GAL – Sem 5 28.05.2024

Completare seminar 4

9) $(\mathbb{R}^4, +, \cdot)|_{\mathbb{R}}$, $V' = \langle\{u, v, w\}\rangle$, $V'' = \langle\{u', v', w'\}\rangle$,

unde $u = (2, 3, 11, 5)$, $v = (1, 1, 5, 2)$, $w = (0, 1, 1, 1)$,

$u' = (2, 1, 3, 2)$, $v' = (1, 1, 3, 4)$, $w' = (5, 2, 6, 2)$

a) Să se arate că $V' \oplus V'' = \mathbb{R}^4$

b) Descrieți V', V'' printr-un sist. de ec. liniare.

$$V': \begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 11 & 5 & 1 \\ 5 & 2 & 1 \end{pmatrix}$$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 11 & 5 & 1 \end{vmatrix} \underset{\text{=====}}{\overset{L_2 - L_3}{}} \begin{vmatrix} 2 & 1 & 0 \\ -8 & -4 & 0 \\ 11 & 5 & 1 \end{vmatrix} = 1 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 2 & 1 \\ -8 & -4 \end{vmatrix} =$$

$$= \cancel{-8} - (\cancel{-8}) = 0$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 5 & 2 & 1 \end{vmatrix} \underset{\text{=====}}{\overset{L_2 - L_3}{}} \begin{vmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

$rg(\text{matrice}) = 2 \Rightarrow R^2 = \{u, v\}$ reper în V'

$\dim V' = 2$

$V'' : \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 5 \\ 2 \end{matrix} \\ \begin{matrix} 3 & 3 \\ 2 & 4 \end{matrix} & \begin{matrix} 6 \\ 2 \end{matrix} \end{pmatrix}$

$$\Delta_1' = \begin{vmatrix} 2 & 1 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{vmatrix} = 0$$

$$\Delta_2' = \begin{vmatrix} 2 & 1 & 5 \\ 1 & 1 & 2 \\ 2 & 4 & 2 \end{vmatrix} = 4 + 20 + 4 - 10 - 16 - 2 = 28 - 28 = 0$$

$\Rightarrow rg = 2 \Rightarrow R' = \{u', v'\}$ reper în V''

$\dim V'' = 2$

Dem. că $V' \cap V'' = \{0_{\mathbb{R}^4}\}$

Fie $x \in V' \cap V'' \Rightarrow \exists\, a, b, a', b' \in \mathbb{R}$ a.î. $x = au + bv = a'u' + b'v'$

$\Leftrightarrow (x_1, x_2, x_3, x_4) = a(2, 3, 11, 5) + b(1, 1, 5, 2) = a'(2, 1, 3, 2) + b'(1, 1, 3, 4) \Rightarrow (2a + b, 3a + b, 11a + 5b, 5a + 2b) =$

$= (2a' + b', a' + b', 3a' + 3b', 2a' + 4b')$

$$\begin{cases} 2a + b - 2a' - b' = 0 \\ 3a + b - a' - b' = 0 \\ 11a + 5b - 3a' - 3b' = 0 \\ 5a + 2b - 2a' - 4b' = 0 \end{cases}$$

$$A = \left(\begin{array}{cccc|c} 2 & 1 & -2 & -1 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 11 & 5 & -3 & -3 & 0 \\ 5 & 2 & -2 & -4 & 0 \end{array}\right)$$

$\det A \neq 0$

SL 0 compatibil determinat $\Rightarrow$

$\Rightarrow \exists! \ (a, b, a', b') = (0,0,0,0) \Rightarrow x = 0_{\mathbb{R}^4} \Rightarrow V' \oplus V''$

$$\left.\begin{array}{l} \dim V' \oplus V'' = \dim V' + \dim V'' = 4 \\ V' \oplus V'' \subset \mathbb{R}^4 \end{array}\right| \Rightarrow V' \oplus V'' = \mathbb{R}^4$$

b) Fie $x \in V' \Rightarrow \exists a, b \in \mathbb{R}$ a.î. $x = a \cdot u + b \cdot v$

$$\begin{cases} 2a + b = x_1 \\ 3a + b = x_2 \\ 11a + 5b = x_3 \\ 5a + 2b = x_4 \end{cases}$$

$$(*) \ A' = \left(\begin{array}{cc|c} \boxed{\begin{matrix} 2 & 1 \\ 3 & 1 \end{matrix}} & & x_1 \\ & & x_2 \\ 11 & 5 & x_3 \\ 5 & 2 & x_4 \end{array}\right)$$

$(*)$ e S.C. $\Rightarrow rg \, A' = rg \, \bar{A} = 2$

$$\Delta_{c1} = \begin{vmatrix} 2 & 1 & x_1 \\ 3 & 1 & x_2 \\ 11 & 5 & x_3 \end{vmatrix} = 0 \Rightarrow 4x_1 + x_2 - x_3 = 0$$

$$\Delta_{c2} = \begin{vmatrix} 2 & 1 & x_1 \\ 3 & 1 & x_2 \\ 5 & 2 & x_4 \end{vmatrix} = 0 \Rightarrow x_1 + x_2 - x_4 = 0$$

$$V' = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ \middle| \ \begin{cases} 4x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \right\}$$

V'' analog

③ $S_1 = \{ x \in \mathbb{R}^4 \mid x_2 + x_3 + x_4 = 0 \} = S(A_1);\ A_1 = (0\ 1\ 1\ 1);\ \dim S_1 = 3$ $(rg A_1 = 1)$

$S_2 = \{ x \in \mathbb{R}_4 \mid x_1 + x_2 = 0,\ x_3 + 2x_4 = 0 \};\ A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix};\ \dim S_2 = 2$ $(rg A = 2)$

a) $\dim S_1$, $\dim S_2$, $\dim(S_1 + S_2)$, $\dim(S_1 \cap S_2)$

$$S_1 \cap S_2 = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{cases} x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \right\} = S(A),\quad A = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$rg A = 3$

$\dim(S_1 \cap S_2) = 4 - rg A = 4 - 3 = 1$

$\dim(S_1 + S_2) = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2) = 3 + 2 - 1 = 4$

$$\left.\begin{array}{l} S_1 + S_2 \subset \mathbb{R}^4 \\ \dim(S_1 + S_2) = 4 \end{array}\right| \Rightarrow S_1 + S_2 = \mathbb{R}^4$$

$R_0 = \{ e_1, e_2, e_3, e_4 \}$

b) $S_1 : x_4 = -x_2 - x_3$

$S_1 = \{ (x_1, x_2, x_3, -x_2 - x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$x_1 \cdot \underbrace{(1, 0, 0, 0)}_{u} + x_2 \underbrace{(0, 1, 0, -1)}_{v} + x_3 \underbrace{(0, 0, 1, -1)}_{w}$

$R_1 = \{ u, v, w \}$

$\dim S_1 = |R_1| = 3$

$S_2 : \begin{cases} x_2 = -x_1 \\ x_3 = -2x_4 \end{cases}$

$S_2 = \{ (x_1, -x_1, -2x_4, x_4) \mid x_1, x_4 \in \mathbb{R} \}$

$x_1 \underbrace{(1, -1, 0, 0)}_{u'} + x_4 \underbrace{(0, 0, -2, 1)}_{v'}$

$$\left.\begin{array}{l} R_2 = \{ u', v' \}\ SG\ pt\ S_2 \\ \dim S_2 = |R_2| = 2 \end{array}\right| \Rightarrow R_2 \text{ reper în } S_2$$

$$A = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{cases} x_2 + x_3 = -x_4 \\ x_1 + x_2 = 0 \\ x_3 = -2x_4 \end{cases} \Leftrightarrow \begin{cases} x_2 = -x_4 + 2x_4 = x_4 \\ x_1 = -x_2 = -x_4 \end{cases}$$

$rg\, A = 3$

$\dim(S_1 \cap S_2) = 4 - rg\, A = 1$

$$S_1 \cap S_2 = \{(-x_4, x_4, -2x_4, x_4) \mid x_4 \in \mathbb{R}\}$$

$$\underset{\underset{w'}{\parallel}}{x_4(-1,1,-2,1)}$$

$\{w'\}$ e reper în $S_1 \cap S_2$

Ex. sem 5

③ $(\mathbb{R}^3, +, \cdot)\,/\mathbb{R}$

$V_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2 + x_3 = 0\}$

$V_2 = \langle\{(1,-1,2), (3,1,0)\}\rangle$

③ $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

$V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 0\}$

$V_2 = \langle\{(1,-1,2), (3,1,0)\}\rangle$

a) Să se descrie V_2 printr-un sistem de ec. liniare.

b) Precizați câte un reper în $V_1, V_2, V_1+V_2, V_1 \cap V_2$

c) Este sumă directă $V_1 + V_2$?

$$\operatorname{rg}\begin{pmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 0 \end{pmatrix} = 2 \text{ (maxim)}$$

$R_2 = \{u, v\}$ reper în V_2

Fie $x \in V_2 \Rightarrow \exists\, a, b \in \mathbb{R}$ a.î. $x = au + bv$

$$(x_1, x_2, x_3) = (a + 3b, -a + b, 2a)$$

$$\begin{cases} a + 3b = x_1 \\ -a + b = x_2 \\ 2a = x_3 \end{cases}$$

Sistem compatibil (SC) $\Leftrightarrow$

$\Leftrightarrow \operatorname{rg} A = \operatorname{rg} \bar{A} = 2$

$$A = \left(\begin{array}{cc|c} 1 & 3 & x_1 \\ -1 & 1 & x_2 \\ 2 & 0 & x_3 \end{array}\right)$$

$$\Delta_c = \begin{vmatrix} 1 & 3 & x_1 \\ -1 & 1 & x_2 \\ 2 & 0 & x_3 \end{vmatrix} = x_3 + 6x_2 - 2x_1 + 3x_3 = -2x_1 + 6x_2 + 4x_3 = 0$$

$$-x_1 + 3x_2 + 2x_3 = 0$$

$$V_2 = \{x \in \mathbb{R}^3 \mid -x_1 + 3x_2 + 2x_3 = 0\}$$

b) $V_1 : x_3 = -2x_1 + x_2$

$$V_1 = \{(x_1, x_2, -2x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$$

$$V_1 = \{x_1 \underbrace{(1, 0, -2)}_{u'} + x_2 \underbrace{(0, 1, 1)}_{v'} \mid x_1, x_2 \in \mathbb{R}\}$$

$R_1 = \{u', v'\}$ e SG pt. V_1

$\dim V_1 = 2 = |R_1|$

R_1 - reper în V_1

$R_2 = \{u', v'\}$ - reper în V_2

$$V_1 \cap V_2 = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{cases} 2x_1 - x_2 + x_3 = 0 \\ -x_1 + 3x_2 + 2x_3 = 0 \end{cases} \right\}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 3 & 2 \end{array} \right) \Bigg| \begin{array}{c} 0 \\ 0 \end{array}$$

$$\begin{cases} 2x_1 - x_2 = -x_3 \\ -x_1 + 3x_2 = -2x_3 \;/\cdot 2 \end{cases} \Rightarrow \begin{cases} 2x_1 - x_2 = -x_3 \\ -2x_1 + 6x_2 = -4x_3 \end{cases} +$$

$$/ \quad 5x_2 = -5x_3 \Rightarrow \boxed{x_2 = -x_3}$$

$$2x_1 - (-x_3) = -x_3 \quad \Rightarrow 2x_1 = -2x_3 \Rightarrow \boxed{x_1 = -x_3}$$

$$= \{ (-x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \}$$

$$x_3(-1, -1, 1) = w'$$

$R = \{w'\}$ - reper în $V_1 \cap V_2$

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) = 2 + 2 - 1 = 3$$

$$V_1 + V_2 \subset \mathbb{R}^3 \quad \Big| \Rightarrow$$

$$V_1 + V_2 = \mathbb{R}^3$$

$R_0 = \{e_1, e_2, e_3\}$ - reper în $V_1 + V_2$

$<\{v_1, \dots, v_n\}> = <\{v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n\}>$

$V \subseteq U$ (din construcție)

„$\subseteq$" $v_1 = v_1 - v_2 + v_2 - v_3 + \dots + v_{n-1} - v_n + v_n$

$v_2 = \dots$

$\dots\dots$

$v_{n-1} = v_{n-1} - v_n + v_n$

$v_n = v_n$

Obs: $R = \{u_1, \dots, u_n\}$ SLI

R reper pe U

$R \xrightarrow{A} R'$

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & 0 \\ \vdots & \vdots & & 1 & 0 & 0 \\ \vdots & \vdots & & -1 & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}$$

$\det A = 1 \Rightarrow R' - SLI \Rightarrow$

$\Rightarrow R'$ reper în $U = V$

$R' \xrightarrow{A^{-1}} R$

$$A^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & 0 & 0 \\ 1 & 1 & & 1 & 0 \\ & & & 1 & 1 \end{pmatrix}$$