1. Studiați convergența simplă și uniformă pentru următoarele șiruri de funcții:

a) $f_n : [\frac{1}{2}, 1] \to \mathbb{R}$, $f_n(x) = \frac{(1+x)^n}{e^{2nx}}$, $(\forall)\, n \in \mathbb{N}^*$

Sol.:

C.s.:

$$f_n(x) = \left(\frac{1+x}{e^{2x}}\right)^n, (\forall)\, x \in [\tfrac{1}{2}, 1], (\forall)\, n \in \mathbb{N}^*$$

$$e^{2x} > e^{2\cdot\frac{1}{2}} = e > 2 > \frac{1}{x}, (\forall)\, x \in [\tfrac{1}{2}, 1]$$

Deci $0 < \frac{1+x}{e^{2x}} < 1$, $(\forall)\, x \in [\frac{1}{2}, 1]$

Fie $x \in [\frac{1}{2}, 1]$.

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \left(\frac{1+x}{e^{2x}}\right)^n = 0 \Rightarrow f_n \xrightarrow[n\to+\infty]{s} f,$$

unde $f : [\frac{1}{2}, 1] \to \mathbb{R}$, $f(x) = 0$

C.u.:

1) $[\frac{1}{2}, 1]$ mulțime compactă

2) f_n, f continue, $(\forall)\, n \in \mathbb{N}^*$

3) $0 < \frac{1+x}{e^{2x}} < 1 \Rightarrow \underbrace{\left(\frac{1+x}{e^{2x}}\right)^n}_{f_n(x)} > \underbrace{\left(\frac{1+x}{e^{2x}}\right)^{n+1}}_{f_{n+1}(x)}$

$f_n(x) > f_{n+1}(x)$, $(\forall)\, x \in [\frac{1}{2}, 1]$, $(\forall)\, n \in \mathbb{N}^*$

$\Rightarrow (f_n)_n$ strict descrescătoare

4) $f_n \xrightarrow[n\to+\infty]{s} f$

Conform Teoremei lui Dini rezultă că

$f_n \xrightarrow[n\to+\infty]{u} f$

2c) $f_n : \left[\frac{1}{2}, \frac{\pi}{2}\right] \to \mathbb{R},\ f_n(x) = (\cos x)^n,\ (\forall)\ n \in \mathbb{N}^*$

Sol.:

C.s.:

$0 \leq \cos x < 1,\ (\forall)\ x \in \left[\frac{1}{2}, \frac{\pi}{2}\right]$

Fie $x \in \left[\frac{1}{2}, \frac{\pi}{2}\right]$.

$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} (\cos x)^n = 0 \Rightarrow f_n \xrightarrow[n\to+\infty]{s} f,$

$f : \left[\frac{1}{2}, \frac{\pi}{2}\right] \to \mathbb{R},\ f(x) = 0$

C.u.:

1) $f_n, f : \left[\frac{1}{2}, \frac{\pi}{2}\right] \to \mathbb{R},\ (\forall)\ n \in \mathbb{N}^*$

2) f continuă

3) $\underset{\left[\frac{1}{2}, \frac{\pi}{2}\right]}{x} \longrightarrow \underset{[0,1]}{\cos x}$ (strict) descrescătoare $\Rightarrow$

$\Rightarrow f_n$ (strict) descrescătoare

4) $f_n \xrightarrow[n\to+\infty]{s} f$

Conform Teoremei lui Polya, rezultă că

$$f_n \xrightarrow[n\to+\infty]{u} f \quad \square$$

2. Studiați convergența simplă și uniformă pentru $(f_n)_n$ și $(f_n')_n$, unde:

a) $f_n : [0,\pi] \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n}$, $(\forall)\, n \in \mathbb{N}^*$

Sol.:

Pentru $(f_n)_n$:

C.s.:

$$0 \le \cos nx \le 1,\ (\forall)\, x \in [0,\pi],\ (\forall)\, n \in \mathbb{N}^* \Rightarrow$$

$$\Rightarrow 0 \le \underbrace{\frac{\cos nx}{n}}_{n\to+\infty} \le \frac{1}{n},\ (\forall)\, x\in[0,\pi],\ (\forall)\, n \in \mathbb{N}^*$$

$$0$$

Conform Criteriului Cleștelui, avem că

$$\lim_{n\to+\infty} \frac{\cos nx}{n} = 0,\ (\forall)\, x \in [0,\pi]$$

Deci, $f_n \xrightarrow[n\to+\infty]{s} f$, unde $f : [0,\pi] \to \mathbb{R}$, $f(x) = 0$

C.u.:

$$\sup_{x\in[0,\pi]} |f_n(x) - f(x)| = \sup_{x\in[0,\pi]} \left| \frac{\cos nx}{n} - 0 \right| =$$

$$= \sup_{x\in[0,\pi]} \frac{|\cos nx|}{n} \le \frac{1}{n} \xrightarrow[n\to+\infty]{} 0 \Rightarrow$$

$$\Rightarrow f_n \xrightarrow[n\to+\infty]{u} f$$

Pentru $(f_n')_n$:

$$f_n'(x) = \left(\frac{\cos nx}{n}\right)' = \frac{-\cancel{n}^{2} \sin nx - 0}{\cancel{n^2}} = -\sin nx,$$

$(\forall)\ x \in [0, \pi]$, $(\forall)\ n \in \mathbb{N}^*$

C.S.:

Alegem $x = \frac{\pi}{2}$.

$$f_{n_{4k}}'\left(\frac{\pi}{2}\right) = -\sin\left(\cancel{4}^{2}k \cdot \frac{\pi}{\cancel{2}}\right) = -\sin 2k\pi = 0 \xrightarrow[k \to +\infty]{} 0$$

$$f_{n_{4k+1}}'\left(\frac{\pi}{2}\right) = -\sin\left[(4k+1) \cdot \frac{\pi}{2}\right] = -\sin\left(2k\pi + \frac{\pi}{2}\right) =$$

$$= -\sin\frac{\pi}{2} = -1 \xrightarrow[k \to +\infty]{} -1$$

$$0 \neq -1 \Rightarrow (\nexists) \lim_{n \to +\infty} f_n'\left(\frac{\pi}{2}\right)$$

Deci $(f_n')_n$ nu converge simplu.

C.u.:

$(f_n')_n$ nu converge simplu $\Rightarrow (f_n')_n$ nu converge uniform

b) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{\operatorname{arctg} nx}{n}$, $(\forall)\ n \in \mathbb{N}^*$

3. Arătați că seria de funcții $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2 + n^4}$ converge uniform.

Sol.:

Fie $f_n:\mathbb{R}\to\mathbb{R}$, $f_n(x)=\operatorname{arctg}\frac{2x}{x^2+n^4}$, $(\forall)\ n\in\mathbb{N}^*$

$$\frac{x^2+n^4}{2}\geq\sqrt{x^2\cdot n^4}=|x|\cdot n^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\geq\frac{|x|}{x^2+n^4}\cdot n^2\Leftrightarrow$$

$$\Leftrightarrow \frac{1}{n^2}\geq\frac{2|x|}{x^2+n^4}\Leftrightarrow\left|\frac{2x}{x^2+n^4}\right|\leq\frac{1}{n^2}\Leftrightarrow$$

$$\left.\begin{array}{l}\Leftrightarrow -\frac{1}{n^2}\leq\frac{2x}{x^2+n^4}\leq\frac{1}{n^2}\\ \operatorname{arctg}\text{ strict crescătoare}\end{array}\right\}\Rightarrow$$

$$\Rightarrow -\operatorname{arctg}\frac{1}{n^2}\leq\operatorname{arctg}\frac{2x}{x^2+n^4}\leq\operatorname{arctg}\frac{1}{n^2}\Leftrightarrow$$

$$\Leftrightarrow \underbrace{\left|\operatorname{arctg}\frac{2x}{x^2+n^4}\right|}_{=|f_n(x)|}\leq\operatorname{arctg}\frac{1}{n^2},\ (\forall)\ n\in\mathbb{N}^*,\ (\forall)\ x\in\mathbb{R}$$

Fie $a_n=\operatorname{arctg}\frac{1}{n^2}$, $(\forall)\ n\in\mathbb{N}^*$

Arătăm că $\sum_{n=1}^{\infty}a_n$ este convergentă.

Avem $\lim_{x\to 0}\frac{\operatorname{arctg}x}{x}=1$

Fie $b_n=\frac{1}{n^2}$, $(\forall)\ n\in\mathbb{N}^*$

$$\lim_{n\to+\infty}\frac{a_n}{b_n}=\lim_{n\to+\infty}\frac{\operatorname{arctg}\frac{1}{n^2}}{\frac{1}{n^2}}=1\in(0,+\infty)$$

Conform Criteriului de comparație cu limită

$$\left.\begin{array}{l} \Rightarrow \displaystyle\sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n \\ \displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ convergentă} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergentă}$$

Conform Teoremei lui Weierstrass, rezultă că $\sum_{n=1}^{\infty} f_n$ converge uniform. $\square$

4. Determinați mulțimea de convergență pentru următoarele serii de puteri:

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot x^n$

Sol.:

$a_n = \frac{(-1)^n}{n \cdot 2^n}, (\forall)\, n \in \mathbb{N}^*$

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \sqrt[n]{\left|\frac{(-1)^n}{n \cdot 2^n}\right|} = \lim_{n\to+\infty} \sqrt[n]{\frac{|(-1)^n|}{n \cdot 2^n}} =$$

$$= \lim_{n\to+\infty} \frac{1}{(\sqrt[n]{n})(\sqrt[n]{2^n})} = \lim_{n\to+\infty} \frac{1}{\underbrace{\sqrt[n]{n}}_{\to 1} \cdot 2} = \frac{1}{2}$$

$$R = \frac{1}{\frac{1}{2}} = 2$$

Fie M mulțimea de convergență a seriei de puteri din enunț.

Avem $(-R,R) \subset M \subset [-R,R]$, i.e. $(-2,2) \subset M \subset [-2,2]$

Studiem dacă $-2 \in M$ și $2 \in M$.

Dacă $x = -2$, seria devine $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} (-2)^n =$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} [(-1)2]^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n \cdot \cancel{2^n}} \cdot \cancel{2^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

divergentă (serie armonică generalizată, $\alpha = 1$)

Deci, $-2 \notin M$.

Dacă $x = 2$, seria devine $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot \cancel{2^n}} \cdot \cancel{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$,

convergentă conform Criteriului lui Leibniz

Deci, $2 \in M$.

Prin urmare, $M = (-2, 2]$.

b) $\sum_{n=1}^{\infty} \frac{n! \cdot x^n}{(a+1) \cdot \ldots \cdot (a+n)}$, $a > 1$

Sol.:

$$a_n = \frac{n!}{(a+1) \cdot \ldots \cdot (a+n)}, \ (\forall)\ n \in \mathbb{N}^*$$

$$\lim_{n \to +\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to +\infty} \left(\frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{(a+1) \cdot \ldots \cdot (a+n)}(a+n+1)} \cdot \frac{\cancel{(a+1) \cdot \ldots \cdot (a+n)}}{\cancel{n!}} \right) = \lim_{n \to +\infty} \frac{n+1}{a+n+1} = 1$$

$$R = \frac{1}{1} = 1$$

Fie M mulțimea de convergență a seriei de puteri din enunț.

Avem $(-1, 1) \subset M \subset [-1, 1]$.

Studiem dacă $-1 \in M$ și $1 \in M$.

Dacă $x = 1$, seria devine $\sum_{n=1}^{\infty} \frac{n! \cdot 1^n}{(a+1) \cdot \ldots \cdot (a+n)} =$

$$= \sum_{n=1}^{\infty} \frac{n!}{(a+1) \cdot \ldots \cdot (a+n)}$$

Fie $x_n = \frac{n!}{(a+1) \cdot \ldots \cdot (a+n)}$, $(\forall)\ n \in \mathbb{N}^*$

$$\lim_{n \to +\infty} n \left(\frac{x_n}{x_{n+1}} - 1 \right) = \lim_{n \to +\infty} n \left(\frac{\frac{\cancel{n!}}{\cancel{(a+1) \cdot \ldots \cdot (a+n)}}}{\frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{(a+1)} \cdot \ldots \cdot (a+n+1)}} - 1 \right)$$

$$= \lim_{n\to+\infty} n\left(\frac{a+n+1}{n+1}-1\right) = \lim_{n\to+\infty} n\left(\frac{a+\cancel{n}+\cancel{1}-\cancel{n}-\cancel{1}}{n+1}\right)$$

$$= \lim_{n\to+\infty} \frac{na}{n+1} = a > 1$$

Conform Criteriului Raabe-Duhamel, avem că $\sum_{n=1}^{\infty} x_n$ este convergentă.

Deci $1 \in M$.

Dacă $x=-1$, seria devine $\sum_{n=1}^{\infty} \frac{n!\cdot(-1)^n}{(a+1)\cdot\ldots\cdot(a+n)}$

Fie $y_n = \frac{n!\cdot(-1)^n}{(a+1)\cdot\ldots\cdot(a+n)}$

$\sum_{n=1}^{\infty} |y_n| = \sum_{n=1}^{\infty} x_n$, convergentă (vezi mai sus)

$\Rightarrow \sum_{n=1}^{\infty} y_n$ absolut convergentă $\Rightarrow \sum_{n=1}^{\infty} y_n$ convergentă

Deci $-1 \in M$.

Așadar, $M=[-1,1]$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}(x+3)^n$

Rez.:

Notăm $x+3=y$.

Seria devine $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}\cdot y^n$

$a_n = \frac{(-1)^n}{\sqrt[3]{n}}$

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty}\left(\left|\frac{(-1)^{n+1}}{\sqrt[3]{n+1}}\right|\cdot\left|\frac{\sqrt[3]{n}}{(-1)^n}\right|\right)$$

$= \lim_{n \to +\infty} \left(\frac{1}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{1} \right) = \lim_{n \to +\infty} \sqrt[3]{\frac{n}{n+1}} = 1$

$R = \frac{1}{1} = 1$

Fie N mulțimea de convergență a seriei de puteri

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot y^n$

Avem $(-1,1) \subset N \subset [-1,1]$

Studiem dacă $-1 \in N$ și $1 \in N$.

Dacă $y=-1$, seria devine $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} (-1)^n =$

$= \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} =$

$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$, divergentă (serie armonică generalizată, $\alpha = \frac{1}{3}$)

Deci $-1 \notin N$.

Dacă $y = 1$, seria devine

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \cdot 1^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ convergentă, conform

Criteriului lui Leibniz

Deci $1 \in N$.

Așadar, $N = (-1,1]$

Fie M mulțimea de convergență a seriei de puteri

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} (x+3)^n$

$$y \in N = (-1,1] \iff -1 < \underset{x+3}{\underset{\parallel}{y}} \le 1 \iff -1 < x+3 \le 1 \quad \Big| -3$$

$$\iff -4 < x \le -2$$

Așadar, $M = (-4,-2]$

d) $$\sum_{n=1}^{\infty} \frac{2^n}{2n+1} \cdot (x-2)^n$$