Theory of Interest - Homework #9

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(4.2) Level annuities with payments less frequent than each interest period

Problem 4.2.1

An annuity pays \$100 at the end of each quarter for ten years. The payments are made directly to a savings account with a nominal interest rate of 4.85% payable monthly, and they are left in the account.

(a) We want to find the effective interest rate for a quarter, and use it to compute the balance in the savings account immediately after the last payment.

We are given the nominal interest rate payable monthly $i^{(12)}$, however we need to use the effective interest rate per quarter j. We compute for j to find

$$j = (1 + \frac{i^{(12)}}{12})^{12} - 1 \approx 1.2174071\%$$

Now that we have found the effective interest rate per quarter we may solve for our accumulated payments. We calculate the accumulated value to find

$$100 s_{\overline{40}|j} = 100 \left(\frac{(1+j)^{40} - 1}{j} \right) \approx \$5,114.05$$

(b) We want to use formula (4.2.4) to recalculate the balance in the savings account immediately after the last payment. Make sure that your answer agrees with your answer to part (a), and note which method you found easier.

Recall formula (4.2.4)

$$s_{\overline{r}|I} = \frac{s_{\overline{n}|i}}{s_{\overline{k}|i}} \tag{1}$$

for the annuity lasting n interest periods and paying 1 at the end of k interest period. (Also note $(1+I)^r = (1+i)^n$ and $rk = n$)

Using the above equation we can convert between the interest rates to find that

$$100 s_{\overline{40}|j} = 100 \frac{s_{\overline{120}|i}}{s_{\overline{3}|i}} \approx \$5,114.05$$

(c) We want to recalculate the balance in the savings account immediately after the last payment by setting P/Y = 4 and C/Y = 12. We input the following into our BA II Plus Calculator

$$N = 40 \qquad I/Y = 4.85 \qquad PV = 0 \qquad PMT = -100 \qquad FV = ?$$

We then press, super excited, [CPT] + [FV] to find that $FV = \$5,114.05$

Problem 4.2.2

Anurag receives an annuity that pays $\$1,000$ at the end of each month. He wishes to replace it with an annuity that has the same term and has only one payment each year, and that payment should be at the betginning of the year. We want to find how much the payments should be if the exchange is based on a nominal discount rate of 3% payable quarterly.

Note*: as in the previous problem there are three ways to solve for the solution. Furthermore we have the three calculations for the monthly effective interest rate I, quarterly effective interest rate i, and nominal interest rate $i^{(4)}$ we find that:

$$1000 a_{\overline{12}|I} = 1000 \frac{a_{\overline{4}|i}}{s_{\overline{1/3}|i}} \approx \$11,806.30$$

or using the BA II Plus calculator with the conditions

$$\begin{array}{ccccc} C/Y = 4 & P/Y = 12 & & & \\ N = 12 & I/Y = i^{(4)} & PV = ? & PMT = -1000 & FV = 0 \end{array}$$

gives us the result ([CPT] + [PV]) $PV \approx \$11,806.30$.

(4.3) Level annuities with payments more frequent than each interest period

Problem 4.3.1

We want to calculate the annuity symbols $\ddot{s}^{(4)}_{\overline{23}|2.25\%}$ and $a^{(12)}_{\overline{\infty}|\,4\%}$. Carefully describe what each one measures.

First we want to identify what $\ddot{s}^{(4)}_{\overline{23}|2.25\%}$ actually means. In order to full understand this symbol let us identify an almost similar symbol $s_{\overline{23}|2.25\%}$. The latter symbol represents the accumulation of 23 end of the term payments at an effective interest rate of 2.25%; that is, the future value of the sum of the 23 payments. Now what happens whenever we add the subscript $^{(4)}$? Well it splits every single payment into more payments; that is, the payments happen 4 times as frequent and the payments are $\frac{1}{4}$. These payments are spaced to where if we accumulated the 4 payments of $\frac{1}{4}$ we would get an accumulated payment equal to the original, and we start the payments at time 0 (it is now an annuity-due with (4)(23) payments of 1/4. Mathematically speaking we have the following equation

$$\ddot{s}^{(4)}_{\overline{23}|2.25\%} = \frac{1}{4}\ddot{s}^{(4)}_{\overline{92}|j}$$

where j is the quarter effective percent of our original or also written

$$j = (1 + .0225)^{1/4} - 1 \approx 0.005578153$$

Furthermore giving us the value

$$\frac{1}{4}\ddot{s}^{(4)}_{\overline{92}|j} = \frac{1}{4}s^{(4)}_{\overline{92}|j}(1+j) \approx 30.11565502$$

*Note: whenever dealing with double dot it indicates that our first payment at time $t = 0$ there is a payment of 1/4 instead of zero (versus the basic annuity-immediate with payments of 1/4)

Next we want to identify the symbol $a^{(12)}_{\overline{\infty}|\,4\%}$. This annuity, just like the last annuity takes an infinite amount of payments of 1 at the end of the term and splits them into smaller payments of 1/4 at the end of smaller terms of 1/12 starting at time $t = 1/12$, and the payments last forever! Mathematically speaking we have

$$a^{(12)}_{\overline{\infty}|\,4\%} = \frac{1}{12}a_{\overline{\infty}|j}$$

where j is equivalent to the effective interest of 1/12 of the length of the original; that is

$$j = (1.04)^{1/12} - 1 \approx 0.00327374$$

Furthermore we have the value

$$\frac{1}{12}a_{\overline{\infty|}j} = \frac{1}{12j} \approx 25.45508772$$

(4.6) Continuously paying annuities

4.6.3

An annuity is continuously varying and payable for ten years. The rate of payment at time t is $(2+t)^2$ and the force of interest is $(1+t)^{-1}$. We want to find the present value of this annuity.

Recall from previous chapters the limit (force of interest)

$$\lim_{m\to\infty} i^{(m)} = \delta$$

where for the annual interest i we have

$$e^{\delta} = 1 + i$$

$$a(t) = \exp(\int_0^t \delta(r)dr)$$

Next we establish the equation of continuous annuities (much needed for the problem set)

$$\overline{a}_{\overline{n|}i} = \lim_{m\to\infty} a_{\overline{n|}}^{(m)}{}_i = \lim_{m\to\infty} \frac{1-v^n}{i^{(m)}} = \frac{1-v^n}{\delta} \tag{2}$$

The present value (as stated in fact 4.6.9 in the book) is stated to equal

$$PV = \int_0^n f(t)v(t)dt$$

and we are given that

$$\delta(t) = (1+t)^{-1}$$

furthermore we can solve for $a(t)$, and $v(t)$ follows. We calculate the accumulation function using the force of interest to find

$$a(t) = \exp(\int_0^t \delta(r)dr) = \exp(\int_0^t (1+r)^{-1}dr) = \exp(\ln(1+r)|_0^t) = 1+t$$

and so it follows that

$$v(t) = \frac{1}{a(t)} = (1+t)^{-1}$$

It seems fairly odd that we ended up where we started!

Lastly, we solve for our present value to find

$$PV = \int_0^n \frac{(2+t)^2}{(1+t)}dt = \int_0^n \frac{t^2+4t+4}{1+t}dt$$

In order to reduce this fraction we use Polynomial Long Division and the linearity of our integral. This is the same as regular division, so don't be frightened. However it is expected that the reader understands this technique. And we continue our calculations

$$= \int_0^n \frac{t^2+4t+4}{1+t}dt = \int_0^n \frac{(t+1)(t+3)+1}{(1+t)}dt = \int_0^n [t+3\frac{1}{1+t}]dt =$$

$$[\frac{t^2}{2}+3t+\ln(t+1)]\ |_0^{10} \approx 82.39789527$$

4.6.3

Stacey and her husband David have a joint savings account that earns 3.5% interest payable continuously and has a current balance of \$58,458. Each year, David wishes to withdraw \$4,000 payable continuously at a level rate. Stacey wishes to deposit X at the beginning of each year (for thirty years) so that the account will last for thirty years. We want to find the least X that will that allow this to happen.

We point out that the first payment will be an amount of X into the savings account of \$58,458. Then for the next 29 payments/withdrawals there will be a value of $X - 4000$ taken from the account. Furthermore we calculate the present value to solve for X to find

$$-58458 + 4000\overline{a}_{\overline{30}|3.5\%} = X\ddot{\overline{a}}_{\overline{29}|3.5\%}$$

and solving for X we find that

$$X = \frac{15,834.82}{18.9} \approx \$837.82$$

(5.2) Amortized loans and amortization schedules

5.2.1

We want to copy and complete the following amortization table. Be careful to note that there is no payment at $t = 2$.

Filling out the table we get the following results

TIME	PAYMENT	INTEREST	PRINCIPAL	BALACE
0	–	–	–	$29,119.00
1	$8000	$1,223.00	$6,777.00	$22,342.00
3	$14,350.14	$1916.14	$12,434.00	$9,908.00
4	$10,324.14	$416.14	$9,908.00	$0

5.2.2

Ellen has a thirty-year mortgage with level monthly payments. The amount of principal in her 82nd payment is $259.34, and the amount of principal in her 56th payment is $230.19. We want to find the amount of interest in her 133rd payment.

Recall the equation for principal P

$$P = OLB(t-1) - OLB(t) \tag{3}$$

where $OLB(t)$ is the outstanding loan balance at time t. and we use the prospective method for the oustanding loan balance for n payments of Q.

$$OLB(t) = Qa_{\overline{n-t|}j} \tag{4}$$

We want to first solve for j the interest and then Q the payments. Once we have found Q we can calculate the outstanding balance of $t = 132$ to find the interest $i(OLB(132))$ on the 133rd payment. Furthermore, we are given the two equations

$$P_{56} = Qa_{\overline{360-56|}} - Qa_{\overline{360-55|}} = 230.19$$

$$P_{82} = Qa_{\overline{360-82|}} - Qa_{\overline{360-81|}} = 259.34$$

And the previous equations can be reduced through algebra to get the new and improved equations

$$P_{56} = \frac{Q}{(1+j)^{305}} = 230.19$$

$$P_{82} = \frac{Q}{(1+j)^{279}} = 259.34$$

and so it follows that

$$\frac{P_{56}}{P_{82}} = \frac{(1+j)^{279}}{(1+j)^{305}} = \frac{230.19}{259.34}$$

and solving for j we get

$$j \approx 0.004596489$$

Now that we have the value of j we can solve for Q. Using either equation above for P_{56} or P_{82} we can plug in j and sovle for Q to find

$$Q \approx \$932.26957611$$

and it follows that the interest in the 133rd payment is the product of the outstanding loan balance at time $t = 132$ times the interest j. Calculating we find

$$OLB(132)(j) = Qa_{\overline{360-132}|j}(j) \approx \$604.59$$

5.2.4

A fifteen-year adjustable-rate mortgage of $117,134.80 is being repaid with monthly payments of $988.45 based upon a nominal interest rate of 6% convertible monthly. Immediately after the 60th payment, the interest rate is increased to a nominal interest rate of 7.5% convertible monthly. The monthly payments remain at $988.45, and there will be an additional balloon payment at the end of the fifteen years to pay the outstanding laon balance.

(a) We want to calculate the loan balance immediately after the 84th payment.

In order to calculate the loan balance immediately after the 84th payment we will use the retrospective method. We have already converted the interest and the solution to the interest will be in the subscript of the annuity. Using the retrospective method for $OLB(84)$ we find that

$$OLB(84) = L[a(84)] - P[s_{\overline{60}|.5\%}v(60)a(84) + s_{\overline{24}|.625\%}] \approx$$

$$183,480.9951 - [80,087.5595 + 25,508.65525] \approx \$77,884.78$$

(b) We want to calculate the amount of interest in the 84th payment. We can find the interest in the 84th payment by finding the product of the outstanding loan balance at $t = 83$ and the effective interest per period 0.625%. Furthermore, we can calculate for $OLB(83)$ by going back a year in payment and interest to find that

$$OLB(83)(0.00625) = [OLB(84) - 988.45](1.00625)^{-1}(0.00625) \approx \$489.90$$

(c) We want to calculate the amount of the balloon payment. We will treat all our previous hard work with the BA II Plus calculator; that is, we will make our BA II Plus do all the hard work for this calculation. We input the following values on our calculator

$$C/Y = 12 = P/Y$$
$$N = 96 \qquad I/Y = 7.5 \qquad PV = OLB(84) \qquad PMT = 988.45 \qquad FV = ?$$

followed by that [CPT] + [FV] button to find that

$$FV \approx \$12,168.43$$

5.2.5

Arlen buys a home for $\$328,000$ and makes a down payment of $\$33,000$. The balance he finances with a fifteen-year mortgage with monthly payments and an annual effective interest rate of 5.8%. There will be level payments followed by a final slightly reduced payment. We want to calculate the amount of interest that Arlen pays in the first five years of the loan.

First we solve for j

$$j = (1+i)^{1/12} - 1 \approx 0.004709416$$

Next we solve for Q using the information above we calculate the equality

$$328,000 = 33,000 + Qa_{\overline{180}|j}$$

and we find that

$$Q \approx \$2,434.15$$

Recall that the amount of interest I for payment Q, time t, and effective monthly interest j is

$$I(t) = Q(1 - v_j^{n-t+1}) \tag{5}$$

In order to calculate the total interest paid for in the first 5 years we sum up the interest for the first 60 payments

$$\sum_{k=0}^{60} I(k) = 60Q - Q\sum_{k=121}^{180} v_j^k = 60Q - Qv_j^{120}\sum_{k=1}^{60} v_j^k =$$

$$60Q - Q\frac{v_j^{121}[1 - v_j^6 0]}{1 - v_j} \approx \$73,797.79$$

(5.3) The sinking fund method

5.3.1

A $\$14,000$ loan is to be repaid by the sinking fund method, with irregular payments into the sinking fund. The table below is a partially completed sinking fund table for this situation. We want to find the missing entries, noting that there was no payment at the end of the 3rd year. Filling in the table we get the following results

YEAR	INTEREST ON LOAN	SINKING FUND DEPOSIT	INTEREST ON SINKING FUND	SINKING FUND BALANCE	NET BALANCE
0	$\$0$	$\$0$	$\$0$	$\$0$	$\$14,000.00$
1	$\$889.00$	$\$5,200.00$	$\$0$	$\$5,200.00$	$\$8,800.00$
2	$\$889.00$	$\$3,000.00$	$\$218.40$	$\$8,418.40$	$\$5,581.60$
4	$\$1,834.45$	$\$4,859.60$	$\$722.00$	$\$14,000.00$	$\$0$

5.3.2

The borrower in a $\$238,000$ loan makes interest payments at the end of each six months for eight years. These are computed using an annual effective discount rate of 6.5%. Each time he makes an interest payment, the borrower also makes a deposit into a sinking fund earning a nominal rate of 4.2% convertible monthly. The amount of each sinking fund deposit is D in the first three years and $2D$ in the remaining five years, and the sinking fund balance at the end of the eight years is equal to the loan amount. We want to find D.

First we solve for the effective semiannual interest rate j (Note*: this is the interest rate for the interest to be paid by the borrower)

$$j \approx (1-d)^{-1/2} - 1 \approx 0.03417538$$

Now we may calculate the interest of how much must be paid, however if we do we would be wasting our time. We actually didn't even need to solve for the previous interest rate. Nevertheless, lets move forward.

Next we solve for the sinking fund interest rate δ

$$\delta = (1 + \frac{i^{(12)}}{12})^6 - 1 \approx 0.02118461$$

Using the information given in the problem we formulate the following equality

$$2Ds_{\overline{10}|\delta} + Ds_{\overline{6}|\delta}(1+\delta)^{10} = 238,000$$

Solving for D we find that

$$D = \frac{238,000}{2s_{\overline{10}|\delta} + s_{\overline{6}|\delta}(1+\delta)^{10}} \approx \$7,980.98$$

5.3.3

Alan borrows \$18,000 for eight years and agrees to make quarterly payments of \$770. Each of these payments consists of interest for the just completed quarter and a deposit to a sinking fund that has a nominal interest rate of 6% convertible quarterly. For the first six years, each year the lender receives 8% nominal interest convertible quarterly. For the remaining two years, the lender receives 12% nominal interest convertible quarterly. We want to find the amount by which the sinking fund is short of repaying the loan at the end of the eight years.

First we calculate the interest for the first 24 payments and then the interest for the remaining 8 payments. For the first 24 payments we have that the interest to be paid is

$$(0.02)(18,000) = 360$$

and the remaining 8 payments the interest to be paid is

$$(0.03)(18,000) = 540$$

Furthermore we formulate the equality of the sinking fund with the remaining deposits after the interest is paid to solve for ϵ the remaining net balance

$$(770-360)s_{\overline{32}|1.5\%}(1.015)^8 + (770-540)s_{\overline{8}|1.5\%} + \epsilon = 18,000$$

and solving for ϵ we find

$$\epsilon = 18,000 - (770-360)s_{\overline{32}|1.5\%}(1.015)^8 - (770-540)s_{\overline{8}|1.5\%} \approx \$2,835.71$$

5.3.5

Bob and Barbara are friends. Bob takes out a \$10,000 loan and agrees to repay it over twelve years by makin gannual level payments at an effective rate of 5.62499%. At the same time, Barbara takes out a \$10,000 loan and agrees to repay it by making annual interest payments at an annual effective interest rate of i. She also agrees to make annual level deposits into a sinking fund that earns 4% annual effective interest so as to accumulate $10,000 at the end of the twelve years. Bob and Barbara discover they have the same damn total annual expenditures resulting from their loans. We want to find the rate i.

First we will look at Bob's account. We need to solve for his payments, P. We have

$$Pa_{\overline{12}|5.62499\%} = 10,000$$

Solving for P we find

$$P \approx \$1,168.37$$

Next we want to look at Barbara's sinking fund account. We want to calculate how much the deposits D into the account are. We have

$$Ds_{\overline{12}|4\%} = 10,000$$

Solving for D we find that

$$D \approx \$665.52$$

Now we make the connection that they both are making payments. In fact, the information tells us that the payments they make sum to be the same value. From this we can gather that the interest paid from the 10,000 is the difference between Bob's payments with Barbara's deposits. Furthermore we get the equation and solve for the interest rate i

$$P - D = 10,000i$$

Solving for i we get

$$i = \frac{P - D}{10,000} \approx 0.050284845$$