Integrală triplă

i) Dacă $A=[a,b]\times[c,d]\times[k,p]$ şi $f:A\to\mathbb{R}$ este continuă atunci A e mulţime comp şi măs J

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=$$
$$=\int_a^b\left(\int_c^d\left(\int_k^p f(x,y,z)\,dz\right)dy\right)dx$$

ii) Fie $B\subset\mathbb{R}^2$ o mult comp şi măs J şi $A=\{(x,y,z)\in\mathbb{R}^3\mid(x,y)\in B$ şi $\varphi(x,y)\le z\le\psi(x,y)\}$ unde $\varphi,\psi:B\to\mathbb{R}$ cont.

Fie $f:A\to\mathbb{R}$ cont. Atunci A este măs J şi mult comp

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=$$
$$=\iint_B\left(\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz\right)dx\,dy$$

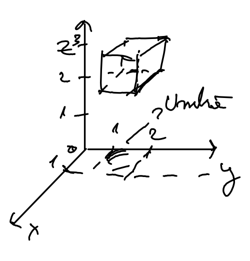
Exemple:

a) $\iiint_A y\,dx\,dy\,dz$, unde $A=[0,1]\times[1,2]\times[2,3]$

("i"): $A=[0,1]\times[1,2]\times[2,3]\Rightarrow A\in J(\mathbb{R}^3)$ şi compactă

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=y$

f cont



$$\iiint_A f(x,y,z)\,dx\,dy\,dz=$$
$$=\int_0^1\left(\int_1^2\left(\int_2^3 y\,dz\right)dy\right)dx=$$
$$=\int_0^1\left(\int_1^2 yz\Big|_2^3\,dy\right)dx=$$
$$=\int_0^1\left(\int_1^2 y\,dy\right)dx=$$
$$=\int_0^1\frac{y^2}{2}\Big|_1^2\,dx$$
$$=\frac{1}{2}\int_0^1(4-1)\,dx=\frac{1}{2}3x\Big|_0^1=\frac{3}{2}$$

b) $\iiint_A(x^2+y^2)z\,dx\,dy\,dz$, $A=\{(x,y,z)\in\mathbb{R}^3\mid(x,y)\in B$
$x^2+y^2\le z\le\sqrt{6-x^2-y^2}\}$
şi $B=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\le 2\}$

("ii"): $x^2+y^2\le z\le\sqrt{6-x^2-y^2}$, unde $(x,y)\in B$

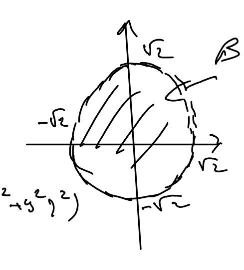
Fie $\begin{cases}\varphi(x,y)=x^2+y^2\\ \text{şi}\\ \psi(x,y)=\sqrt{6-x^2-y^2}\end{cases}\Rightarrow\varphi(x,y)\le z\le\psi(x,y)$

$\left.\begin{array}{l}\varphi(x,y),\psi(x,y)\text{ cont}\\ \varphi(x,y),\psi(x,y)\in B\\ B\text{ compactă}, B\in J(\mathbb{R}^2)\end{array}\right|\Rightarrow A$ comp şi $A\in J(\mathbb{R}^3)$

Fie $f(x,y,z)=(x^2+y^2)z$, $f:A\to\mathbb{R}$, cont

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=$$
$$=\iint_B\left(\int_{\varphi(x,y)}^{\psi(x,y)}(x^2+y^2)z\,dz\right)dx\,dy=$$
$$=\iint_B\left(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}}(x^2+y^2)z\,dz\right)dx\,dy=$$
$$=\iint_B(x^2+y^2)\frac{z^2}{2}\Big|_{x^2+y^2}^{\sqrt{6-x^2-y^2}}\,dx\,dy=$$
$$=\iint_B\frac{(x^2+y^2)}{2}\left(6-x^2-y^2-(x^2+y^2)^2\right)dx\,dy$$

$B\in J(\mathbb{R}^2)$
B compactă



Fie $g:B\to\mathbb{R}$, $g(x,y)=\frac{x^2+y^2}{2}\left(6-x^2-y^2-(x^2+y^2)^2\right)$

Facem schimbare de variabilă

$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$

$(x,y)\in B\Rightarrow x^2+y^2\le 2\Rightarrow r\in[0,\sqrt{2}]$
$\theta\in[0,2\pi]$

Fie $C=[0,\sqrt{2}]\times[0,2\pi]\Rightarrow C\in J(\mathbb{R}^2)$, şi comp

$$\iint_B g(x,y)\,dx\,dy=\iint_C r\,g(r\cos\theta,r\sin\theta)\,d\theta\,dr$$
$$=\int_0^{\sqrt{2}}\left(\int_0^{2\pi}r\cdot\frac{\overbrace{r^2\cos^2\theta+r^2\sin^2\theta}^{1}}{2}\left(6-\overbrace{r^2\cos^2\theta-r^2\sin^2\theta}^{1}-(\underbrace{r^2\cos^2\theta-r^2\sin^2\theta}_{1})^2\right)d\theta\right)dr$$
$$=\int_0^{\sqrt{2}}\left(\int_0^{2\pi}r\cdot\frac{r^2}{2}(6-r^2-r^4)\,d\theta\right)dr$$
$$=\int_0^{\sqrt{2}}\frac{r^3}{2}(6-r^2-r^4)\,\theta\Big|_0^{2\pi}\,dr$$
$$=2\pi\int_0^{\sqrt{2}}3r^3-\frac{r^5}{2}-\frac{r^7}{2}\,dr$$
$$=2\pi\left(\frac{3}{4}r^4\Big|_0^{\sqrt{2}}-\frac{1}{12}r^6\Big|_0^{\sqrt{2}}-\frac{1}{16}r^8\Big|_0^{\sqrt{2}}\right)$$
$$=\frac{8}{3}\pi$$

Schimbarea de variabilă

SV: $\begin{cases}x=h_1(u,v,w)\\ y=h_2(u,v,w)\\ z=h_3(u,v,w)\end{cases}$

Jacobianul $\overset{\text{not}}{=}J=\begin{vmatrix}\frac{\partial h_1}{\partial u}&\frac{\partial h_1}{\partial v}&\frac{\partial h_1}{\partial w}\\ \frac{\partial h_2}{\partial u}&\frac{\partial h_2}{\partial v}&\frac{\partial h_2}{\partial w}\\ \frac{\partial h_3}{\partial u}&\frac{\partial h_3}{\partial v}&\frac{\partial h_3}{\partial w}\end{vmatrix}$

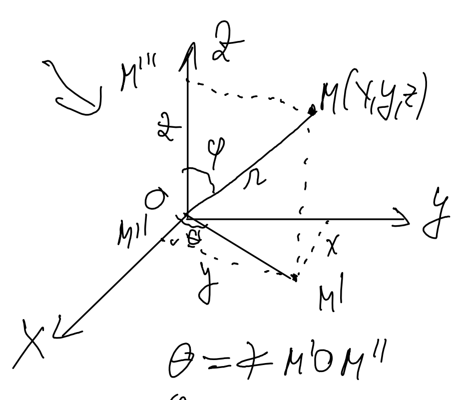
$$\iiint_A f(x,y,z)\,dx\,dy\,dz=$$
$$=\iiint_B|J|\cdot f(h_1(u,v,w),h_2(u,v,w),h_3(u,v,w))\,du\,dv\,dw,$$

unde B se găseşte din condiţia $(x,y,z)\in A$.

1) Trecerea la coordonate sferice

$\begin{cases}x=\alpha+r\cos\theta\sin\varphi\\ y=\beta+r\sin\theta\sin\varphi\\ z=\gamma+r\cos\varphi\end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$, $\varphi\in[0,\pi]$
$\alpha,\beta,\gamma\in\mathbb{R}$

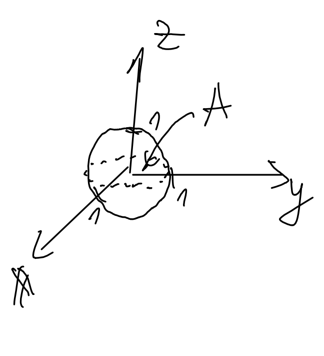
$J=r^2\sin\varphi$



$\theta=\sphericalangle M'OM''$
$\varphi=\sphericalangle MOM'''$
$r=OM$

Ex:
$\iiint_A\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3/x^2+y^2+z^2\le 1\}$

$A\in J(\mathbb{R}^3)$ compactă



$f:A\to\mathbb{R}$, $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ cont

Fie $\begin{cases}x=r\cos\theta\sin\varphi & r\in(0,\infty)\\ y=r\sin\theta\sin\varphi & \theta\in[0,2\pi]\\ z=r\cos\varphi, & \varphi\in[0,\pi]\end{cases}$

$(x,y,z)\in A\Leftrightarrow x^2+y^2+z^2\le 1\Leftrightarrow$

$\Leftrightarrow r^2\cos^2\theta\sin^2\varphi+r^2\sin^2\theta\sin^2\varphi+r^2\cos^2\varphi\le 1\Leftrightarrow$

$\Leftrightarrow(\underbrace{r^2\cos^2\theta+r^2\sin^2\theta}_{1})\sin^2\varphi+r^2\cos^2\varphi\le 1\Leftrightarrow$

$\Leftrightarrow\left.\begin{array}{l}\underbrace{r^2\sin^2\varphi+r^2\cos^2\varphi}_{1}\le 1\Leftrightarrow r^2\le 1\\ \text{Dar } r\in(0,\infty)\end{array}\right\}\Rightarrow r\in[0,1]$

Fie $B=[0,1]\times[0,2\pi]\times[0,\pi]$. Avem că

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\iiint_B r^2\sin\varphi\,f(r\cos\theta\sin\varphi,r\sin\theta\sin\varphi,r\cos\varphi)\,dr\,d\theta\,d\varphi=$$
$$=\int_0^1\int_0^{2\pi}\int_0^{\pi}r^3\sin\varphi\,d\varphi\,d\theta\,dr=\int_0^1\int_0^{2\pi}r^3(-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi}d\theta\,dr=$$
$$=\int_0^1\int_0^{2\pi}r^3(1+1)\,d\theta\,dr=\int_0^1\int_0^{2\pi}2r^3\,d\theta\,dr=\int_0^1 2r^3\cdot\theta\Big|_{\theta=0}^{\theta=2\pi}dr$$
$$=4\pi\int_0^1 r^3\,dr=4\pi\cdot\frac{1}{4}=\pi\ \square$$

2) Trecerea la coordonate sferice generalizate

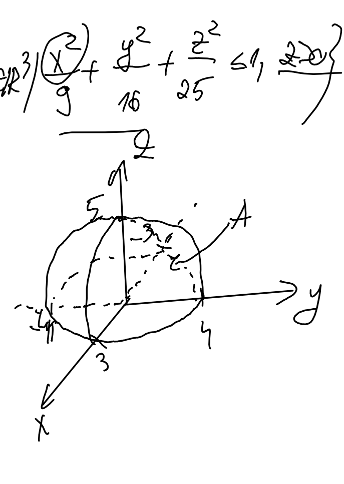
$\begin{cases}x=\alpha+a\,r\cos\theta\sin\varphi\\ y=\beta+b\,r\sin\theta\sin\varphi\\ z=\gamma+c\,r\cos\varphi\end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$, $\varphi\in[0,\pi]$
$a,b,c>0$, $\alpha,\beta,\gamma\in\mathbb{R}$

Jacobianul este $abc\,r^2\sin\varphi$.

Ex:
$\iiint_A 1\,dx\,dy\,dz$, $A=\{(x,y,z)\in\mathbb{R}^3/\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}\le 1, z\ge 0\}$

$A\in J(\mathbb{R}^3)$ compactă

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=1$ cont



Fie $\begin{cases}x=3r\cos\theta\sin\varphi\\ y=4r\sin\theta\sin\varphi\\ z=5r\cos\varphi\end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$, $\varphi\in[0,\pi]$

$(x,y,z)\in A\Leftrightarrow\begin{cases}\frac{\cancel{9}r^2\cos^2\theta\sin^2\varphi}{\cancel{9}}+\frac{\cancel{16}r^2\sin^2\theta\sin^2\varphi}{\cancel{16}}+\frac{\cancel{25}r^2\cos^2\varphi}{\cancel{25}}\le 1\\ 5r\cos\varphi\ge 0\end{cases}\Leftrightarrow$

$\Leftrightarrow\begin{cases}r^2\le 1\\ 5r\cos\varphi\ge 0\end{cases}\Leftrightarrow r\in[0,1]$, $\theta\in[0,2\pi]$, $\varphi\in[0,\frac{\pi}{2}]$

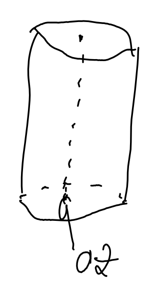
Fie $B=[0,1]\times[0,2\pi]\times[0,\frac{\pi}{2}]\in J(\mathbb{R}^3)$ compactă

Avem că $\iiint_A f(x,y,z)\,dx\,dy\,dz=\iiint_B 60r^2\sin\varphi\,f(\dots)\,dr\,d\theta\,d\varphi$

$$=\iiint_B 60r^2\sin\varphi\,dr\,d\theta\,d\varphi=\int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{2}}60r^2\sin\varphi\,d\varphi\,d\theta\,dr=$$
$$=60\int_0^1 r^2\,dr\cdot\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\cdot\int_0^{2\pi}1\,d\theta=$$
$$=60\frac{r^3}{3}\Big|_0^1\cdot(-\cos\varphi)\Big|_0^{\frac{\pi}{2}}\cdot\theta\Big|_0^{2\pi}=60\cdot\frac{1}{3}\cdot 1\cdot 2\pi=40\pi\ \square$$

3) Trecerea la coordonate cilindrice

$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}$, $r>0$, $\theta\in[0,2\pi]$, $z\in\mathbb{R}$

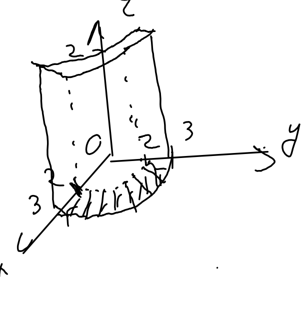


Jacobianul este r.

Ex:
$\iiint_A xz\,dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3/4\le x^2+y^2\le 9, x\ge 0, y\ge 0, 0\le z\le 2\}$,

$A\in J(\mathbb{R}^3)$ compactă

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=xz$ cont



Fie $\begin{cases}x=r\cos\theta\\ y=r\sin\theta, & r\in[2,3];\\ z=z & \theta\in[0,\frac{\pi}{2}];\\ & z\in[0,2]\end{cases}$

Fie $B=[2,3]\times[0,\frac{\pi}{2}]\times[0,2]$. Atunci

$$\iiint_A xz\,dx\,dy\,dz=\iiint_B r\cdot(r\cos\theta)\cdot z\,dr\,d\theta\,dz=$$
$$=\int_2^3\int_0^{\frac{\pi}{2}}\int_0^2 z\cdot r^2\cos\theta\,dz\,d\theta\,dr=$$
$$=\int_2^3 r^2\,dr\cdot\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\cdot\int_0^2 z\,dz=\frac{r^3}{3}\Big|_2^3\cdot\sin\theta\Big|_0^{\frac{\pi}{2}}\cdot\frac{z^2}{2}\Big|_0^2=$$
$$=\left(\frac{27}{3}-\frac{8}{3}\right)\cdot 1\cdot 2=\frac{38}{3}\ \square$$