Theory of Interest - Homework #1

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Section 1.3

Problem 1.3.1

Given $A_K(t) = \frac{1000}{100-t}$ for $0 \leq t < 100$, we want to find K and $a(20)$. Recall that

$$K = A_K(0) = \frac{1000}{100} = 10$$

Next we want to compute $a(20)$,

$$a(20) = A_{10}(20)/10 = \frac{1000}{(100-t)10} = 1.25$$

Problem 1.3.3

Suppose that an account is governed by a quadratic accumulation function $a(t) = \alpha t^2 + .01t + \beta$ and the interest rate for the first year is 2%. We want to find α, β, and the interest rate for the fourth year i_4. We can solve for beta by using the fact that $a(0) = 1$, and on the other hand we have

$$a(0) = \beta = 1$$

Next we want to solve for α. We can do this by using the formula for the n^{th} period

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} \tag{1}$$

more specifically we have that $i_1 = .02$ and

$$i_1 = a(1) - 1 = \alpha + 1.01 - 1 = .02$$

doing the algebra we find that

$$\alpha = .01$$

Next we want to find i_4. Using the formula above we have that

$$i_4 = \frac{1.2 - 1.12}{1.12} \approx .07142857$$

Problem 1.3.5

It is known that for each positive integer k, the amount of interest earned by an investor in the k-th period is k. We want to find the amount of interest earned by the investor from time 0 to time n, where $n \in \mathbb{Z}$. Adding up each period up to n we get that the interest is equal to

$$\sum_{k=0}^{n} k = \frac{(n+1)n}{2}$$

Problem 1.3.6

It is known that for each positive integer k, the amount of interest earned by an investor in the k-th period is 2^k. We want to find the amount of interest earned by the investor from time 0 to time n, where $n \in \mathbb{Z}$. Adding up each period up to n we get that the interest is equal to

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Section 1.4

Problem 1.4.1

We want to find how much interest is earned in the fourth year if $1000 is invested under simple interest at an annual rate of 5%, and the balance at the end of the fourth year. Using equation (1) from problem 1.3.3 we find that

$$i_4 = \frac{1.2 - 1.15}{1.15} \approx .043473$$

Next to find out the balance at the end of the fourth year we solve for

$$A_{1000}(4) = 1000(1 + (.05)4) = 1200$$

and thus the balance is $1200.

Problem 1.4.2

We want to find in how many years will \$500 accumulate to \$800 at 6% simple interest. We can do this by setting up the equality

$$A_{500}(t) = 800 = 500(1 + .06t)$$

solving the equation for t gives us that $t = 10$.

Problem 1.4.4

We want to find the yearly simple interest rate so that \$1000 invested at time 0 will grow to \$1700 in 8 years. Once again we set up the equality

$$A_{1000}(8) = 1700 = 1000(1 + 8i)$$

solving the equation for i gives us that $i = 8.75\%$.

Problem 1.4.6

A loan is made at time 0 at simple interest at a rate of 5%.

(a) We want to find in which period this is equivalent to an effective rate of $\frac{1}{23}$. Using equation (1) from Problem 1.3.3 we can find that for the n-th period and a simple interest rate s,

$$i_n = \frac{s}{1 + s(n-1)}$$

Plugging in our variables we get

$$i_n = \frac{.05}{1 + .05(n-1)}$$

and solving for n we find that $n = 4$.

(b) We want to find the effective interest rate for [4.6]. Using the equation

$$i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)} \tag{2}$$

therefore we have that

$$i_{[4,6]} = \frac{a(6) - a(4)}{a(4)} = \frac{1.3 - 1.2}{1.2} \approx .0833333$$

Section 1.5

Problem 1.5.1

We want to find how long it takes ,t, for \$2200 to grow with compound interest at an annual effective interest rate of 4% to \$8000. For compound interest we use the equation

$$a(t) = (1+i)^t \tag{3}$$

and so our amount function is

$$A_{2200}(t) = 2200(1.04)^t = 8000$$

and solving for t gives us

$$t = \frac{\ln(8000/2200)}{ln(1.04)} \approx 32.915877$$

Problem 1.5.2

After 13 years a value of K has grown to \$32,168 with compound interest at an annual effective interest rate of 6.2%. We want to solve for K. Using equation (3) from problem 1.5.1 above we have that the amount function is

$$A_K(t) = Ka(t) = K(1+i)^t \tag{4}$$

Solving for K we get

$$K = \frac{A_K(13)}{a(13)} = \frac{32168}{(1.062)^{13}} \approx 14,716.53$$

Problem 1.5.3

Invested money in an account earning compound interest at an annual effective interest rate of i doubles in 9 years. We want to solve for i. Using equation for compound interest, (3) in problem 1.5.1, gives us

$$a(9) = (1+i)^9 = 2$$

Solving for i gives us

$$i = \sqrt[9]{2} - 1 \approx .080059739$$

Problem 1.5.4

We want to find out how much interest is earned in the fourth year by $1000 invested under compound interest at an annual effective interest rate of 5%. For compound interest each year earns the same amount of interest.

Side Note* (Just a cool fact about compound interest) : More specifically, using equation (2) from problem 1.4.6 for n-th period interest and (3) from problem 1.5.1 for compound interest yields

$$i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)} = \frac{(1+i)^{t_2} - (1+i)^{t_1}}{(1+i)^{t_1}} = (1+i)^{t_2 - t_1} - 1$$

Therefore

$$i_1 = i_2 = \cdots = i_n$$

for all $n \in \mathbb{N}$ Regardless we have that

$$i_4 = (1.05)^1 - 1 = .05$$

(It ends here)

However we are looking for the interest instead of interest rate so we get that the interest earned is

$$A_K(t_2) - A_K(t_1) = 1000(1.05^4 - 1.05^3) \approx \$57.88125$$

Problem 1.5.6

We have that $826 is deposited into a bank account with compound interest. For the first three years the annual effective interest rate is 3%. For the next two years the effective interest rate is 4%. For the following five years the annual effective interest rate is 5%. We want to find the balance after the ten years. We use our compound interest equation (3) in problem 1.5.1 to get

$$A_{826}(10) = 826(1.03)^3(1.04)^2(1.05)^5 \approx \$1245.962282$$

Problem 1.5.8

Suppose that $2500 is invested in a fund earning 10% simple interest annually. After two years there is the option of moving the money to an account that pays compound interest at an annual effective rate of 7%. We want to know whether we should switch our money (a) if we wish to liquidate in five more years? (b) if we wish to liquidate in eight more years?

(a) First let us identify how much money each amount function yields. Let $S_K(t)$ be the amount function for the simple interest and $A_K(t)$ be the amount function for compound interest. Then we have that

$$S_{2500}(7) = 2500(1 + .10(7)) = \$4250$$

$$A_{S_{2500}(2)}(5) = 3000(1.07)^5 \approx \$4207.655192$$

Therefore assuming our goal is to have more money we should keep our money in the simple interest account.

(b) Next let us see if it was ten years instead. For ten years the values yield

$$S_{2500}(10) = 2500(1 + .10(12)) = \$5000$$

$$A_{S_{2500}(2)}(8) = 3000(1.07)^10 \approx \$5154.558539$$

Although for ten years we would have to go with the compound interest option, since it yields more money.

Problem 1.5.10

In 1993, \$4200 is deposited into an account that used 4% annual effective interest rate when the balance was under \$5000 and a 5.5% annual effective interest rate when the balance is at least \$5000. In 1999, \$1000 is withdrawn from the account. We want to find what the balance is in 2003. First we must see how long it takes for the balance to achieve \$5000. We can do this by using our amount function and solving for t_1, or more clearly

$$A_4200(t_1) = 4200(1.04)^{t_1} = 5000$$

Solving for t_1 we get

$$t_1 = \frac{\ln(5000/4200)}{\ln(1.04)} \approx 4.445441531$$

Next we know that \$1000 is take out at $A_{4200}(6)$, but first let us find out what this value is with our new annual effective interest rate of 5.5%

$$A_{4200}(6) = 5000(1.055)^{6-t_1} \approx \$5433.97$$

And so we have the new value of \$4433.97 and once more we want to find t_2 the time it takes to get to \$5000 once again. Solving for t_2 gives us

$$t_2 = \frac{\ln(5000/4433.97)}{\ln(1.04)} \approx 3.0632428$$

We then compute our final value

$$A_{4200}(10) = 5000(1.055)^{10-6-t_2} = \$5257.170235$$

Section 1.6

1.6.1

An amount of \$3000 is borrowed for one year at an annual discount rate of 8%. We want to find the extra money is able to be used. We use the equation where d denotes the discount rate K denotes the extra money able to be used and L is the loan amount.

$$K = (1 - d)L \tag{5}$$

Solving for K we get

$$K = (1 - .08)3000 = \$2760$$

1.6.2

An amount of \$$X$ is borrowed for one year at a discount rate of 6% and there is an extra use of \$2400. We want to solve for \$$X$. Using equation (5) in problem 1.6.1 we can find that for \$$X = L$

$$L = \frac{K}{(1-d)} = \frac{2400}{(1-.06)} \approx \$2553.191489$$

1.6.3

An amount of \$1450 is borrowed for one year at a discount rate of d, and there is an extra use of \$1320. We want to solve for d. Using equation (5) in problem 1.6.1 we can find solving for d gives us

$$d = 1 - \frac{K}{L} = 1 - \frac{1320}{1450} \approx 0.089655172$$

1.6.4

The amount of interest earned on \$$K$ for one year is 256. The amount of discount paid on a one year loan for \$$K$ transacted on a discount rate that is equivalent to the interest rate of the first interest is \$236. We want to find the value of \$$K$. We have that

$$K(1 + i) = K + 256$$

$$K(1 - d) = K - 236$$

Also since the interest and discount rates are equivalent it is sufficient to use the equation

$$1 = (1 + i)(1 - d) \tag{6}$$

Therefore when we multiply the above equations together we get

$$K^2(1+i)(1-d) = (K+256)(K-236)$$

and expanding the terms and solving for K gives us

$$K = \$3020.8$$

1.6.5

A savings account earns compound interest at an annual effective interest rate i. Given that $i_{[2,4.5]} = .20$ we want to find $d_{[1,3]}$. First we must solve for the annual effective interest rate i. We can do this by using equation (2) above from problem 1.3.3. Therefore we have that

$$i_{[2,4.5]} = (1+i)^{4.5-2} - 1 = .2$$

and solving for i gives us

$$i = .075653757$$

Next we know that the discount of a period for compound interest is

$$d_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_2)} = \frac{(1+i)^{t_2} - (1+i)^{t_1}}{(1+i)^{t_2}} = 1 - (1+i)^{t_1 - t_2} \tag{7}$$

Furthermore we have that

$$d_{[1,3]} = 1 - (1+i)^{1-3} = 1 - (1.075653757)^{-2} = .135718926$$