1. Fie $m \in \mathbb{N}^*$ şi $d_2 \overset{\text{not.}}{=\!=} d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $d(x,y) =$

$= \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}$. Arătaţi că d_2 este metrică pe $\mathbb{R}^m$.

Sol.: Fie $x, y, z \in \mathbb{R}^m$.

1) $d(x,y) = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2} \geq 0$ (evident)

2) $d(x,y) = 0 \Leftrightarrow \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2} = 0 \Leftrightarrow$

$\sum_{i=1}^{m} (x_i - y_i)^2 = 0 \Leftrightarrow (x_i - y_i)^2 = 0 \Leftrightarrow x_i - y_i = 0 \Leftrightarrow$

$\Leftrightarrow x_i = y_i$, $(\forall)\, i = \overline{1,m} \Leftrightarrow x = y$

3) $d(x,y) = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{m} [-(y_i - x_i)]^2} =$

$= \sqrt{\sum_{i=1}^{m} (y_i - x_i)^2} = d(y,x)$

4) Arătăm că $d(x,z) \leq d(x,y) + d(y,z)$

Folosim inegalitatea Cauchy-Buniakovski-Schwartz (C.B.S.):

Pentru orice $m \in \mathbb{N}^*$ şi orice $a_1, \dots, a_m, b_1, \dots, b_m \in \mathbb{R}$ avem:

$$\left(\sum_{i=1}^{m} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{m} a_i^2\right) \cdot \left(\sum_{i=1}^{m} b_i^2\right)$$

$$\Updownarrow$$

$$\left|\sum_{i=1}^{m} a_i b_i\right| \leq \left(\sqrt{\sum_{i=1}^{m} a_i^2}\right)\left(\sqrt{\sum_{i=1}^{m} b_i^2}\right)$$

$$d(x,z)=\sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}=\sqrt{\sum_{i=1}^{n}(\underbrace{x_i-y_i}+\underbrace{y_i-z_i})^2}$$

$$=\sqrt{\sum_{i=1}^{n}\left[(x_i-y_i)^2+(y_i-z_i)^2+2(x_i-y_i)(y_i-z_i)\right]}$$

$$=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2+\sum_{i=1}^{n}(y_i-z_i)^2+2\sum_{i=1}^{n}(x_i-y_i)(y_i-z_i)}\leq$$

$$\leq\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2+\sum_{i=1}^{n}(y_i-z_i)^2+2\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\right)\cdot\left(\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right)}$$

C.B.S.

$$\Rightarrow\sum_{i=1}^{n}(x_i-y_i)(y_i-z_i)\leq\left|\sum_{i=1}^{n}(x_i-y_i)(y_i-z_i)\right|\leq$$

$$\leq\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\right)\left(\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right)$$

$$=\sqrt{\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right)^2}$$

$$=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}=d(x,y)+d(y,z)$$

Deci d este metrică pe $\mathbb{R}^n$.

2. Fie $n \in \mathbb{N}^*$, $d_2 \overset{\text{not.}}{=\!=} d$ ca mai sus şi $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$. Arătaţi că $(\exists)\ a, b \in (0,+\infty)$ a.î.

$a\, d_1(x,y) \leq d(x,y) \leq b\, d_1(x,y),\ (\forall)\ x,y \in \mathbb{R}^n$.

Sol.:

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \cdot 1 \underset{\text{C.B.S.}}{\leq} \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \cdot \sqrt{\sum_{i=1}^{n} 1^2} =$$

$$= d(x,y) \cdot \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}} d_1(x,y) \leq d(x,y),\ (\forall)\ x,y \in \mathbb{R}^n$$

Alegem $a = \frac{1}{\sqrt{n}}$

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \leq \sqrt{\left(\sum_{i=1}^{n} |x_i - y_i|\right)^2}$$

$$= \left|\sum_{i=1}^{n} |x_i - y_i|\right| = \sum_{i=1}^{n} |x_i - y_i| = d_1(x,y),\ (\forall)\ x,y \in \mathbb{R}^n$$

Alegem $b = 1$. $\square$

3. Fie $n \in \mathbb{N}^*$, $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $d_\infty(x,y) = \max\{|x_i - y_i| \mid i = \overline{1,n}\}$ şi d_1 ca mai sus. Arătaţi că $(\exists)\ a, b \in (0,+\infty)$ a.î. $a\, d_1(x,y) \leq d_\infty(x,y) \leq b\, d_1(x,y),\ (\forall)\ x,y \in \mathbb{R}^n$.

Sol.:

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \leq n \max\{|x_i - y_i| \mid i = \overline{1,n}\} =$$

$$= n \cdot d_\infty(x,y) \Rightarrow \frac{1}{n} \cdot d_1(x,y) \leq d_\infty(x,y),\ (\forall)\ x,y \in \mathbb{R}^n$$

Alegem $a = \frac{1}{n}$.

$$d_\infty(x,y) = \max\{ |x_i - y_i| \mid i=\overline{1,n}\} \le \sum_{i=1}^{n} |x_i-y_i|$$

$$= d_1(x,y),\ (\forall)\ x,y \in \mathbb{R}^n$$

Alegem $b=1$. $\square$

Observație! Conform ultimelor două exerciții și ultimei observații din seminarul precedent, dacă $(x^k)_k \subset \mathbb{R}^n$, $x^k = (x_1^k, \dots, x_n^k)$, $x \in \mathbb{R}^n$, $x = (x_1,\dots,x_n)$ ($n\in\mathbb{N}^*$ fixat în mod arbitrar), avem echivalențele:

$$\lim_{k\to+\infty} x^k \overset{d_1}{=} x \iff \lim_{k\to+\infty} x^k \overset{d}{=} x \iff \lim_{k\to+\infty} x^k \overset{d_\infty}{=} x \iff \lim_{k\to+\infty} x_i^k = x_i,\ (\forall)\ i=\overline{1,n}$$

4. Faceți analiza topologică a mulțimii $A\subset\mathbb{R}$, unde:

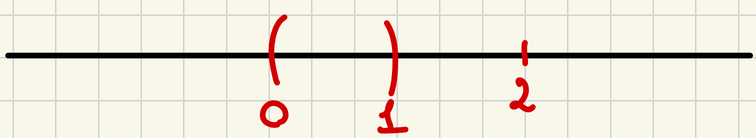
a) $A=(0,1)\cup\{2\}$

b) $A=\mathbb{N}$

c) $A=\{\frac{1}{n} \mid n\in\mathbb{N}^*\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Sol.:

a) $A=(0,1)\cup\{2\}$



1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow (\exists)\ r > 0$ a.î. $(x-r, x+r) \subset A$

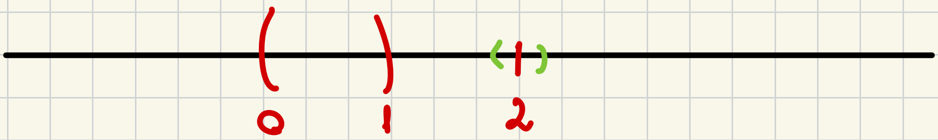
$\mathring{A} \subset A$

$\left.\begin{array}{l}(0,1) \subset A \\ (0,1) \text{ deschisă}\end{array}\right\} \Rightarrow (0,1) \subset \mathring{A}$

Deci $(0,1) \subset \mathring{A} \subset (0,1) \cup \{2\}$

Studiem dacă $2 \in \mathring{A}$.

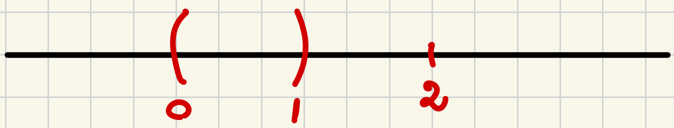
$2 \in \mathring{A} \Leftrightarrow (\exists)\ r > 0$ a.î. $(2-r, 2+r) \subset A$



Deci $2 \notin \mathring{A}$

2) $\bar{A} = ?$

$x \in \bar{A} \Leftrightarrow (\forall)\ r > 0$, avem $(x-r, x+r) \cap A \neq \emptyset$



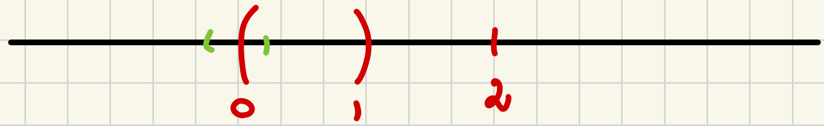
$A \subset \bar{A}$

$\left.\begin{array}{l}[0,1] \cup \{2\} \supset A \\ [0,1] \cup \{2\} \text{ închisă}\end{array}\right\} \Rightarrow \bar{A} \subset [0,1] \cup \{2\}$

Deci $(0,1) \cup \{2\} \subset \bar{A} \subset [0,1] \cup \{2\}$

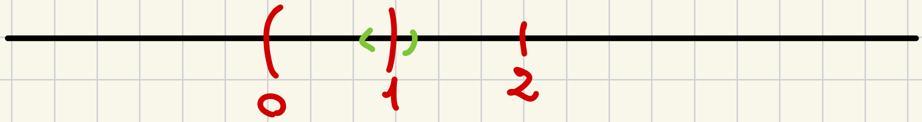
Studiem dacă $0 \in \bar{A}$ și $1 \in \bar{A}$.

$0 \in \bar{A} \Leftrightarrow (\forall)\ r > 0$, avem $(0-r, 0+r) \cap A \neq \emptyset$



Deci $0 \in \bar{A}$.

$1 \in \bar{A} \iff (\forall)\ r>0$, avem $(1-r, 1+r) \cap A \neq \emptyset$

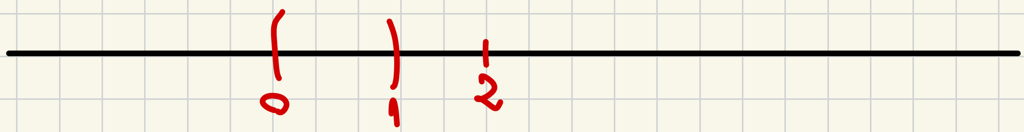


Deci $1 \in \bar{A}$.

Așadar, $\bar{A} = [0,1] \cup \{2\}$

3) $A' = ?$

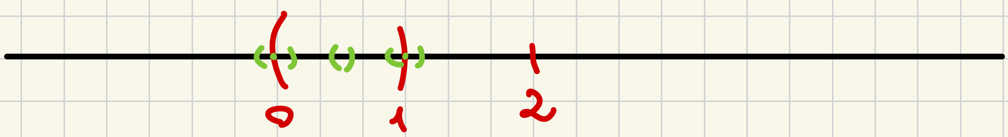
$x \in A' \iff (\forall)\ r>0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq 0$



$A' \subset \bar{A} = [0,1] \cup \{2\}$

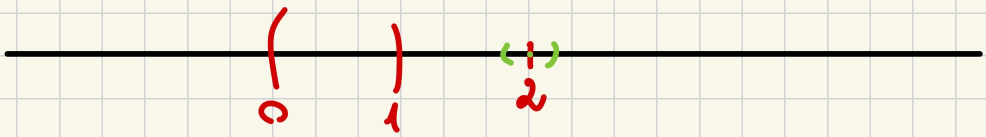
Fie $x \in [0,1]$.

$x \in A' \iff (\forall)\ r>0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq 0$



Deci $x \in A'$.

$2 \in A' \iff (\forall)\ r>0$, avem $(2-r, 2+r) \cap (A \setminus \{2\}) \neq \emptyset$



Deci $2 \notin A'$.

Deci $A' = [0,1]$.

4) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = ([0,1] \cup \{2\}) \setminus (0,1) = \{0, 1, 2\}$

5) $Izo(A) = {}'A = \bar{A} \setminus A' = ([0,1] \cup \{2\}) \setminus [0,1] = \{2\}$ $\square$

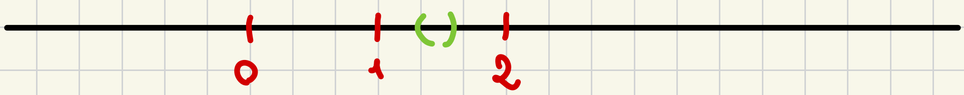
2) $\mathring{A} = ?$

$x \in \mathring{A} \iff (\exists)\ r>0$ a.î. $(x-r, x+r) \subset A$

$\mathring{A} = \emptyset$, deoarece între orice două numere reale $(\exists)$ o infinitate de numere raționale și o infinitate de numere iraționale

$\bar{A} = ?$

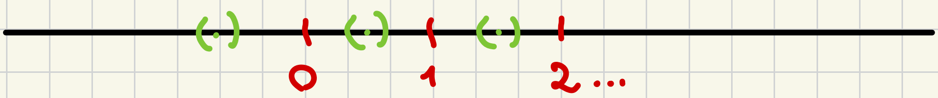
$x \in \bar{A} \iff (\forall)\ r>0$, avem $(x-r, x+r) \cap A \neq \emptyset$



$A \subset \bar{A} \Rightarrow \mathbb{N} \subset \bar{A}$

Fie $x \in \mathbb{R} \setminus \mathbb{N}$

$x \in \bar{A} \iff (\forall)\ r>0$, avem $(x-r, x+r) \cap A \neq \emptyset$



Deci $x \notin \bar{A}$

Așadar, $\bar{A} = \mathbb{N}$

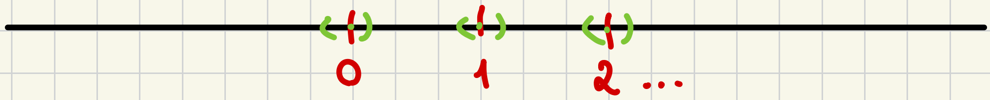
$A' = ?$

$x \in A' \iff (\forall)\ r>0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq 0$

$A' \subset \bar{A} = \mathbb{N}$

Fie $x \in \mathbb{N}$.

$x \in A' \iff (\forall)\ r>0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq 0$



Deci $x \notin A'$.

Prin urmare, $A' = \emptyset$

$Fr(A) = \bar{A} \setminus \mathring{A} = \mathbb{N} \setminus \emptyset = \mathbb{N}$

$Izo(A) = \bar{A} \setminus A' = \mathbb{N} \setminus \emptyset = \mathbb{N}$ □

c)

1) $\mathring{A} = ?$

$x \in \mathring{A} \iff (\exists)\ r>0$, a.î. $(x-r, x+r) \subset A$

$\mathring{A} = \emptyset$, deoarece între orice două numere reale, există o infinitate de numere raționale și o infinitate de numere iraționale

2) $A' = ?$

$x \in A' \iff (\exists)\ (x_k)_k \subset A \setminus \{x\}$ a.î. $\lim\limits_{k \to +\infty} x_k = x$

Orice șir de elemente din A poate avea drept limită fie un element din A (șirurile constante), fie 0.

Deci $A' = \{0\}$

3) $\overline{A} = A \cup A' = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}$

4) $\partial A = \overline{A} \setminus \mathring{A} = (\{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}) \setminus \emptyset = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}$

5) ${}'A = \overline{A} \setminus A' = (\{ \frac{1}{n} \mid n \in \mathbb{N}^* \} \cup \{0\}) \setminus \{0\} = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \}$

$\square$