ERDOS-SIERPINSKI PROBLEM

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Abstract. The Sum of Divisors Function of an integer n, denoted $\sigma(n)$ is the sum of positive divisors n. The Erdos-Seirpenski Problem asks: Are there infinitely many solutions to $\sigma(n) = \sigma(n+1)$? Erdos claimed that there were, however it has yet to be proven. Spira defined the Sum of Divisors Function for the Gaussian Integers, denoted $g\sigma(n)$. Combining the two, we look into the problem: Are there infinitely many solutions to $g\sigma(n) = g\sigma(n+1)$?

1. The Integers

The Sum of Divisors Function, or sigma function $\sigma(n)$, is the sum of all the positive divisors of an integer n. If n has the prime factorization $n = {p_1}^{m_1}{p_2}^{m_2}\cdots{p_N}^{m_N}$

$$\sigma(n) = \sum_{d|n} d = \left(\frac{{p_1}^{m_1+1}-1}{p_1-1}\right)\left(\frac{{p_2}^{m_2+1}-1}{p_2-1}\right)\cdots\left(\frac{{p_N}^{m_N+1}-1}{p_N-1}\right)$$

The Erdos-Sierpinski Problem is to find solutions to the equations $\sigma(n) = \sigma(n+1)$. The first few solutions are:

$$\sigma(206) = \sigma(207) = 312$$
$$\sigma(957) = \sigma(958) = 1440$$
$$\sigma(1334) = \sigma(1335) = 2160$$
$$\sigma(1364) = \sigma(1365) = 2688$$
$$\sigma(1634) = \sigma(1635) = 2688$$
$$\sigma(2685) = \sigma(2686) = 4320$$
$$\sigma(2974) = \sigma(2975) = 4464$$
$$\sigma(4364) = \sigma(4365) = 7644$$

Guy and Shanks noted that some solutions have the form

$$n = 2p, \;\; n+1 = 3^m q$$

when q and p are both prime with

$$q = 3^{m+1} - 4, \;\; p = \frac{3^m q - 1}{2}$$

Yields the solution $n = 14, 206, 19358$ for $m = 1, 2,$ and 4 Similarly,

$$n = 3^m q, \quad n + 1 = 2p$$

when q and p are both prime for

$$q = 3^{m+1} - 10, \quad p = \frac{3^m q + 1}{2}$$

yield the solutions

2. The Sigma Fucntion in the Gaussian Integers

Gaussian Integers (complex integers), also known as the Integer Ring $\mathbb{Z}[\sqrt{-1}]$, can be defined as the Complex Divisor Function. Spira defined the Complex Divisor Function for Gaussian Integers very well, denoted $g\sigma$. In his definition he demonstrated that we need to use the first quadrant of the form of the complex numbers, excluding the imaginary axis. After rotated to the first quadrant it is the same as the formula above, except for Gaussian Integers for all the variables instead regular integers.

For example, if we were trying to find $g\sigma(2)$ we would have the prime factorization of $2 = (1 + i)(1 - i)$. We need all the primes in the first quadrant, so we can multiply $(1 + i)$ by i and we get $2 = i(1 + i)(1 + i)$. Adding up the primes we get $g\sigma(2) = g\sigma((1 + i)^2) = 1 + (1 + i) + (1 + i)^2 = 2 + 3i$

3. Gaussian Integers Containing a Single Prime

Let n be prime, and $n + 1$ be the product of two distinct primes p and q, then $n = (n + 1) - 1 = pq - 1$. If p and q are of odd parody, then $n = (1 + i)$. This would force $pq = 2 + i$, a contradiction since $2 + i$ is prime and can't be the product of two primes. So either p or q must be of even parody, namely $(1 + i)$, WLOG let $p = 1 + i$ and $q = a + bi$, then $n = (1 + i)q - 1$. Assume $g\sigma(n) = n + 1 = g\sigma(n + 1) = (1 + i) = (2 + i)(a + bi + 1)$, then we have the equations:

$$1 = 2a + 2$$

and

$$i(1) = i(a + 2b + 1)$$

By Spira's definition of Complex Divisor Function we are unable to have negative numbers in the domain, therefore leads to a contradiction, so there exists no solutions in this form.

This time let $n + 1$ be prime, and n be the product of two distinct primes p and q, then $n = (n + 1) - 1 = pq$. For $n + 1$ to be of even parody, then $n = i$, and i is not the product of two primes. That leaves n to be of even parody, so either p or q is $1 + i$. WLOG let $p = 1 + i$, $q = a + bi$, and $n = c + di$, then $(1 + i)(a + bi) = (c + di + 1) - 1 =$

$c+di$. Assume $g\sigma((1+i)(a+bi)) = (2+i)(a+1+bi) = g\sigma(n+1) = n+2 = c+di+1$, then we have the four equations:

$$a - b = c$$
$$i(a+b) = i(d)$$
$$2a + 2 - b = c + 1$$
$$i(a + 1 + 2b) = i(d)$$

Taking $d = a+1+2b = a+b$ we get $b = -1$, however the input may not be negative leading to our contradiction. Thus there are no consecutive integers in this form or any form.

4. Gaussian Integers Containing Two primes

Furthermore, using Guy and Shanks method with Gaussian Integers we have n and $n+1$ the form of two distinct primes. For the distinct primes q and p (either not equal to $1+i$ or $2+i$), let $n = (1+i)p$ and $n+1 = (2+i)q$. If we let $p = a+bi$ and $q = c+di$, then $n = (n+1)-1 = (1+i)(a+bi) = (2+i)(c+di)-1$. Assume $g\sigma(n) = g(n+1)$, then $g\sigma(1+i)g\sigma(p) = (2+i)(a+bi+1) = g\sigma(2+i)g\sigma(q) = (3+i)(c+di+1)$. Splitting the real and imaginary parts we have the four equations:

$$a - b = 2c - d - 1$$
$$(a+b)i = i(c+2d)$$
$$2a + 2 - b = 3c + 3 - d$$
$$i(a + 2b + 1) = i(c + 3d + 1)$$

. By the four equations we have: $a = 5$, $b = 2$, $c = 3$, and $d = 2$. As you can see $a + bi$ is prime because $5^2 + 2^2 = 29$, and same for $2 + 3i$ similarly $2^2 + 3^2 = 13$.

Suppose we know a, b, e, f, then we can solve for c, d, g, h. Let

$$n = (a+bi)(c+di), \quad n+1 = (e+fi)(g+hi)$$

and by $g\sigma$ of primes

$$g\sigma(n) = (a+bi+1)(c+di+1) = (ac+a+c-bd) + i(ad+d+bc+b)$$
$$g\sigma(n+1) = (e+fi+1)(g+hi+1) = (eg+e+g-fh) + i(eh+fg+h+f)$$

and we have the four equations

$$ac - bd = eg - fh - 1$$
$$bc + ad = fg + eh$$
$$ac + a + c - bd = eg + e + g - fh$$
$$ad + d + bc + b = eh + fg + h + f$$

Thus our solutions:

$$c = \frac{ea^2 - 2e^2a - ea - a + eb^2 - bf - 2ebf + ef^2 + f^2 + e^3 + e^2 + e}{a - e^2 + b - f^2}$$

$$d = \frac{a^2b - 2eab - af + b^3 - 2b^2f + bf^2 + e^2b + eb + b - f}{a - e^2 + b - f^2} - b + f$$

$$g = \frac{a^3 - 2ea^2 - a^2 + ab^2 - 2abf + af^2 + e^2a + ea - a - b^2 + bf + e}{a - e^2 + b - f^2}$$

$$h = \frac{a^2b - 2eab - af + b^3 - 2b^2f + e^2b + eb + b - f}{a - e^2 + b - f^2}$$

Examples of where $g\sigma(n) = g\sigma(n+1)$

$$g\sigma(3+7i) = 10 + 10i = g\sigma(4+7i)$$
$$g\sigma(19+25i) = -20 + 60i = g\sigma(20+25i)$$
$$g\sigma(19+75i) = -100 + 100i = g\sigma(20+75i)$$
$$g\sigma(40+85i) = -100 + 140i = g\sigma(41+85i)$$
$$g\sigma(90+245i) = -320 + 480i = g\sigma(91+245i))$$

5. Non-Consecutive Gaussian Integers Containg Two Primes

For $g\sigma(n) = g\sigma(n + k + qi)$ can also be solved using the previous formula. It is easy to show using the previous equations.

$$n = (a + bi)(c + di), \quad n + k + qi = (e + fi)(g + hi)$$

and by $g\sigma$ of primes

$$g\sigma(n) = (a + bi + 1)(c + di + 1) = (ac + a + c - bd) + i(ad + d + bc + b)$$

$$g\sigma(n + k + qi) = (e + fi + 1)(g + hi + 1) = (eg + e + g - fh) + i(eh + fg + h + f)$$

and we have the four equations

$$ac - bd = eg - fh - k$$
$$bc + ad = fg + eh - q$$
$$ac + a + c - bd = eg + e + g - fh$$
$$ad + d + bc + b = eh + fg + h + f$$

Giving the solutions:

$$c = \frac{-ea^2-afq+eak+ak+2e^2a-eb^2+bfk+2ebf+bq+ebq-f^2k-ef^2-fq-e^2k-ek-e^3}{a^2-2ea+b^2-2bf+f^2+e^2}$$

$$d = \frac{-a^2b+a^2q+2eab+afk+aq-eaq-b^3+2b^2f+b^2q-bf^2-bfq-ebk-bk-e^2k+fk-ea}{a^2-2ea+b^2-2bf+f^2+e^2} + b - f - q$$

$$g = \frac{-a^3+a^2k+2ea^2-ab^2+2abf-af^2-afq-eak+ak-e^2a+b^2k-bfk+bq+ebq-fq-ek}{a^2-2ea+b^2-2bf+f^2+e^2}$$
$$h = \frac{-a^2b+a^2q+2eab+afk+aq-eaq-b^3+2b^2f+b^2q-bf^2-bfq-ebk-bk-e^2b+fk-ea}{a^2-2ea+b^2-2bf+f^2+e^2}$$

A few solutions for $g\sigma(n) = g\sigma(n+i)$

$$g\sigma(15+9i) = 40i = g\sigma(15+10i)$$
$$g\sigma(25+39i) = -40+80i = g\sigma(25+40i)$$
$$g\sigma(25+90i) = -120+120i = g\sigma(25+91i)$$
$$g\sigma(65+60i) = -40+160i = g\sigma(65+61i)$$

Lastly, solutions for $g\sigma(n) = g\sigma(n+1+i)$

$$g\sigma(8+16i) = 10+10i = g\sigma(3+12i)$$
$$g\sigma(5+5i) = 20i = g\sigma(6+6i)$$
$$g\sigma(13+4i) = 20i = g\sigma(14+5i)$$
$$g\sigma(84+5i) = 8+96i = g\sigma(85+6i)$$
$$g\sigma(90+6i) = -160+160i = g\sigma(91+7i))$$
$$g\sigma(138+102i) = -400+80i = g\sigma(139+103i)$$