Homework 4 | Thomas Lockwood

Real Analysis II | December 17, 2016

Exercise 1. *For $0 < \alpha < \beta$, we want to show that*

$$(\sum_{i=1}^{\infty} |x_i|^{\beta})^{\alpha/\beta} \leq \sum_{i=1}^{\infty} |x_i|^{\alpha}$$

Then we want to determine the limit on the left-hand side as $\beta \to \infty$, and explain why we know this limit exists.

Proof. Let us evaluate this summation. To get rid of the trivial case we suppose that $\sum_{i=1}^{\infty} |x_i|^{\alpha} = \infty$, then it would follow that this inequality holds. So let us instead assume that $\sum_{i=1}^{\infty} |x_i|^{\alpha} = k^{\alpha}$ for some $k \in \mathbb{R}$ then we have that

$$\frac{\sum_{i=1}^{\infty} |x_i|^{\alpha}}{k^{\alpha}} = 1$$

Furthermore, we have that $\frac{|x_i|}{k} \leq 1$ for all $i \in \mathbb{N}$. So it follows that

$$0 \leq \left(\frac{|x_i|}{k}\right)^{\beta} \leq \left(\frac{|x_i|}{k}\right)^{\alpha}$$

for all $i \in \mathbb{N}$. Adding up all of the terms we find that:

$$\sum_{i=1}^{\infty} \left(\frac{|x_i|}{k}\right)^{\beta} \leq \sum_{i=1}^{\infty} \left(\frac{|x_i|}{k}\right)^{\alpha}$$

and we can also see that since $\alpha/\beta < 1$

$$\left(\sum_{i=1}^{\infty} \left(\frac{|x_i|}{k}\right)^{\beta}\right)^{\alpha/\beta} \leq \sum_{i=1}^{\infty} \left(\frac{|x_i|}{k}\right)^{\beta}$$

Now we let $|x_i|/k = |z_i|$ and use the above inequalities to finalize that

$$\left(\sum_{i=1}^{\infty} |z_i|^{\beta}\right)^{\alpha/\beta} \leq \sum_{i=1}^{\infty} |z_i|^{\alpha}$$

We next want to show where the limit goes as $\beta \to \infty$. We know this limit exists since it is a monotonically decreasing and bounded on both sides. Let $M = \max\{S\}$ where $S = \{|x_i| : i \in \mathbb{N}\}$. Let $\alpha = 1$. Then we have that

$$\frac{|x_i|}{M} \leq 1 \text{ for all } i \in \mathbb{N}$$

and so

$$\lim_{\beta \to \infty} \left(\frac{|x_i|}{M}\right)^{\beta} \leq 1 \text{ for all } i \in \mathbb{N}$$

Which gives us two cases, this term either converges to zero or the term is equal to one. Suppose that there are N numbers of terms that converge to 1. (Note*: if there were infinite amount of terms then we would have that this series diverges. Throughout this problem we assume the series converges, otherwise we wouldn't be able to show anything). Furthermore we would have that

$$\lim_{\beta\to\infty}\left(\sum_{i=1}^{\infty}\left(\frac{|x_i|}{M}\right)^{\beta}\right)^{1/\beta} = \lim_{\beta\to\infty}\left(N\left(\frac{x_j}{M}\right)^{\beta}\right)^{1/\beta} = \lim_{\beta\to\infty}(N)^{1/\beta} = 1$$

However we want the limit of the non-normalized terms; that is, the terms that are not divided by the constant M. Pulling out the constant M we find that the homogeneity of this operation is one, and this results the limit to be equal to M. In other words we have that:

$$\lim_{\beta\to\infty}\left(\sum_{i=1}^{\infty}|x_i|^{\beta}\right)^{1/\beta} = M$$

where $M = \max\{S\}$, such that $S = \{|x_i| : i \in \mathbb{N}\}$. □

Exercise 2. *Let f be a bounded real-valued function on $\mathbb{R}^n$. We define*

$$M_\delta f(a) = \sup\{f(x) : |x - a| < \delta\}$$

$$m_\delta f(a) = \inf\{f(x) : |x - a| < \delta\}$$

Proof. We want to show that $M_\delta f(x) - m_\delta f(x)$ is a decreasing function as δ decreases.

First we want to show that $M_\delta f(x) \leq M_{\delta+\epsilon} f(x)$ for some $\epsilon > 0$. We will use contradiction. Suppose instead that $M_\delta f(x) > M_{\delta+\epsilon} f(x)$. Then there would exist some point α in $B_\delta(x)$ for some x such that $\alpha > x$ for all $x \in B_{\delta+\epsilon}(x)$, however $\alpha \in B_{\delta+\epsilon}(x)$ since $B_\delta(x) \subset B_{\delta+\epsilon}(x)$, so we have that $\alpha > \alpha$ our contradiction.

By the same argument we can say that $m_{\delta+\epsilon} f(x) \leq m_\delta f(x)$ for some $\epsilon > 0$.

Furthermore we have the following:

$$M_\delta f(x) \leq M_{\delta+\epsilon} f(x)$$

$$m_{\delta+\epsilon} f(x) \leq m_\delta f(x)$$

and by summing these inequalities up we find that,

$$M_\delta f(x) + m_{\delta+\epsilon} f(x) \leq m_\delta f(x) + M_{\delta+\epsilon} f(x)$$

and subtracting each side by $m_\delta f(x)$ and $m_{\delta+\epsilon} f(x)$ we reach our conclusion:

$$M_\delta f(x) - m_\delta f(x) \leq M_{\delta+\epsilon} f(x) - m_{\delta+\epsilon} f(x)$$

□

For the next problem we define the oscillation function:

$$(\Omega f)(x) = \lim_{\delta\to 0}[(M_\delta f)(x) - (m_\delta f)(x)]$$

Exercise 3. *We want to show that $(\Omega f)(x) = 0$ if and only if f is continuous at x. Let M be a metric space with metric $d(\cdot,\cdot)$. We want to show that for $x, y \in M$ and w any other point in M*

$$|d(x, w) - d(y, w)| \leq d(x, y)$$

Proof. First our goal is to show the if and only if.

$\Rightarrow$ Suppose that $(\Omega f)(x) = 0$. We want to show that f is continuous at x. We have that for all $\epsilon_n > 0$ there exist an $\delta_n > 0$ such that $a, b \in |x - p| < \delta_n$ implies that

$$|M_\delta f(a) - m_\delta f(b)| < \epsilon_n$$

however we can notice that:

$$|f(x) - f(p)| \leq |M_\delta f(a) - m_\delta f(b)| < \epsilon_n$$

Furthermore for all $\epsilon_n > 0$ there exist some $\delta_n > 0$ such that if

$$|x - p| < \delta_n$$

then

$$|f(x) - f(p)| < \epsilon_n$$

$\Leftarrow$ Suppose instead that $f(x)$ is a continuous function. That is; for ever $\epsilon_n/2 > 0$ there exist $\delta_n/2 > 0$ such that if $a, b \in \{p : |x - p| < \delta_n/2\}$ then $|f(x) - f(p)| < \epsilon_n/2$. So we can see that

$$|M_\delta f(a) - m_\delta f(b)| = |\sup |f(x) - f(a)| - \inf |f(x) - f(b)|| < \epsilon_n/2 + \epsilon_n/2 = \epsilon_n$$

for $a, b \in \{p : |x - p| < \delta_n/2\}$. And as we let $\delta_n/2 \to 0$ we have that $\epsilon_n/2 \to 0$ and so $\epsilon_n \to 0$ and we conclude that:

$$(\Omega f)(x) = lim_{\delta \to 0}[(M_\delta f)(x) - (m_\delta f)(x)] = 0$$

Next we want to show that $|d(x, w) - d(y, w)| \leq d(x, y)$. We can show this by using the triangle inequality. Recall that:

$$d(x, w) \leq d(x, y) + d(y, w)$$

Furthermore we have,

$$|d(x, w) - d(y, w)| \leq |[(d(x, y) + d(y, w)] - d(y, w)| = |d(x, y)| = d(x, y)$$

and so it follow that

$$d(x, w) \leq d(x, y) + d(y, w)$$

$\square$

Exercise 4. *Suppose that $\mu(\Omega) < \infty$, $\{f_n\}$ is a sequence of bounded complex valued measurable functions defined on Ω, and $f_n \to f$ as $n \to \infty$ uniformly on Ω. We want to prove that*

$$\lim_{k \to \infty} \text{int}_\Omega f_k d\mu = \text{int}_\Omega f d\mu$$

and that the hypothesis $\mu(\Omega) < \infty$ is necessary.

Proof. Let $f_n = u_n + iv_n$ and $f = u + iv$, and $u_n \to u$ and $v_n \to v$. We have that

$$\lim_{n \to \infty} \text{int}_\Omega f_n d\mu = \lim_{n \to \infty} \text{int}(u_n + iv_n) d\mu = \lim_{n \to \infty} \text{int}_\Omega u_n + i \lim_{n \to \infty} \text{int}_\Omega v_n$$

And we use the property of finite measures to use bounded convergence theorem to get $\square$

$$\lim_{n \to \infty} \text{int}_\Omega u_n + i \lim_{n \to \infty} \text{int}_\Omega v_n = \text{int}_\Omega u + i \, \text{int}_\Omega v = \text{int}_\Omega f$$

We conclude that

$$\lim_{k \to \infty} \text{int}_\Omega f_k d\mu = \text{int}_\Omega f d\mu$$