Theory of Interest - Homework #2

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Section 1.7

Problem 1.7.1

Suppose $a(t) = 1 + .05t$. We want to find how much one would need to invest at time 3 in order to have \$3200 at time 8. In order to find this value we must first bring the value back to time 0 by using our discount function $v(t) = a(t)^{-1}$, and then push it forward with our accumulation function stated above. This gives us

$$P_{t_2} v(t_2) a(t_1) = P_{t_1} \tag{1}$$

Where P_{t_n} is the value of the investment at time t_n, and K is the original value of the investment. Plugging in our values and solving for P_{t_1} we get:

$$P_3 = P_8 v(8) a(3) = 3200 \frac{(1.15)}{(1.4)} \approx \$2628.571429$$

Problem 1.7.2

We want to find the value at $t = 6$ of \$4850 to be paid at time 12 if $a(t) = (1 - .04t)^{-1}$. Using the equation (1) above we get that

$$P_6 = P_{12} v(12) a(6) = 4850 \frac{(.52)}{(.70)} \approx \$3602.857143$$

Problem 1.7.3

A house was purchased on July 31, 2002 for $\$156,000$. If the price rises at a compound interest rate of 6.5% annually, then how much was the home worth on July 31, 1998? Using equation (1) above and $a(t) = (1.065)^t$ we get that:

$$P_{1998} = P_{2002}v(4) = 156000(1.065)^{-4} \approx \$121,262.4022$$

Problem 1.7.5

We want to find the present value of $5000 due in ten years assuming money grows according to compound interest and the annual effective rate of interest is 4% for the first three years, 5% for the next two years, and 5.5% for the final five years. We can work backwards using our discount function, in other words:

$$P_0 = P_10V_(10) = 5000(1.04)^{-3}(1.05)^{-2}(1.055)^{-5} \approx \$3084.814759$$

Problem 1.7.7

We have two options to repay a loan. Option 1: Repay $6000 now and $5940 in one year. Option 2: Repay $12000 in 6 months. We want to find the annual effective interest rate(s) i at which both options have the same present value. Our options yield the present values:

$$PV_1 = 6000v(0) + 5940v(1)$$

$$PV_2 = 12000v(.5)$$

Setting up our equality gives us

$$PV_1 = PV_2$$

$$6000v(0) + 5940v(1) - 12000v(.5) = 0$$

which is equivalent to by re-writting with the discount function

$$6000 + 5940(1+i)^{-1} - 12000(1+i)^{-.5} = 0$$

And if we let $x = (1+i)^{-.5}$ our equation turns into a quadratic, and furthermore:

$$5940x^2 - 12000x + 6000 = 0$$

which yields the two values $x \approx 1.\overline{1}$ or $x = .\overline{90}$ and thus gives us the values of $i = -.19$ or $i = .21$. Assuming our interest is positive we use the value $i = .21$.

Problem 1.7.8

Two projects have equal net present values when calculated using a 6% annual effective interest rate. Project 1: initial investment of \$20,000 immediately, and will return \$8000 in one year and \$15,000 in two years. Project 2: initial investment of \$10000 immediately and \$$X$ in two years, and will return \$3000 in one year and \$14,000 in three years. We also want to find the difference in the net present values of the two projects if they are calculated using a 5% annual effective interest rate. Our options yield the present values:

$$PV_1 = -20000v(0) + 8000v(1) + 15000v(2)$$

$$PV_2 = -10000v(0) + -Xv(2) + 3000v(1) + 14000v(3)$$

Using the equation $v(t) = (1.06)^{-t}$, we let $PV_1 = PV_2$ and find that $X \approx \$897.1164115$. Now instead, with the value of X we find that the difference with the new discount function $v'(t) = (1.05)^{-t}$ we find that the difference D, is

$$D \approx 31.94$$

Section 1.8

Problem 1.8.1

If money grows according to simple discount at an annual rate of 5%. We want to find the value at time 4 of \$3,460 to be paid at time 9. Using our equation (1) and the accumulation function $a(t) = v(t)^{-1}$ where $v(t) = 1 - dt$ we get that

$$P_4 = P_9 v(9) a(4) = 3460 \frac{(.55)}{(.80)} \approx \$2502.50$$

Problem 1.8.2

An investment in account earns interest based on simple discount at a 2% annual rate. We want to find the effective interest rate in the fifth year. The equation for effective interest rate of the $n-$th year gives us

$$i_n = \frac{a(t) - a(t-1)}{a(t)} \tag{2}$$

Using $a(t) = (1 - .02t)^{-1}$ we get that

$$i_5 = \frac{a(5) - a(4)}{a(4)} = \frac{(1 - .10)^{-1} - (1 - .08)^{-1}}{(1 - .08)^{-1}} \approx .0\overline{2}$$

Problem 1.8.3

Suppose one can invest \$1000 in a fund earning simple discount at an annual rate of 8% or in a fund earning simple interest at an annual rate of 12%. We want to know how long must one invest their money in order for the simple discount account be preferable. We have the two accumulation functions:

$$a(t) = (1 - .08t)^{-1}$$

$$b(t) = (1 + .12t)$$

If we solve for t whenever $a(t) = b(t)$ we find that

$$t \approx 4.1\overline{6}$$

As we evaluate the function of simple discount grows uncontrollably at a higher amount while the simple interest rate is linear. Therefore, the simple discount rate will be higher for $t > 4.1\overline{6}$.

Problem 1.8.5

Suppose one invests \$300 in a fund earning simple interest at 6%. Three years later one withdraws the investment (everything) and invests it in another fund earning 8% simple discount.

(a) We want to find how much time, including the three years, it will take to get to \$650.

(b) We also want to find at what annual effective rate of compound interest would \$300 accumulate to \$650 in the same amount of time.

(a) First let us find how much time it takes our amount function

$$A_{300}(t+3) = 300(1.18)(1 - .08t)^{-1} = 650$$

Solving for t we find that

$$t + 3 \approx 8.692307692$$

(b) Plugging our previous $t + 3$ value and solving for the interest rate on the new amount function

$$B_{300}(t) = (1 + i)^{t+3} = 650$$

we find that the interest rate is

$$i \approx .090302715$$

Section 1.9

Problem 1.9.1

A savings account starts with $1000 and has an annual effective discount rate of 6.4%. We want to find the accumulated value at the end of five years. To solve for the accumulated value at the end of 5 years, denoted P_5 we use the accumulation function $a(t) = (1-d)^{-1}$ and thus generate the formula:

$$P_5 = P_0 a(5) = 1000(1-.064)^{-5} \approx \$1,391.940773$$

Problem 1.9.2

One wishes to obtain $4000 to pay their college tuition now. With a loan with an annual effective discount rate of 3.5% how much (a) will one have to repay if the loan term is 6 years. (b) We also want to find the annual effective interest rate of the loan.

(a) For $a(t) = (1-d)^{-t}$ we use the formula

$$P_6 = P_0 a(6) = 4000(1-.035)^{-6} \approx \$4953.31687$$

(b) We then solve for the annual effective interest using the formula:

$$i = \frac{d}{(1-d)} = .035/(1-.035) \approx .03626943$$

Problem 1.9.4

An account is governed by compound interest. The interest for three years on $480 is $52. We want to find the amount of discount for two years on $1000. First we must solve for our annual effective discount rate using the accumulation as used in the previous two problems:

$$P_0 a(3) = P_0 + 52$$

Which gives us, by shifting the terms around

$$d = 1 - \sqrt[3]{\frac{480}{532}} \approx .033704697$$

Now we must compute the amount of discount for two years by using the formula:

$$P_0 - P_0 v(2) = 1000 - 1000(1-.033704697)^2 \approx \$66.27338799$$

Problem 1.9.7

The amount of interest on \$$X$ for two years is \$320. The amount of discount on \$$X$ for one year is \$148. We want to find the annual effective interest rate i and the value of X. We have two equations and two unknowns:

$$Xa(2) = X + 320$$

$$Xv(1) = X - 148$$

If we square the second equation (Notice: $v(1)^2 = (1-d)^2 = v(2) = a(2)^-1)$ and multiply it by the first equation we get:

$$X^3 a(2)v(1)^2 = (X + 320)(X - 148)^2$$

Which further reduces to

$$X^3 = (X + 320)(X - 148)^2$$

Now we can put it in the form of a quadratic by factoring out the terms and we find that

$$24X^2 - 72816X + 7009280 = 0$$

and after using the handy dandy quadratic formula we get the solutions $X \approx \$2934.475103$ or $X \approx \$99.52489734$, however the latter solution does not make sense in the scenario we are in. Therefore we conclude that

$$X = \$2934.48$$

Furthermore when we solve for d we use the second equation to find that $d = 148/X$ and so we have $d \approx .050434914$ and thus using the formula

$$i = \frac{d}{1-d} \approx .053113699$$

Section 1.10

Problem 1.10.1

Suppose we have compound interest and $d^{(4)} = 8\%$. We want to find equivalent rates d, $d^{(3)}$, i, and $i^{(6)}$. Using the formula for equivalent rates:

$$(1 + \frac{i^{(m)}}{m})^m = 1 + i = 1 - d = (1 - \frac{d^{(m)}}{m})^m \tag{3}$$

We can solve for each of the rates giving us the values $d \approx .07763184$, $d^{(3)} \approx .07973213816$, $i \approx .084165785$, and $i^{(6)} \approx .08135747987$

Problem 1.10.2

The annual effective interest rate on one's loan is 6.6%. We want to find the equivalent nominal discount rate convertible monthly on the loan. We also want to find the effective monthly discount rate. We have that $i = .066$ and using equation (3) above we can find that

$$d = \frac{i}{1+i} = \frac{.066}{1.066} \approx .061913696$$

and we use the value of d and $m = 12$ to solve for

$$d^{(m)} = m[(1-d)^{\frac{1}{m}} - 1] \approx .0637434228$$

Once we divide this value by twelve we find the monthly effective monthly discount rate to be

$$\frac{d^{(12)}}{12} \approx .0053119519$$

Problem 1.10.3

Suppose we have compound interest and an effective monthly interest rate of 0.5%. We want to find equivalent rates $i^{(12)}$, i, and d. We are given that

$$\frac{i^{(12)}}{12} = 0.005$$

therefore we get the value

$$i^{(12)} = 0.06$$

and using the formula

$$i = (1 + \frac{i^{(12)}}{12})^{12} - 1 = (1.005)^{12} - 1 \approx .06167781186$$

And once more using the above formula we can solve for d:

$$d = \frac{i}{1+i} \approx .05809466$$

Problem 1.10.4

We want to find the accumulated value of \$2480 at the end of twelve years if the nominal interest rate was 2% convertible monthly for the first three years, the nominal rate of discount was 3% convertible semiannually for the next two years, the nominal rate of interest was 4.2% convertible once every two years for the next four years, and the annual effective rate of discount was .058 for the last three years. Inputting the interest values gives us the equation for the accumulative functions $a_k(t)$:

$$P_0 a_1(3) a_2(2) a_3(4) a_4(3) = P_{12}$$

$$2480(1+\frac{.02}{12})^{(12)(3)}(1-\frac{.03}{2})^{(-2)(2)}(1+\frac{.04}{1/2})^{(1/2)(4)}(1-.058)^{-3} \approx \$3932.317655$$