Theory of Interest - Homework #6

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(3.3) Annuities-due

Problem 3.3.1

April received an inheritance from her grandmother in the form of an annuity. The annuity pays $3,000 on January 1st from 1966 through 1984. We want to find the value of this annuity on January 1, 1966 using an annual effective interest rate of 5% and representing this value by an appropriate annuity symbol.

We have that

$$\ddot{a}_{\overline{19|}5} = a_{\overline{19|}5}(1.05) = \frac{[1-(1.05)^{-19}](1.05)}{.05} \approx 12.6895869$$

and so it follows from the basic annuity that the present value,

$$PV = 3000\ddot{a}_{\overline{19|}5} \approx \$38,068.76071$$

Problem 3.3.2

Suppose $i = 3\%$. We want to find the value one month before the first payment of a level annuity-due paying $200 at the beginning of each month for five years.

First we must convert our annual interest to monthly interest (this author is really trying to trick us without specifying what i is, but nevertheless we will assume it is annual), and we have that for the monthly effective interest rate j

$$j = (1+i)^{1/12} - 1 = (1.03)^{1/12} - 1 \approx 0.00246626977$$

Next we must find one month before our lovely annuity

$$(1+j)^{-1}\ddot{a}_{\overline{60|}j} = a_{\overline{60|}j} = \frac{[1-(1+j)^{-60}]}{j} \approx 55.70810507$$

Note*: we used the property that the annuity-immediate is one month behind the annuity-due. Furthermore it follows from our basic annuity that the value one month before the annuity level payments

$$S_{-1} = 200a_{\overline{60|}j} \approx \$11,141.62101$$

Problem 3.3.4

Steven Wong wishes to save for his retirement by depositing $\$1,200$ at the beginning of each year for thirty years. Exactly one year after his last deposit, he wishes to begin making annual level withdrawals until he has made twenty withdrawals and exhausted the savings. We want to find the amount of each withdrawal if the effective interest rate is 5% during the first thirty years but only 4% after that.

We set up Mr. Wong's deposits of $1200 for the first 30 years with interest of 5% on the left side of the equation and push the time forward to 30. On the right hand side we take the withdrawals and use the annuity to pull the value back to time 30 and it follows from our equality that

$$1200\ddot{a}_{\overline{30|}5}(1.05)^{30} = P\ddot{a}_{\overline{20|}4}$$

Solving for P we find that

$$P = \frac{1200\ddot{a}_{\overline{30|}5}(1.05)^{30}}{\ddot{a}_{\overline{20|}4}} \approx \$5922.83195$$

Problem 3.3.5

Starting on his 25th birthday and continuing through his 60th birthday, Fred deposits $\$7,500$ each year on his birthday into a retirement fund earning an annual effective rate of 5%. Immediately after the last deposit, the accumulated value of the fund is transferred to a fund earning an annual effective rate j. Five years later, a twenty-five year annuity-due paying $\$5,800$ each month is purchased with the funds. The purchase price of the annuity was determined using annual effective rate of interest 4%. We want to find j.

First we calculate the monthly interest m for the second annuity

$$m = (1.04)^{1/12} - 1 \approx 0.00327374$$

Next we set up our equality we have that

$$7500\ddot{a}_{\overline{36|}5}(1.05)^{35}(1+j)^5 = 5800\ddot{a}_{\overline{300|}m}$$

and rearranging our terms and solving for j we find that

$$j \approx 0.090943214$$

(3.4) Perpetuities

Problem 3.4.1

Athlete Kalen wishes to retire at age forty-five and receive annual birthday payments of $\$40,000$ beginning on his forty-fifth birthday. After his death the payments will get carried on. In order for Kalen to be able to carry out his plan, he makes contribuitions to a savings account with a guaranteed annual effective interest rate of 4%. We want to find how much money, K, Kalen will need.

Setting up our perpetuity we find that

$$K = 40000\ddot{a}_{\overline{\infty|}4} = \frac{40,000(1.04)}{.04} \approx \$1,040,000$$

Problem 3.4.2

Grahan receives $\$640,000$ at his retirement. He invests X in a twenty-year annuity-immediate with annual payments and the remaining $\$640,000 - X$ is used to purchase a perpetuity-immediate with annual payments. His total annual payments received during the first twenty years are twice as large as those received thereafter. The annual effective interest rate is 5%. we want to find X.

We get the following two equations

$$640,000 - X = Pa_{\overline{\infty|}5}$$

$$X = Pa_{\overline{20|}5}$$

First we solve for P to find that

$$P = \frac{640,000}{(a_{\overline{\infty|}5} + a_{\overline{20|}5})} \approx \$19,715.23175$$

and it follows that

$$X \approx \$245,695.365$$

Problem 3.4.3

Suppose $\$40,000$ was invested on January 1, 1980 at an annual effective interest rate of 7% in order to provide an annual (calendar-year) scholarship of $\$5,000$ each year forever, the scholarships paid out each January 1. We want to find

(a) In what year can the $5,000 scholarship be made.

We use the following equations

$$X = 5000\ddot{a}_{\overline{\infty}|7}$$

$$40,000(1.07)^t = X$$

Solving for X we use the first equation to find that

$$X = \frac{5000(1.07)}{.07} \approx \$76,428.57143$$

and next we use the value of X to solve for t and get

$$t = \frac{\ln(\frac{X}{40,000})}{\ln(1.07)} \approx 9.569761713 \text{ years}$$

(b) What smaller scholarship can be awarded the year prior to the first $5,000 scholarship.

We notice that there is about half a year where the interest goes beyond the perpetuity. If we calculate the difference at time 10 and pull it back a year to time 9 we can give away a cheaper scholarship a year sooner. We get the equation

$$[40,000(1.07)^{10} - 40,000(1.07)^t](1.07)^{-1} \approx \$2109.797068$$

Problem 3.4.6

Cheyenne wishes to endow a professorship at her alma mater. She learns that to do so she must contribute $1,000,000. We want to find what annual salary would a $1,000,000 gift support forever if salary payments are made monthly, starting one month after the gift and the money's growth is governed by an annual effective discount rate of 6%.

First we want to convert our annual effective discount rate to monthly effective discount rate, j. We get

$$j = (1 - .06)^{-1/12} - 1 \approx 0.0051696$$

We next notice that our annual salary is derived from the monthly interest from the $1,000,000

$$P = 1,000,000(1 + j) \approx \$5,169.600152$$

and our annual salary is just 12 of these payments giving us

$$S = 12P \approx \$62,035.20182$$

(3.5) Deferred annuities and annuity values on any date

Problem 3.5.1

Sydney wins a prize. She has a choice of receiving a payment of $\$160,000$ immediately or receiving a deferred perpetuity with $\$10,000$ annual payments, the first payment occurring in exactly four years. Which has a greater present value if the calculation is based on an annual effective interest rate of 5%? How about if the annual effective rate used is 6%? What real life considerations should enter into Sydney's choice besides maximizing her present value?

At 5% we have that the two options give us the current present values:

Option 1: $\$160,000$

Option 2: $(1+i)^{-3}10,000a_{\overline{\infty}|5} = (1.05)^{-3}\frac{10,000}{.05} \approx \$172,767.5197$

So we will take the second option (\$_\$) for the money.

At 6% we have that the two options give us the current present values:

Option 1: $\$160,000$

Option 2: $(1+i)^{-3}10,000a_{\overline{\infty}|6} = (1.06)^{-3}\frac{10,000}{.06} \approx \$139,936.5472$

So we will take the first option if there is 6% interest.

Problem 3.5.2

A level perpetuity-immediate is to be shared by three charities providing medical research and a fourth charity providing assistance to children of veterans. For n years, the three research charities will receive the payments equally. Thereafter, all the payments will go to the charity aiding children of veterans. It is reported that the present value of each of the charities' bequest is equal to a common amount when calculated with an annual effective interest rate of 12.25%. We want to find n. We also want to find if the interest rate is more modest 6%, the proportion of the total bequest directed to the charity aiding children of veterans.

In order to solve the problem we use the following equation

$$Pa_{\overline{n}|12.25} = 3Pa_{\overline{\infty}|12.25}(1+i)^n$$

and solving for n we find that

$$n = \frac{\ln(4)}{\ln(1.1225)} \approx 11.99648899 \text{ years}$$

Next we simply solve for the proportion by solving the following

$$Pa_{\overline{n|}12.25} : 3Pa_{\overline{\infty|}12.25}(1+i)^{-n}$$

which reduces to

$$(1.06)^{-12} : 1 - (1.06)^{-12}$$

and furthermore we find that our percentage in a decimal form is

$$(1.06)^{-12} \approx 0.496969364$$

Problem 3.5.3

Alice owned an annuity which had level annual payments for twelve consecutive years, the first of these being in exactly twelve years. She sold it, and the selling price of \$21,092.04 was based on a yield rate for the investor of 7.8%. We want to find what the amount of the level payments are.

We use the following equation to solve for the payments P

$$21,092.04(1.078)^{12} = P\ddot{a}_{\overline{12|}7.8}$$

and solving for P we find that

$$P = \frac{21,092.04(1.078)^{12}}{\ddot{a}_{\overline{12|}7.8}} \approx \$6,328.000374$$

(3.6) Outstanding loan balances

Problem 3.6.2

Olena loans her sister Irini \$8000. The loan is to be repaid at a nominal interest rate of 4.8% payable monthly. The monthly payments are to be for \$100 except for a final smaller payment. We want to find how much Irini owes Olena at the end of one year.

Recall the equation for retrospective outstanding loans

$$OLB_k = L(1+j)^k - QS_{\overline{k|}} \tag{1}$$

for a loan amount L and payments Q. First we solve for our monthly effective interest rate j

$$j = .048/12 = 0.004$$

Next, using the above equation we find our outstanding balance at time $k = 12$

$$OLB_{12} = 8000(1 + .004)^{12} - 100\frac{(1.004)^{12} - 1}{.004} \approx \$7,165.806472$$

Problem 3.6.3

Mr. Bell buys a home for an unspecified amount. He pays a down payment of $20,000 and finances the remainder for 15 years with level end-of-month payments of $1,692. The annual effective interest rate for the first five years is 4%, and thereafter it is 6%. Mr Bell sells the house just after making his 100th mortgage payment. The selling price is $258,000. We want to find out how much money will Mr. Bell get at closing.

In order to solve this we first solve for the outstanding balance at $k = 120$ (Recall we are paying monthly instead of annually). First we must calculate the monthly effective interest rate j

$$j = (1.06)^{1/12} - 1 \approx 0.004867551$$

Now we calculate the remaining payments after the 100th payment using the prospective method and subtract that value from how much Mr. Bell sold the house, namely $258,000 to find our closing balance B

$$B = 258,000 - OLB_{100} = 258,000 - 1692(a_{\overline{80|}j}) \approx \$146,105.2173$$

Problem 3.6.5

Alice purchases a boat. The purchase price is $18,300 and she makes a down payment of $3,800. She finances the balance for six years. There are level end-of-the-month payments (except for a slightly reduced final payment) and the loan is made at an annual effective rate of 5.2%. We want to find

(a) the amount of the first seventy-one payments. First we calculate the monthly interest rate j

$$j = (1.052)^{1/12} - 1 \approx 0.004233362$$

We use the following equation

$$(18,300 - 3800) = Pa_{\overline{72|}j}$$

Solving for P we find that

$$P = \frac{(18,300 - 3800)}{a_{\overline{72|}j}} \approx \$234.07$$

(b) the outstanding loan balance just after the twenty-fourth payment. we use the retrospective method to get the following equation

$$OLB_{24} = [(18,300 - 3800) - (234.07)a_{\overline{24|}j}](1 + j)^{24} = \$10,147.35647$$

Problem 3.6.7

Janelle receives a home improvement loan of \$14,508.97. The loan has a nominal interest rate convertible monthly of 4.8%. The term of the loan is two years and Janelle is expected to make level end-of-the-month payments, except that she is allowed to miss one payment so long as she then pays higher level payments for the remainder of the two years, so as to have repaid the loan at the end of the two-year period. Suppose Janelle misses the payment at the end of the ninth month. We want to find what her new level payments will be.

First we wish to solve for the original payments P. We use the following equation

$$14,508,97 = Pa_{\overline{24|}.4}$$

Solving for P we find that

$$P \approx 635.2299231$$

Next, we want to find out how much the new level payments X are. We use the following equation

$$Xa_{\overline{24-9|}.4} = Pa_{\overline{24-8|}.4}(1 + j)$$

And solving for X we find that

$$X \approx \$678.946364$$

Note*: we didn't have to round off the last payment since the values corresponded well, however in other scenarios we would have to reduce for a final payment.