Theory of Interest - Homework #7

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(3.7) Nonlevel annuities

Problem 3.7.1

Stuart wishes to have $\$14,000$ to buy a used car three years from now. He plans to accomplish this, through an account with a nominal interest rate of 3% compounded monthly, by depositing \$300 at the end of each month during the first year, and Q at the end of each month during the second two years. We want to find what the smallest Q that will suffice.

First we set up the equation, for the monthly interest $j = .03/12$, to solve for the value of the annuity at time 36 (months)

$$300a_{\overline{12}|j}(1+j)^{36} + Qa_{\overline{24}|j}(1+j)^{24} = 14,000$$

solving for Q we find

$$Q = \frac{14,000 - 300a_{\overline{12}|j}(1+j)^{36}}{a_{\overline{24}|j}(1+j)^{24}} \approx \$409.8592047 \approx \$409.86$$

Problem 3.7.2

On January 1, 1980, Suzanne received a twenty-year annuity-due that paid \$100 each January 1 and \$300 each July 1. We want to find what the value of this annuity on January 1, 1980, calculated using an effective rate of interest of 6%.

We set up the equation and solve for the present value

$$PV = 100\ddot{a}_{\overline{20}|6} + 300\ddot{a}_{\overline{20}|6}(1.06)^{-1/2} \approx \$4758.513648 \approx \$4758.51$$

Problem 3.7.4

Bill deposits \$100 at the end of each year for thirteen years into fund A. Seth deposits \$100 at the end of each year for thirteen years into fund B. Fund A earns an annual effective rate of 15% for the first five years and an annual effective rate of 6% thereafter. Fund B earns an annual effective rate of i throughout the thirteen years. The two funds have equal accumulated values at the end of the thirteen years. We want to find i.

We can split these payments into two different annuities giving us the following values to input into our BA II Plus Calculator

$$\begin{aligned} N &= 13 \\ PV &= 0 \\ PMT &= -100 \\ FV &= 100S_{\overline{5}|15}(1.06)^8 + 100S_{\overline{8}|6} \end{aligned}$$

Next we solve for the interest rate by pressing the buttons [CPT] + [I/Y] to find that

$$i \approx 7.38\%$$

Problem 3.7.7a

Lucy received a gift of twenty-one year annuity on the day she was born. The annuity pays \$500 on her odd birthdays and \$700 on her even birthdays.

(a) If the nominal rate of interest is 8% payable monthly, we want to find the value of this annuity on the day she was born.

We can solve for the present value of this annuity by splitting it up into the sum of two biannual annuities

$$PV = 700a_{\overline{10}|j} + 500\ddot{a}_{\overline{11}|j}(1+j)^{-1/2}$$

where

$$j = (1 + \frac{.08}{12})^{24} - 1$$

so it follows that

$$PV \approx \$5817.131322 \approx \$5817.13$$

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(3.8) Annuities with payments in geometric progressions

Problem 3.8.1

Al and Sal are twins. Al is given a fifteen-year annuity with end-of-year payments. The first payment Al receives, precisely one year from the date he is given the annuity, is for \$100, and then subsequent payments decrease by 4% annually. Sal is given an n-year level annuity that has the same present value as Al's when the present values are calculated using $i = 5\%$. Again calculated using $i = 5\%$, the accumulated value at the end of n years of Sal's annuity is \$1,626.29. We want to find the common present value of the two annuities and then find n.

Using our imaginary number line (hopefully you have one written in front of you, since it helps a ton) we can find the present value

$$V_0 = \sum_{t=1}^{15} 100 \frac{0.96^{t-1}}{1.05^t} = \frac{100}{1.05} \sum_{t=1}^{15} \left(\frac{0.96}{1.05} \right)^{t-1}$$

*Note: I will use V_t as the value at time t. Using our geometric partial equation

$$S_n = a + ar + ar^2 + \cdots + ar^n = \frac{a(1 - r^n)}{(1 - r)} \tag{1}$$

Using equation (1) we can calculate the present value

$$V_0 = \frac{100}{1.05} \sum_{t=1}^{15} \left(\frac{0.96}{1.05} \right)^{t-1} = \frac{100(1 - \left(\frac{0.96}{1.05}\right)^{15})}{(1.05 - 0.96)} \approx \$821.39$$

Problem 3.8.2

On January 1, 1988, Wanda received a deferred perpetuity paying \$3,000 on July 1 of even numbered years beginning on July 1, 1996 and \$1,200 on July 1 of odd years beginning on July 1, 1997. The interest rate is 4% effective for even numbered numbered years and 5% on odd numbered years. We want to find the value of this perpetuity on January 1, 1988.

We calculate the value at July 1, 1994 (I will denote as x).

$$V_x = 3000 a_{\overline{\infty}|j} + 1200 a_{\overline{\infty}|j} (1.04)^{-1/2} (1.05)^{-1/2} \approx \$45,150.50167$$

for a biannual interest rate

$$j = (1.04)(1.05) - 1 = 0.092$$

Furthermore if we want to find the value at January 1, 1988 (I will denote as y) it follows that the value at y is

$$V_y = V_x(1+j)^{-3}(1.04)^{-1/2} \approx \$33,954.85$$

Problem 3.8.3

Consider an annuity-immediate with monthly payments for twenty years. The payments are level in the course of each year, then increase by 2% for the next year. We want to find the present value of this annuity if the initial payment is \$1000 and $i = 4\%$.

We calculate the present value

$$V_0 = \sum_{t=1}^{20}(1000)a_{\overline{12|}j}\left(\frac{1.02}{1.04}\right)^{t-1} = (1000)a_{\overline{12|}j}\frac{1-\left(\frac{1.02}{1.04}\right)^{20}}{(1-\frac{1.02}{1.04})} \approx \$196,614.90$$

Problem 3.8.5

On January 1, 1988, Felix inherited a perpetuity with annual payments beginning in six months. The first payment was \$3,000, and after that the payments increased by 3% each year. We want to find the value of this perpetuity on January 1, 1995 if the annual effective interest rate was 6% from January 1, 1988 through January 1, 1996 and 4% thereafter.

We will split this perpetuity up into the sum of an annuity and a perpetuity. We have that the value we desire at x (January 1, 1995)

$$V_x = A(1.06)^{1/2}(1.06)^6 + B(1.04)^{-1/2}(1.06)^{-1}$$

Where A is the value of the first annuity with 6% interest for the first 8 payments and B is the perpetuity after the first 8 payments.

We will denote $\ddot{g}_{\overline{n|}j}$ as the annuity with geometric non-level annuity payments where $j = (1+g)/(1+i)$

Calculating A and B we have

$$A = (3000)\ddot{g}_{\overline{8|}6} = \frac{3000(1-\left(\frac{1.03}{1.06}\right)^8)}{1-\frac{1.03}{1.06}} \approx \$21,752.5546$$

$$B = 3000(1.08)^8\ddot{g}_{\overline{\infty|}4} = \frac{3000(1.08)^8}{1-\frac{1.03}{1.04}} \approx \$395,232.2654$$

Furthermore it follows that

$$V_x \approx \$397,388.55$$

(3.9) Annuities with payments in arithmetic progressions

Problem 3.9.1

Suppose that the effective interest rate per interest period is 3%. We want to describe what the following annuity symbols mean and calculate them to the nearest $\frac{1}{100}$.

(a) $(DS)_{\overline{28|}}$ indicates a decreasing arithmetic progression that starts at \$28 for the first payment at the end of the year, and calculates the accumulated value after 28 end of the year payments, where each next consecutive payment is \$1 less than the last. Furthermore algebraically we have for $v = \frac{1}{1.03}$,

$$(Ds)_{\overline{28|}} = (I_{1,-1}s)_{\overline{28|}} = (1+i)^{28}(I,a)_{\overline{28|}} = 1a_{\overline{28|}.03} + \frac{-1}{.03}(a_{\overline{28|}.03} - 28v^{28}) \approx \$704.37$$

(b) $(I\ddot{a})_{\overline{\infty|}}$ is an perpetuity due with the initial payment at \$1 and each payment grows by \$1 more; that is, the second payment would be \$2 and the nth payment would be \$$n$. This happens forever, and we calculate the value of this perpetuity at the time of the first payment. Furthermore, algebraically we have

$$(I\ddot{a})_{\overline{\infty|}} = [\lim_{n\to\infty}(I_{1,1}a)_{\overline{n|}}](1.03) = [\lim_{n\to\infty}(1a_{\overline{n|}.03} + \frac{1}{.03}(a_{\overline{n|}.03} - 28v^{28}))](1.03)$$

$$= [1a_{\overline{\infty|}.03} + \frac{1}{.03}(a_{\overline{\infty|}.03}](1.03) \approx \$1,178.78$$

(c) $(I_{100,10}a)_{\overline{15|}}$ This is an increasing arithmetic annuity-immediate with the first payment of \$100 and each payment after increases by \$10; that is, the second payment is \$110 and the nth payment is $\$100 + \$(n-1)10$. For this we are calculating the value of all these payments one year before the first payment. Furthermore we calculate

$$(I_{100,10}a)_{\overline{15|}} = 100a_{\overline{15|}.03} + \frac{10}{.03}(a_{\overline{15|}.03} - 15v^{15}) \approx \$1,963.80$$

Problem 3.9.2

Payments of \$5000 are made into a fund at the beginning of each year for ten years. The fund is invested at an annual effective rate of i. The interest generated is reinvested at 10%. The total accumulated value at the end of the ten years is \$100,000. We want to find i.

We start off with the value at time 10 to give us the following equality

$$(I_{P,Q}S)_{\overline{10|}} + 5000(10) = 100,000$$

Expanding our terms we find that

$$i5000S_{\overline{10|}10} + \frac{i5000}{.10}(S_{\overline{10|}10} - 10) = 50,000$$

Solving for i we find

$$i = \frac{10}{S_{\overline{10|}\,10}} + 10(S_{\overline{10|}10} - 10) = \frac{10}{(11S_{\overline{10|}10} - 100)} \approx 0.13278155$$

Problem 3.9.3

A perpetuity has annual payments. The first payment is for \$320 and then payments increase by \$30 each year until they become level at \$980. We want to find the value of this annuity at the time of the first payment using an annual effective interest of 4%.

We must calculate two things: the perpetuity after the arithmetic non-level annuity rises to \$980 and and the arithmetic non-level annuity that rises the payments by \$30 until \$980. We calculate the first 22 payments (not including any of the payments of \$980 then calculate the perpetuity of payments starting at time 23. Furthermore we find at time of the first payment we get

$$\begin{aligned}[(I_{320,30}a)_{\overline{23|}} + \frac{980}{.04}](1.04) &= [320a_{\overline{23|}.04} + \frac{30}{.04}(a_{\overline{23|}.04} - 23v^{23}) + \frac{980}{.04}](1.04) \\ &\approx \$19,591.87\end{aligned}$$

(3.10) Yield rate examples involving annuities

Problem 3.10.1

An investor invests \$58,000 and receives an annuity of \$7,000 at the end of each year for twelve years and an additional payment of \$15,000 at the end of the thirteenth year. Each time he gets a \$7,000 payment, he immediately deposits \$4,000 in a savings account that earns 9%. We want to find the annual yield received by the investor over the thirteen years.

For this problem we shall use our magical cashflow worksheet on our BA II Plus calculator. In the [CF] function we input

$$\begin{aligned}&CF_0 = -58,000 \\ &C_{01} = 3000 && F_{01} = 12 \\ &C_{02} = 15,000 + 4000S_{\overline{12|}.09}(1.09) \approx 102,813.5383 && F_{02} = 1\end{aligned}$$

We then press [IRR] + [CPT] to find

$$i \approx 8.432601909\%$$

Problem 3.10.3

This dude named John invests a total of $10,000 (all of his money he has been saving since he was 3 years old and he is 43 now, we blame George Bush). He purchases an annuity with payments of $1000 at the beginning of each year for ten years at an effective interest rate of 8%. As annuity payments are received, they are reinvested at an effective annual rate of 7%. The balance of the $10,000 is invested in a ten-year certificate of deposit with a nominal annual interest rate of 9% compounded quarterly. We want to calculate the annual effective yield on the entire $10,000 investment over the ten-year period.

We calculate the value at time 10

$$10,000(1+i)^{10} = [10,000 - 1000\ddot{a}_{\overline{10|}.08}]\left(1 + \frac{.09}{4}\right)^{40} + [1000\ddot{S}_{\overline{10|}.07}]$$

solving for i we find that

$$i \approx 7.949212452\%$$