Real Analysis - Problem Set #4

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Problem 1

Let $\{f_n\}$ be a decreasing sequence of measurable non-negative integrable functions, $f_n(x) \geq f_{n+1}(x) \geq 0$, and $f_n(x) = \lim_{n\to\infty} f_n$. We want to show that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

and that the conclusion does not follow if the condition that for some k, $\int f_k d\mu < \infty$.

First we look at this sequence of functions $\{f_n\}$ all of which are integrable, and so it follows that $\int f_n d\mu < \infty$ for all $n \in \mathbb{N}$. This comes directly from the definition of integrability. Furthermore we can conclude that there exist some support E for all $f_n(x)$ such that $m(E) < \infty$. Suppose that there wasn't, then it would contradict the fact that these functions are integrable. We proceed to use our Bounded Convergence Theorem by letting $M = \int f_0(x) d\mu < \infty$. Now we have the conditions of the Bounded Convergence Theorem.

However to be more clear, since there is some support E we know that for some $\epsilon > 0$ we can find some A_ϵ of E such that

$$m(E - A_\epsilon) < \epsilon$$

And since $f_n(x) \to f(x)$ as $n \to \infty$ we can find some large enough $N \in \mathbb{N}$ such that for all $n \geq N$

$$|f_n(x) - f(x)| < \epsilon$$

that is supported on a set of A_ϵ and furthermore we have that

$$\int |f_n(x) - f(x)| d\mu \leq \int_{A_\epsilon} |f_n(x) - f(x)| d\mu + \int_{E - A_\epsilon} |f_n(x) - f(x)| d\mu \leq \epsilon m(E) + 2Mm(E - A_\epsilon)$$

The right side of this equation converges to zero as we let $\epsilon \to 0$. Therefore we conclude that

$$\int |f_n(x) - f(x)| d\mu \to 0$$

and consequently

$$\lim_{n\to\infty}\int f_n d\mu = \int f d\mu$$

Problem 2

We want to determine the limits as $n \to \infty$ for

$$\text{(a)} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}dx \qquad \text{(b)} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}dx$$

(a) We wish to show that the conditions for problem 1 holds, and if not that condition the bounded convergence condition holds, so we can use the result.

Let $\{f_n(x)\}$ be the sequence such that $f_n(x) = (1 - \frac{x}{n})^n e^{x/2}dx$. We can the sequence alternates positive and negative values until $n \geq x$ then becomes an increasing sequence. Let $E_n = [0, n]$. However for sufficiently large enough n we have that $f_n(x) \leq f_{n+1}(x) \leq f(x)$ for all x in E_n and where $\lim_{n\to\infty} f_n(x) = f(x)$ for all x in E_n.

Furthermore for all x we know there exist some $\epsilon_n = \frac{1}{n^2} > 0$ such that there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $|f_n(x) - f(x)| < \frac{1}{n^2}$. Also

$$\int_0^n f_n(x) - f(x)dx \leq \int_0^n |f_n(x) - f(x)|dx = \int_0^n |\left(1 - \frac{x}{n}\right)^n e^{x/2} - e^{-x/2}|dx$$

and we can evaluate the right integral to be

$$\lim_{n\to\infty}\int_0^n |\left(1 - \frac{x}{n}\right)^n e^{x/2} - e^{-x/2}|\, dx \leq e_n m(E_n) = \frac{1}{n}$$

And once we let $n \to \infty$ we get that

$$\int_0^n f_n(x) - f(x)dx \leq \int_0^n |f_n(x) - f(x)|dx \to 0$$

Consequently,

$$\lim_{n\to\infty}\int \left(1 - \frac{x}{n}\right)^n e^{x/2}dx = \int e^{-x/2}$$

Furthermore we have that the integral of (a) follows as a problem from calculus

$$\int_0^\infty e^{-x/2}dx = 2$$

(b) We use the previous proof to justify that the same holds for this integral as well. Furthermore we get that

$$\lim_{n\to\infty}\int_0^n (1 + \frac{x}{n})^n e^{-2x}dx = \int_0^\infty e^{-x}dx = 1$$

Problem 3

We want to prove the following continuity property for integrable functions for $f \in L^1(d\mu)$, then for each $\epsilon > 0$ there exist a $\delta > 0$ such that $\int_E |f|d\mu < \epsilon$ for any measurable set E with $\mu(E) < \delta$.

Let *epsilon* > 0, and $\sup_E(f) = \alpha$. We let $\delta = \epsilon/\alpha$ and so we find some $m(E) < \epsilon/\alpha$ and it follows that:

$$\int_E |f|d\mu < m(E)\alpha \leq \epsilon$$

We conclude that for ever ϵ there exist a δ such that the above property holds.

Note*: it has to be that E has finite measure on the support of f otherwise it would contradict that f is a member of $L^1(d\mu)$.

Problem 4

We want to prove that any infinite σ-algebra of sets must have uncountably many members.

Let F be our sigma algebra with the infinite distinct members $A_1, A_2, \dots$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Furthermore we know that the sigma algebra generated by these A_i lives inside F. We let $S = \sigma(A_1, A_2, \dots) \subset F$. Next we want to show that there exist a bijection between S and $\{0,1\}^{\mathbb{N}}$. We define the map:

$$f : S \to \{0,1\}^{\mathbb{N}}$$

where $s = A_{i_1} \cup A_{i_2} \cup \dots$ and for $f(s)$ we put a 1 for each i_j coordinate and 0 everywhere else. For example if we had $s = A_1 \cup A_2 \cup A_4$ then $f(s) = \{1, 1, 0, 1, 0, 0, \dots, 0, \dots\}$ This map is a bijective map, and furthermore we conclude that

$$|S| = |\{0,1\}^{\mathbb{N}}| = |\mathbb{R}| \leq |F|$$

Problem 5

We want to construct a sequence of continuous functions $\{f_n\}$ on $[0,1]$ such that $0 \leq f_n \leq 1$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

but the sequence $\{f_n\}$ does not converge at any point in $[0,1]$.

We will construct a sequence with characteristic functions. We define our function to be

$$f_1(x) = \chi_{[0,1]}$$

$$f_2(x) = \chi_{[0,1/2]} \qquad f_3 = \chi_{[1/2,1]}$$

$$f_4(x) = \chi_{[0,1/3]} \qquad f_5 = \chi_{[1/3,2/3]} \qquad f_6 = \chi_{[2/3,1]}$$

$$\cdots$$

We can see that this function decreases in width and oscillates from left to right. Therefore for all $N \in \mathbb{N}$ there exist some $n \geq N$ such that $|f_n(x) - L| > \epsilon$ for any given $1 > \epsilon > 0$ and $L \in [0,1]$.