# Tutoriat 10 - Exam training - Seria 23

## Contents



# 1 Exam from 2020-2021

Already solved here

# 2 Exam from 2022-2023

## 2.1 Probabilities

### 2.1.1 Exercise 1

First exercise from here!

# Exercise 2

Let $X$ and $Y$ be random variables with joint support

$$(x, y) \in [0, i] \times [0, i],$$

and joint pdf

$$f_{X,Y}(x, y) = cxy^2.$$

## (a) Determination of the constant $c$

The normalization condition gives

$$\int_0^i \int_0^i cxy^2 \, dx \, dy = 1.$$

First,

$$\int_0^i x \, dx = \frac{i^2}{2}, \qquad \int_0^i y^2 \, dy = \frac{i^3}{3}.$$

Thus,

$$c \cdot \frac{i^2}{2} \cdot \frac{i^3}{3} = 1 \quad \Longrightarrow \quad \boxed{c = \frac{6}{i^5}}.$$

## (b) Marginal pdf of $X$

$$f_X(x) = \int_0^i f_{X,Y}(x,y)\,dy = \frac{6x}{i^5}\int_0^i y^2\,dy = \boxed{\frac{2x}{i^2}}, \quad 0 \le x \le i.$$

## (c) Marginal CDF of $Y$

First, compute the marginal pdf of $Y$:

$$f_Y(y) = \int_0^i f_{X,Y}(x,y)\,dx = \frac{6y^2}{i^5}\int_0^i x\,dx = \frac{3y^2}{i^3}, \quad 0 \le y \le i.$$

Then, the marginal CDF is

$$F_Y(y) = \int_0^y f_Y(t)\,dt = \int_0^y \frac{3t^2}{i^3}\,dt = \boxed{\frac{y^3}{i^3}}, \quad 0 \le y \le i.$$

## (d) Probability that $X + Y < i$ (Extra)

We compute

$$\mathbb{P}(X + Y < i) = \int_0^i \int_0^{i-x} f_{X,Y}(x,y)\,dy\,dx.$$

Substitute the joint pdf:

$$\mathbb{P}(X + Y < i) = \int_0^i \int_0^{i-x} \frac{6}{i^5}xy^2\,dy\,dx.$$

First integrate with respect to $y$:

$$\int_0^{i-x} y^2\,dy = \frac{(i-x)^3}{3}.$$

Thus,

$$\mathbb{P}(X + Y < i) = \frac{6}{i^5}\int_0^i x \cdot \frac{(i-x)^3}{3}\,dx = \frac{2}{i^5}\int_0^i x(i-x)^3\,dx.$$

Expand the integrand:

$$x(i-x)^3 = x(i^3 - 3i^2x + 3ix^2 - x^3) = i^3x - 3i^2x^2 + 3ix^3 - x^4.$$

Integrate term by term:

$$\int_0^i i^3x\,dx = \frac{i^5}{2}, \quad \int_0^i 3i^2x^2\,dx = i^5,$$
$$\int_0^i 3ix^3\,dx = \frac{3i^5}{4}, \quad \int_0^i x^4\,dx = \frac{i^5}{5}.$$

Putting everything together:

$$\int_0^i x(i-x)^3 dx = i^5\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = \frac{i^5}{20}.$$

Therefore,

$$\boxed{\mathbb{P}(X + Y < i) = \frac{2}{i^5} \cdot \frac{i^5}{20} = \frac{1}{10}.}$$

## (e) Checking independence of $X$ and $Y$ (Extra)

Two random variables are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

From earlier results,

$$f_X(x) = \frac{2x}{i^2}, \qquad f_Y(y) = \frac{3y^2}{i^3}.$$

Their product is

$$f_X(x)f_Y(y) = \frac{2x}{i^2} \cdot \frac{3y^2}{i^3} = \frac{6}{i^5}xy^2.$$

Since

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

we conclude that

$$\boxed{X \text{ and } Y \text{ are independent.}}$$

### 2.1.2 Exercise 3

Let $X$ be a discrete random variable with distribution

$$X \sim \begin{pmatrix} -1 & 0 & i \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

**Expectation**

$$\mathbb{E}[X] = (-1)\frac{1}{3} + 0 + i\frac{1}{3} = \frac{i-1}{3}.$$

**Second Moment**

$$\mathbb{E}[X^2] = \frac{1+i^2}{3}.$$

**Variance**

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \boxed{\frac{2(i^2+i+1)}{9}}.$$

## 2.2 Statistics

### 2.2.1 Exercise 4

Calculate the odds of choosing a natural number between 0 and 20 that is divisible by 3.

**Solution**

Let

$$\Omega = \{1, 2, \dots, 20\}.$$

Let $A$ be the event that the chosen number is divisible by 3.
The numbers divisible by 3 in this interval are:

$$3, 6, 9, 12, 15, 18,$$

so

$$|A| = 6, \quad |\Omega| = 20.$$

Hence,

$$P(A) = \frac{6}{20} = \frac{3}{10}, \quad P(A^c) = 1 - P(A) = \frac{7}{10}.$$

The odds in favor of $A$ are defined as

$$O(A) = \frac{P(A)}{P(A^c)}.$$

Therefore,

$$\boxed{O(A) = \frac{3/10}{7/10} = \frac{3}{7}.}$$

#### 2.2.2 Exercise 5:

**Given** We observe a single data point

$$x = i,$$

generated from a normal distribution with unknown mean $\theta$ and unit variance:

$$x \mid \theta \sim \mathcal{N}(\theta, 1).$$

The prior distribution for $\theta$ is

$$\theta \sim \mathcal{N}(50, 1).$$

Estimate the parameter $\theta$

**Solution**

**Likelihood** The likelihood function is

$$f(x \mid \theta) = c_1 \exp\left(-\frac{1}{2}(x-\theta)^2\right), \quad c_1 = \frac{1}{\sqrt{2\pi}}.$$

Substituting $x = i$:

$$f(i \mid \theta) = c_1 \exp\left(-\frac{1}{2}(i-\theta)^2\right).$$

**Prior** The prior density is

$$f(\theta) = c_2 \exp\left(-\frac{1}{2}(\theta-50)^2\right), \quad c_2 = \frac{1}{\sqrt{2\pi}}.$$

**Bayes' Theorem** Applying Bayes' theorem:

$$f(\theta \mid i) = \frac{f(i \mid \theta) f(\theta)}{f(i)}.$$

Let

$$c_3 = \frac{c_1 c_2}{f(i)}.$$

Then

$$f(\theta \mid i) = c_3 \exp\left(-\frac{1}{2}(i-\theta)^2 - \frac{1}{2}(\theta-50)^2\right).$$

**Completing the Square** Expanding and simplifying the exponent gives

$$f(\theta \mid i) = c_4 \exp\left(-\frac{1}{2}\left(\theta - \frac{i+50}{2}\right)^2\right).$$

**Posterior Distribution** Therefore, the posterior distribution is

$$\boxed{\theta \mid i \sim \mathcal{N}\left(\frac{i+50}{2}, \frac{1}{2}\right)}$$

The posterior mean is

$$\boxed{\mathbb{E}[\theta \mid i] = \frac{i+50}{2}.}$$

## 2.3 Exercise 6:

Let

$$X \sim \mathcal{N}(i, 1).$$

### (a) Symmetric $\frac{1}{2}$-Probability Interval for $X$

A symmetric probability interval $(-a, a)$ for a standard normal random variable $Z \sim \mathcal{N}(0, 1)$ satisfies

$$P(-a \le Z \le a) = \frac{1}{2}.$$

From the standard normal table,

$$a \approx 0.674.$$

The standardization of $X$ is

$$Z = \frac{X - i}{1} = X - i.$$

Thus,

$$P(i - a \le X \le i + a) = \frac{1}{2}.$$

Therefore, the symmetric $\frac{1}{2}$-probability interval for $X$ is

$$\boxed{(i - 0.674,\ i + 0.674)}.$$

### (b) Symmetric $\frac{1}{2}$-Probability Interval for $Y = 3X - 5$ (Extra)

Define the random variable

$$Y = 3X - 5.$$

Since $X \sim \mathcal{N}(i, 1)$, it follows that

$$Y \sim \mathcal{N}(3i - 5,\ 9).$$

Using the linear transformation of intervals, the symmetric $\frac{1}{2}$-probability interval for $Y$ is obtained by applying the transformation $y = 3x - 5$ to the interval for $X$.

Thus,

$$Y \in \big(3(i - 0.674) - 5,\ 3(i + 0.674) - 5\big).$$

Therefore, the required interval is

$$\boxed{(3i - 7.022,\ 3i - 2.978).}$$

## 2.4 Exercise 7 (Extra)

Let $X_1$ and $X_2$ be independent observations from a normal distribution

$$X_i \sim \mathcal{N}(\mu, \sigma^2), \quad i = 1, 2,$$

where both $\mu$ and $\sigma^2$ are unknown.

The observed values are

$$x_1 = 4, \qquad x_2 = 8.$$

Estimate the parameters $\mu$ and $\sigma$ using the method of maximum likelihood.

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# Solution

## Likelihood Function

The likelihood function is

$$L(\mu, \sigma^2) = \prod_{i=1}^{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right).$$

Taking the logarithm:

$$\ell(\mu, \sigma^2) = -\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{2} (x_i - \mu)^2.$$

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## MLE of the Mean

Differentiate with respect to $\mu$ and set equal to zero:

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{2} (x_i - \mu) = 0.$$

Thus,

$$\hat{\mu}_{\text{MLE}} = \frac{x_1 + x_2}{2} = \frac{4+8}{2} = \boxed{6}.$$

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## MLE of the Variance

Differentiate with respect to $\sigma^2$ and set equal to zero:

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^{2} (x_i - \mu)^2 = 0.$$

Substitute $\hat{\mu}_{\text{MLE}} = 6$:

$$\sum_{i=1}^{2} (x_i - 6)^2 = (4-6)^2 + (8-6)^2 = 4 + 4 = 8.$$

Thus,

$$\hat{\sigma}^2_{\text{MLE}} = \frac{1}{2} \cdot 8 = 4.$$

Therefore,

$$\boxed{\hat{\sigma}_{\text{MLE}} = 2.}$$

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# Final Estimates

$$\boxed{\hat{\mu}_{\text{MLE}} = 6, \qquad \hat{\sigma}_{\text{MLE}} = 2.}$$