1. Determinați $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și precizați dacă $(\exists)$

$\lim\limits_{n\to+\infty} x_n$, unde:

a) $x_n = 1+2(-1)^{n+1}+3(-1)^{\frac{n(n+1)}{2}}, \ (\forall)\, n\in\mathbb{N}$

Sol.:

$$x_{4k} = 1+2(-1)^{4k+1}+3(-1)^{\frac{\cancel{4}^{2}k(4k+1)}{\cancel{2}}}$$
$$= 1-2+3$$
$$= 2 \xrightarrow[k\to+\infty]{} 2$$

$$x_{4k+1} = 1+2(-1)^{4k+2}+3(-1)^{\frac{(4k+1)(4k+2)}{2}}$$
$$= 1+2+3(-1)^{(4k+1)(2k+1)}$$
$$= 1+2-3$$
$$= 0 \xrightarrow[k\to+\infty]{} 0$$

$$x_{4k+2} = 1+2(-1)^{4k+3}+3(-1)^{\frac{(4k+2)(4k+3)}{2}}$$
$$= 1-2+3(-1)^{(2k+1)(4k+3)}$$
$$= 1-2-3$$
$$= -4 \xrightarrow[k\to+\infty]{} -4$$

$$x_{4k+3} = 1+2(-1)^{4k+4}+3(-1)^{\frac{(4k+3)(4k+4)}{2}}$$
$$= 1+2+3(-1)^{(4k+3)(2k+2)}$$
$$= 1+2+3$$
$$= 6 \xrightarrow[k\to+\infty]{} 6$$

$$\mathbb{N} = 4\mathbb{N} \cup (4\mathbb{N}+1) \cup (4\mathbb{N}+2) \cup (4\mathbb{N}+3)$$

$$\mathcal{L}((x_n)_n) = \{-4, 0, 2, 6\}$$

$$\left.\begin{aligned}\underline{\lim}\, x_n &= -4\\ \overline{\lim}\, x_n &= 6\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \underline{\lim}\, x_n \neq \overline{\lim}\, x_n \Rightarrow \nexists \lim_{n\to+\infty} x_n \quad \square$$

2c) $x_n = \left(1+\frac{1}{n}\right)^n \sin\frac{n\pi}{3}$, $(\forall) n \in \mathbb{N}^*$.

Sol.:

$$x_{6k} = \left(1+\frac{1}{6k}\right)^{6k} \sin\frac{6k\pi}{3} = \left(1+\frac{1}{6k}\right)^{6k} \sin(2k\pi)$$

$$\xrightarrow[k\to+\infty]{} e\cdot 0 = 0$$

$$x_{6k+1} = \left(1+\frac{1}{6k+1}\right)^{6k+1} \sin\frac{(6k+1)\pi}{3} = \left(1+\frac{1}{6k+1}\right)^{6k+1} \cdot \underbrace{\sin\left(2k\pi+\frac{\pi}{3}\right)}_{\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}}$$

$$\xrightarrow[k\to+\infty]{} \frac{e\sqrt{3}}{2}$$

$$x_{6k+2} = \left(1+\frac{1}{6k+2}\right)^{6k+2} \cdot \sin\frac{(6k+2)\pi}{3} = \left(1+\frac{1}{6k+2}\right)^{6k+2} \cdot$$

$$\sin\left(2k\pi+\frac{2\pi}{3}\right) = \left(1+\frac{1}{6k+2}\right)^{6k+2} \sin\left(\frac{2\pi}{3}\right)$$

$$= \left(1+\frac{1}{6k+2}\right)^{6k+2} \sin\left(\pi-\frac{\pi}{3}\right) = \left(1+\frac{1}{6k+2}\right)^{6k+2} \cdot$$

$$\sin\frac{\pi}{3} \xrightarrow[k\to+\infty]{} \frac{e\sqrt{3}}{2}$$

$$x_{6k+3} = \left(1+\frac{1}{6k+3}\right)^{6k+3} \sin\frac{(6k+3)\pi}{3} = \left(1+\frac{1}{6k+3}\right)^{6k+3} \cdot$$

$$\sin(2k\pi+\pi) = \left(1+\frac{1}{6k+3}\right)^{6k+3} \sin\pi \xrightarrow[k\to+\infty]{} e\cdot 0$$

$$= 0$$

$$x_{6k+4} = \left(1+\frac{1}{6k+4}\right)^{6k+4} \sin\left(\frac{(6k+4)\pi}{3}\right) =$$

$$\left(1+\frac{1}{6k+4}\right)^{6k+4} \sin\left(2k\pi+\frac{4\pi}{3}\right) =$$

$$\left(1+\frac{1}{6k+4}\right)^{6k+4} \cdot \sin\left(\frac{4\pi}{3}\right) = \left(1+\frac{1}{6k+4}\right)^{6k+4} \cdot$$

$$\sin\left(\pi+\frac{\pi}{3}\right) = \left(1+\frac{1}{6k+4}\right)^{6k+4} \cdot \left(-\sin\frac{\pi}{3}\right)$$

$$\xrightarrow[k\to+\infty]{} -\frac{e\sqrt{3}}{2}$$

$$x_{6k+5} = \left(1+\frac{1}{6k+5}\right)^{6k+5} \sin\left(\frac{(6k+5)\pi}{3}\right) =$$

$$= \left(1+\frac{1}{6k+5}\right)^{6k+5} \cdot \sin\left(2k\pi+\frac{5\pi}{3}\right)$$

$$= \left(1+\frac{1}{6k+5}\right)^{6k+5} \cdot \sin\left(\frac{5\pi}{3}\right) = \left(1+\frac{1}{6k+5}\right)^{6k+5} \cdot$$

$$\sin\left(2\pi-\frac{\pi}{3}\right) = \left(1+\frac{1}{6k+5}\right)^{6k+5} \cdot \left(-\sin\frac{\pi}{3}\right)$$

$$\xrightarrow[k\to+\infty]{} -\frac{e\sqrt{3}}{2}$$

$$\mathbb{N}^* = (6\mathbb{N}^*) \cup (6\mathbb{N}+1) \cup (6\mathbb{N}+2) \cup (6\mathbb{N}+3) \cup$$

$$\cup (6\mathbb{N}+4) \cup (6\mathbb{N}+5)$$

$$\mathcal{L}((x_n)_n) = \left\{-\frac{e\sqrt{3}}{2}, 0, \frac{e\sqrt{3}}{2}\right\}$$

$$\underline{\lim}\, x_n = -\frac{e\sqrt{3}}{2}$$

$$\overline{\lim}\, x_n = \frac{e\sqrt{3}}{2}$$

$$\underline{\lim}\, x_n \neq \overline{\lim}\, x_n \Rightarrow (\nexists) \lim_{n\to+\infty} x_n \quad \square$$

c) $x_n = \dfrac{n \cos \frac{n\pi}{2}}{n^2+1}$, $(\forall)\, n \in \mathbb{N}$

Sol.:

$$\left.\begin{array}{l} \lim\limits_{n\to+\infty} \dfrac{n}{n^2+1} = 0 \\ -1 \le \cos \dfrac{n\pi}{2} \le 1, \\ (\forall)\, n\in\mathbb{N} \end{array}\right\} \overset{C.M.}{\Rightarrow} \lim_{n\to+\infty}\left(\frac{n}{n^2+1}\cdot \cos\frac{n\pi}{2}\right) = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\to+\infty} x_n = 0 \Rightarrow \underline{\lim}\, x_n = \overline{\lim}\, x_n = 0 \quad \square$$

2. Determinați suma seriei $\sum\limits_{n=1}^{\infty} \dfrac{n}{(n+1)!}$ și precizați dacă este convergentă.

Sol.:

$x_n = \dfrac{n}{(n+1)!}$, $(\forall)\, n \in \mathbb{N}^*$

$$s_n = x_1 + x_2 + \dots + x_n$$

$$= \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!}$$

$$= \frac{2-1}{2!} + \frac{3-1}{3!} + \dots + \frac{(n+1)-1}{(n+1)!}$$

$$= \frac{2}{2!} - \frac{1}{2!} + \frac{3}{3!} - \frac{1}{3!} + \dots + \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}$$

$$= \frac{1}{1!} - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} + \cancel{\dots} + \cancel{\frac{1}{n!}} - \frac{1}{(n+1)!}$$

$$= 1 - \frac{1}{(n+1)!}, \quad (\forall)\, n \in \mathbb{N}^*$$

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(1 - \frac{1}{(n+1)!}\right) = 1$$

Deci, $\sum\limits_{n=1}^{\infty} x_n = 1 \in \mathbb{R}$.

Prin urmare, $\sum_{n=1}^{\infty} x_n$ este convergentă. $\square$

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3. Studiați convergența (natura) următoarelor serii:

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n^2}$

Sol.:

Fie $x_n = \frac{\sqrt{n-1}}{n^2}, (\forall) n \in \mathbb{N}^*$

$y_n = \frac{\sqrt{n}}{n^2}, (\forall) n \in \mathbb{N}^*$

$x_n < y_n, (\forall) n \in \mathbb{N}^*$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{\cancel{n^{\frac{1}{2}}}}{\cancel{n^2}_{\,n^{2-\frac{1}{2}}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}},$$

convergentă (serie armonică generalizată, $\alpha = \frac{3}{2}$)

Conform criteriului de comparație cu inegalități, avem că $\sum_{n=1}^{\infty} x_n$ este convergentă $\square$

b) $\sum_{n=1}^{\infty} \frac{a^n}{\sqrt[n]{n}}, \ a > 0$

Sol.:

Fie $x_n = \frac{a^n}{\sqrt[n]{n}}, (\forall) n \in \mathbb{N}^*$.

Aplicăm criteriul raportului.

$$\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to+\infty} \frac{a^{\cancel{n+1}}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{\cancel{a^n}} = \lim_{n\to+\infty} a \cdot \underbrace{\frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}}}_{\xrightarrow[n\to+\infty]{} 1} = a$$

1) Dacă $a<1$ (i.e. $a\in(0,1)$), atunci $\sum_{n=1}^{\infty}$ este convergentă.

2) Dacă $a>1$ (i.e. $a\in(1,+\infty)$), atunci $\sum_{n=1}^{\infty} x_n$ este divergentă.

3) Dacă $a=1$, acest criteriu nu decide.

Pentru $a=1$, avem $x_n=\frac{1}{\sqrt[n]{n}}$, $(\forall)\, n\in\mathbb{N}^*$

$$\lim_{n\to+\infty} x_n=\lim_{n\to+\infty}\frac{1}{\sqrt[n]{n}}=\frac{1}{1}=1\neq 0$$

Conform criteriului suficient de divergență, avem

$$\sum_{n=1}^{\infty} x_n \begin{cases} \text{conv. pt. } a\in(0,1) \\ \text{div. pt. } a\in[1,+\infty) \end{cases} \square$$

c) $\sum_{n=1}^{\infty}\left(\frac{an^2+3n+2}{2n^2+n+1}\right)^n$, $a>0$

Sol.:

Fie $x_n=\left(\frac{an^2+3n+2}{2n^2+n+1}\right)^n$, $(\forall)\, n\in\mathbb{N}^*$

Aplicăm criteriul radicalului:

$$\lim_{n\to+\infty}\sqrt[n]{x_n}=\lim_{n\to+\infty}\sqrt[n]{\left(\frac{an^2+3n+2}{2n^2+n+1}\right)^n}=\lim_{n\to+\infty}\left(\frac{an^2+3n+2}{2n^2+n+1}\right)=\frac{a}{2}$$

1) Dacă $\frac{a}{2}<1$ (i.e. $a\in(0,2)$), atunci $\sum_{n=1}^{\infty} x_n$ este convergentă.

2) Dacă $\frac{a}{2}>1$ (i.e. $a\in(2,+\infty)$), atunci $\sum_{n=1}^{\infty} x_n$ este divergentă.

3) Dacă $\frac{a}{2}=1$ (i.e. $a=2$), atunci criteriul nu decide.

Fie $a=2$.

$$x_n=\left(\frac{2n^2+3n+2}{2n^2+n+1}\right)^n, \forall n\in\mathbb{N}^*$$

$$\lim_{n\to+\infty} x_n=\lim_{n\to+\infty}\left(\frac{2n^2+3n+2}{2n^2+n+1}\right)^n=\lim_{n\to+\infty}\left(1+\frac{2n^2+3n+2}{2n^2+n+1}-1\right)^n$$

$$=\lim_{n\to+\infty}\left(1+\frac{\cancel{2n^2}+3n+2-\cancel{2n^2}-n-1}{2n^2+n+1}\right)^n=\lim_{n\to+\infty}\left(1+\frac{2n+1}{2n^2+n+1}\right)^n=$$

$$=\lim_{n\to+\infty}\left[\underbrace{\left(1+\frac{2n+1}{2n^2+n+1}\right)^{\frac{2n^2+n+1}{2n+1}}}_{\xrightarrow{n\to+\infty} e}\right]^{\frac{2n+1}{2n^2+n+1}\cdot n}=e^{\lim\limits_{n\to+\infty}\frac{n(2n+1)}{2n^2+n+1}}=$$

$$=e^1=e\neq 0$$

Conform criteriului suficient de divergență, avem că $\sum\limits_{n=1}^{\infty} x_n$ este divergentă $\square$

d) $\sum\limits_{n=1}^{\infty}\frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$

Sol.:

Fie $x_n=\frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}, \forall n\in\mathbb{N}^*$

$$y_n=\frac{\sqrt{n^2}}{\sqrt{n^3}}, \forall n\in\mathbb{N}^*$$

$$\lim_{n\to+\infty} \frac{x_n}{y_n} = \lim_{n\to+\infty} \left(\frac{\sqrt{n^2+1}}{\sqrt{n^3+1}} \cdot \frac{\sqrt{n^3}}{\sqrt{n^2}} \right) = \lim_{n\to+\infty} \left(\sqrt{\frac{n^5+n^3}{n^5+n^2}} \right) =$$

$$= \sqrt{1} = 1 \in (0, +\infty)$$

Conform criteriului de comparație cu limită, avem că

$$\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n.$$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \sqrt{\frac{n^2}{n^3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} =$$

$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, divergentă (serie armonică generalizată, $\alpha = \frac{1}{2}$)

Deci $\sum_{n=1}^{\infty} x_n$ este divergentă $\square$