Real Analysis - Problem Set #7

Thomas Lockwood

March 31, 2016

Problem 1

Suppose f is integrable on $[0,1]$ and we define

$$g(x) = \int_x^1 \frac{f(t)}{t} dt \quad \text{for } 0 < x \leq 1$$

We want to prove that g is integrable on $[0,1]$ and

$$\int_0^1 g(x)dx = \int_0^1 f(t)dt$$

In order to show that $g(x)$ is integrable for all x we need to show that

$$\int_0^1 |g(x)|dx < \infty$$

First let us make the observation that $g(x) \leq g(0)$ for any $x \in [0,1]$ by the monotonicity property of measures. The next observation we make is $\frac{f(t)}{t}$ at $t = 0$ may not be a finite value, and instead may "blow up" to infinity.

On the other hand we have by using Fubini's Theorem

$$\int_0^1 |g(x)| = \int_0^1 \int_x^1 \frac{|f(t)|}{t} dtdx = \int_0^1 \int_x^1 \frac{|f(t)|}{t} dxdt$$

We changed the order of integration! It can be seen as the swap $dxdt = dtdx$. So now we measure with respect to x before we measure with respect to t to find that

$$\int_0^1 \int_x^1 \frac{|f(t)|}{t} dxdt = \int_0^1 \frac{|f(t)|}{t} \int_x^1 dxdt = \int_0^1 \frac{|f(t)|}{t} dt \int_x^1 dx$$

We use the reflection property of integrable integrals

$$\int_{[0,1]} f(x) = \int_{[0,1]} f(-x)$$

and find that

$$\int_0^1 \frac{|f(t)|}{t} dt \int_x^1 dx = \int_0^1 \frac{|f(t)|}{t} dt \int_0^x dx$$

for any $x \in (0,1]$. We let $x = t$ to find

$$\int_0^1 \frac{|f(t)|}{t} dt \int_0^x dx = \int_0^1 \frac{|f(t)|}{t} dt \int_0^t dt = \int_0^1 |f(t)| \left(\frac{t}{t}\right) dt = \int_0^1 |f(t)| dt$$

and as was assumed $\int_0^1 |f(t)|dt < \infty$ furthermore we conclude that

$$\int_0^1 g(x)dx = \int_0^1 f(t)dt < \infty$$

Problem 2

Let E be a set of finite measure in $[0,1]$. We want to show that

$$\lim_{n\to\infty} \int_E \cos(nx)dx = \lim_{n\to\infty} \int_E \sin(nx)dx = 0$$

We can prove this using the properties of outer measures. Recall

$$m_*(E) = \inf \sum_j^\infty |Q_j|$$

and

$$0 \leq m(E) \leq m_*(E)$$

Where Q_i are the countable coverings of the set $E \subset \cup_j^\infty Q_i$. We notice that as n increases, $\sin(nx)$ and $\cos(nx)$ have higher frequency and cross the x-axis in the 2π interval more times. We can also notice that $\sin(nx)$ and $\cos(nx)$ are bounded by 1, and therefore by dominated convergence theorem we may pass the limit through the integral. once we show that the outer measure is the square $Q = [-1,1] \times [0, 2\pi]$ everything is complete since $m(Q) = 0$ for the Lebesgue measure in $\mathbb{R}^2$.

Claim: Q is the outer measure. If we take any open set inside this interval, then we notice that it intersects $\cos(nx)$ and $\sin(nx)$ for some n and x. Furthermore any open sets and all open sets contained in Q must be in the cover. We conclude that Q is a cover for E and if there existed any smaller cover it would miss a point of $\cos(nx)$ and $\sin(ny)$ for some x and y respectively.

We conclude by squeeze theorem that

$$0 \leq \lim_{n\to\infty} \int_E \cos(nx)dx = \lim_{n\to\infty} \int_E \sin(nx)dx \leq \int_Q 1dx = 0$$

Problem 3

Let E be a measurable set in $\mathbb{R}$ with $m(E) < \infty$. We want to show that

$$\lim_{n\to\infty} \int_E \cos^2(nx + a_n)dx = \frac{1}{2}m(E)$$

We use our trig identities to find that

$$\begin{aligned}\cos(nx + a_n)\cos(nx + a_n) &= \frac{1}{2}\left[\cos(nx + a_n - nx - a_n) + \cos(nx + a_n + nx + a_n)\right] \\ &= \frac{1}{2} + \frac{\cos(2nx + 2a_n)}{2}\end{aligned}$$

Furthermore we conclude by using problem 2

$$\lim_{n\to\infty} \int_E \cos^2(nx + a_n)dx = \lim_{n\to\infty} \int_E \frac{\cos(2nx + 2a_n)}{2}dx + \int_E \frac{1}{2}dx = 0 + \frac{1}{2}m(E)$$

Problem 4

Suppose f is integrable. We want to prove that

$$\mu\{x : |f(x)| > \lambda\} \leq \frac{1}{\lambda}\int |f|d\mu$$

Once a picture is drawn the intuition is straight forward. We have the measure of intervals on the x-axis. Whenever we multiply these intervals by a number greater than one it increases. Furthermore let $A = \{x : |f(x)| > \lambda\}$ then we let $\mu(A) = k$. We chose a simple function for all $g = \inf\{f(x) : |f(x)|/\lambda > 1\} = m$ and we conclude that

$$\mu(A) = k = \int_A 1d\mu \leq \int_A g = mk \leq \frac{1}{\lambda}\int |f|d\mu$$