Theory of Interest - Homework #8

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March 24, 2016

Formulas referenced to

$$a_{\overline{n}|}a(n) = s_{\overline{n}|}$$
$$a_{\overline{n}|i\%} = \frac{1-v^n}{i}$$
$$v = (1+i)^{-1} = 1/a(1)$$
$$s_{\overline{n}|i\%} = \frac{(1+i)^n - 1}{i}$$
$$\ddot{a}_{\overline{n}|i\%} = (1+i)a_{\overline{n}|i\%}$$
$$\ddot{s}_{\overline{n}|i\%} = (1+i)a_{\overline{s}|i\%}$$

(3.11) Annuity-symbols for nonintegral terms

3.11.1

Alex loans Nomar \$200,000 at a rate of 4% nominal interest convertible quarterly. They agree that Nomar will repay the loan by making quarterly payments or he will be beat with a potato in a sock. These payments will each be \$25,000 except for the last payment which will be a drop payment (completely legal). We want to find the total legnth of the loan and the aount of the final payment.

We establish the equality

$$a_{\overline{n}|.01}25,000 = 200,000$$

Solving for n we have

$$n = -\frac{\ln(1-.08)}{\ln(1.01)} \approx 8.37$$

What this means is that if we make annuity-due payments of \$25,000 it would take us approximately 8.37 quarter-years. Furthermore it will take us 9 payments with one deduced payment for the last quarter. In order to calculate the last payment we find the difference in the future value (at time 9) of the \$200,000 loan and how much we have payed before the 9th payment. Calculating the 9th payment we find

$$200,000(1.01)^9 - 25,000a_{\overline{8}|.01}(1.01)^9 \approx \$9,523.87$$

3.11.2

In the previous problem, suppose that the parties decide they would prefer to decrease the loan term by one period by opting for a balloon payment rather than a drop payment. We want to find the amount of the balloon payment.

In order to make a balloon payment we will make, instead of 9 payments, 8 payments! This means our 8th payment will be a nice big juicy payment. Furthermore we can calculate the excess we need to make at the 8th payment by finding the difference at the present value of our annuity and our loan amount of \$200,000. We establish the equality to find the 8th payment X

$$200,000 - 25,000a_{\overline{7}|.01} = X(1.01)^{-8}$$

Solving for X we find that

$$X \approx \$34,429.58$$

3.11.3

Brandt loans Lori \$5,000 at an annual effective interest rate of 5%. They agree that Lori will make annual end-of-year payments of \$600 to repay the loan, then realize that they must figure the term of the loan. If the term T is to be such that $\$5,000 = \$600a_{\overline{T}|5\%}$, we want to find T and the amount of the last payment at time T.

First, since we are given the equality, we solve for T

$$T = -\frac{\ln(1 - \frac{5000}{600}(.05))}{\ln(1.05)} \approx 11.04723687$$

In order to find out how much is owed we calculate for the loan L

$$L(1.05)^T - S_{\overline{11}|.05}(1.05)^{T-11} \approx \$27.69$$

(3.12) Annuities governed by general accumulation functions

3.12.1

Suppose that the accumulation in an investment fund is governed by the accumulation function $a(t) = 1 + .04t$ where time is measured in years from January 1, 1990.

(a) We want to find the accumulation $\$100s_{\overline{6}|}$ at the time of the last deposit if deposits of \$100 are made at the end of each year for six years, the first deposit being on December 31, 1990.

In order to calculate the future value we must first find the present value and push it forward to the date we are interested in. We notice that $v(t) = 1/a(t)$ and use $v(t)$ to calculate the present value PV

$$PV = \frac{100}{1.04} + \frac{100}{1.08} + \frac{100}{1.012} + \frac{100}{1.16} + \frac{100}{1.20} + \frac{100}{1.24} \approx 528.2175442$$

However this is the value of $\$100a_{\overline{6}|}$ not our desired value $\$100s_{\overline{6}|}$. We recall the equality from chapter 2

$$a_{\overline{n}|} \cdot a(n) = s_{\overline{n}|} \tag{1}$$

So it follows that

$$s_{\overline{6}|} = a_{\overline{6}|} \cdot a(6) = a_{\overline{6}|}(1.24) \approx \$654.99$$

(b) We want to find what effective rate of compound interest would produce the same accumulated value (on December 31, 1995) for the sequence of six \$100 deposits.

Calculating this value without a calculator requires solving a 5th degree polynomial! In order to keep our sanity, we rely on our BA II Plus calculator and input the following values to solve for our interest rate:

$$N = 6 \qquad PV = 0 \qquad PMT = 100 \qquad FV = -100s_{\overline{6}|} \approx -654.99$$

We proceed to find the interest rate by pressing the buttons [CPT] + [I/Y] to find

$$i \approx 3.49845\%$$

3.12.2

Suppose that accumulation is governed by compound interest and the annual effective rate of interest in the k-th year is $.02 + .01k$. We want to find $a_{\overline{5}|}$ and $s_{\overline{5}|}$. We also want to find T such that $s_{\overline{5}|}/a_{\overline{5}|} = a(T)$

In order to solve for $a_{\overline{5}|}$ we must first identify that the interest changes each year. Furthermore pulling back each payment to time 0 we find that the present value of all these payments, or the annuity after 5 payments, is equivalent to

$$a_{\overline{5}|} = PV = (1.03)^{-1} + [(1.03)(1.04)]^{-1} + [(1.03)(1.04)(1.05)]^{-1} + [(1.03)(1.04)(1.05)(1.06)]^{-1}$$
$$+[(1.03)(1.04)(1.05)(1.06)(1.07)]^{-1} \approx 4.416118832$$

Recall the identity

$$s_{\overline{n}|} = a_{\overline{n}|}a(n)$$

Furthermore this solves for T where $s_{\overline{5}|}/a_{\overline{5}|} = a(T) = a(5)$, and so $T = 5$. And it follows that

$$s_{\overline{5}|} = a_{\overline{5}|}a(5) = a_{\overline{5}|}(1.03)(1.04)(1.05)(1.06)(1.07) \approx 5.6336364$$

3.12.3

We want to find the value at $t = 0$ of a perpetuity that pays $1,000$ at the end of each year starting at $t = 3$ assuming that $a(t) = \frac{1}{3}(t+1)(t+3)$.

In order to calculate the present value we must find the perpetuity of the infinite number of payments and bring it back to time $t = 0$. Furthermore we get the following equation

$$PV = 1000a_{\overline{\infty}|} = 1000\sum_{t=4}^{\infty}\frac{1}{a(t)}$$

Let us focus on the most rigorous part, the infinite series. Our series can be split by partial fractions

$$\sum_{t=4}^{\infty}\frac{1}{a(t)} = \sum_{t=4}^{\infty}\frac{3}{(t+1)(t+3)} = \sum_{t=4}^{\infty}[\frac{3}{2(t+1)} - \frac{3}{2(t+3)}]$$
$$= [\frac{3}{10} - \frac{3}{16}] + [\frac{3}{12} - \frac{3}{18}] + [\frac{3}{14} - \frac{3}{20}] + [\frac{3}{16} - \frac{3}{20}] + \ldots$$
$$= \frac{3}{10} + \frac{3}{12} + \frac{3}{14} + [\frac{3}{16} - \frac{3}{16}] + [\frac{3}{18} - \frac{3}{18}] + [\frac{3}{20} - \frac{3}{20}] \ldots$$
$$= \frac{3}{10} + \frac{3}{12} + \frac{3}{14}$$

Furthermore we have solved the summation by canceling out the excessive terms! (A telescoping series). We go back to the above equation to solve for our present value

$$PV = 1000\sum_{t=4}^{\infty}\frac{1}{a(t)} = 1000[\frac{3}{10}+\frac{3}{12}+\frac{3}{14}] = \$675$$

(3.13) The investment year Method

3.13.1

Frank invests \$1000 at the beginning of 1989, 1990, and 1991 in Experience Investment Fund which credits money according to the investment year method using the chart (seen on page 176). there are no withdrawals or further deposits. We want to find what would his balance have been on January 1, 1994.

We can do this by summing up the accumulated values for the \$1,000 put in at 1989, 1990, and 1991 respectively. We find that the values give us a future value

$$FV = 1000(1.06)(1.055)(1.05)(1.045)(1.05) + 1000(1.065)(1.06)(1.055)(1.05) + \\ 1000(1.06)(1.055)(1.05) \approx \$3713.16$$

3.13.2

Suppose that money was equally likely to be invested in the Experience Investment Fund of Problem (3.13.1) in any of the years 1988-1993. We want to find the average rate of return paid by the fund during 1993.

I have found something interesting with this problem; that is, if we calculate the geometric average rate of return and the arithmetic average rate of return (both rounded to the thousandths place we get the same value. Never the less we find the value of interest at time 1993 for each of the possibilities. We get (respectively)

1.05 1.05 1.05 1.05 1.055 1.06

If we chose to find the arithmetic average (the way to actually solve the problem) we get

$$\frac{(4)(1.05) + 1.055 + 1.06}{6} = 1.0525$$

Giving us the average rate of 5.25%.

If we chose to find the geometric average (the abstract way to solve the problem) we get

$$\sqrt[6]{(1.05)^4(1.055)(1.06)} \approx 1.052493091$$

Giving us an average rate of 5.25%.

Chapter 3 review problems

Note: we may refer to an account. By account we mean a time line with a particular present value (or future value) and particular interest rate (or accumulation function). A time line can be drawn to represent each account.

3.R.2

Mrs. Chen has saved $22,000 for a down payment and qualifies for a thirty-year mortgage at an annual effective interest rate of 6%. In addition, Mrs. Chen is opening a retirement account that has a 5.2% annual effective interest rate. Mrs. Chen wants to be able to retire in thirty-eight years, at which point she needs to be able to withdraw $4,100 at the beginning of each month for twenty-five years. Mrs. Chen has $2,600 at the end of each month to use for a combination of mortgage payments and retirement contributions. We want to find how expensive a house she can buy.

The price of the house will be at time 0; that is, $t = 0$ with respect to the retirement account. Let us sort out what is happening in this problem. We have that there is an annuity-immediate of payments $2600 - M$ per month for 30 years, where M is the payment for the mortgage. There is also another annuity-immediate for the next 8 years of $2600 per month at time 30 to 38. Lastly there is another annuity-due starting at time 38 for 25 years $4100 is returned to the user. Wow three annuities! We can calculate for time 38 with respect to the retirement account as for

$$j = (1.04)^{1/12} - 1$$

we have

$$T_{38} = (2600 - M)S_{\overline{360}|j}(1.052)^8 + 2600s_{\overline{96}|j} = 4100\ddot{a}_{\overline{300}|j}$$

So we solve for M to find

$$M = 2600 + \frac{2600s_{\overline{96}|j} - 4100\ddot{a}_{\overline{300}|j}}{s_{\overline{300}|j}(1.052)^8} \approx \$2290.97$$

Next we can look closer into the mortgage account (the one with 6% interest). We have the payments M per month for 30 years, and an initial value of \$22,000. Furthermore we calculate the present value for

$$z = (1.06)^{1/12} - 1$$

we have the biggest mortgage Mrs. Chen can afford is

$$PV = Ma_{\overline{360|}z} + 22,000 = \$410,714.79$$

3.R.6

Derek receives a perpetuity. It has annual payments with the first payment in exactly eight years. The first payment is \$10,000, the second is \$15,000, and then the payments alternate between \$10,000 and \$15,000 until there have been a total of thirty payments. After that, the payments are all \$10,000. We want to find the present value of this perpetuity if $i = 4\%$.

We can write this account in terms of an annuity and perpetuity. Calculating the present value we have

$$PV = [5000a_{\overline{15|}j} + 10000a_{\overline{\infty|}4\%}](1.04)^{-7} = \$222,186.62$$

3.R.7

Serena receives a fifty-year annuity-due that has payments that start at \$2,000, and increase by 3% per year through the twenty-fourth payment, then stay level at \$4,000. We want to find the accumulated value of this annuity at the end of fifty years if the annual effective rate of interest remains 4.2% throughout the time of the annuity.

Looking into the account with 4.2% annual effective interest we have a future value at time 50 of

$$FV = [2000\sum_{t=0}^{23}\frac{(1.03)^t}{(1.042)^t}](1.042)^{50} + 4000s_{\overline{26|}4.2}(1.042)$$

$$= 2000\left[\frac{(1 - \frac{(1.03)^{24}}{(1.042)^{24}})}{(1 - \frac{(1.03)}{(1.042)})}\right](1.042)^{50} + 4000s_{\overline{26|}4.2}(1.042) \approx \$519,729.09$$

Note: we converted the above summation by using geometric series with a ratio of $\frac{1.03}{1.042}$ ($=$ r) and an initial value of \$2,000($=$ a).