Real Analysis - Problem Set #8

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Problem 1

We return to problem #5 on Problem Set #7:

Suppose we add the hypothesis that $f_n \geq 0$ to the conditions: $\{f_n\}$, $f \in L^2(d\mu)$, $f_n \to f$ a.e., and $\int |f_n|^2 d\mu \to \int |f|^2 d\mu$ as $n \to \infty$. We want to find an elementary argument to show that $f_n \to f$ in $L^2(d\mu)$

Problem 2

Let A and B be two Lebesgue measurable subsets of the line $\mathbb{R}$ with finite positive measure $0 < m(A)$, $m(B) < \infty$. Define the sum set $A + B = \{x + y : x \in A, y \in B\}$. We want to use the notion of convolution to show that the set A + B must contain an interval.

First let us observe what we have. We have that $0 < m(A)$, so there must exist some interval in A let us call I_a, where $0 < m(I_a) < m(A)$. We can then observe that for some fixed $y_0 \in B$ the interval $I_{a+y_0} = \{x + y_0 : x \in I_a\}$ is contained in A + B, and $m(I_{x+y_0}) = m(I_a)$, since the Lebesgue measure is translation invariant. Furthermore there must exist some interval in $A + B$, namely I_{a+y_0}.

Problem 3

We want to use properties of the Fourier transform to show that there does not exist a function $I \in L^1(\mathbb{R}^n)$ such that $f \star I = f$ for all $f \in L^1(\mathbb{R}^n)$

If we had such a function it would be a straight vertical line, so that as it passes through any function the intersection of the vertical line and the function would be the value of that function evaluated at that point. However a vertical line/plane/higherdiminsional plan as we have been taught since high school this would not be a function, and surely has no lebesgue measure since it is only a point. So we can see that this I would not be in $L^1(\mathbb{R}^n)$.

Problem 4

Consider for $f \in L^1[0, \infty)$ the function

$$(Tf)(x) = \frac{1}{x} \int_0^x f(t)dt \ , \ x > 0$$

We want to show that for $x > 0$, $(Tf)(x)$ is continuous and that $(Tf)(x) \to 0$ as $x \to \infty$ so that $Tf \in C_0((0, \infty))$