Examen 2023 - 2024
GAL

I. $f: \mathbb{R}^3 \to \mathbb{R}^3$ apl liniară $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$

a) ? $w \in Ker(f)$ $\quad w = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$
$\in Im f$

r: $w \in Ker(f) \Leftrightarrow f(w) = 0$

$\Leftrightarrow A \cdot w = 0$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 9 \end{pmatrix} \neq 0_{\mathbb{R}_3}$$

$\Rightarrow w \notin Ker(f)$

r: $w \in Im f \Leftrightarrow \exists x \in \mathbb{R}^3$ a.î. $f(x) = w$

$$\Rightarrow Ax = w \Rightarrow \begin{cases} y - z = 1 \\ x + 3z = 2 \\ \cancel{3x + 3y = -1} \end{cases}$$

$$\det A = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 3 & 3 & 0 \end{vmatrix} = 6 \neq 0 \Rightarrow A \text{ SLI}$$

$\Rightarrow \dim A = \dim \mathbb{R}^3 \Rightarrow A$ surjectivă

$\Rightarrow w \in \mathbb{R}^3 \in Im f$

b) w vector propriu $\Leftrightarrow \exists \lambda \in \sigma(A)$

a.î. $A \cdot w = \lambda \cdot w \Rightarrow \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ -\lambda \end{pmatrix}$

$\lambda = 3$, $\lambda = -5 \Rightarrow \overset{3 \neq -5}{\cancel{\lambda_0}} \Rightarrow w$

nu e vec. propriu

c) $\lambda \in K$, $\lambda \in \sigma(f) \Leftrightarrow \det(A - \lambda I_3) = 0$

$\Leftrightarrow \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & 3 \\ 3 & 3 & -\lambda \end{vmatrix} = -\lambda^3 - 3 + 9 - 3\lambda + 9\lambda + \lambda = 0$

$\Leftrightarrow -\lambda^3 + 7\lambda + 6 = 0$

$\Leftrightarrow \lambda^3 - 7\lambda - 6 = 0$

$\Leftrightarrow (\lambda+1)(\lambda-3)(\lambda+2) = 0$

$\Rightarrow \sigma(f) = \{-2, -1, 3\}$

$$\begin{array}{l|l} \lambda^3 - 7\lambda - 6 & \lambda + 1 \\ \hline \lambda^3 + \lambda^2 & \lambda^2 - \lambda - 6 \\ -\lambda^2 - 7\lambda - 6 & \\ -\lambda^2 - \lambda - \cancel{6} & \\ -6\lambda - 6 & \\ -6\lambda - 6 & \\ \hline 0 & \end{array}$$

I $\lambda = -2$

$A * V = \lambda \cdot v \Rightarrow (A - I\cdot\lambda)v = 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left|\begin{matrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ 3 & 3 & 2 \end{matrix}\right| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \left|\begin{matrix} L_1 - L_2 \\ \longrightarrow \\ L_3 - 3L_2 \end{matrix}\right| \begin{matrix} 1 & -1 & -4 \\ 1 & 2 & 3 \\ 0 & -3 & -4 \end{matrix} \left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right|$$

$$\xrightarrow{L_2 + L_1} \left|\begin{matrix} 1 & -1 & -4 \\ 0 & 1 & -1 \\ 0 & -3 & -4 \end{matrix}\right| \begin{matrix} L_3 + 3L_2 \\ \longrightarrow \\ L_1 + L_2 \end{matrix} \left|\begin{matrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{matrix}\right|$$

$$\left|\begin{matrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ 3 & 3 & 2 \end{matrix}\right| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \left|\begin{matrix} L_1 - 2L_2 \\ \longrightarrow \\ L_3 - 3L_2 \end{matrix}\right| \begin{matrix} 0 & -3 & -7 \\ 1 & 2 & 3 \\ 0 & -3 & -7 \end{matrix} \left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right|$$

$$\Rightarrow \left|\begin{matrix} 1 & 2 & 3 \\ 0 & -3 & -7 \end{matrix}\right| \begin{matrix} 0 \\ 0 \end{matrix} \Big| \Leftrightarrow \begin{cases} x + 2y + 3z = 0 \\ -3y - 7z = 0 \end{cases}$$

$$\Rightarrow y = -\frac{7}{3}z \Rightarrow x = \frac{14}{3} - \frac{9}{3}z$$

$$x = \frac{5}{3}z$$

$$\Rightarrow v = \left\langle \left(\frac{5}{3}, -\frac{7}{3}, 1 \right) \right\rangle$$

$\underline{II}$ $\lambda = -1$

$$\Rightarrow \left|\begin{matrix} 1 & 1 & -1 \\ 1 & 1 & 3 \\ 3 & 3 & 1 \end{matrix}\right| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \left| \xrightarrow[L_3-3L_2]{L_1-L_2} \right| \begin{matrix} 0 & 0 & -4 \\ 1 & 1 & 3 \\ 0 & 0 & -8 \end{matrix} \left| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right|$$

$\Rightarrow z = 0$, $x = -y \Rightarrow V = \langle (1, -1, 0) \rangle$

$\underline{III}$ $\lambda = 3$

$$\left|\begin{matrix} -3 & 1 & -1 \\ 1 & -3 & 3 \\ 3 & 3 & -3 \end{matrix}\right| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \left| \xrightarrow[L_1+3L_2]{L_3+L_2} \right| \begin{matrix} 0 & -8 & 8 \\ 1 & -3 & 3 \\ 4 & 0 & 0 \end{matrix} \left| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right|$$

$\Rightarrow x = 0$, $y = z$

$\Rightarrow V = \langle (0, 1, 1) \rangle$

d) $A_{diag} = P D P^{-1}$

$$P = \begin{pmatrix} \frac{5}{3} & 1 & 0 \\ -\frac{7}{3} & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad P^{-1} = \left(\begin{matrix} \frac{5}{3} & 1 & 0 \\ -\frac{7}{3} & -1 & 0 \\ 1 & 0 & 1 \end{matrix} \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right.\right)$$

$$\left(\begin{matrix} 5 & 3 & 0 \\ -7 & -3 & 0 \\ 1 & 0 & 1 \end{matrix} \left| \begin{matrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{matrix} \right.\right) \xrightarrow{L_1+L_2} \left(\begin{matrix} -2 & 0 & 0 \\ -7 & -3 & 0 \\ 1 & 0 & 1 \end{matrix} \left| \begin{matrix} 3 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{matrix} \right.\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 \\ \cancel{-7} & \cancel{-3} & 0 & 0 & -3 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 - L_1]{L_2 - 7L_1} \left(\begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{array}\right.$$

$$\left.\begin{array}{ccc} -\frac{3}{2} & -\frac{3}{2} & 0 \\ \frac{21}{2} & \frac{15}{2} & 0 \\ \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \text{—} & 4 & \text{—} \\ 0 & 1 & 0 & \frac{21}{6} & \frac{15}{6} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right)$$

$$\Rightarrow D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = P \cdot D \cdot P^{-1}$$

A diag. $\Leftrightarrow$ are valori proprii $| = \mathrm{rg}\, A$

inclusiv radacini duble

II a) $\det |A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & -4 \\ -1 & 2 & 3 \end{vmatrix} \neq 0 \Rightarrow$ SLI

$\Rightarrow$ Bază

$\Rightarrow \dim V = \mathrm{rg}\, A = 3 \Rightarrow \mathbb{R}^3 = V$

b) $\det B = \begin{vmatrix} 3 & 5 \\ -1 & 29 \\ 3 & 13 \end{vmatrix} \neq 0 \Rightarrow$ baza

$\Rightarrow \dim V = rg\, B = 2$

c) $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

$= 3 + 2 - 2 = 3$

$\left.\begin{array}{l} U = \mathbb{R}^3 \\ V \subset \mathbb{R}^3 \end{array}\right| \Rightarrow \dim(U \cap V) = \dim(V)$

o bază în $U \cap V$ ar putea fi $B_0 = \{e_1, e_2, e_3\}$

d) $\dim(U \cap V) = \dim V = 2$ B e bază în $U \cap V$

III $f(x) = 9x^2 + 24xy + 16y^2 - 40x + 30y$

$A = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$ $\quad \delta = 9 \cdot 16 - 144 = 0$

$\Rightarrow$ parabolă

$\tilde{A} = \begin{pmatrix} 9 & 12 & -20 \\ 12 & 16 & 15 \\ -20 & 15 & 0 \end{pmatrix}$

$\sigma(A) = ?$

$$\det(A - \lambda J_2) = \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 144 - 144 + \lambda^2 - 16\lambda - 9\lambda$$

$$= \lambda(\lambda - 25) = 0 \Rightarrow \sigma(A) = \{0, 25\}$$

vec. pp = ?

i) $\lambda = 0 \Rightarrow \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{matrix} 3u + 4v = 0 \\ 4u + 3v = 0 \end{matrix} \quad \cancel{\Rightarrow u - v = 0} \quad \Leftrightarrow u = v$$

$3x = -4y \quad x = -\frac{4}{3} y \Rightarrow vec = \left(-\frac{4}{3}, 1\right)$

$\Leftrightarrow (-4, 3)$

ii) $\lambda = 25 \Rightarrow \begin{pmatrix} -16 & 12 \\ 12 & 9 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

$-4u + 3v = 0 \Rightarrow u = \frac{3}{4} v$

$4u + 3v = 0$

$\Rightarrow x = \left(\frac{3}{4}, 1\right) \Leftrightarrow (3, 4)$

$$\begin{array}{cc} \lambda=0 & \lambda=25 \end{array}$$

$$P = \begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \qquad \tilde{u} = \frac{u}{\|u\|} = \left(-\frac{4}{5}, \frac{3}{5}\right) \quad \tilde{x}$$

$$u \qquad v \qquad \tilde{v} = \frac{v}{\|v\|} = \left(\frac{3}{5}, \frac{4}{5}\right) \quad \tilde{y}$$

$$\Rightarrow f(x) = \cancel{9x'^2} + \overset{25}{\cancel{16}}y'^2 - 40\left(-\frac{4}{5}x' + \frac{3}{5}y'\right)$$

$$+ 30\left(\frac{3}{5}x' + \frac{4}{5}y'\right) = \cancel{9x'^2} + 50x' + \overset{25}{\cancel{16}}y'^2$$

$$\cancel{= 9\left(x' + \frac{25}{9}\right)^2 - \frac{25^2}{9^2}} + 25y'^2 + 50x$$

$$\cancel{9(x'')^2 + 16(y'')^2 = \frac{625}{81}}$$

$$\cancel{\frac{81\cdot 9}{625}(x'')^2 + \frac{81\cdot 16}{625}(y'')^2} = 1$$

$$25(y'')^2 = -50x$$

$$(y'')^2 = -2x''$$

Focar: ec canonica de tip parabolic

$4p = -2 \Rightarrow p = -\frac{1}{2}$

$(y'')^2 = -2x''$ axa de simetrie este o

$\Rightarrow F'' = \left(0, -\frac{1}{2}\right)$

$x' = \frac{1}{5}(-4x + 3y)$

$y' = \frac{1}{5}(3x + 4y)$

$$\begin{pmatrix} x_F \\ y_F \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

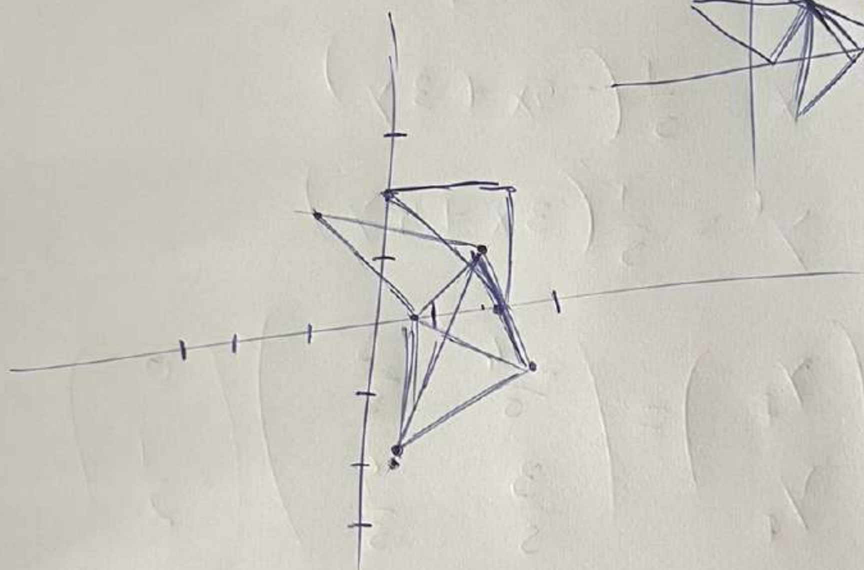
$$= \begin{pmatrix} -\frac{3}{10} \\ -\frac{2}{5} \end{pmatrix}$$

$\Rightarrow F\left(-\frac{3}{10}, -\frac{2}{5}\right)$

IV $A' = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = (\sqrt{3}, 1)$

$B' = (-1, \sqrt{3})$ $\quad C' = (\sqrt{3}-1, \sqrt{3}+1)$

$A'' = (\sqrt{3}, 1)$ $\quad B'' = (1, -\sqrt{3})$ $\quad C'' = (\sqrt{3}+1, 1-\sqrt{3})$



$|M_{B_0} f| = \cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6} = 1 \in SO(2) \subset O(2)$

$|M_{B_0} g| = -\left(\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}\right) = -1 \in O(2)$

$\Rightarrow$ f, g pastreaza distantele

$\Rightarrow [AB] \equiv [A'B'] \equiv [A''B'']$, $[AC] \equiv [A'C'] \equiv [A''C'']$

$[BC] \equiv [B'C'] \equiv [B''C''] \overset{L.L.L}{\Rightarrow} \triangle ABC \equiv \triangle A'B'C' \equiv \triangle A''B''C''$

3) $h = ~~g~~$

ii)

$M(h) = \cancel{M_f \cdot M(g)}\ M(g) \cdot M(f)$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

cum $g, f \in SO(2)$ ~~s~~ sau $O(2)$

$\Rightarrow$ sunt bijective $\Rightarrow g, f$ ~~endom~~ automorf

stim endomorfe

$\Rightarrow g, f \in Aut(E_0^2) \to g \circ f \in Aut(E_0^2)$

reprezentarea in coord.

$f(x,y) = M(f) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{\sqrt{3}}{2} - \frac{1}{2} y, \frac{\sqrt{3}}{2} y + \frac{1}{2} x \right)$

$= \frac{1}{2} (\sqrt{3} x - y, \cancel{\sqrt{3}x}\ x + \sqrt{3} y)$

$g(x,y) = \frac{1}{2} (\sqrt{3} x + y, x - \sqrt{3} y)$

$h(x,y) = (x, -y)$

5) ?

V $A=\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{25} \end{pmatrix}$ $\tilde{A}=\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{25} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

9·25) 100) 36)

$I = \frac{1}{4}+\frac{1}{9}+\frac{1}{25} = \frac{311}{900}$

II $\tilde{I} = -\frac{589}{900}$

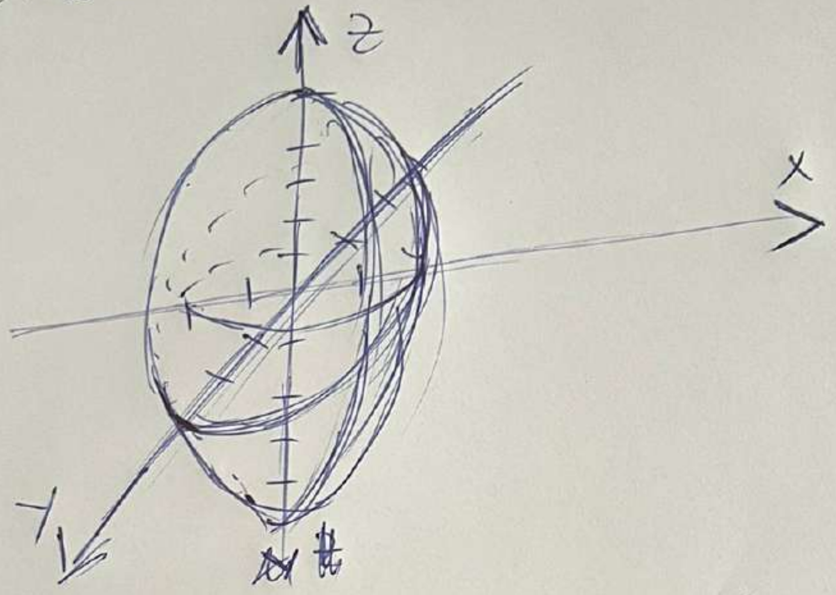
$\delta = \frac{1}{900} \neq 0 \Rightarrow$ are centru unic

$\Delta = -\frac{1}{900}$

$$\left.\begin{aligned} \frac{\partial Q}{\partial x}(x,y,z) &= \frac{x}{2}=0 \\ \frac{\partial Q}{\partial y}(x,y,z) &= \frac{2}{9}y=0 \\ \frac{\partial Q}{\partial z}(x,y,z) &= \frac{2}{25}z=0 \end{aligned}\right\} \Rightarrow C(0,0,0)$$

$\delta > 0$, $\Delta < 0$ $\Rightarrow$ elipsoid

ec. canonică



VI a) $\langle u, v \rangle = 18$

$u \times v = (4-0)e_3 + (10-8)e_1 + (0-4)e_2 = (2, -4, 4)$

b) $\langle u \times v, w \rangle = \begin{vmatrix} 2 & 0 & 2 \\ 5 & 2 & 0 \\ 4 & 2 & 2 \end{vmatrix} = 8 + 20 - 16 = 12$

c) $\|u \times v\| = \sqrt{4+16+16} = 6$

d) $\cos(\widehat{u, v}) = \dfrac{\langle u, v \rangle}{\|u\| \cdot \|v\|} = \dfrac{\cancel{18}\,3}{3\sqrt{5} \cdot 2\sqrt{2}} = \dfrac{3}{\sqrt{10}}$