Homework 1 — Thomas Lockwood

Real Analysis II — December 17, 2016

Exercise 1. *What is a Cauchy Sequence?*

Solution. For a sequence $\{x_i\} = x_1, x_2, \ldots$, it is called a Cauchy Sequence if for every $\epsilon > 0$ there exist some $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all integers $n, m > N$. □

Exercise 2. *What is a compact set?*

Solution. A set S of a topological space $\mathcal{T}$ is compact if for every open cover there exist a finite sub cover. □

Exercise 3. *What does it mean for a metric space to complete?*

Solution. A metric space is complete if every cauchy sequence converges. □

Exercise 4. *State the Bolzano-Weierstrass Theorem.*

Solution. Each bounded sequence in $\mathbb{R}^n$ has a convergent sub-sequence. □

Exercise 5. *Sketch the critical steps in the proof of the Heine-Borel Theorem.*

Solution. The Heine-Borel Theorem states that a subset S of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

$\Rightarrow$ Assume that S is compact. We want to show that S is closed. We then can show that it is closed by using the fact that $\mathbb{R}^n$ is a Hausdorff space. We can then prove that any point not in S is an interior point of S^c, thus making S^c open and S closed. We prove this by taking the cover of the union of disjoint sets from each point in S, and any arbitrarily chosen point in S^c. We proceed with the given information that S is compact so there is a finite cover, and then take the intersection of the finite union. We then conclude that the arbitrarily chosen point of S^c is an interior point of S^c.

Next we want to show that S is bounded. This can be proved by contradiction, for if there is a set that is compact and not bounded it would lead to a contradiction since there must be a finite subcover for each cover on a compact space.

$\Leftarrow$ Assume instead that S is closed and bounded. We want to prove that S is compact. We prove this by creating a set in which it is true that the set is compact on the interval from $\{b : b \in S$ such that $[\min S, b]$ is compact $\}$. We then proceed to show that this set is non-empty and has a greatest value. Then we show by contradiction that the greatest value cannot be below b. □

Exercise 6. *Show that for $p > 2$*

$$\sum_{n=1}^{\infty} \left[\sqrt{n+1} - \sqrt{n}\right]^p < \infty$$

Solution. We have

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

once we multiply $\sqrt{n+1} - \sqrt{n}$ by it's conjugate. Also,

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n+1}} = \left[\frac{1}{n+1}\right]^{1/2}$$

Therefore,

$$\left[\frac{1}{\sqrt{n+1}+\sqrt{n}}\right]^p < \left[\frac{1}{n+1}\right]^{p/2}$$

is true for all $n \in \mathbb{N}$, and by using the squeeze theorem and convergence of the p series we can conclude that for $p > 2$

$$\sqrt{n+1}-\sqrt{n} = \sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{n+1}+\sqrt{n}}\right]^p < \sum_{n=1}^{\infty}\left[\frac{1}{n+1}\right]^{p/2} < \infty$$

$\square$

Exercise 7. *Let $a_1 = \sqrt{2}$ and let a_n for $n \geq 2$ be defined recursively by the formula*

$$a_{n+1} = \sqrt{2+\sqrt{a_n}}$$

Solution. a) We want to prove by induction that $\sqrt{2} \leq a_n \leq 2$ for all $n \in \mathbb{N}$.

Base Case: By definition of a_1 we have that $\sqrt{2} \leq a_1 \leq 2$.

Inductive Step: Assume $\sqrt{2} \leq a_k \leq 2$ is true for some $k \in \mathbb{N}$. We want to show that $\sqrt{2} \leq a_{k+1} \leq 2$ is as well true. Certainly,

$$0 \leq a_k \leq 2$$

therefore assuming $\sqrt{a_k} > 0$ we have that,

$$0 \leq \sqrt{a_k} \leq 2$$

and once we add two we get the inequality

$$2 \leq \sqrt{a_k}+2 \leq 4$$

and we conclude that

$$\sqrt{2} \leq \sqrt{2+\sqrt{a_k}} = a_{k+1} \leq 2$$

We conclude that a_k implies that a_{k+1} is true, and therefore is true for all $k \in \mathbb{N}$

b) We want to prove that $\{a_n\}$ is a Cauchy sequence, and we want to conclude that $\{a_n\}$ converges. We can show that this sequence is Cauchy by showing that it converges, and use this to show that the cauchy sequence converges. We will do this by showing that the sequence is monotone and bounded. We have already shown this sequence is bounded from part a), so all we need to show is that this sequence is monotone. We will do this by induction.

Base Case: The case where $n = 1$ is trivial. Inductive Hypothesis: Assume $a_{k+1} \geq a_k$. We want to show $a_{k+2} \geq a_{k+1}$. Inductive step: We have that

$$a_{k+1} = \sqrt{2+\sqrt{a_k}} \geq a_k \geq \sqrt{a_k}$$

so we have that

$$\sqrt{2+\sqrt{a_k}} \geq \sqrt{a_k}$$

or equivalently

$$2+\sqrt{2+\sqrt{a_k}} \geq 2+\sqrt{a_k}$$

and once we square root each side we get

$$a_{k+2} = \sqrt{2+\sqrt{2+\sqrt{a_k}}} \geq \sqrt{2+\sqrt{a_k}} = a_{k+1}$$

Therefore we conclude that this sequence is monotone and bounded and therefore converges. Furthermore we can use the triangle inequality to show that this sequence is a Cauchy sequence. □

Exercise 8. *Suppose the terms $\{a_n\}$ satisfy $|a_{n+1} - a_n| \leq b_n$ for all n where $\sum_{n=1}^{\infty} b_n < \infty$.*

Solution. We want to prove that $\{a_n\}$ is a Cauchy sequence. We have that $\sum_{n=1}^{\infty} b_n < \infty$ which implies that $b_n \to 0$, otherwise its sum would diverge. This implies that $|a_{n+1} - a_n| \to 0$, and as a result we get that $\{a_n\}$ is a Cauchy sequence. To be more rigorous we could have used the triangle inequality to verify its convergence. □

Exercise 9. *Prove that for $x_j \geq 0$, $j = 1, \ldots, n$*

$$\left[\prod_{j=1}^{n} x_j\right]^{1/n} \leq \frac{1}{n}\sum_{j=1}^{n} x_j$$

Solution. We want to show that

$$\left[\prod_{j=1}^{n} x_j\right]^{1/n} \leq \frac{1}{n}\sum_{j=1}^{n} x_j$$

or equivalently

$$\ln(\left[\prod_{j=1}^{n} x_j\right]^{1/n}) \leq \ln(\frac{1}{n}\sum_{j=1}^{n} x_j)$$

and using the properties of logarithms we rearrange the terms to get that the above is equivalent to

$$\frac{1}{n}\sum_{j=1}^{n} \ln(x_j) \leq \ln(\frac{1}{n}\sum_{j=1}^{n} x_j)$$

However this function is convex. In other words, let $f(x) = x\ln(x)$ and all $\epsilon > 0$ we want to show that $f(x) + f(\epsilon) \leq f(x + \epsilon)$. we have that

$$f(x) + f(\epsilon) = x\ln(x) + \epsilon\ln(\epsilon)$$

$$f(x+\epsilon) = (x+\epsilon)\ln(x+\epsilon) = x\ln(x+\epsilon) + \epsilon\ln(x+\epsilon)$$

and so we have that

$$x\ln(x) \leq x\ln(x+\epsilon)$$

and

$$\epsilon\ln(x+\epsilon)$$

or furthermore

$$f(x) + f(\epsilon) = x\ln(x) + \epsilon\ln(\epsilon) \leq x\ln(x+\epsilon) + \epsilon\ln(x+\epsilon)$$

From here I think we can show that the above inequality is true, however I do not know. It follows from the log sum inequality and Jensen's inequality.

□

Exercise 10. *Find the value of the integral* $\mathrm{int}_0^{\infty} \frac{1}{\sqrt{x}}\frac{1}{1+x}dx$ *using only paper and pencil calculations and change of variables.*

Solution. Let $u = \sqrt{x}$ $u = \tan(\theta)$ and $du = \sec^2(\theta)$. (Recall that $\tan^2(\theta) + 1 = \sec^2(\theta)$). Then our new integral is

$$\text{int}_0^\infty \frac{1}{\sqrt{x}} \frac{1}{1+x} dx = \text{int}_0^\infty 2 \frac{1}{u^2+1} du = \text{int}_0^\infty \frac{2}{\sec^2(\theta)} \sec^2(\theta) d\theta = \text{int}_0^\infty 2 d\theta = 2\theta$$

On the other hand we have that $u = \tan\theta$ and so

$$2\theta = 2\tan^{-1}(u) = 2\tan^{-1}(\sqrt{x})$$

needed to be evaluated for all non-negative x. We then compute

$$\lim_{x \to \infty} \tan^{-1}(x) - \tan^{-1}(0) = \frac{\pi}{2}$$

□