1. Determinați mulțimea de convergență pentru seria de puteri $\sum_{n=1}^{\infty} \frac{(-2)^n}{2n+1} \cdot (x-2)^n$.

Sol.:

Notăm $x-2=y$

Seria devine $\sum_{n=1}^{\infty} \frac{(-2)^n}{2n+1} \cdot y^n$

Fie $a_n = \frac{(-2)^n}{2n+1}$, $(\forall)\, n \in \mathbb{N}^*$

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \left|\frac{(-2)^{n+1}}{2n+3}\right| \cdot \left|\frac{2n+1}{(-2)^n}\right| =$$

$$= \lim_{n\to+\infty} \left(\frac{2^{\cancel{n+1}}}{2n+3} \cdot \frac{2n+1}{\cancel{2^n}} \right) = \lim_{n\to+\infty} \frac{4n+2}{2n+3} = 2$$

Deci $R = \frac{1}{2}$

Notăm cu N mulțimea de convergență a seriei de puteri $\sum_{n=1}^{\infty} \frac{(-2)^n}{2n+1} \cdot y^n$

Avem $\left(-\frac{1}{2}, \frac{1}{2}\right) \subset N \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Studiem dacă $-\frac{1}{2} \in N$ și $\frac{1}{2} \in N$.

Dacă $y = -\frac{1}{2}$, seria devine $\sum_{n=1}^{\infty} \frac{(-2)^n}{2n+1} \cdot \left(-\frac{1}{2}\right)^n =$

$$= \sum_{n=1}^{\infty} \frac{\left((-2)\cdot\left(-\frac{1}{2}\right)\right)^n}{2n+1} = \sum_{n=1}^{\infty} \frac{1^n}{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

Fie $x_n = \frac{1}{2n+1}$, $(\forall)\, n \in \mathbb{N}^*$

Fie $y_n = \frac{1}{n}$, $(\forall)\, n \in \mathbb{N}^*$

$$\lim_{n\to+\infty} \frac{x_n}{y_n} = \lim_{n\to+\infty} \frac{1}{2n+1} \cdot n = \lim_{n\to+\infty} \frac{n}{2n+1} = \frac{1}{2} \in (0,+\infty)$$

Conform Criteriului de comparație cu limită, avem că $\sum_{n=1}^{\infty} \sim \sum_{n=1}^{\infty} y_n$

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n}$, divergentă (serie armonică)

Deci, $\sum_{n=1}^{\infty} x_n$ divergentă.

Prin urmare, $-\frac{1}{2} \notin N$.

Dacă $y = \frac{1}{2}$, seria devine $\sum_{n=1}^{\infty} \frac{(-2)^n}{2n+1} \cdot \left(\frac{1}{2}\right)^n =$
$= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cancel{2^n}}{2n+1} \cdot \frac{1}{\cancel{2^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$, convergentă

(Criteriul lui Leibnitz)

Deci $\frac{1}{2} \in N$.

Așadar, $N = \left(-\frac{1}{2}, \frac{1}{2}\right]$.

Fie M mulțimea de convergență a seriei de puteri $\sum_{n=1}^{\infty} \frac{(-2)^n}{2n+1} \cdot (x-2)^n$

$$y \in N = \left(-\frac{1}{2}, \frac{1}{2}\right] \Leftrightarrow -\frac{1}{2} < \underset{\substack{\| \\ x-2}}{y} \leq \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow -\frac{1}{2} + 2 < x \leq \frac{1}{2} + 2 \Leftrightarrow \frac{3}{2} < x \leq \frac{5}{2}$$

Deci $M=\left(\frac{3}{2}, \frac{5}{2}\right]$ $\square$

2. Arătați că:

a) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, $(\forall)\, x\in\mathbb{R}$

Sol.:

Fie $I=\mathbb{R}$

$a=0$

$f: I\to\mathbb{R}$, $f(x)=\sin x$

$f\in C^{\infty}(I)$

$f(x)=\sin x$, $(\forall)\, x\in\mathbb{R} \Rightarrow f(0)=\sin 0=0$

$f'(x)=\cos x$, $(\forall)\, x\in\mathbb{R} \Rightarrow f'(0)=\cos 0=1$

$f''(x)=-\sin x$, $(\forall)\, x\in\mathbb{R} \Rightarrow f''(0)=-\sin 0=0$

$f'''(x)=-\cos x$, $(\forall)\, x\in\mathbb{R} \Rightarrow f'''(0)=-\cos 0=-1$

$f^{(4)}(x)=+\sin x$, $(\forall)\, x\in\mathbb{R} \Rightarrow f^{(4)}(0)=\sin 0=0$

$\vdots$

Conform formulei lui Taylor cu rest Lagrange, $(\forall)\, n\in\mathbb{N}$, $(\forall)\, x\in I$, $x\neq 0$, $(\exists)\, c$ între 0 și x a.î.

$$f(x)=f(0)+\frac{f'(0)}{1!}(x-0)+\frac{f''(0)}{2!}(x-0)^2+\ldots+$$

$$+\frac{f^{(n)}(0)}{n!}(x-0)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}$$

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}, (\forall)\, n\in\mathbb{N}, (\forall)\, x\in\mathbb{R}^*$$

Fie $x\in\mathbb{R}^*$.

Avem echivalența:

$$\lim_{n\to+\infty} R_n(x)=0 \iff \lim_{n\to+\infty}|R_n(x)|=0$$

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\cdot x^{n+1}\right| = \frac{|f^{(n+1)}(c)|\cdot|x|^{n+1}}{(n+1)!} \le$$

$$\le \frac{1\cdot|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}, (\forall)\, n\in\mathbb{N}$$

$$\lim_{n\to+\infty}\left(\frac{\cancel{|x|^{n+2}}^{\,|x|}}{\cancel{(n+2)!}_{\,(n+2)}}\cdot\frac{\cancel{(n+1)!}}{\cancel{|x|^{n+1}}}\right) = \lim_{n\to+\infty}\frac{|x|}{n+2} = 0$$

Conform Criteriului raportului pentru șiruri cu termeni strict pozitivi, avem că:

$$\lim_{n\to+\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$$

Avem $\underset{\substack{\downarrow\, n\to+\infty\\ 0}}{0} \le \underset{\substack{\downarrow\, n\to+\infty\\ 0}}{|R_n(x)|} \le \underset{\substack{\downarrow\, n\to+\infty\\ 0}}{\frac{|x|^{n+1}}{(n+1)!}}, (\forall)\, n\in\mathbb{N}$

Conform Criteriului Cleștelui, rezultă că

$$\lim_{n\to+\infty}|R_n(x)| = 0$$

Deci, $\lim\limits_{n\to+\infty} R_n(x) = 0$

Așadar, $\underset{\substack{\parallel\\ \sin x}}{f(x)} = \sum\limits_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot(x-0)^n$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\cdot x^{2k+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot x^{2n+1}, (\forall)\, n\in\mathbb{N}^*$$

$$\left.\begin{array}{l} f(0)=\sin 0=0 \\ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\cdot 0^{2n+1}=0 \end{array}\right\} \Rightarrow \underset{\sin 0}{\overset{\|}{f(0)}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\cdot 0^{2n+1}$$

Așadar, $\underset{\sin x}{\overset{\|}{f(x)}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\cdot x^{2n+1}$, $(\forall)\, x\in\mathbb{R}$ $\square$

e) $\ln(1-x) = \sum_{n=0}^{\infty}\left(-\frac{1}{n+1}\right)x^{n+1}$, $(\forall)\, x\in[-1,1)$

Sol.:

Fie $f: [-1,1)\to\mathbb{R}$, $f(x)=\ln(1-x)$

$$f'(x) = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(-1)\cdot x^n,\ (\forall)\, x\in(-1,1)$$

Integrăm „termen cu termen" și obținem că există $c\in\mathbb{R}$ a.î.:

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)}{(n+1)}\cdot x^{n+1}+c,\ (\forall)\, x\in[-1,1)$$

$$\left.\begin{array}{l} f(0)=\ln 1=0 \\ \sum_{n=0}^{\infty}\frac{(-1)}{n+1}\cdot 0^{n+1}+c=0+c \end{array}\right\}\Rightarrow c=0$$

Deci, $\underset{\ln(1-x)}{\overset{\|}{f(x)}} = \sum_{n=0}^{\infty}\frac{(-1)}{n+1}\, x^{n+1}$, $(\forall)\, x\in(-1,1)$

Dacă $x=-1$, atunci $\sum_{n=0}^{\infty}\frac{(-1)}{n+1}\cdot x^{n+1} = \sum_{n=0}^{\infty}\frac{(-1)}{n+1}\cdot(-1)^{n+1} =$

$= \sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$, convergentă

(Criteriul lui Leibniz)

Conform Teoremei a II-a a lui Abel, avem că

$$\lim_{\substack{x \to -1 \\ x > -1}} f(x) = \sum_{n=0}^{\infty} \frac{(-1)}{n+1} \cdot (-1)^{n+1} \Leftrightarrow$$

$$\Leftrightarrow \lim_{\substack{x \to -1 \\ x > -1}} \ln(1-x) = \sum_{n=0}^{\infty} \frac{(-1)}{n+1} \cdot (-1)^{n+1} \Leftrightarrow$$

$$\Leftrightarrow \underset{f(-1)}{\underset{\parallel}{\ln 2}} = \sum_{n=0}^{\infty} \frac{(-1)}{n+1} \cdot (-1)^{n+1}$$

Deci, $\underset{\ln(1-x)}{\underset{\parallel}{f(x)}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$, $(\forall)\, x \in [-1, 1)$ $\square$

3.

a) Studiați continuitatea lui f.

b) Determinați $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

c) Studiați diferențiabilitatea lui f, unde:

i) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$

Sol.:

a) Vezi Seminar 6

b) Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y)=\frac{(xy)'_x\sqrt{x^2+y^2}-\left(\sqrt{x^2+y^2}\right)'_x(xy)}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$=\frac{y\sqrt{x^2+y^2}-\frac{2x}{2\sqrt{x^2+y^2}}\cdot xy}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{x\sqrt{x^2+y^2}-\frac{2y}{2\sqrt{x^2+y^2}}\cdot xy}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to0}\frac{f((0,0)+te_1)-f(0,0)}{t}$$

$$=\lim_{t\to0}\frac{f((0,0)+(t,0))-f(0,0)}{t}=$$

$$=\lim_{t\to0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to0}\frac{\frac{t\cdot0}{\sqrt{t^2+0^2}}-0}{t}=$$

$$= \lim_{t \to 0} \frac{0-0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+(0,t)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{0\cdot t}{\sqrt{0+t^2}} - 0}{t} = 0$$

c) $\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ sunt continue pe } \mathbb{R}^2 \setminus \{(0,0)\} \\ \text{(operații cu funcții elementare)} \\ \mathbb{R}^2 \setminus \{(0,0)\} \text{ deschisă} \end{array}\right\} \Rightarrow$

f diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$:

$\underbrace{df(0,0)}_{T} : \mathbb{R}^2 \to \mathbb{R}$, $\underbrace{df(0,0)}_{T}(u,v) =$

$$= \left[\left(\underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0} \quad \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right]^t = 0\cdot u + 0 \cdot v = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Alegem $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$, $\forall n \in \mathbb{N}^*$

Avem $\lim_{n\to+\infty} (x_n, y_n) = (0,0)$ şi

$$\lim_{n\to+\infty} \frac{x_n y_n}{x_n^2 + y_n^2} = \lim_{n\to+\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{1}{n^2}\cdot\frac{n^2}{2} = \frac{1}{2} \neq 0$$

Deci $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \neq 0$

Prin urmare, f nu este diferenţiabilă în $(0,0)$

iii) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{x^5 y^2}{x^8 + y^4} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Sol.:

a) f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operaţii cu funcţii elementare)

Studiem continuitatea lui f în $(0,0)$.

Varianta 1

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$|f(x,y) - f(0,0)| = \left| \frac{x^5 y^2}{x^8 + y^4} - 0 \right| = \frac{|x^5|\, y^2}{x^8 + y^4} =$$

$$= |x| \cdot \frac{x^4 y^2}{x^8 + y^4} \leq \frac{1}{2}|x| \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow f \text{ continuă în } (0,0)$$

$\leq \frac{1}{2}$ (Explicaţie: $\frac{x^8 + y^4}{2} \geq \sqrt{x^8 y^4} = |x^4 y^2| = x^4 y^2 : (x^8 y^4) \Rightarrow$

$$\Rightarrow \frac{1}{2} \geq \frac{x^4 y^2}{x^8+y^4}\Big)$$

Varianta 2

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$|f(x,y) - f(0,0)| = \left| \frac{x^5 y^2}{x^8+y^4} - 0 \right| = \frac{|x^5| \cdot y^2}{x^8+y^4} = \frac{|x|^5 \cdot y^2}{x^8+y^4}$$

$$= \left(\frac{|x|^8}{x^8+y^4}\right)^{\frac{5}{8}} \cdot \left(\frac{y^4}{x^8+y^4}\right)^{\frac{2}{4}} \cdot (x^8+y^4)^{\frac{5}{8}+\frac{2}{4}-1}$$

$$= \underbrace{\left(\frac{x^8}{x^8+y^4}\right)^{\frac{5}{8}}}_{\leq 1} \underbrace{\left(\frac{y^4}{x^8+y^4}\right)^{\frac{2}{4}}}_{\leq 1} \cdot (x^8+y^4)^{\frac{1}{8}} \leq (x^8+y^4)^{\frac{1}{8}} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\frac{1}{8} > 0$

$\Rightarrow f$ continuă în $(0,0)$

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(x^5y^2)'_x (x^8+y^4) - (x^8+y^4)'_x}{(x^8+y^4)^2}$$

$$= \frac{5x^4y^2(x^8+y^4) - x^5y^2 \cdot 8x^7}{(x^8+y^4)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2yx^5(x^8+y^4) - x^5 \cdot y^2 \cdot 4y^3}{(x^8+y^4)^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

c) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continue pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operații cu funcții elementare) $\Big\} \Rightarrow$

$\mathbb{R}^2 \setminus \{(0,0)\}$ deschisă

$\Rightarrow f$ diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Studiem diferențiabilitatea în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$, atunci

$\underbrace{df(0,0)}_{T} : \mathbb{R}^2 \to \mathbb{R}$, $\underbrace{df(0,0)}_{T}(u,v) =$

$$= {}^t\left[\begin{pmatrix} \underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0} & \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right] = 0\cdot u + 0\cdot v =$$

$= 0$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|}$$

$$= \frac{\frac{x^5y^2}{x^8+y^4} - 0 - 0}{\sqrt{x^2+y^2}} = \frac{x^5y^2}{(x^8+y^4)\sqrt{x^2+y^2}}$$

Alegem $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right)$, $(\forall)\, n \in \mathbb{N}^*$

Avem $\lim\limits_{n\to+\infty} (x_n, y_n) = (0,0)$ și

$$\lim_{n\to+\infty} \frac{x_n^5 y_n^2}{(x_n^8+y_n^4)\sqrt{x_n^2+y_n^2}} = \lim_{n\to+\infty} \frac{\frac{1}{n^9}}{\frac{2}{n^8}\cdot\sqrt{\frac{1}{n^2}+\frac{1}{n^4}}} =$$

$$= \lim_{n \to +\infty} \frac{\frac{1}{n^9}}{\frac{2}{n^8} \cdot \sqrt{\frac{n^2+1}{n^4}}} = \lim_{n \to +\infty} \frac{1}{\cancel{n^9}} \cdot \frac{\cancel{n^8}^{\,n} \cdot n^2}{2\sqrt{n^2+1}} = \frac{1}{2} \neq 0$$

Deci $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^5 y^2}{(x^8+y^4)\sqrt{x^2+y^2}} \neq 0$

Prin urmare, f nu este diferențiabilă în $(0,0)$