Homework 2 | Thomas Lockwood
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Exercise 1. *We want to show that the following three conditions on a metric space K are equivalent:*
(a) K is compact.
(b) K is sequentially compact.
(c) K is totally bounded and complete.

Solution. First we will show that (a) $\Rightarrow$ (b). Assume that K is compact. We want to show that K is sequentially compact. Let $\{x_i\}$ for $i \in \mathbb{N}$ be an infinite sequence.

Before we prove this let us see what is happening here. We have shown in Homework 1 that compact implies that the set is closed and bounded. Therefore our metric space K is closed and bounded. We have an infinite sequence on this metric. If this infinite sequence doesn't have a convergent sub-sequence that forces every point to be an isolated point. However this can't be possible since this space is bounded.

Back to our proof, suppose that $\{x_i\}$ doesn't have a convergent sub-sequence. Then we can take the closed set $\bigcup_i \{x_i\} = E \subset K$ and define it as E, which is also compact (closed subsets of compact spaces are compact). Next since there is no convergent sub-sequence it suffices to say that there exist an open set U_i that contains $\{x_i\}$ and no other points of $\{x_j\}$ for all $i \neq j$, and $\{U_i\}$ is a cover of E. However we know that E is compact therefore there exist a finite subcover, which leads to a contradiction since our sequence is infinite. We conclude that E is sequentially compact.

Next we want to show that (b) $\Rightarrow$ (c). First we want to show that sequentially compact implies totally bounded. We will do this by proof by contradiction. Assume a set K is sequentially compact. Assume also that K is not totally bounded. Then there exist an infinite sequence $\{x_i\}$ such that there exist some $\epsilon > 0$ such that it cannot be covered by a finite number of balls with radius less that ϵ call it $\{B_\epsilon(x_i)\}$. We then construct our new sequence by taking one point out of each $B_\epsilon(x_i)$. Since there are infinite number of balls to choose from this sequence is an infinite collection of isolated points, and thus does not have a convergent sub-sequence a contradiction to being sequentially compact. We conclude that K is totally bounded.

Now we want to prove that K is complete. We have that all infinite sequences in K have a convergent sub-sequence, since K is sequentially compact. Therefore all Cauchy sequences in K converge, and K is complete.

Next we want to show that (c) $\Rightarrow$ (b). Assume that K is totally bounded and complete. we want to show that K is sequentially compact. Let $\{C_i\}_{i=1}^k$ be a cover of balls of radius ϵ for K and $\{x_j\}$ be an infinite sequence in K. We know there exist a finite number of C_i, so at least one C_i contains infinitely many points of $\{x_j\}$. We let this sequence be our sub-sequence $\{x_j^*\}$, and with our properties of totally bounded we find by taking smaller ϵ balls that this sequence is Cauchy. Furthermore, K is complete and $\{x_j^*\}$ converges. We conclude that K is sequentially compact.

Lastly we want to show that $(b) \Rightarrow (a)$. Assume that K is sequentially compact. We want to show that K is compact. We will prove this by contradiction. Suppose there exist a cover $\{C_\alpha\}$ such that had no finite sub-cover. Then we can construct a sequence without a convergent sub-sequence a contradiction. We conclude that $\{C_\alpha\}$ has a finite sub-cover and K is compact. □

Exercise 2. *Let X be a metric space.*

(1) We want to show that a compact set is closed and bounded.

(2) We want to determine what additional assumptions are needed to show that a closed and bounded set is compact.

(3) We want to give an example of a complete metric space where a closed and bounded set is not compact.

(4) We want to explain how these results relate to the Heine-Borel Theorem and the Bolzano Weierstrass Theorem on $\mathbb{R}^n$.

(5) We want to find the key aspect of a metric space that is used in this argument.

Proof. (1) It is not always the case that a compact space is closed. For example take the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1,2\}\}$ on the space $\{1,2\}$. The set $\{1\}$ is a compact space, however it not closed. We can check since it is not the complement of an open set.

Also if a compact space is not bounded, then we can find a contradiction by finding an open cover that has no finite sub-cover (as we did in the proof for the first problem for sequences).

(2) & (3) We need that the space is not totally disconnected or not incomplete. For example, when we look at the Cantor set with the Standard topology we can see that this set is closed and bounded however we can find an open cover that has no finite sub-cover by covering it with balls with ϵ such that ϵ becomes arbitrarily small.

(4) These results are ignored on $\mathbb{R}^n$ since this set is complete. However when taking subset of $\mathbb{R}^n$ we can find incomplete spaces. Furthermore with the Bolzano Weierstrass Theorem we have to include that a set that is Totally Bounded is also complete, otherwise we could find this contradiction.

(5) The key aspect of a metric goes beyond defining an open set in order to keep these properties, but it is also the case that we need to verify that if we partition a set we can turn it into a measure (similar to cutting up a line or a shape into squares and adding up the pieces to find the whole). It could also be re-defining a structure that can be well-defined, for example, the Cantor set was not defined well enough to show it is enough to be compact. However it is a subset of $\mathbb{R}^n$ which is a complete space and has the properties of compactness. Therefore if a set does not meet the requirements it might be the case that it is a subset of a set that does. That or just becuase a set holds a definition (like closed and bounded implies compactness) doesn't mean its subsets do as well. Some definitions are overlooked, and the small details mean much more than we expected. □

Exercise 3. *Uniform convergence is an important concept. We want to give an example of function classes that are preserved under uniform convergence.*

Proof. Taken from wikipedia "In the mathematical field of analysis, uniform convergence is a type of convergence stronger than point wise convergence. A sequence $\{f_n\}$ of functions converges uniformly to a limiting function f if the speed of convergence of $f_n(x)$ to $f(x)$ does not depend on x.

The concept is important because several properties of the functions f_n, such as continuity and Riemann integrability, are transferred to the limit f if the convergence is uniform, but not necessarily if the convergence is not.

Uniform convergence to a function on a given interval can be defined in terms of the uniform norm."

Some example of equivalent classes of functions are functions with a countable number of points of discontinuity. We can limit this set as a null set and set the measure to 0. We can also use to prove that int $f_n(x)dx \to$ int $f(x)dx$. So we can use this notion of equivalent classes to define a measure on the functions $\{f_n\}$. ☐

Exercise 4. *Let X be an uncountable set and $E = \{$collection of all sets $A \subset X$ such that either A or A^c is at most countable$\}$. We want to define a set function μ so that $\mu(A) = 0$. We also want to evaluate the case where $\mu(A) = 1$. In the second case, we want to prove that E is a $\sigma-$algebra and μ is a measure.*

Proof. We define the measure

$$\mu(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Suppose first that x is not in A. Now we have that $\mu(A) = 0$. Next instead, assume that $x \in A$, then we have that $\mu(A) = 1$.

We want to show that E is a $\sigma-$ algebra. We will do this by showing that E is i) closed under complements, ii) countable unions, and iii) finite intersections.

i) Suppose that A is in E. We want to show that A^c is in E. However, since A is in E we have that A or A^c is countable, and furthermore A and A^c are in E.

ii) Suppose that A_i are in E for all $i \in \mathbb{N}$. We want to show that $\cup_i A_i$ is in E.First suppose that all A_i are countable then we have that $\cup_i A_i$ is countable and $\cup_i A_i \in E$. Suppose that at A_k is uncountable, then we have that $\cap_i A_i^c$ is countable and so $\cap_i A_i^c \in E$ and furthermore we use property i) to show that $(\cap_i A_i^c)^c = \cup A_i \in E$.

iii) Suppose that A and B are in E. We want to show that $A \cap B$ is in E. Using property ii) we can see that if we let $A_i = \emptyset$ for all $i > 2$, and let $A_1 = A^c$ and $A_2 = B^c$ we have $A^c \cup B^c \in E$. By using property i) we know the complement is in E, other words $(A^c \cup B^c)^c = A \cap B \in E$. ☐