Theory of Interest - Homework #3

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Section 1.11

Problem 1.11.1

Suppose $d^{(4)} = 3.2\%$. We want to find δ. We use the formula

$$d = 1 - (1 - \frac{d^{(m)}}{m})^m = 1 - (1 - \frac{d^{(4)}}{4})^4 \approx .031618$$

and

$$\delta = \ln(1 - d) \approx .032128687$$

Problem 1.11.2

Given that $\delta = .04$, we want to find the accumulated value of \$300 five years after it is deposited. We use the accumulation function $a(t) = e^{\delta(t)}$ to get the result of the following amount function:

$$A_{300}(5) = 300a(5) \approx \$366.42$$

Problem 1.11.3

There is a choice of depositing money in account A which has an annual effective interest rate of 5.2%, account B which has an effective monthly rate of .44%, or account C that is governed by force of interest $\delta = .0516$. We want to order the accounts based on annual interest rates. We have the following interest rates:

$$i_A = .052$$

$$i_B = (1.0044)^{12} - 1 \approx .054096687$$

$$i_C = e^{.0516} - 1 \approx .052954476$$

Therefore we conclude that the interest rates ordered yield the following inequality:

$$i_A < i_C < i_B$$

Section 1.12

Problem 1.12.1

Given that the force of interest is $\delta_t = .05 + .006t$, we want to find the accumulated value after three years of an investment of \$300 made at: (a) time 0, (b) time 4. We shall use the formula:

$$a(t) = \exp(\int_0^t \delta_r dr) \tag{1}$$

(a) Furthermore our amount function yields:

$$A_{300}(3) = 300\exp(\int_0^3 .05{+}.006t dr) = 300\exp(.05t{+}.003t^2 \mid_0^3) = 300\exp(.177) \approx \$358.0893279$$

(b) In this case we have the same except we must change the bounds. For example

$$a(t_1) - a(t_0) = \exp(\int_0^{t_1} \delta_r dr) - \exp(\int_0^{t_0} \delta_r dr) = \exp(\int_{t_0}^{t_1} \delta_r dr)$$

If we let $A(t+s|t)$ be the amount function for the time interval $[t, t+s]$ then we may use the above equation to solve for each value and we get:

$$A_{300}(7|4) = 300(a(t_1) - a(t_0)) = 300\exp(.05t + .003t^2 \mid_4^7) \approx 300\exp(.249) \approx \$384.8226099$$

Problem 1.12.2

Given that the force of interest is $\delta_t = \frac{2t}{1+t^2}$, we want to find the effective rate of discount for the sixth year. Recall that:

$$d_6 = \frac{a(6) - a(5)}{a(6)}$$

And using the equation (1) above for our accumulation function we find that:

$$a(t) = 1 + t^2$$

Now we can solve for the rate of discount for the sixth year:

$$d_6 = \frac{a(6) - a(5)}{a(6)} = \frac{6^2 - 5^2}{37} \approx .\overline{297}$$

Problem 1.12.5

Given that $a(t) = (1.02)^t(1 + .03t)(1 - .05t)^{-1}$, we want to find δ_3. Recall that

$$\delta_t = \frac{d}{dt}\ln[a(t)] = \frac{d}{dt}\ln[(1.02)^t(1 + .03t)(1 - .05t)^{-1}]$$

And after solving the derivative we find that:

$$\delta_3 = \ln(1.02) + \frac{.03}{1 + .03t} + \frac{.05}{1 - .05t} = \ln(1.02) + \frac{.03}{1 + .09} + \frac{.05}{.85} \approx .106149092$$

Problem 1.12.7

Suppose one invested \$2500 in a fund earning 10% simple interest. Further suppose that one has the option at any time to swap to an account earning compound interest at an annual effective interest rate of 7%. We want to find the time t it takes to maximize our money. We can do this by first solving whenever the compound interest accumulation function, $b(t)$ grows higher than simple interest accumulation function, $a(t)$. We can do this by solving the below equation for t:

$$a(t) = (1 + .10t) = (1.07)^t = b(t)$$

However we can see that this equation is not easy to solve, so instead we can use the force of interest for each accumulation function and set them equal to find t:

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{(.10)}{(1 + .10t)} = \ln(1.07)\frac{(1.07)^t}{(1.07)^t} = \frac{b'(t)}{b(t)}$$

Once we solve for t we find that

$$t \approx 4.78 \text{ years}$$

1 Section 1.14

Problem 1.14.1

Inflation is forecast to be at an annual rate of 3% for the next year. We want to find: (a) what the real rate of interest will be if the stated effective rate for the next year is 4.2%, (b) what the real rate will be if the actual rate of inflation is 4.6%.

(a) In order to find the real rate of interest we use the equation

$$1 + j = \frac{1 + i}{1 + r} \tag{2}$$

for j the real interest rate, r the inflation rate, and i the interest rate. Furthermore we find that

$$j = \frac{1+i}{1+r} - 1 = \frac{1.042}{1.03} - 1 \approx .011650485$$

(b) Again we use equation (2) to find that:

$$j = \frac{1+i}{1+r} - 1 = \frac{1.042}{1.046} - 1 \approx -.003824092$$

Problem 1.14.2

We want to show that the nominal interest rate i and the real interest rate j are equal if and only if the inflation rate is zero. Let $i = j$, then equation (2) gives us:

$$1 + j = \frac{1+j}{1+r}$$

which implies that $1 + r = 1$ and $r = 0$. To prove the other direction we can simply plug in $r = 0$ to verify that $j = i$. Therefore we conclude that $i = j$ if and only if $r = 0$.

Problem 1.14.3

The nominal rate of discount is 3% convertible quarterly. The inflation-adjusted (effective) rate of interest is 1.24%. We want to find the rate of inflation. Recall that

$$(1 - \frac{d^{(4)}}{4})^{-4} = 1 + i \tag{3}$$

And so by equation (2) and (3) we find that

$$r = \frac{1+i}{1+j} - 1 = \frac{(1 - \frac{d^{(4)}}{4})^{-4}}{1+j} - 1 = \frac{(1 - \frac{.03}{4})^{-4}}{1.0124} - 1 \approx .017948488$$

Problem 1.14.4

We want to find out using the real interest rate when the two present values are equal; that is, $PV_{(1+j)^t}(\$X \text{ at time } n) = PV_{(1+i)^t}(\$Y \text{ at time } n)$. Recall that:

$$PV_{(1+j)^t}(\$X \text{ at time } n) = X(1+j)^{-t}$$

$$PV_{(1+i)^t}(\$Y \text{ at time } n) = Y(1+i)^{-t}$$

So setting up our equality gives us:

$$X(1+j)^{-t} = X(\frac{1+i}{1+r})^{-t} = Y(1+i)^{-t}$$

and reducing gives us:

$$X = Y(1+r)^t$$

Section 2.2

Problem 2.2.1

An amount of $\$K$ is deposited into an account paying 4% annual effective discount. The balance at the end of three years is \$982. We want to find K.We can do this by using the accumulation function $a(t) = (1-.04)^{-t}$ for the amount function:

$$A_K(3) = K(1-.04)^{-t} = 982$$

and solving for K we find that

$$K \approx \$868.810752$$

Problem 2.2.2

An amount of \$2000 is deposited in a five-year certificate of deposit. At maturity the balance is \$2580.64. We want to find the annual effective rate of interest governing the account. We use the accumulation function $a(t) = (1+i)^t$ to solve the amount function:

$$A_{2000}(5) = 2000(1+i)^5 = 2580.64$$

and solving for i we find that:

$$i \approx .0522998$$

Problem 2.2.3

An amount of \$1800 is deposited into an account with a constant nominal interest rate of 3.2% convertible monthly. The balance after t years is \$1965.35. We want to find t. We use the accumulation function $a(t) = (1+\frac{.032}{12})^{12t}$ for the amount function:

$$A_{1800}(t) = 1800(1+\frac{.032}{12})^{12t} = 1965.35$$

Solving for t we find that:

$$t \approx 2.75003$$

Problem 2.2.4

We want to use the rule of seventy-two to approximate the length of time it takes for money to double at an annual effective interest rate of 5% and then at an annual effective rate of 10%. Then we want to find the exact time at each of these interest rates respectively. For 5% and 10% we find that the seventy-two rule gives us:

$$t_{5\%} = 72/5 = 14.4 \text{ years}$$

$$t_{10\%} = 72/10 = 7.2 \text{ years}$$

However once we find the actual values we find that:

$$t_{5\%} = \frac{\ln(2)}{\ln(1.05)} \approx 14.2 \text{ years}$$

$$t_{10\%} = \frac{\ln(2)}{\ln(1.10)} \approx 7.27 \text{ years}$$

2 Section 2.3

Problem 2.3.1

Sidney borrows \$12000. The loan is governed by compound interest and the annual effective rate of discount is 6%. Sidney repays \$4000 at the end of one year, X at the end of two years, and \$3000 at the end of three years in order to exactly pay off the loan. We want to find X. We use the equality between present values and get the equation:

$$12000 = 4000v(1) + Xv(2) + 3000v(3)$$

where $a(t) = (1 - .06)^{-t}$ and $v(t) = 1/a(t)$ we find that:

$$X = \frac{[12000 - 4000(.94) - 3000(.94)^3]}{(.94)^2} \approx \$6505.486648$$

Problem 2.3.2

Rafael opens a savings account with a deposit of \$1500. He deposits \$500 one year later and \$1000 a year after that. Just after Rafael's deposit of \$1000, the balance in his account is \$3078. We want to find the annual effective interest rate governing the account. By using our handy dandy BAII Plus calculator we use the [CF] button and plug in the following values:

$$\begin{aligned}
CF_0 &= 1500 \\
C_{01} &= 500 \\
F_{01} &= 1 \\
C_{02} &= -2078 \\
F_{02} &= 1
\end{aligned}$$

Next we press the [IRR] button and then [CPT] to find that:

$$i \approx .02207683431$$

Problem 2.3.3

Esteban borrows \$20000, and the loan is governed by compound interest at an annual effective interest rate of 6%. Estaban agrees to repay the loan by making a payment of \$10000 at the end of T years and a payment of \$12000 at the end of $2T$ years. We want to find T. Using our equality of present values we use the equation:

$$20000 = 10000v(T) + 12000v(2T)$$

where we let $v(t) = 1.06^{-t} = v^t$. Substituting we get the polynomial of degree 2:

$$f(v) = 12000v^2 + 10000v - 20000$$

Solving for the roots to $f(v) = 0$ we find that $v \approx .939901716$ which we use to solve for the value T:

$$T = \frac{\ln(v)}{-\ln(1.06)} \approx 1.063688479$$

*Note: there are two values that can be found, however the other will result in a negative year which we will assume is impossible.

Problem 2.3.4

Shakari opens a savings account with a deposit of \$3500. She deposits \$500 six months later and \$800 nine months after opening the account. The balance in Shakari's account one year after she opened it is \$5012. Assuming that the account grows by compound interest at a constant annual effective interest rate i, we want to find i. We use our calculator [CF] function and plug in the values for the interval for one-fourth years:

$$\begin{aligned}
CF_0 &= 3500 \\
C_{01} &= 0 \\
F_{01} &= 1 \\
C_{02} &= 500 \\
F_{02} &= 1 \\
C_{03} &= 800 \\
F_{03} &= 1 \\
C_{03} &= -5012 \\
F_{03} &= 1
\end{aligned}$$

Next we press our [IRR] and [CPT] to find that our $i_4 = 1.317932969$, however this is not our annual effective interest. So we must convert it to annual using the equation:

$$i = (1 + i_4)^4 - 1 \approx .053768674$$

Problem 2.3.5

A loan is negotiated with the lender agreeing to accept \$8000 after one year, \$9000 after two years, and \$20000 after four years in full repayment of the loan. The loan is renegotiated so that the borrower makes a single payment of \$37000 at time T and this results in the same total present value of payments when calculated using an annual effective rate of 5%. We want to estimate T using the method of equated line (we will denote $\overline{T}$), and we want find the exact value of T. We have using the method of equated line gives us the equation:

$$\overline{T} = \frac{8000}{37000} + 2\frac{9000}{37000} + 4\frac{20000}{37000} \approx 2.\overline{864} \text{ years}$$

Now to compute T exactly. Solving for the exact value of T gives us the equation:

$$T = \frac{\ln[(1.05)^{-1}\frac{8000}{37000} + (1.05)^{-2}\frac{9000}{37000} + (1.05)^{-4}\frac{20000}{37000}]}{-\ln(1.05)} \approx 2.824807661$$