Theory of Interest - Homework #5

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Section 3.0

Problem 3.0.1

We want to give two real life examples of annuities. Recall that an annuity is a series of payments. Some real life examples include:

- Car Payments
- House Payments
- Food Stamps (payed by the government)
- Social Security
- Utility Bills (Electricity, Water, and/or Gas)

Section 3.2

Problem 3.2.1

We want to find the present value (one period before the first payment) of an annuity-immediate that lasts five years and pays $3000 at the end of each month, using a nominal interest rate of 3% convertible monthly. Then we want to repeat the problem using an annual effective discount rate of 3%. We want to explain why which one is higher.

We will start off by defining an annuity-immediate. Recall that a basic annuity-immediate is equivalent to

$$a_{\overline{n}|i} = \frac{1 - v^n}{i} \tag{1}$$

Furthermore for our annuity of 3% nominal annually convertible monthly we find that

$$Pa_{\overline{n}|}i = 3000a_{\overline{60}|\frac{.03}{12}} = 3000\frac{1-(1+.03/12)^{-60}}{.03/12} \approx \$166,957.0731$$

On the other hand our annuity of 3% annual effective discount rate yields

$$i = \frac{d}{1-d} \approx 0.030927835$$

However we need the monthly effective interest rate and so we get

$$j = (1+i)^{1/12} - 1 \approx 0.002541491$$

and solving for our annuity we find that

$$Pa_{\overline{n}|}j = 3000a_{\overline{60}|0.002541491} = 3000\frac{1-(1+j)^{-60}}{j} \approx \$166,751.664$$

We can see that the first annuity-immediate yielded a higher result. This is because the interest rate on the first is less, which means whenever we bring the payments the the present value we reduce the payments less the less the interest rate is. This is our v^t value.

Problem 3.2.2

Tracy receives payments of $\$X$ at the end of each year for n years. The present value of her annuity is \$493. Gary receives payments of $\$3X$ at the end of each year for $2n$ years. The present value of his annuity is \$2748. Let the annual effective interest rate for both annuities be the same. We want to find the value v^n.

We let

$$A = Xa_{\overline{n}|}i = \frac{X(1-v^n)}{i} = 493$$

$$B = 3Xa_{\overline{2n}|}i = \frac{3X(1-v^n)}{i} = 2748$$

then we have that

$$\frac{A}{B} = \frac{(1-v^n)}{3(1-v^{2n})} = \frac{493}{2748}$$

Also we let $t = v^n$ and by moving the terms around we get the polynomial

$$493t^2 - 916t + 423 = 0$$

Solving the polynomial we find the roots

$$v^n = t = 0.85801217 \quad \text{or} \quad v^n = t = 1$$

Problem 3.2.3

The Browns wish to accumulate at least $150,000 at the time of their last deposit in a college fund for their daughter by contributing an amount A into the account at the end of each year for eighteen years. We want to find the smallest annual payment A that will suffice if the college fund earns a level annual effective interest rate of 5%. If at the end of ten years, it is announced that the annual effective interest rate will drop to 4.5%, we want to find how much must the Browns increase their payments in order to reach their accumulation goal. We will assume the only payment we reduce is the final payment when rounding off cents.

If we want to calculate the present value we first let $v = (1.05)^{-1}$, and calculating the annuity-immediate we find

$$S_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n$$

Fortunately for us we have our already derived our annuity-immediate above (Equation (1)) in problem 3.2.1. This gives us

$$a_{\overline{18}|5} = A\frac{(1-(1.05)^{-18})}{.05} = 150,000(1.05)^{-18}$$

Rearranging the terms and solving for A we find that

$$A = \frac{(150,000)(1.05)^{-18}(.05)}{(1-(1.05)^{-18})} \approx \$5331.933348$$

However we can't pay with a fraction of a cent, so we must round our monthly payment to be $5331.94. However once we calculate how much we have actually paid, we will find that we have paid $150,000.1871 and not $150,000, so we must make our last monthly payment $5331.94 - 0.19 = 5331.75$, therefore we will have a monthly payment of $5331.94 for the first 17 months and $5331.75 for the last month.

Next let us take in account that the interest changes the tenth year. For this we can calculate it two different ways. We can do the same as above except switch the $150,000 to the remaining amount we need after the ten years, but that is no fun. Instead we will use our handy dandy BA II Plus calculator. We input the following values

$$N = 10 \qquad I/Y = 5 \qquad PV = 0 \qquad PMT = 5331.94 \qquad FV = ?$$

Then we press [CPT] + [FV] to find that $FV = 67,064.56833$ we go back to our worksheet and input the following new values:

$$N = 8 \qquad I/Y = 4.5 \qquad PV = -67,064.56833 \qquad PMT = ? \qquad FV = 150,000$$

Then we press [CPT] + [PMT] to find that $PMT = 5823.812043$. So we have found the estimated monthly payments, however we will need to round and find how much it will cost for the last month. In order to do this we plug in our calculator:

$$N = 8 \qquad I/Y = 4.5 \qquad PV = -67,064.56833 \qquad PMT = 5823.82 \qquad FV = ?$$

Then we press [CPT] + [FV] to find that $FV = 150,000.0746$ a value 0.0746 greater than what we were supposed to pay, so we decrease the last payment by this amount, and we conclude that our payments will be \$5823.82 per month for 7 months, and the last payment will be \$5823.77. This is \$491.88 increase from the last payment for the first 7 months, and \$491.81 for the last month.

Problem 3.2.4

Elwood wishes to purchase a home. She has saved up \$13,200 for a down payment. Based on her earnings, she qualifies for a thirty-year mortgage with level monthly payments of \$820 including escrow and a nominal interest rate convertible monthly of 5.84%. Her payments are due at the end of each month. From each payment, \$240 will be put aside in an escrow account for the payment of taxes and homeowners insurance. We want to find what is the most expensive house Elwood can buy.

(Let $v = (1 + \frac{.0584}{12})^{-1}$.) Furthermore, we calculate her annuity-immediate with payments P, and add her down payment D, to find the most expensive house Elwood can buy

$$Pa_{\overline{n}|i} + D = (820 - 240)\frac{(1 - v^{360})}{.0585/12} + 13,200 \approx \$111,514.9272$$

Problem 3.2.6

Mrs. Williams finds that she has two options for investing \$32,000.02 for fifteen years. The first option is to deposit the \$32,000.02 into a fund earning a nominal rate of discount $d^{(4)}$ payable quarterly. The second option is to purchase an annuity-immediate with 15 level annual payments, the annuity payments computed using an annual effective rate of 7%, and then when she gets an annuity payment, to immediately invest it into a fund earning an annual effective rate of 5%. Mrs. Williams calculates that the second option produces an accumulated value that is \$1,500 more than the accumulated value yielded by the first option. We want to calculate $d^{(4)}$

First let us calculate the second option for investing. For this we first calculate for how much each of the payments P are

$$a_{\overline{15}|7} = 32,000.02 = P\frac{1 - 1.05^{-15}}{.05}$$

Solving for P we find that

$$P = 3513.430186$$

And using this P we find how much the payments are with investments by summing up the terms

$$FV_2 = \sum_{t=0}^{14} P(1.05)^t$$

However we notice that

$$FV_2 = a_{\overline{n}|i}(1.05)^{15} = \frac{(1.05)^{15} - 1}{.05} \approx \$75,814.77669$$

Next we calculate $d^{(4)}$ for the first payment by using the previous value

$$32,000.02(1 - \frac{d^{(4)}}{4})^{-60} + 1500 = 75,814.77669$$

and by solving for the quarterly discount rate we find that

$$d^{(4)} = 0.055778987$$

Problem 3.2.7

A buyer of a 2003 Protege S Hatchback has a choice of 0% financing for 60 months or a $\$3,600$ rebate. He plans to make no down payment. The buyer is able to qualify for 7% annual effective financing through his credit union and thereby take advantage of the rebate. Let Y denote his negotiated price for the Protege S Hatchback. We want to find how large must Y be in order for the 0% dealer financing to be preferable.

Before we solve this problem let us try to grasp what whoever made this problem is trying to say. For the first scenario we have 0% financing which means we will be making 60 monthly payments of $\$Y/20$. In the next scenario we will be making payments to make sure that the present value of the sum of the payments is \$3600 less than $\$Y$, the cost of the car. (We can notice that with 0% financing we get that the first payment has a NPV of $\$Y$.) Furthermore, can find out when the payment of $\$Y/60$ is less that the payment with the rebate. Let us find the payment of the rebate.

We have that the second payment P_2 (with the rebate), with an effective monthly interest rate j comes out to

$$Y - 3600 = P_2 a_{\overline{n}|j}$$

and rearranging the terms to solve for P_2 we find that

$$P_2 = \frac{Y - 3600}{a_{\overline{n}|j}}$$

Furthermore now that we have defined P_2 we can find out when it is equal to P_1, and this will be the point they intersect and P_1 starts to become less that P_2. We have

$$P_1 = \frac{Y}{60} = \frac{Y - 3600}{a_{\overline{n}|j}} = P_2$$

Next we solve for Y to find that

$$Y = \frac{(-3600)(60)}{(a_{\overline{n}|j} - 60)}$$

However we are not able to solve this equation until we solve for $a_{\overline{n}|j}$. Recall that for $j = (1.07)^{1/12} - 1 \approx 0.00565415$ we have that

$$a_{\overline{n}|j} = a_{\overline{60}|j} = \frac{1 - (1 + j)^{-60}}{j} \approx 50.7616626$$

and so it follows that

$$Y = \$23,830.830438978 \approx \$23,830.83$$