CALCULUS

with Early Transcendentals

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Cover Design: Tee Jay Zajac

Cover Sculpture:
Arabesque XXIX
12″ H × 10½″ W × 9½″ D
Bubinga Wood
by Robert Longhurst
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Chapter Opening Artwork:
Calculus Series
4.625″ H × 8.315″ W
Digital Painting
by Jameson Deichman
www.jamesondeichman.com

HAWKES LEARNING

A division of Quant Systems, Inc.
546 Long Point Road, Mount Pleasant, SC 29464

Library of Congress Control Number: 2014957010

ISBN: 978-1-935782-21-6

ABOUT THE COVER

Arabesque XXIX

The sculpture on the cover of this text is a piece entitled *Arabesque XXIX* by the American artist Robert Longhurst (b. 1949). Although not a mathematician, Longhurst explains his work in this way: "It just so happens that what I produce sometimes turns out with an orientation toward math. My interest is in creating pieces that have appealing form. Proportion and scale are very important as is the craft aspect of my work." His pieces, which can be found at www.robertlonghurst.com, have captured the attention of many mathematicians over the years, and his work has appeared or been discussed in several math texts.

The object depicted by *Arabesque XXIX* is an example of an **Enneper surface**, a simpler version of which was introduced by the German mathematician Alfred Enneper (1830–1885) in 1864. Enneper, with his contemporary Karl Weierstrass (1815–1897), made great strides in understanding and characterizing *minimal surfaces*. A minimal surface is one that spans a given boundary curve with, locally, the least surface area, meaning that the surface area cannot be decreased by deforming any part of the surface slightly (a soap film spanning a boundary formed by a simple closed loop of wire is an example of a naturally occurring minimal surface). Except for the simplest cases, the area of a surface is a concept that is difficult to even define, let alone calculate, without the benefit of calculus; you will learn how calculus is used to define and determine surface area in Chapter 6.

An alternate, but equivalent, characterization is that a minimal surface is one for which each point has zero *mean curvature*, meaning that the maximum curvature of the surface at the point is equal in magnitude (but opposite in sign) to its minimum curvature. The *curvature* of a given curve at a particular point is a characteristic with precise meaning, which you will learn about in Chapter 12. The surface in Figure 1, a portion of the original Enneper surface, has zero mean curvature at every point; this is illustrated for the central point shown in red, as the curvatures of the two dashed curves add to zero. This particular configuration is also an example of a *saddle point*, which you will study further in Chapter 13.

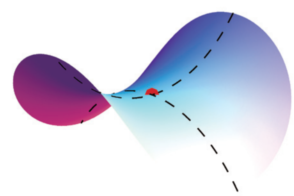


Figure 1

The characterization of minimal surfaces that Enneper and Weierstrass discovered is expressed in terms of integrals of functions of the complex plane, and their work falls into the categories of *differential geometry* and *calculus of variations*, two branches of mathematics that build upon the calculus you'll learn in this text. But the formulas resulting from their integrals are examples of *parametric surfaces*, which you'll study in detail in Chapter 15.

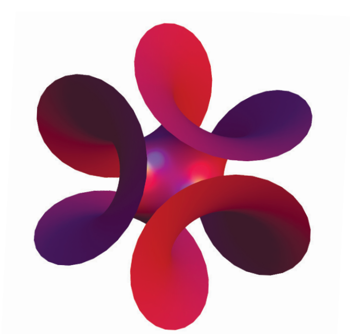


Figure 2

Specifically, the surface of *Arabesque XXIX* is similar to that defined parametrically by

$$x = r\cos\theta - \frac{r^5}{5}\cos 5\theta$$

$$y = -r\sin\theta - \frac{r^5}{5}\sin 5\theta$$

$$z = \frac{2r^3}{3}\cos 3\theta$$

where $0 \le r \le 1.38$ and $0 \le \theta \le 2\pi$. The graph of the surface defined by this parametrization appears in Figure 2.

1.5 Calculus, Calculators, and Computer Algebra Systems

TOPICS

1. Graphs via calculators and computers
2. Animations and models
3. Least-squares curve fitting

The technology available at any given time inevitably colors the ways in which mathematics is learned and used. The historical interplay between calculus and technology is extensive, and no discourse on calculus would be complete without a discussion of computational tools. The tools available on graphing calculators and computers are the focus of this section.

TOPIC 1 Graphs via Calculators and Computers

The graphing capabilities of modern calculators and mathematical software are an especially useful technological addition. While pictures can be misleading and must be used with a small amount of skepticism, there is no denying that the ability to quickly sketch curves and surfaces greatly speeds up the process of solving many problems. The details on the use of a particular calculator or software package are best left to the user's manual, but some features are common across all graphing technology.

One of the most basic features is the ability to choose the **display** or **viewing window** when graphing a function. When graphing functions from $\mathbb{R}$ to $\mathbb{R}$, this is simply the choice of the minimum and maximum values for the horizontal and vertical axes. For the purposes of illustration, we will refer to these values as *xMin*, *xMax*, *yMin*, and *yMax* (on some calculators, these are the exact labels for these quantities). The display window is then a rectangle in the plane bounded by these values, $[xMin, xMax]$ by $[yMin, yMax]$, and their choice determines the portion of the function being graphed. It is important to note that this choice effectively gives the user the ability to zoom in (or out) on a particular part of a graph.

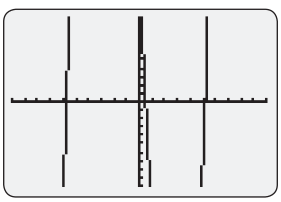


Figure 1

$f(x) = x^3 - 30x + 15$ on $[-10,10]$ by $[-10,10]$

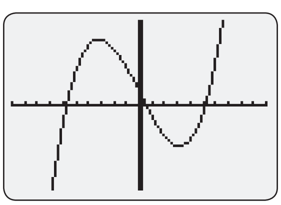


Figure 2

$f(x) = x^3 - 30x + 15$ on $[-10,10]$ by $[-100,100]$

Example 1

Graph the function $f(x) = x^3 - 30x + 15$ in the following viewing windows using a graphing calculator.

a. $[-10,10]$ by $[-10,10]$ **b.** $[-10,10]$ by $[-100,100]$

Solution

a. We set $xMin = -10$, $xMax = 10$, $yMin = -10$, and $yMax = 10$. The resulting graph is displayed in Figure 1. Note that it appears to be cut off at the top and bottom, indicating that the graph continues vertically beyond the viewing window.

b. For this viewing window we just need to change the range of y-values to be from $yMin = -100$ to $yMax = 100$. Figure 2 shows the graph in this new viewing window. This is a more complete picture of the graph of $f(x)$, revealing the significant parts of the graph.

The next example illustrates the use of a graphing calculator and a computer algebra system to graph an interesting part of the function $f(x) = x^2 \sin(1/x)$.

Example 2

We will illustrate the graph of the function $f(x)=x^2\sin(1/x)$, showing the graph at different magnifications or viewing windows.

The first observation we wish to make is that since $\sin(1/x)$ cannot take any values above 1 or below -1, the graph of $f(x)$ cannot go above x^2 or below $-x^2$, as illustrated in Figure 3.

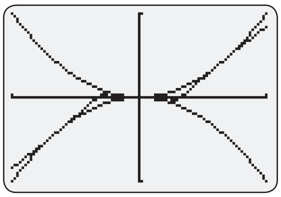


Figure 3a Graphs of x^2, $f(x)$, and $-x^2$

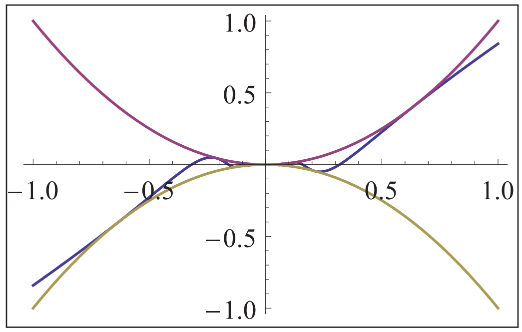


Figure 3b Graphs of x^2, $f(x)$, and $-x^2$

In order to better understand the graph of $f(x)$, let us take a closer look at $g(x)=\sin(1/x)$. As you might recall from your experience with precalculus, $g(x)$ oscillates between -1 and $+1$ (like any well-mannered sine function would), but it does so in a surprising way (see Figure 4): it makes infinitely many oscillations near 0 on both sides of the y-axis. This is most easily checked by noting that $g(x)=\sin(1/x)=0$ for $x=1/(n\pi)$, $n=\pm1,\pm2,\ldots$. In other words, $g(x)=0$ for infinitely many values of x that approach 0 as n grows large. In a similar fashion, one can show that $g(x)=1$ and $g(x)=-1$ infinitely many times on both sides of 0.

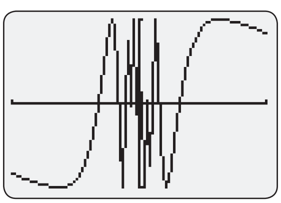


Figure 4a $g(x)=\sin(1/x)$

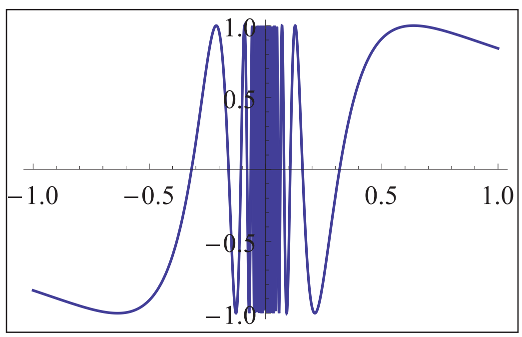


Figure 4b $g(x)=\sin(1/x)$

The effect of multiplying $g(x)$ by x^2 is that $f(x)=x^2\sin(1/x)$ will still oscillate in a similar fashion, but now between x^2 and $-x^2$ (again, you might recall "damped trigonometric graphs" from your prior studies). The oscillation of $f(x)$ is not very clear in Figure 3 but is well illustrated if we ask our technology to "zoom in" or, equivalently, choose a "smaller" viewing window, as shown in Figure 5.

Example 1

As an example of the use of the exponential growth model, suppose a strain of bacteria being cultured in a petri dish is observed to double in count every hour. That information alone is sufficient to determine the *growth constant* k, under the assumption that t is measured in hours, as follows.

$$
\begin{aligned}
P(1) &= 2P(0) && \text{Population at 1 hour is twice the initial population.}\\
P_0 e^{k(1)} &= 2P_0 && \text{Use } P(t) = P_0 e^{kt}.\\
e^k &= 2 && \text{Cancel } P_0.\\
k &= \ln 2 && \text{Solve for } k.
\end{aligned}
$$

If the culture began with, say, $P_0 = 15$ bacteria, the population of bacteria (rounded to the nearest integer) and the rate of population growth (rounded to one decimal place) at the 0, $\frac{1}{2}$, 1, and 2 hour marks are as follows.

Population $P(t) = 15e^{(\ln 2)t}$	Rate of population growth $P'(t) = (\ln 2)P(t)$
$P(0) = 15e^0 = 15$	$P'(0) = (\ln 2)P(0) \approx 10.4$ bacteria/hour
$P\left(\frac{1}{2}\right) = 15e^{(1/2)\ln 2} = 15e^{\ln\left(2^{1/2}\right)} = 15\left(2^{1/2}\right) \approx 21$	$P'\left(\frac{1}{2}\right) = (\ln 2)P\left(\frac{1}{2}\right) \approx 14.7$ bacteria/hour
$P(1) = 15e^{\ln 2} = 15(2) = 30$	$P'(1) = (\ln 2)P(1) \approx 20.8$ bacteria/hour
$P(2) = 15e^{2\ln 2} = 15e^{\ln\left(2^2\right)} = 15\left(2^2\right) = 60$	$P'(2) = (\ln 2)P(2) \approx 41.6$ bacteria/hour

TOPIC 2 Physics Applications

Newton's Second Law of Motion often appears as $F = ma$, where F represents force, m mass, and a acceleration. This is a simplified version of Newton's actual observation, however, which is that the net force on an object is equal to the rate of change (with respect to time) of its momentum: $F = dP/dt$, where momentum P is the product of the object's mass m and velocity v. The Product Rule allows us to rewrite the second law as follows.

Theorem

Newton's Second Law of Motion

$$F = \frac{d}{dt}(mv) = \frac{dm}{dt}v + m\frac{dv}{dt}$$

The fact that mass is constant in many elementary applications accounts for the frequent reduction to the familiar $F = m(dv/dt)$, or $F = ma$.

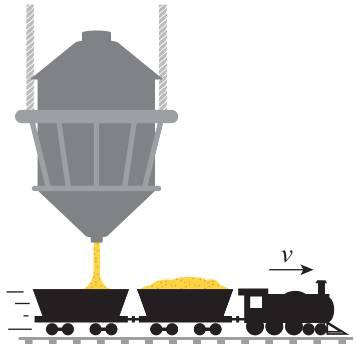


Figure 1

Example 2

A freight train is slowly moving forward at a constant velocity v under a hopper that is dropping grain at a constant rate into the train's open-topped freight cars. In order to maintain its constant velocity, the train's engine must exert a force (beyond that necessary to counter friction) equal to the rate of change of its momentum. What is that force?

Solution

In this setting, v is constant and hence the dv/dt (or acceleration) factor is 0. But the mass is changing, and so Newton's Second Law tells us that the force necessary to overcome the increasing load is

$$F = \frac{dm}{dt}v + m \cdot 0 = \frac{dm}{dt}v.$$

Using units of kilograms (kg) for mass, meters (m) for distance, seconds (s) for time, and newtons (N) for force, if the train has a constant forward velocity of $\frac{1}{2}$ m/s and is being loaded at a rate of 300 kg/s, the force the engine must exert is

$$F = \frac{dm}{dt}v = (300\ \text{kg/s})\left(\frac{1}{2}\ \text{m/s}\right) = 150\,(\text{kg}\cdot\text{m})/\text{s}^2 = 150\ \text{N}.$$

Newton's laws of motion are remarkable for their brevity and versatility—their principles accurately describe the behavior of seemingly unrelated physical situations. As another example of the use of his second law, consider the following experiment.

Example 3

A rope of length L and mass M is held vertically over a scale so that its lower end just touches the scale, and is then allowed to drop onto the scale. What force does the scale register as the rope drops onto it?

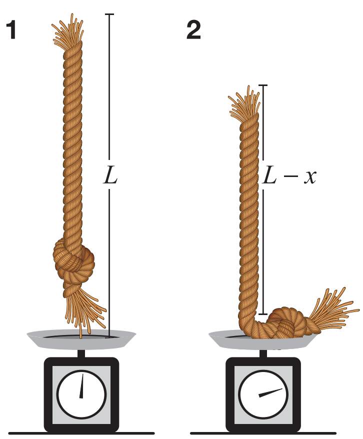


Figure 2

Solution

The answer to this question *after* the rope has fully dropped onto the scale and is at rest is easily found with an elementary application of Newton's Second Law: the force registering at that point is the mass of the rope M times the acceleration due to gravity g. The product Mg is what we normally call *weight*, and Mg is indeed what would show on the scale's dial. But the more interesting question is what registers on the scale during intermediate stages when, say, a segment of length x has landed on the scale and the segment of length $L - x$ has yet to hit.

For convenience, let m denote the mass of the segment of length x that has dropped and is lying at rest on the scale. At any moment in time we know that $m = M(x/L)$ (the total mass times the fraction of the rope on the scale), but it is important to keep in mind that m is continuously changing over time from the moment the rope is dropped to the time when $x = L$. The

13–24 Determine the differential dy for the given values of x and dx.

13. $y = 3x^2 + x; \quad x = 1, \quad dx = 0.2$

14. $y = x\sqrt{x-5}; \quad x = 6, \quad dx = 0.01$

15. $y = \dfrac{4x+1}{x-3}; \quad x = 2, \quad dx = 0.1$

16. $y = \sec x; \quad x = \dfrac{\pi}{4}, \quad dx = \dfrac{1}{8}$

17. $y = x^{3/2} + x^{-3/2}; \quad x = 4, \quad dx = \dfrac{1}{16}$

18. $y = \ln x + \dfrac{1}{\ln x}; \quad x = e, \quad dx = 0.01$

19. $y = x\tan x; \quad x = -\dfrac{\pi}{4}, \quad dx = \dfrac{1}{4}$

20. $y = e^{\sqrt{x^2+3}}; \quad x = 1, \quad dx = 0.001$

21. $y = \sqrt{\ln(x+1)}; \quad x = e-1, \quad dx = \dfrac{-1}{e^2}$

22. $y = \arctan x; \quad x = -1, \quad dx = \dfrac{-1}{2^5}$

23. $y = \dfrac{\tan x}{x^2+1}; \quad x = \dfrac{\pi}{3}, \quad dx = -0.1$

24. $y = \cos(\arcsin x); \quad x = 0.6, \quad dx = -0.16$

25–28 Calculate the values of dy and Δy and then use graph paper to draw the curve near the given point, indicating all three of the line segments dx, dy, and Δy.

25. $y = \dfrac{1}{2}x^2; \quad x = 1, \quad dx = \dfrac{1}{2}$

26. $y = \tan x; \quad x = 0, \quad dx = \dfrac{\pi}{6}$

27. $y = 2^x; \quad x = 1, \quad dx = \dfrac{1}{4}$

28. $y = \dfrac{1}{x^2}; \quad x = 1, \quad dx = -\dfrac{1}{4}$

29–40. Find the values of Δy and compare them with dy at the indicated points for the curves given in Exercises 13–24.

41–48 Use linear approximation to approximate the given number. Compare this approximation to the actual value obtained using a calculator or computer algebra system. Round your answer to four decimal places. (**Hint:** First identify $f(x)$ and c; then find and appropriately evaluate $L(x)$.)

41. $\sqrt{9.1}$

42. $(1.01)^3$

43. $(7.9)^{2/3}$

44. $\dfrac{1}{10.1}$

45. $\sqrt[5]{31}$

46. $\cos 1$

47. $\ln 2.7$

48. $e^{1.05}$

49. Prove the power and quotient rules for differentials.

a. $d(x^n) = nx^{n-1}dx$

b. $d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}$

50. Use the equations for V and dV from Example 4 to prove that the propagated error in the calculated volume of a sphere, in percentage terms, is three times larger than the margin of error in the measured radius; that is,

$$\frac{dV}{V} = 3\frac{dr}{r}.$$

51. Prove or disprove that an analogous equation to that obtained in Exercise 50 is true for a cube; that is, if the measured side length of a cube is a units with a margin of error of da, then

$$\frac{dV}{V} = 3\frac{da}{a}.$$

52–71 Use differentials or linearization to provide the requested approximations.

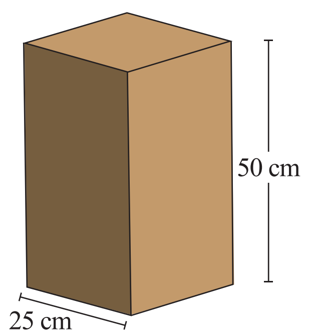
52. The side of a square was measured to be 9.5 cm with a possible error of 0.5 mm. Approximate the propagated error in the calculated area of the square. Express your answer as a percentage error.

53. The radius of a circular disk was measured to be $10\frac{1}{8}$ inches. Estimate the maximum allowable error in the measurement of the radius if the percentage error in the calculated area of the disk cannot exceed 2.5 percent.

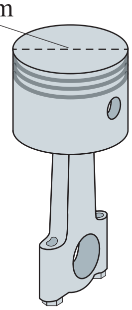
54. The base and altitude of a triangle were measured to be 7 and 9 inches, respectively. If the possible error in both cases is $\frac{1}{16}$ inches, approximate the propagated error when computing the area of the triangle.

55. Two sides of a triangle were measured to be 60 and 80 mm, respectively, while the included angle is 60 degrees. If the margin of error of the linear measurements is 0.1 mm, while that of the angle measurement is 0.1 degrees, find the possible propagated error in the calculated area of the triangle.

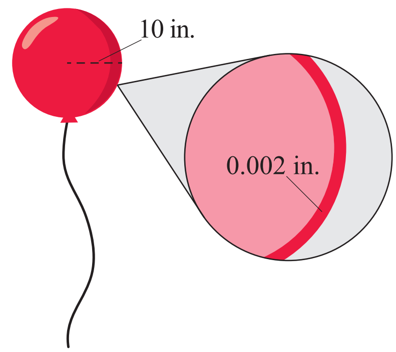
56. A box in the shape of a rectangular prism has a square base. If the edge of the base is 25 cm and the height is 50 cm, both with a possible measurement error of 0.2 mm, estimate the propagated errors in both the computed volume and surface area of the box. Express both answers as percentage errors.



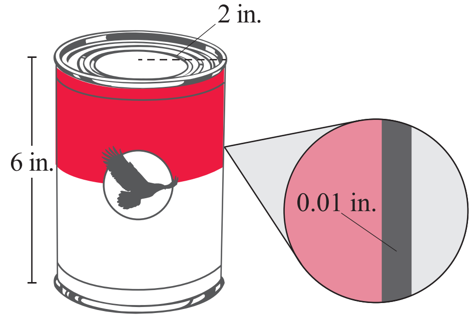
57. A piston of diameter 84 mm is being manufactured for an automobile engine. If the maximum percentage error in the measurement of the diameter is 0.05%, estimate the greatest possible value of the propagated error in the computed cross-sectional area of the piston. Express your answer as a percentage error.



58. If the radius of an inflated balloon is 10 inches and the thickness of its wall is 0.002 inches, estimate the volume of the material it is made of. (Assume the balloon is perfectly spherical.)



59. A tin can has a circular base of radius 2 inches and a height of 6 inches. If the thickness of its walls is 0.01 inches, estimate the volume of the material it is made of.



60. The exterior of a small private observatory needs to be painted. The building is approximately a circular cylinder with a hemisphere on top. The radius of the base is 3.5 feet and the height of the entire structure is 10 feet. Express the volume as a function of the radius of the base and use linearization to estimate the amount of paint that will provide a coat that is $\frac{1}{32}$ inches thick.