci $\sum\limits_{n=1}^{\infty} y_n$ $\begin{cases} \text{convergentă, dacă } a \in (0,3) \\ \text{divergentă, dacă } a \in [3,+\infty) \end{cases}$ $\square$

e) $\sum\limits_{n=1}^{\infty} \dfrac{\sin\left(\frac{1}{n(n+1)}\right)}{\left(\cos\frac{1}{n}\right)\left(\cos\frac{1}{n+1}\right)}$

Sol.:

Fie $x_n = \dfrac{\sin\left(\frac{1}{n(n+1)}\right)}{\left(\cos\frac{1}{n}\right)\left(\cos\frac{1}{n+1}\right)}$, $(\forall)\, n \in \mathbb{N}^*$

$$x_n = \frac{\sin\left(\frac{1}{n} - \frac{1}{n+1}\right)}{\cos\frac{1}{n} \cdot \cos\frac{1}{n+1}} = \frac{\sin\frac{1}{n}\cdot\cos\frac{1}{n+1} - \sin\frac{1}{n+1}\cdot\cos\frac{1}{n}}{\cos\frac{1}{n}\cdot\cos\frac{1}{n+1}}$$

$$= \frac{\sin\frac{1}{n}}{\cos\frac{1}{n}} \cdot \frac{\sin\frac{1}{n-1}}{\cos\frac{1}{n+1}} = \operatorname{tg}\frac{1}{n} - \operatorname{tg}\frac{1}{n+1},\ \forall n \in \mathbb{N}^*$$

$$s_n = x_1 + x_2 + \ldots + x_n$$

$$= \left(\operatorname{tg}\frac{1}{1} - \cancel{\operatorname{tg}\frac{1}{2}}\right) + \left(\cancel{\operatorname{tg}\frac{1}{2}} - \cancel{\operatorname{tg}\frac{1}{3}}\right) + \cancel{\ldots} + \left(\cancel{\operatorname{tg}\frac{1}{n}} - \operatorname{tg}\frac{1}{n+1}\right)$$

$$= \operatorname{tg} 1 - \operatorname{tg}\frac{1}{n+1},\ \forall n \in \mathbb{N}^*$$

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(\operatorname{tg}1 - \operatorname{tg}\frac{1}{n+1}\right) = \operatorname{tg}1 - \operatorname{tg}0 = \operatorname{tg}1$$

Deci, $\sum_{n=1}^{\infty} x_n = \operatorname{tg} 1 \in \mathbb{R}$

Prin urmare, $\sum_{n=1}^{\infty} x_n$ convergentă $\square$

f) $\sum_{n=1}^{\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\left(\cos\frac{1}{n}\right)\left(\cos\frac{1}{n+1}\right)}$

Sol.:

Fie $x_n = \frac{\sin\frac{1}{\sqrt{n}}}{\cos\frac{1}{n}\cos\frac{1}{n+1}}, (\forall) n \in \mathbb{N}^*$

$$\frac{1}{\sqrt{n}}, \frac{1}{n}, \frac{1}{n+1} \in (0,1] \subset \left(0, \frac{\pi}{2}\right) \Rightarrow \begin{cases} \sin\frac{1}{\sqrt{n}} > 0 \\ \cos\frac{1}{n} > 0 \\ \cos\frac{1}{n+1} > 0 \end{cases}, (\forall) n \in \mathbb{N}^*$$

Deci $x_n > 0, (\forall) n \in \mathbb{N}^*$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Fie $y_n = \frac{1}{\sqrt{n}}, (\forall) n \in \mathbb{N}^*$

$$\lim_{n \to +\infty} \frac{x_n}{y_n} = \lim_{n \to +\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\cos\frac{1}{n} \cdot \cos\frac{1}{n+1}} \cdot \sqrt{n} =$$

$$= \lim_{n \to +\infty} \left(\frac{1}{\underbrace{\cos\frac{1}{n}}_{\to 1} \underbrace{\cos\frac{1}{n+1}}_{\to 1}} \cdot \underbrace{\frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}}_{\xrightarrow{n \to \infty} 1} \right) = 1 \in (0, +\infty)$$

Conform Criteriului de comparație cu limită, avem că $\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$.

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$, divergentă (serie armonică generalizată, $\alpha = \frac{1}{2}$)

Deci, $\sum_{n=1}^{\infty} x_n$ este divergentă $\square$