Homework 3 — Thomas Lockwood

Real Analysis II — December 17, 2016

Exercise 1. *Let m_* be the outer measure on $\mathbb{R}^n$. Suppose E is a subset of $\mathbb{R}^n$ such that $m_*(E) < \infty$. Let $\mathscr{O}_m$ be the open set*

$$\mathscr{O}_m = \{x : d(x, E) < 1/m\}$$

(a) We want to show that if E is compact, then

$$m_*(E) = \lim_{m\to\infty} m_*(\mathscr{O}_m)$$

(b) Give examples to show that this property may not hold for cases where E is closed and unbounded or E is open and bounded.

Solution. (a) We have that $E \subset \mathscr{O}_m \subset \mathbb{R}^n$ for all $m \in \mathbb{N}$. Also we can see that the sequence of sets $\{\mathscr{O}_m\}$ are decreasing $\mathscr{O}_{k+1} \subset \mathscr{O}_k \subset \cdots \subset \mathscr{O}_2 \subset \mathscr{O}_1$ for all $k \in \mathbb{N}$. Furthermore, we can use observation 3 of outer measures to see that

$$m_*(E) = \inf(\mathscr{O}_m) = \lim_{m\to\infty} m_*(\mathscr{O}_m)$$

We just need to show that there exist some $\mathcal{O}_m$ is contained in every open set that contains E. However, E is totally bounded, therefore for any open set that contains E we find some ϵ that is small enough to be inside. There exist one since this converges to 0.

However, I am unable to think of a case where E is open and bounded, and this property doesn't hold. In the book it also states that open sets are measurable. One idea for a set that is open and bounded would be a dense set, say $\mathbb{Q} \cap [0, 1]$. It seems like it might raise some contradictions when making a countable union of disjoint sets.

□

Exercise 2. *Let E be a subset of $\mathbb{R}^n$ with $0 < m_*(E) < \infty$ (m_* is the notation of outer measure in the book). Prove that for each α, $0 < \alpha < 1$, there exist an open ball B so that:*

$$m_*(E \cap B) \geq \alpha \ vol(B)$$

Proof. Let $1 > \alpha > 0$. We want to show that there exist some ball B such that the above inequality holds true. We we use the notion of $\mathrm{vol}(B)$ we refer to its outer Lesbegue measure, denoted above as $m_*(B)$. Also we can deduct that $E \cap B \subset B$, since intersections always are a subset of the set. Thus using our Monotinicity property of exterior measure, we can say that $m_*(E\cap B) \leq m_*(B)$. Recall that $B = (B\cap E) \cup (B \cap E^c)$, so we can even split this inequality up into

$$m_*(E \cap B) \leq m_*(B) = m_*(B \cap E) + m_*(B \cap E^c)$$

since these sets are disjoint, that is $(B \cap E) \cap (B \cap E^c) = \emptyset$ We can also see by rearranging the inequality for some $m_*(B) \neq 0$ that

$$\frac{m_*(E \cap B)}{m_*(E \cap B) + m_*(B \cap E^c)} \leq 1$$

This is equivalent to the function $f(x) = \frac{a}{a+x}$ a decreasing function for some $a \in \mathbb{R}$. Surely it is evident that this function converges to 0 as $x \to \infty$. However what is x in our scenario? x seems to be the $\mathrm{vol}(B \cap E^c)$; that is, the volume of the ball B outside of the set E. We also have that the outer measure of $\mathbb{R}^n$ is ∞ and the outer measure of E is some finite value, thus it is bounded by some open set. That being said we can let this ball go as far outside of E as we need to ensure it is less than any given $1 > \alpha > 0$. In other words, we can let $m_*(B \cap E^c) \to \infty$ and as a result

$$\frac{m_*(E \cap B)}{m_*(E \cap B) + m_*(B \cap E^c)} \to 0$$

It seems I have proved the wrong result!

Suppose that there existed some α such that there was no B such that $m_*(E \cap B) \geq \alpha \operatorname{vol}(B)$. We let B be the interior of E, and see that $m_*(B) \leq \alpha \operatorname{vol}(B)$, however this implies that $\operatorname{vol}(B) = 0$ and furthermore E has no interior. Say E is a bunch of isolated points, then it would contradict the fact that $0 < m_*(B)$ in which we assumed. Therefore E is a closed set with no interior. Therefore E is closed and bounded, and therefore compact. We use part 1 to show that this set $m * (E) = \lim_{m\to\infty} m_*(\prime_m)$. However this contradicts the fact that $m_*(E) = m_*(\mathcal{O}_i) \leq \alpha \operatorname{vol}(\mathcal{O}_i)$ since there doesn't exist some open set O_i such that this is true. $\square$

Exercise 3. *Let X be a complete metric space. Suppose k is a closed and bounded subset with the property that it contains a countably infinite set of points $\{x_k\}$ with the property that*

$$d(x_m, x_n) \geq 2\alpha > 0 \textit{ for all } n \neq m$$

Proof. We want to show that the set k is not compact. We can do this by showing that this set is not sequentially compact. We can do this by contradiction. Suppose that k is sequentially compact, then we have that this sequence has a convergent sub-sequence. However we can see that for $2\alpha > 0$ there doesn't exist any $N \in \mathbb{N}$ such that for all $j, i \geq N$ that $d(x_j, x_i) < 2\alpha$ $\square$

Exercise 4. *Let f be a continuous function with compact support on $\mathbb{R}^n$ e.g. $f \in C_c(\mathbb{R}^n$. We want to show that*

$$\lim_{h\to 0} \mathrm{int}_{\mathbb{R}^n} |f(x+h) - f(x)| dx = 0$$

Proof. We have that f has a compact support, and furthermore it is bounded by some measurable compact set. Let $g(x) = f(x+h) - f(x)$. Furthermore we can see that the non-negative function has a limit $\lim_{h\to 0} |f(x+h) - f(x)| = \lim_{h\to 0} |g(x)| \to 0$ and is 0. We take the $max(|g(x)|) = K$, and the dominated convergence theorem follows:

$$\lim_{h\to 0} \mathrm{int}\, |g(x)| \to 0 = \mathrm{int} \lim_{h\to 0} |g(x)| \to 0$$

$\square$