

Tutoriat 3: Discrete Random Variables and Conditional Probabilities (Revision)

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Exercise 1

Determine the probability of selecting a random two-digit number that is co-prime with 12, i.e.:

$$\gcd(X, 12) = 1,$$

where X denotes the selected number.

Solution:

1. **Total possible outcomes:** The two-digit numbers range from 10 to 99, inclusive. The total number of such integers is $99 - 10 + 1 = 90$.
2. **Favorable outcomes:** We need $\gcd(X, 12) = 1$. Since $12 = 2^2 \times 3$, a number X is co-prime with 12 if and only if it is **not** divisible by 2 and **not** divisible by 3.
3. **Counting using Inclusion-Exclusion:** Let S be the set of 90 two-digit numbers.
 - Let A be the subset of numbers divisible by 2.
 - Let B be the subset of numbers divisible by 3.

We are looking for the size of the set $S \setminus (A \cup B)$, which is $|S| - |A \cup B|$. By the Principle of Inclusion-Exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$.

- $|A|$ (divisible by 2): The numbers are $\{10, 12, \dots, 98\}$. The count is $\frac{98-10}{2} + 1 = \frac{88}{2} + 1 = 44 + 1 = 45$.
 - $|B|$ (divisible by 3): The numbers are $\{12, 15, \dots, 99\}$. The count is $\frac{99-12}{3} + 1 = \frac{87}{3} + 1 = 29 + 1 = 30$.
 - $|A \cap B|$ (divisible by 6): The numbers are $\{12, 18, \dots, 96\}$. The count is $\frac{96-12}{6} + 1 = \frac{84}{6} + 1 = 14 + 1 = 15$.
4. **Number of non-favorable outcomes** (numbers not co-prime with 12): $|A \cup B| = 45 + 30 - 15 = 60$.
 5. **Number of favorable outcomes** (numbers co-prime with 12): $|S| - |A \cup B| = 90 - 60 = 30$.
 6. **Probability:**

$$P(\gcd(X, 12) = 1) = \frac{\text{Favorable Outcomes}}{\text{Total Outcomes}} = \frac{30}{90} = \frac{1}{3}.$$

Exercise 2

A bowl contains 4 black balls (B) and 7 white balls (W). Total = 11 balls. Solve the following subproblems (assuming draws are without replacement):

1. Four balls are drawn. Find the probability that exactly two of them are white, given that the first two balls drawn were of different colors.
2. Write the probability distribution of the random variable X that represents the number of white balls drawn in two random extractions.
3. Find the probability that fewer than two white balls are drawn.
4. Compute the variance of $2X$, i.e. $\text{Var}[2X]$.

Solution:

1. Let E be the event "exactly 2W and 2B are drawn in 4 draws". Let F be the event "the first two balls drawn were of different colors" (W_1B_2 or B_1W_2). We need $P(E|F) = \frac{P(E \cap F)}{P(F)}$.

- $P(F) = P(W_1B_2) + P(B_1W_2) = \left(\frac{7}{11} \times \frac{4}{10}\right) + \left(\frac{4}{11} \times \frac{7}{10}\right) = \frac{28}{110} + \frac{28}{110} = \frac{56}{110} = \frac{28}{55}$.
- $P(E \cap F)$ is the probability that the first two are different AND the total is 2W, 2B.

- Case 1: W_1B_2 . For a total of 2W, 2B, the next two draws ($3^{rd}, 4^{th}$) must be 1W and 1B.

After W_1B_2 , we have 6W and 3B left (Total 9).

$$P(1W, 1B \text{ on draws } 3, 4) = P(W_3B_4) + P(B_3W_4) = \left(\frac{6}{9} \times \frac{3}{8}\right) + \left(\frac{3}{9} \times \frac{6}{8}\right) = \frac{18}{72} + \frac{18}{72} = \frac{36}{72} = \frac{1}{2}.$$

$$\text{So, } P(\text{Case 1}) = P(W_1B_2) \times P(1W, 1B \text{ next} | W_1B_2) = \frac{28}{110} \times \frac{1}{2} = \frac{14}{110}.$$

- Case 2: B_1W_2 . For a total of 2W, 2B, the next two draws must be 1W and 1B.

After B_1W_2 , we have 6W and 3B left (Total 9).

$$P(1W, 1B \text{ on draws } 3, 4) = \frac{1}{2} \text{ (same as above). So, } P(\text{Case 2}) = P(B_1W_2) \times P(1W, 1B \text{ next} | B_1W_2) = \frac{28}{110} \times \frac{1}{2} = \frac{14}{110}.$$

$$P(E \cap F) = P(\text{Case 1}) + P(\text{Case 2}) = \frac{14}{110} + \frac{14}{110} = \frac{28}{110} = \frac{14}{55}.$$

- $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{14/55}{28/55} = \frac{14}{28} = \frac{1}{2}.$

2. X = number of white balls in 2 draws. X can take values $\{0, 1, 2\}$.

- $P(X = 0) = P(B_1B_2) = \frac{4}{11} \times \frac{3}{10} = \frac{12}{110} = \frac{6}{55}.$
- $P(X = 1) = P(W_1B_2) + P(B_1W_2) = \left(\frac{7}{11} \times \frac{4}{10}\right) + \left(\frac{4}{11} \times \frac{7}{10}\right) = \frac{56}{110} = \frac{28}{55}.$
- $P(X = 2) = P(W_1W_2) = \frac{7}{11} \times \frac{6}{10} = \frac{42}{110} = \frac{21}{55}.$

The probability distribution is:

$$X = \begin{pmatrix} 0 & 1 & 2 \\ 6/55 & 28/55 & 21/55 \end{pmatrix}$$

(Check: $6 + 28 + 21 = 55$, so probabilities sum to 1).

3. $P(X < 2) = P(X = 0) + P(X = 1) = \frac{6}{55} + \frac{28}{55} = \frac{34}{55}$. (Alternatively: $1 - P(X = 2) = 1 - \frac{21}{55} = \frac{34}{55}$).

4. $\text{Var}[2X] = 2^2 \text{Var}[X] = 4 \text{Var}[X]$. We need $\text{Var}[X] = E[X^2] - (E[X])^2$.

- $E[X] = (0 \times \frac{6}{55}) + (1 \times \frac{28}{55}) + (2 \times \frac{21}{55}) = \frac{0+28+42}{55} = \frac{70}{55} = \frac{14}{11}$.
- $E[X^2] = (0^2 \times \frac{6}{55}) + (1^2 \times \frac{28}{55}) + (2^2 \times \frac{21}{55}) = \frac{0+28+4 \times 21}{55} = \frac{28+84}{55} = \frac{112}{55}$.
- $\text{Var}[X] = \frac{112}{55} - \left(\frac{14}{11}\right)^2 = \frac{112}{55} - \frac{196}{121} = \frac{112 \times 11 - 196 \times 5}{605} = \frac{1232 - 980}{605} = \frac{252}{605}$.

$$\text{Var}[2X] = 4 \times \text{Var}[X] = 4 \times \frac{252}{605} = \frac{1008}{605}.$$

Exercise 3 (Hat Problem - EX#1)

Before a big party, n people leave their hats in a cloakroom. After the party, each person randomly picks one hat. What is the average number of people who get their own hat back?

Solution:

Solution 1 (Using Indicator Variables): Let X be the random variable representing the number of people who get their own hat back. Let X_i be an indicator variable for the i -th person:

$$X_i = \begin{cases} 1, & \text{if person } i \text{ gets their own hat} \\ 0, & \text{otherwise} \end{cases}$$

The probability that person i gets their own hat is $P(X_i = 1) = \frac{1}{n}$, as there is one correct hat for them out of n total hats.

The expected value of an indicator variable is the probability of the event it indicates:

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1) = \frac{1}{n}$$

The total number of people who get their own hat is the sum of these indicator variables:

$$X = \sum_{i=1}^n X_i$$

By the linearity of expectation, the expected number of people who get their own hat is:

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

So, the average number of people who get their own hat back is 1, regardless of the number of people n .

Solution 2 (Using Permutations): We can model the way people get their hats back using a random permutation σ of the set $\{1, 2, \dots, n\}$. Here, $\sigma(i) = j$ means that person i took the hat belonging to person j . We are looking for the number of "fixed points," i.e., the number of people i such that $\sigma(i) = i$. Let X be the random variable for the number of fixed points in a random permutation. We want to find $E[X]$.

There are $n!$ total possible permutations. Let's count the total number of fixed points across all $n!$ permutations. Consider a specific person i . How many permutations have i as a fixed point (i.e., $\sigma(i) = i$)? If i is fixed, the remaining $n - 1$ people and $n - 1$ hats can be arranged in $(n - 1)!$ ways. Since there are n people who could be a fixed point, the total number of fixed points across all permutations is:

$$\sum_{i=1}^n (\text{Number of permutations where } \sigma(i) = i) = \sum_{i=1}^n (n - 1)! = n \cdot (n - 1)! = n!$$

The expected value $E[X]$ is the total number of fixed points divided by the total number of permutations:

$$E[X] = \frac{\text{Total fixed points}}{\text{Total permutations}} = \frac{n!}{n!} = 1$$

Exercise 4 (Computer Problem - EX#2)

In a computer lab, there are 10 computers. Each computer has a 0.5 probability of working (independently). What is the probability that the 6th computer tested is the 4th one that works?

Solution: This is a problem that can be solved using the Negative Binomial distribution, or more simply, by breaking it down into two independent events.

Let E be the event that "the 6th computer tested is the 4th one that works." This event E can be split into two necessary and independent conditions:

1. Let A be the event that "exactly 3 of the first 5 computers work."
2. Let B be the event that "the 6th computer works."

We need both of these events to occur, so we are looking for $P(E) = P(A \cap B)$. Since the computers are independent, $P(A \cap B) = P(A) \times P(B)$.

- **Calculating $P(B)$:** The probability that any single computer works is given as 50%.

$$P(B) = 0.5 = \frac{1}{2}$$

- **Calculating $P(A)$:** This is a binomial probability problem. We have $n = 5$ trials (the first 5 computers) and we want exactly $k = 3$ successes (working computers). The probability of success p is 0.5. The probability mass function for a Binomial distribution is $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$.

$$P(A) = \binom{5}{3} (0.5)^3 (1 - 0.5)^{5-3} = \binom{5}{3} (0.5)^3 (0.5)^2$$

We know that $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4}{2 \times 1} = 10$.

$$P(A) = 10 \cdot (0.5)^5 = 10 \cdot \frac{1}{32} = \frac{10}{32} = \frac{5}{16}$$

- **Calculating $P(E)$:**

$$P(E) = P(A) \times P(B) = \frac{5}{16} \times \frac{1}{2} = \frac{5}{32}$$

The probability that the 6th computer is the 4th one that works is $\frac{5}{32}$.

Exercise 5 (Insurance Problem - EX#3)

An insurance company divides its clients into two categories: 20

1. Andrei, a client of this company, has not had an accident in the last year. What is the probability that he is a high-risk client?
2. What if he drives for 3 years without an accident?

Solution: This is a Bayesian inference problem. Let's define our events:

- R : The event that Andrei is a High-Risk client.
- S : The event that Andrei is a Low-Risk (Scăzut) client.
- A_k : The event that Andrei has k accidents in a given period.

We are given the prior probabilities:

- $P(R) = 0.2$
- $P(S) = 0.8$

The number of accidents is modeled by a Poisson distribution, $P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$. The likelihood of having $k = 0$ accidents is $P(X = 0) = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda}$.

The average number of accidents is the parameter λ :

- For High-Risk: $\lambda_R = 1$ accident/year.
- For Low-Risk: $\lambda_S = 0.1$ accidents/year.

a) Probability after 1 year with 0 accidents

We want to find $P(R|A_0)$ in 1 year). Using Bayes' theorem:

$$P(R|A_0) = \frac{P(A_0|R) \cdot P(R)}{P(A_0)}$$

First, let's find the conditional probabilities (likelihoods) for 1 year:

- $P(A_0|R) = e^{-\lambda_R} = e^{-1}$
- $P(A_0|S) = e^{-\lambda_S} = e^{-0.1}$

Next, we find the total probability of having 0 accidents, $P(A_0)$, using the law of total probability:

$$\begin{aligned} P(A_0) &= P(A_0|R) \cdot P(R) + P(A_0|S) \cdot P(S) \\ &= (e^{-1})(0.2) + (e^{-0.1})(0.8) \end{aligned}$$

Now we can calculate the posterior probability:

$$P(R|A_0) = \frac{0.2 \cdot e^{-1}}{0.2 \cdot e^{-1} + 0.8 \cdot e^{-0.1}}$$

Using the values $e^{-1} \approx 0.36788$ and $e^{-0.1} \approx 0.90484$:

$$P(R|A_0) \approx \frac{0.2 \cdot (0.36788)}{0.2 \cdot (0.36788) + 0.8 \cdot (0.90484)} = \frac{0.073576}{0.073576 + 0.723872} = \frac{0.073576}{0.797448} \approx 0.09226$$

So, there is approximately a ****9.2%**** chance he is a high-risk client.

b) Probability after 3 years with 0 accidents

For a 3-year period, the Poisson parameter λ scales linearly (assuming accidents in one year are independent of another).

- New High-Risk parameter: $\lambda'_R = 1/\text{year} \times 3 \text{ years} = 3$
- New Low-Risk parameter: $\lambda'_S = 0.1/\text{year} \times 3 \text{ years} = 0.3$

We want to find $P(R|A_0 \text{ in 3 years})$. The likelihoods for 3 years are:

- $P(A_0 (3y)|R) = e^{-\lambda'_R} = e^{-3}$
- $P(A_0 (3y)|S) = e^{-\lambda'_S} = e^{-0.3}$

Using Bayes' theorem again:

$$P(R|A_0 (3y)) = \frac{P(A_0 (3y)|R) \cdot P(R)}{P(A_0 (3y)|R) \cdot P(R) + P(A_0 (3y)|S) \cdot P(S)}$$

$$P(R|A_0 (3y)) = \frac{0.2 \cdot e^{-3}}{0.2 \cdot e^{-3} + 0.8 \cdot e^{-0.3}}$$

Using the values $e^{-3} \approx 0.04979$ and $e^{-0.3} \approx 0.74082$:

$$P(R|A_0 (3y)) \approx \frac{0.2 \cdot (0.04979)}{0.2 \cdot (0.04979) + 0.8 \cdot (0.74082)} = \frac{0.009958}{0.009958 + 0.592656} = \frac{0.009958}{0.602614} \approx 0.01652$$

After 3 years with no accidents, the probability that he is a high-risk client drops to approximately ****1.65%****.