Real Analysis - Problem Set #5

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Problem 1

Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for almost every x e.g. $m(\{x : |f_n(x)| = \infty\}) = 0$. We want to show that there exist a sequence $\{c_n\}$ of positive real numbers such that

$$g_n(x) = \frac{f_n(x)}{c_n} \to 0 \text{ a.e.}$$

(Hint: pick c_n such that $m(\{x : |f_n(x)/c_n| > 1/n\}) < 2^{-n}$ then apply the Borell-Cantelli Lemma.)

For each n we pick some c_n such that

$$m(\{x : |f_n(x)| < \frac{c_n}{n}\}) < 2^{-n}$$

and so it follows since c_n is positive that

$$m(\{x : \left|\frac{f_n(x)}{c_n}\right| < \frac{1}{n}\}) < 2^{-n}$$

and so by summing up the n we have

$$\sum_{n=1}^{\infty} m(\{x : |g_n(x)| = \left|\frac{f_n(x)}{c_n}\right| < \frac{1}{n}\}) < \sum_{n=1}^{\infty} 2^{-n} = 1$$

The summation on the right sums to a finite value, so the right side's terms converges to zero, and we apply the Borel-Cantelli Lemma to find that for $E_n = \{x : |f_n(x)/c_n| < 1/n\}$

$$m(\limsup_{k\to\infty} \chi_{E_k} g_n(x) = g_n(x)) = 0$$

and so it follows that

$$g_n(x) \to 0$$

as $n \to \infty$

Problem 2

We want to construct an open set $E \subset [0,1]$ which is dense in $[0,1]$ and $m(E) = \lambda$. We define $E = \mathbb{Q} \cap [0,1]$ and the measure

$$m(x) = \lambda \text{ for } x \in [0,1]$$

$$m(x) = 0 \text{ else}$$

It can be easily proved that this is a measure on $[0,1]$ and E is dense on $[0,1]$ such that $m(E) = \lambda$.

We next want to construct a dense set for the Lebesgue measure m. Let $E = \mathbb{Q} \cap [0,1] \cup [0,\lambda]$. The rational numbers have a measure of zero, and so it follows that $m(E) = m([0,\lambda]) = \lambda$.

Problem 3

We want to construct a Borel set $E \subset \mathbb{R}$ such that

$$0 < m(E \cap I) < m(I)$$

for every non-empty open interval I. We want to find out if it is possible to have $m(E) < \infty$

I cannot think of such a set. As close as I can get is trying an approach of the rational numbers surrounded by balls. According to the Borel-Cantelli Lemma a constructed set like this can not have finite measure.

Another idea is to try the "Fat Cantor Set" as in class it was said to have a measure greater than zero. However it seems that in creating the cantor set we can find some I such that the intersection is the emptyset resulting in a measure of zero.

Problem 4

Suppose ϕ is a continuous real-valued function defined in the open interval $(a,b) \subset \mathbb{R}_s$ with the mid-point property:

$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y)$$

for all $x, y \in (a,b)$. We want to prove that ϕ is convex. Recall that: a function is convex if for all $x, y \in (a,b)$ and for all $t \in [0,1]$

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

We can do this by converting t and $(1-t)$ in binary expansion, and summing up the terms gives us our inequality. We let

$$t = 2^{-t_1} + 2^{-t_2} + \cdots = \sum_{t_i} 2^{-t_i}$$

where $(t_1, t_2, t_3, \ldots)$ is the binary expansion of t. We can then do the same for $t-1$. Furthermore we get

$$\phi(tx+(1-t)y) = \phi(\sum_{t_i} 2^{-t_i}x+\sum_{t_j} 2^{-t_j}y) \leq \sum_{t_i} 2^{-t_i}\phi(x)+\sum_{t_j} 2^{-t_j}\phi(y) = t\phi(x)+(1-t)\phi(y)$$

where i indicates the terms for the binary expansion of t and j indicates the terms for the binary expansion of $1-t$. Note: it was too messy to expand each term, so there is some faith taken by the reader, however it is a recursive formula that we use to make the inequality step true. The binary expansion converts to the outside of the function ϕ and when collecting the terms we get our desired value.

Problem 5

Suppose ϕ is a real-valued function on $\mathbb{R}$ such that

$$\phi\left[\int_0^1 f(x)dx\right] \leq \int_0^1 \phi[f(x)]dx$$

for every real-valued bounded measurable function f on $[0,1]$. We want to prove that ϕ is convex.

Attempt 1: If we take the second derivative on both sides we can try to show that the second derivative is greater than 0. However this seems a bit tricky and I fail to have confidence that this will give us our desired result. Instead we will try another idea.

Attempt 1: If we use the properties of our measures we can split the integral to a form that we need to show convexity. We have that

$$\phi\left[\int_0^t f(x)dx + \int_t^1 f(x)dx\right] \leq \int_0^t \phi[f(x)]dx + \int_t^1 \phi[f(x)]dx$$

However I am unclear how to go from here, since what we need is a more discrete version of what we have. Perhaps if we let $f(x) = x$ then we have that

$$\phi\left[\int_0^t xdx + \int_t^1 xdx\right] = \phi\left[t + (1-t)\right] \leq \int_0^t \phi[x]dx + \int_t^1 \phi[x]dx$$

It still seems the right side of the inequality is not entirely the result we desire.