1.

a) Studiați continuitatea lui f.

b) Determinați $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

c) Studiați diferențiabilitatea lui f, unde:

i) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{y^3}{x^4+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Sol.:

a) f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operații cu funcții elementare)

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$|f(x,y) - f(0,0)| = \left| \frac{y^3}{x^4+y^2} - 0 \right| = \frac{|y^3|}{x^4+y^2} =$$

$$= \frac{|y| \cdot y^2}{x^4+y^2} = |y| \cdot \underbrace{\frac{y^2}{x^4+y^2}}_{\leq 1} \leq |y| \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow$$

(Explicație: $y^2 \leq x^4+y^2 \;/ : (x^4+y^2)$ $\Rightarrow \frac{y^2}{x^4+y^2} \leq 1$)

$\Rightarrow f$ continuă în $(0,0)$

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{0(x^4+y^2) - 4x^3y^3}{(x^4+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{3y^2(x^4+y^2) - 2y \cdot y^3}{(x^4+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f((0,0)+te_1)-f(0,0)}{t}=$$

$$=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{\frac{0}{t^4+0^2}-0}{t}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{f((0,0)+te_2)-f(0,0)}{t}=$$

$$=\lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=\lim_{t\to 0}\frac{\frac{t^3}{0^4+t^2}-0}{t}=$$

$$=\lim_{t\to 0}\frac{t}{t}=1$$

c) $\left.\begin{array}{l}\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2\setminus\{(0,0)\} \\ \quad\quad\text{(operații cu funcții elementare)} \\ \mathbb{R}^2\setminus\{(0,0)\} \text{ deschisă}\end{array}\right\}\Rightarrow$

$\Rightarrow f$ diferențiabilă pe $\mathbb{R}^2\setminus\{(0,0)\}$

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$, atunci

$df(0,0):\mathbb{R}^2\to\mathbb{R}$, $df(0,0)(u,v)=$

$$=\,^t\left[\begin{pmatrix}\underset{\substack{\parallel\\0}}{\dfrac{\partial f}{\partial x}(0,0)} & \underset{\substack{\parallel\\1}}{\dfrac{\partial f}{\partial y}(0,0)}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}\right]=0\cdot u+1\cdot v=v$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)\big((x,y)-(0,0)\big)}{\|(x,y)-(0,0)\|}$$

$$=\lim_{(x,y)\to(0,0)} \frac{\dfrac{y^3}{x^4+y^2}-0-y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\cancel{y^3}-\cancel{y^3}-yx^4}{\sqrt{x^2+y^2}\,(x^4+y^2)} =$$

$$=\lim_{(x,y)\to(0,0)} \frac{(-yx^4)}{\sqrt{x^2+y^2}\,(x^4+y^2)}$$

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$\left|\frac{-yx^2}{\sqrt{x^2+y^2}\,(x^4+y^2)}-0\right| = |x|\,\frac{|x|}{\sqrt{x^2+y^2}}\cdot\frac{|yx^2|}{x^4+y^2} \le$$

≤ 1 (explicație: $|x|=\sqrt{x^2}\le$

$\le\sqrt{x^2+y^2}\ \ /:\sqrt{x^2+y^2}$

$\Rightarrow \dfrac{|x|}{\sqrt{x^2+y^2}}\le 1$)

$$\le |x|\cdot\frac{|yx^2|}{x^4+y^2} \le \frac{|x|}{2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\le \frac{1}{2}$ (explicație: $\dfrac{x^4+y^2}{2}\ge\sqrt{x^4y^2}=|x^2y|\ \ /:(x^4+y^2)$

$\Rightarrow \dfrac{1}{2}\ge\dfrac{|x^2y|}{x^4+y^2}$

Deci $\displaystyle\lim_{(x,y)\to(0,0)}\frac{(-yx^4)}{\sqrt{x^2+y^2}\,(x^4+y^2)}=0$

Prin urmare, f este diferențiabilă în $(0,0)$ $\square$

ii) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\dfrac{x^7y^8}{\sqrt{x^{28}+y^{28}}} & ;\ (x,y)\neq(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$

Sol.:

a) f continuă pe $\mathbb{R}^2\setminus\{(0,0)\}$ (operații cu funcții elementare)

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$|f(x,y)-f(0,0)|=\left|\frac{x^7y^8}{\sqrt{x^{28}+y^{28}}}-0\right|=\frac{|x|^7\cdot y^8}{\sqrt{x^{28}+y^{28}}}=$$

$$=|y|\frac{|x^7y^7|}{\sqrt{x^{28}+y^{28}}}\le\frac{1}{\sqrt{2}}\cdot|y|\xrightarrow[(x,y)\to(0,0)]{}0\Rightarrow f \text{ continuă în } (0,0)$$

$\le\frac{1}{\sqrt{2}}$ (Explicație: $\dfrac{x^{28}+y^{28}}{2}\ge\sqrt{x^{28}\cdot y^{28}}=$

$=|x^{14}\cdot y^{14}|=x^{14}\cdot y^{14}\Rightarrow\dfrac{\sqrt{x^{28}+y^{28}}}{\sqrt{2}}\ge\sqrt{x^{14}\cdot y^{14}}=$

$=|x^7\cdot y^7|\ \Big|:\sqrt{x^{28}+y^{28}}\Rightarrow\dfrac{1}{\sqrt{2}}\ge\dfrac{|x^7\cdot y^7|}{\sqrt{x^{28}+y^{28}}}$

b) Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y)=\frac{7x^6\cdot y^8\cdot\sqrt{x^{28}+y^{28}}-\dfrac{28x^{27}}{2\sqrt{x^{28}+y^{28}}}\cdot x^7y^8}{x^{28}+y^{28}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{8x^7 \cdot y^7 \cdot \sqrt{x^{28}+y^{28}} - \frac{28y^{27}}{2\sqrt{x^{28}+y^{28}}} \cdot x^7 \cdot y^8}{x^{28}+y^{28}}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^7 \cdot 0}{\sqrt{t^{28}+0}}}{t}$$

$$= \lim_{t\to 0} \frac{\frac{0}{t^{14}}}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{0 \cdot t^8}{\sqrt{0+t^{28}}}}{t} =$$

$$= \lim_{t\to 0} \frac{\frac{0}{t^{14}}}{t} = 0$$

c) $\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue pe } \mathbb{R}^2 \setminus \{(0,0)\} \text{ (operații cu funcții elementare)} \\ \mathbb{R}^2 \text{ deschisă} \end{array}\right\} \Rightarrow$

$\Rightarrow f$ este diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$, avem

$df(0,0) : \mathbb{R}^2 \to \mathbb{R}$, $df(0,0)(u,v) =$

$$= {}^{t}\left[\left(\underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0} \quad \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right] = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|}$$

$$=\lim_{(x,y)\to(0,0)} \frac{\frac{x^7y^8}{\sqrt{x^{28}+y^{28}}}-0-0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^7y^8}{\sqrt{x^{28}+y^{28}}\sqrt{x^2+y^2}}$$

Alegem $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$, $(\forall)\, n\in\mathbb{N}^*$

Avem $\lim_{n\to\infty} (x_n, y_n) = (0,0)$ și

$$\lim_{n\to+\infty} \frac{\frac{1}{n^7}\cdot\frac{1}{n^8}}{\sqrt{\frac{1}{n^{28}}+\frac{1}{n^{28}}}\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} =$$

$$=\lim_{n\to+\infty} \frac{\frac{1}{\cancel{n^{15}}}}{\frac{\sqrt{2}}{\cancel{n^{14}}}\cdot\frac{\sqrt{2}}{\cancel{n}}} = \frac{1}{2} \neq 0$$

Deci $\lim_{(x,y)\to(0,0)} \frac{x^7y^8}{\sqrt{x^{28}+y^{28}}\sqrt{x^2+y^2}} \neq 0$

Prin urmare, f nu este diferențiabilă în origine $\square$

2. Fie $\varphi: \mathbb{R}^2\to\mathbb{R}$ o funcție diferențiabilă și $f:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z) = \varphi(xy, x^2+y^2-z^2)$. Arătați că f este diferențiabilă și $xz\,\frac{\partial f}{\partial x}(x,y,z) - yz\,\frac{\partial f}{\partial y}(x,y,z) + (x^2-y^2)\frac{\partial f}{\partial z}(x,y,z) = 0$, $(\forall)\, x,y,z\in\mathbb{R}^3$.

Sol.:

Fie $g:\mathbb{R}^3\to\mathbb{R}^2$, $g(x,y,z)=(xy,\ x^2+y^2-z^2)$

Fie $u,v:\mathbb{R}^3\to\mathbb{R}$

$$u(x,y,z)=xy$$

$$v(x,y,z)=x^2+y^2-z^2$$

Avem $f=\varphi\circ g$

$$\frac{\partial g}{\partial x}(x,y,z)=(y,2x)$$

$$\frac{\partial g}{\partial y}(x,y,z)=(x,2y) \qquad (\forall)\ (x,y,z)\in\mathbb{R}^3$$

$$\frac{\partial g}{\partial z}(x,y,z)=(0,2z)$$

$\left.\begin{array}{l}\dfrac{\partial g}{\partial x},\ \dfrac{\partial g}{\partial y},\ \dfrac{\partial g}{\partial z} \text{ continue pe } \mathbb{R}^3 \text{ (operații cu funcții elementare)} \\ \mathbb{R}^3 \text{ deschisă}\end{array}\right\} \Rightarrow$

$\left.\begin{array}{l}\Rightarrow g \text{ este diferențiabilă pe } \mathbb{R}^3 \\ \varphi \text{ diferențiabilă pe } \mathbb{R}^2\end{array}\right\} \Rightarrow f=\varphi\circ g$ este diferențiabilă pe $\mathbb{R}^3$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "g=(u,v)"] \arrow[rr, bend right, "f=\varphi\circ g"'] & \mathbb{R}^2 \arrow[r, "\varphi"] & \mathbb{R} \\
(x,y,z) \arrow[u, squiggly] & (u,v) \arrow[u, squiggly] &
\end{tikzcd}$$

$$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial(\varphi\circ g)}{\partial x}(x,y,z)=$$

$$=\frac{d\varphi}{du}(g(x,y,z))\cdot\frac{\partial u}{\partial x}(x,y,z)+\frac{d\varphi}{dv}(g(x,y,z))\cdot\frac{\partial v}{\partial x}(x,y,z)$$

$$= \frac{\partial \varphi}{\partial u}(g(x,y,z)\cdot y + \frac{\partial \varphi}{\partial v}(g(x,y,z)\cdot 2x.$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial(\varphi \circ g)}{\partial y} = \frac{\partial \varphi}{\partial u}(g(x,y,z)\, \frac{\partial u}{\partial y}(x,y,z)$$
$$+ \frac{\partial \varphi}{\partial v}(g(x,y,z)\, \frac{\partial v}{\partial y}(x,y,z)$$

$$= \frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot x + \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot 2y$$

$$\frac{\partial f}{\partial z}((x,y,z)) = \frac{\partial(\varphi \circ g)}{\partial z} = \frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot$$

$$\cdot \frac{\partial u}{\partial z}(x,y,z) + \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot \frac{\partial v}{\partial z}(x,y,z) =$$

$$= \frac{\partial \varphi}{\partial u}(g(x,y,z))\, 0 + \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot(-2z)$$

$$xz\,\frac{\partial f}{\partial x}(x,y,z) - yz\,\frac{\partial f}{\partial y}(x,y,z) + (x^2-y^2)\cdot$$

$$\cdot \frac{\partial f}{\partial z}(x,y,z) = xz\Big(\frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot y +$$

$$+ \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot 2x\Big) - yz\Big(\frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot x +$$

$$+ \frac{\partial \varphi}{\partial v}(g(x,y,z))\cdot 2y\Big) + (x^2-y^2)\Big(\frac{\partial \varphi}{\partial u}(g(x,y,z)\cdot 0$$

$$+ \frac{\partial \varphi}{\partial v}(g(x,y,z)(-2z)\Big) = \ldots = 0$$

3. Determinați punctele de extrem local pentru funcțiile de mai jos și precizați natura lor:

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = -x^4 - y^4 + 4xy$

Sol.:

$\mathbb{R}^2$ deschisă

f continuă (operații cu funcții elementare)

$\frac{\partial f}{\partial x}(x,y) = -4x^3 + 4y$

$(\forall)(x,y) \in \mathbb{R}^2$

$\frac{\partial f}{\partial y}(x,y) = -4y^3 + 4x$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ continue (operații cu funcții elementare)} \\ \mathbb{R}^2 \text{ deschisă} \end{array}\right\} \Rightarrow f \text{ este diferențiabilă pe } \mathbb{R}^2$$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} -4x^3 + 4y = 0 \quad /:(-4) \\ -4y^3 + 4x = 0 \quad /:(-4) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \Leftrightarrow \begin{cases} (y^3)^3 - y = 0 \\ x = y^3 \end{cases}$$

$$y^9 - y = 0 \Leftrightarrow y(y^8 - 1) = 0 \Leftrightarrow y \in \{0, 1, -1\} \Rightarrow$$

$$\Rightarrow x \in \{0, 1, -1\}$$

Punctele critice ale lui f sunt $(0,0)$, $(1,1)$, $(-1,-1)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -12x^2$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -12y^2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 4 \qquad (\forall)\ (x,y) \in \mathbb{R}^2$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 4$$

Observăm că f este de clasă C^2.

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} =$$

$$= \begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix},\ (\forall)\ (x,y) \in \mathbb{R}^2$$

$$Hf(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

$$\Delta_1 = 0\ ,\quad \Delta_2 = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$$

Deci, $(0,0)$ nu este punct de extrem local al lui f

$$Hf(1,1)=\begin{pmatrix}-12 & 4\\ 4 & -12\end{pmatrix}$$

$\Delta_1 = -12<0$

$$\Delta_2 = \begin{vmatrix}-12 & 4\\ 4 & -12\end{vmatrix} = 144-16=128>0$$

Deci, $(1,1)$ este punct de maxim local al lui f

$$Hf(-1,-1) = \begin{vmatrix}-12 & 4\\ 4 & -12\end{vmatrix}$$

$\Delta_1 = -12<0$

$$\Delta_2 = \begin{vmatrix}-12 & 4\\ 4 & -12\end{vmatrix} = 128>0$$

Deci, $(-1,-1)$ este punct de maxim local al lui f

$\square$

b) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^3+8y^3-2xy$

c) $f:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z)=x^2+y^2+z^2-xy+x-2z$

Sol.:

$\mathbb{R}^3$ este deschisă

f continuă (operații cu funcții elementare)

$$\frac{\partial f}{\partial x}(x,y,z) = 2x-y+1$$

$$\frac{\partial f}{\partial y}(x,y,z) = 2y-x \quad (\forall)\ (x,y,z)\in\mathbb{R}^3$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2z-2$$

$\left.\begin{array}{l}\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \text{ continuă (operații cu funcții elementare)} \\ \mathbb{R}^3 \text{ deschisă}\end{array}\right\} \Rightarrow$

$\Rightarrow f$ diferențiabilă pe $\mathbb{R}^3$

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y,z)=0 \\ \dfrac{\partial f}{\partial y}(x,y,z)=0 \\ \dfrac{\partial f}{\partial z}(x,y,z)=0 \end{cases} \Leftrightarrow \begin{cases} 2x-y+1=0 \\ 2y-x=0 \\ 2z-2=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 4y-y+1=0 \\ x=2y \\ z=1 \end{cases} \Leftrightarrow \begin{cases} y=-\frac{1}{3} \\ x=-\frac{2}{3} \\ z=1 \end{cases}$$

Singurul punct critic al lui f este $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = 2$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z)=2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = -1 = \frac{\partial^2 f}{\partial y \partial x}(x,y,z)$$ (din Lema lui Schwartz)

$$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 0 = \frac{\partial^2 f}{\partial z \partial x}(x,y,z)$$

$(\forall)\ (x,y,z)\in\mathbb{R}^3$

$$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 0 = \frac{\partial^2 f}{\partial z \partial y}(x,y,z)$$

Observăm că f este de clasă C^2.

$$Hf(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y,z) & \frac{\partial^2 f}{\partial x \partial y}(x,y,z) & \frac{\partial^2 f}{\partial x \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y,z) & \frac{\partial^2 f}{\partial y^2}(x,y,z) & \frac{\partial^2 f}{\partial y \partial z}(x,y,z) \\ \frac{\partial^2 f}{\partial z \partial x}(x,y,z) & \frac{\partial^2 f}{\partial z \partial y}(x,y,z) & \frac{\partial^2 f}{\partial z^2}(x,y,z) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ (\forall)\ (x,y,z)\in\mathbb{R}^3$$

$$Hf\left(-\frac{2}{3}, -\frac{1}{3}, 1\right) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 2 > 0$

$$\Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 + 0 + 0 - 0 - 2 - 0 = 6 > 0$$

Deci, $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ este punct de minim local al lui f. $\square$

d) $f : (0, +\infty)^3 = (0, +\infty) \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}$,

$$f(x, y, z) = \frac{1}{x} + \frac{x}{y} + \frac{y}{z} + z$$