1. Studiați uniform continuitatea funcțiilor:

a) $f:[0,+\infty)\to\mathbb{R},\ f(x)=\sqrt{x}$;

b) $f:[1,2)\to\mathbb{R},\ f(x)=\frac{1}{x}$.

**Sol.:**

a) $f'(x)=\frac{1}{2\sqrt{x}},\ (\forall)\ x\in(0,+\infty)$

$|f'(x)|=\left|\frac{1}{2\sqrt{x}}\right|=\frac{1}{2\sqrt{x}}\leq\frac{1}{2},\ (\forall)\ x\in[1,+\infty)$

$\Rightarrow f|_{[1,+\infty)}$ este u.c.

$$\left.\begin{array}{l} f|_{[0,1]} \text{ este continuă} \\ [0,1] \text{ mulțime compactă} \end{array}\right\} \Rightarrow f|_{[0,1]} \text{ este u.c}$$

$$\left.\begin{array}{l} f|_{[0,1]} \text{ u.c.} \\ f|_{[1,+\infty)} \text{ u.c.} \end{array}\right\} \Rightarrow f \text{ u.c. (pe } [0,+\infty)) \quad \square$$

b) $f'(x)=-\frac{1}{x^2},\ (\forall)\ x\in[1,2)$

**Sol.:**

$|f'(x)|=\left|-\frac{1}{x^2}\right|=\frac{1}{x^2}\leq 1,\ (\forall)\ x\in[1,2)$

$\Rightarrow f$ u.c. (pe $[1,2)$) $\square$

c) $f:(0,+\infty)\to\mathbb{R},\ f(x)=\frac{1}{x}$

**Sol.:**

Alegem $(x_n)_n\subset(0,+\infty),\ x_n=\frac{1}{n},\ (\forall)\ n\in\mathbb{N}^*$ și $(y_n)_n\in$

$\in(0,+\infty)$, $y_n=\frac{1}{2n}$, $(\forall)\, n\in\mathbb{N}^*$

Avem: $\lim\limits_{n\to+\infty}(x_n-y_n)=\lim\limits_{n\to+\infty}\left(\frac{1}{n}-\frac{1}{2n}\right)=0$ și

$$\lim_{n\to+\infty}(f(x_n)-f(y_n))=\lim_{n\to+\infty}\left(\frac{1}{\frac{1}{n}}-\frac{1}{\frac{1}{2n}}\right)$$

$$\lim_{n\to+\infty}(n-2n)=\lim_{n\to+\infty}(-n)=-\infty\neq 0$$

Deci $f$ nu este u.c. $\square$

**2.** Fie $a\geq 0$ și $f:(a,+\infty)\to\mathbb{R}$, $f(x)=\ln x$. Arătați că $f$ este u.c. $\Leftrightarrow a>0$.

**Sol.:**

**„$\Leftarrow$":**

Presupunem că $a>0$. Arătăm că $f$ u.c.

$f'(x)=\frac{1}{x}$, $(\forall)\, x\in(a,+\infty)$

$|f'(x)|=\left|\frac{1}{x}\right|=\frac{1}{x}$, $(\forall)\, x\in(a,+\infty)$

$\frac{1}{x}<\frac{1}{a}$, $(\forall)\, x\in(a,+\infty)\Rightarrow|f'(x)|<\frac{1}{a}$, $(\forall)\, x\in(a,+\infty)$

$\Rightarrow f$ u.c. pe $(a,+\infty)$

**„$\Rightarrow$":**

Presupunem că $f$ este u.c. Arătăm că $a>0$.

Presupunem prin absurd că $a\not>0$. Deci $a\leq 0$.

Cum $a\geq 0$, rezultă că $a=0$.

Alegem $(x_n)_n\subset(0,+\infty)$, $x_n=\frac{1}{n}$, $(\forall)\, n\in\mathbb{N}^*$ și

$(y_n)_n\subset(0,+\infty)$, $y_n=\frac{1}{2n}$, $(\forall)\, n\in\mathbb{N}^*$

Avem $\lim\limits_{n\to+\infty}(x_n-y_n)=\lim\limits_{n\to+\infty}\left(\frac{1}{n}-\frac{1}{2n}\right)=0$ și

$$\lim_{n\to+\infty}(f(x_n)-f(y_n))=\lim_{n\to+\infty}\left(\ln\frac{1}{n}-\ln\frac{1}{2n}\right)=$$

$$=\lim_{n\to+\infty}\ln\left(\frac{\frac{1}{\cancel{n}}}{\frac{1}{2\cancel{n}}}\right)=\ln 2\neq 0$$

Deci $f$ nu este u.c., contradicție!

Rămâne că $a>0$. $\square$

3. Fie $f:(0,\frac{2}{\pi}]\to\mathbb{R}$, $f(x)=\sin\frac{1}{x}$. Arătați că $f$ nu este u.c.

Sol.:

Presupunem prin absurd că $f$ u.c., deci $(\exists)$ $\tilde{f}:[0,\frac{2}{\pi}]\to\mathbb{R}$ continuă a.î. $\tilde{f}\big|_{(0,\frac{2}{\pi}]}=f$

$\tilde{f}$ continuă în $0 \Rightarrow \lim\limits_{x\to 0}\tilde{f}(x)=\tilde{f}(0)\in\mathbb{R}$

Dar, $\lim\limits_{\substack{x\to 0\\ x>0}}\tilde{f}(x)=\lim\limits_{\substack{x\to 0\\ x>0}}f(x)=\lim\limits_{\substack{x\to 0\\ x>0}}\sin\frac{1}{x}$

$\tilde{f}\big|_{(0,\frac{2}{\pi}]}=f$

Deci, $\lim\limits_{\substack{x\to 0\\ x>0}}\sin\frac{1}{x}=\tilde{f}(0)\in\mathbb{R}$

Arătăm că $(\nexists)$ $\lim\limits_{\substack{x\to 0\\ x>0}}\sin\frac{1}{x}$

$\sin y = 0 \Leftrightarrow y = n\pi$, $(\forall) n \in \mathbb{N}$

$\sin y = 1 \Leftrightarrow y = 2n\pi + \frac{\pi}{2}$, $(\forall) n \in \mathbb{N}$

Alegem $x_n = \frac{1}{n\pi}$, $(\forall) n \in \mathbb{N}^*$ și $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, $(\forall) n \in \mathbb{N}^*$

Avem $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} y_n = 0$ și

$$\lim_{n\to+\infty} \sin\frac{1}{x_n} = \lim_{n\to+\infty} \sin(n\pi) = 0,$$

$$\lim_{n\to+\infty} \sin\frac{1}{y_n} = \lim_{n\to+\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$$

$0 \neq 1 \Rightarrow (\nexists) \lim\limits_{\substack{x\to 0\\ x>0}} \sin\frac{1}{x}$, contradicție cu

$\lim\limits_{\substack{x\to 0\\ x>0}} \sin\frac{1}{x} = \tilde{f}(0) \in \mathbb{R}$

Rămâne că $f$ nu este u.c. $\square$

4. Studiați convergența simplă și uniformă pentru următoarele șiruri de funcții:

a) $f_n : [0, +\infty) \to \mathbb{R}$, $f_n(x) = \frac{x}{x+n}$, $(\forall) n \in \mathbb{N}^*$.

Sol.: C.s.

Fie $x \in [0, +\infty)$.

$\lim\limits_{n\to+\infty} f_n(x) = \lim\limits_{n\to+\infty} \frac{x}{x+n} = 0 \Rightarrow f_n \xrightarrow[n\to+\infty]{s} f$, unde

$f : [0, +\infty)$, $f(x) = 0$

C.u.

$$\sup_{x\in[0,+\infty)} \left(|f_n(x)-f(x)|\right) = \sup_{x\in[0,+\infty)} \left|\frac{x}{x+n} - 0\right| =$$

$$= \sup_{x\in[0,+\infty)} \left|\frac{x}{x+n}\right| = \sup_{x\in[0,+\infty)} \frac{x}{x+n} \underset{x=n}{\geq} \frac{n}{n+n} = \frac{\cancel{n}}{2\cancel{n}} =$$

$$= \underset{\neq 0}{\frac{1}{2}} \;\not\xrightarrow[n\to+\infty]{}\; 0 \Rightarrow f_n \;\not\xrightarrow[n\to+\infty]{u}\; f$$

b) $f_n : [2,3] \to \mathbb{R},\ f_n(x) = \dfrac{x}{x+n},\ (\forall)\ n\in\mathbb{N}^*$

Sol.: C.s.

Fie $x\in[2,3]$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{x}{x+n} = 0 \Rightarrow f_n \xrightarrow[n\to+\infty]{s} f, \text{ unde}$$

$f : [2,3] \to \mathbb{R},\ f(x) = 0$

C.u.

$$\sup_{x\in[2,3]} |f_n(x) - f(x)| = \sup_{x\in[2,3]} \left|\frac{x}{x+n} - 0\right| =$$

$$= \sup_{x\in[2,3]} \left|\frac{x}{x+n}\right|$$

Fie $f_n : [2,3] \to \mathbb{R},\ f_n(x) = \dfrac{x}{x+n},\ (\forall)\ n\in\mathbb{N}^*$

$$f_n'(x) = \frac{\cancel{x}+n-\cancel{x}}{(x+n)^2} = \frac{n}{(x+n)^2} > 0,\ (\forall)\ n\in\mathbb{N}^*, (\forall)\ x\in[2,3]$$

| $x$ | $2$ | | | $3$ |
|---|---|---|---|---|
| $f_n'(x)$ | | $+ + + + + + + + + +$ | | |
| $f_n(x)$ | $\frac{2}{2+n}$ | $\nearrow$ | $\nearrow$ | $\frac{3}{3+n}$ |

Deci $\sup\limits_{x\in[2,3]} \left|\frac{x}{x+n}\right| = \frac{3}{3+n} \xrightarrow[n\to+\infty]{} 0$

Prin urmare, $f_n \xrightarrow[n\to+\infty]{u} f$ $\square$

c) $f_n : [0,+\infty) \to \mathbb{R}$, $f_n(x) = \sqrt{x^2+\frac{1}{n}}$, $(\forall)\, n\in\mathbb{N}^*$

Sol.: C.s.

Fie $x\in[0,+\infty)$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \sqrt{x^2+\frac{1}{n}} = \sqrt{x^2} = |x| = x$$

$$\Rightarrow f_n \xrightarrow[n\to+\infty]{s} f, \text{ unde } f:[0,+\infty)\to\mathbb{R},\ f(x)=x$$

C.u.

$$\sup_{x\in[0,+\infty)} |f_n(x)-f(x)| = \sup_{x\in[0,+\infty)} \left|\sqrt{x^2+\frac{1}{n}} - x\right| =$$

$$= \sup_{x\in[0,+\infty)} \left(\sqrt{x^2+\frac{1}{n}} - x\right) = \sup_{x\in[0,+\infty)} \frac{\cancel{x^2}+\frac{1}{n}-\cancel{x^2}}{\sqrt{x^2+\frac{1}{n}}+x}$$

Amplificăm cu conjugata

$$= \sup_{x\in[0,+\infty)} \frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+x} = \frac{\frac{1}{n}}{\sqrt{0^2+\frac{1}{n}}+0} = \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \sqrt{\frac{1}{n}} \xrightarrow[n\to+\infty]{} 0$$

$$\Rightarrow f_n \xrightarrow[n\to+\infty]{u} f \quad \square$$

d) $f_n:[0,+\infty)\to\mathbb{R},\ f_n(x)=\dfrac{n}{n+x},\ (\forall)\ n\in\mathbb{N}^*$

**Sol.:** C.s.

Fie $x\in[0,+\infty)$.

$$\lim_{n\to+\infty} f_n(x)=\lim_{n\to+\infty}\frac{n}{n+x}=1 \Rightarrow f_n \xrightarrow[n\to+\infty]{s} f, \text{ unde}$$

$f:[0,+\infty)\to\mathbb{R},\ f(x)=1$

C.u.

$$\sup_{x\in[0,+\infty)} |f_n(x)-f(x)| = \sup_{x\in[0,+\infty)}\left|\frac{n}{n+x}-1\right| =$$

$$= \sup_{x\in[0,+\infty)}\left(1-\frac{n}{n+x}\right) = \sup_{x\in[0,+\infty)} \frac{\cancel{n}+x-\cancel{n}}{n+x} =$$

$$= \sup_{x\in[0,+\infty)} \frac{x}{n+x}$$

Fie $g_n:[0,+\infty)\to\mathbb{R},\ g_n(x)=\dfrac{x}{n+x},\ (\forall)\ n\in\mathbb{N}^*$

$$g_n'(x)=\frac{x'(n+x)-x(n+x)'}{(n+x)^2}=\frac{n+\cancel{x}-\cancel{x}}{(n+x)^2}=\frac{n}{(n+x)^2}>0,$$

$(\forall)\ n\in\mathbb{N}^*,\ (\forall)\ x\in[0,+\infty)$

| $x$ | $0$ | $+\infty$ |
|---|---|---|
| $g_n'(x)$ | + + + + + + + + | |
| $g_n(x)$ | $0$ $\nearrow$ | |

$$\sup_{x\in[0,+\infty)} \frac{x}{x+n} = \lim_{x\to+\infty} \frac{x}{x+n} = 1 \not\xrightarrow[n\to+\infty]{} 0 \Rightarrow$$

$$\Rightarrow f_n \not\xrightarrow[n\to+\infty]{u} f$$

e) $f_n : (0,1] \to \mathbb{R},\ f_n(x) = x^n,\ (\forall)\ n \in \mathbb{N}^*$

**Sol.:** C.s.

Fie $x \in (0,1]$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} x^n = \begin{cases} 0, & x\in(0,1) \\ 1, & x=1 \end{cases} \Rightarrow$$

$$\Rightarrow f_n \xrightarrow[n\to+\infty]{s} f, \text{ unde } f:(0,1]\to\mathbb{R},\ f(x) = \begin{cases} 0, & x\in(0,1) \\ 1, & x=1 \end{cases}$$

C.u.

$$\left.\begin{array}{l} f_n \text{ continuă în } 1,\ (\forall)\ n\in\mathbb{N}^* \\ f \text{ nu este continuă în } 1 \end{array}\right\} \Rightarrow f_n \not\xrightarrow[n\to+\infty]{u} f \quad \square$$