# Real Analysis - Problem Set #6

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## Problem 1

Let $\mu$ be a probability measure eg $\mu(\omega) = 1$ and $f$, $g$ non-negative measurable functions with $fg \geq 1$. For $0 < p < \infty$ we want to show that

$$\int f^p d\mu \int g^p d\mu \geq 1$$

We will use a direct proof using Hölders inequality. The inequality states that for $1/p + 1/q = 1$ we have

$$\int |f| d\mu \int |g| d\mu \leq \left( \int |f|^p d\mu \right)^{1/p} \left( \int |g|^q d\mu \right)^{1/q}$$

We have that $fg \geq 1$ for all $x \in \Omega$ and $\mu(\Omega) = 1$ so it follows that

$$1 \leq \int |fg| d\mu$$

Let $q$ be the Hölder conjugate of $p$; that is, let $1/p + 1/q = 1$ and by Hölders inequality it follows such that

$$1 \leq \int |fg| d\mu \leq \left( \int |f|^p d\mu \right)^{1/p} \left( \int |g|^q d\mu \right)^{1/q} \leq \int |f|^p d\mu \int |g|^q d\mu$$

We can see that two cases follow; that is, either $p < q$ or $q < p$. If $p > q$ then the last inequality follows straight from our previous result. Instead let us suppose that $p < q$. (We omit the case where $p = q = 1$ since it is trivial, and has already been proven above.) Also it should be noted that the absolute value can be dropped since $fg \geq 1$ for all $x \in \Omega$

Also recall that on our previous homework we showed the following result

$$||f||_{p+1} \leq ||f||_p$$

Furthermore it follows for $p < q$ we have

$$\left(\int |f|^p d\mu\right)^{1/p} \left(\int |g|^q d\mu\right)^{1/q} \leq \int |f|^p d\mu \left(\int |g|^p d\mu\right)^{1/p} \leq \int |f|^p d\mu \int |g|^p d\mu$$

## Problem 2

For $1 \leq p < \infty$ and $f \in L^P(\mathbb{R})$ we define

$$g(x) = \int_x^{x+1} f(t)dt$$

We want to show that $g \in C_o(\mathbb{R})$; that is, $g$ is continuous and $\lim_{|x|\to 0} |g(x)| = 0$.

Note*: we will use $B_r(x)$ as the neighborhood of radius $r$ around the point $x$.

We want to show that $g(x)$ is continuous. We will do a direct proof. Let $\epsilon > 0$ we want to show there exist some $\delta > 0$ such that for all $x, x_o \in \mathbb{R}$ such that $|x - x_0| < \delta$ implies that $|g(x) - g(x_0)| < \epsilon$. First let us rearrange the latter inequality

$$|g(x) - g(x_0)| = |\int_{x_0+1}^{x+1} f(t)dt - \int_{x_0}^{x} f(t)dt| \leq 2 \sup_{t \in B_\epsilon(x)} f(t)(|x - x_0|) \leq \sup_{t \in B_\epsilon(x)} f(t)\delta$$

So we have that $\epsilon / \sup_{t \in B_\epsilon(x)} = \delta$. We conclude that $g(x)$ is continuous.

Next we want to show that $\lim_{|x|\to\infty} = 0$. We will prove this similar to the Borel-Cantelli Theorem. We define the measure for $E \subset \mathbb{R}$

$$\mu(E) = \int_E f(t)dt$$

we can partition $\mathbb{R}$ into countably infinite number of subsets $\mathbb{R} = \cup_i E_i$. Then we have that

$$\mu(\cup_i E_i) = \sum_i \mu(E_i)$$

On the other hand

$$\mu(\cup_i E_i) = \mu(\mathbb{R}) < \infty$$

And so we have that

$$\sum_i \mu(E_i) < \infty$$

it follows that for $E_i$ with large enough intervals (defined on the way we partition $\mathbb{R}$) that $\lim_{|x_i|\to\infty} g(x_i) \leq \lim_{i\to\infty} \mu(E_i) \to 0$. Note*: I have not defined $E_i$ and if it is necessary we can define $E_i = [i, i+1]$ where $i \in \mathbb{N}$

Next, we would like to determine does the previous result hold if $f \in L^\infty(\mathbb{R})$. Yes the following will hold in $L^\infty$ since

$$\lim_{x\to\infty} L^x = L^\infty$$

and on the last homework we showed $L^x$ is a decreasing sequence.

Lastly, we want to construct an example of an unbounded function $f$ such that $\int_{\mathbb{R}} |f|^p dt = \infty$ for all $p$, $0 < p < \infty$. Let $f = x$ then we have that $f$ is unbound and for all $p > 0$

$$\int_{\mathbb{R}} x^p dx = \infty$$

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## Problem 3

If $f$ is integrable on $\mathbb{R}$, we want to show that

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

is uniformly continuous. We also want to prove that $\lim_{x\to\infty} F(x) = 0$

The proof is directly from Problem 2 once we notice that if we sum up the $E_i$ up to $x$ and let $i \in \mathbb{Q}$ then we have the following results.

## Problem 4

We want to prove that if $f$ is integrable on $\mathbb{R}$ and $\delta > 0$, then $f(\delta x)$ converges to $f(x)$ in norm as $\delta \to 1$.

## Problem 5

If $f$ is integrable on $\Omega$ with measure $\mu$. If for each $\alpha > 0$, we let $E_\alpha = \{x \in \Omega : |f(x)| > \alpha\}$, then we want to show that

$$\int |f| d\mu = \int_0^\infty \mu(E_\alpha) d_\alpha$$

First let us see what is happening in this problem. We have that if we cut up the measure of the function into intervals and sum them up it will be the same as the area of the whole function. This should follow from measure theory and order of integration.

Recall that $\mu(E_\alpha) = \int_\omega \chi_{E_\alpha}|f(x)|dx$ then it follows

$$\int_0^\infty \mu(E_\alpha)d_\alpha = \int_0^\infty \int_\Omega \chi_{E_\alpha}|f(x)|dxd\alpha = \int_\Omega \int_0^\infty \chi_{E_\alpha}|f(x)|d\alpha dx = \int_\Omega \chi_{[0,\infty]}|f(x)|dx = \int_\Omega |f|$$