ERDOS-SIERPINSKI PROBLEM

THOMAS LOCKWOOD

Abstract. The Sum of Divisors Function of an integer n, denoted $\sigma(n)$ is the sum of positive divisors n. The Erdos-Seirpenski Problem asks: Are there infinitely many solutions to $\sigma(n) = \sigma(n+1)$? Erdos claimed that there were, however it has yet to be proven. Spira defined the Sum of Divisors Function for the Gaussian Integers, denoted $g\sigma(n)$. Combining the two, we look into the problem: Are there infinitely many solutions to $g\sigma(n) = g\sigma(n+1)$?

1. The Integers

The Sum of Divisors Function, or sigma function $\sigma(n)$, is the sum of all the positive divisors of an integer n. If n has the prime factorization $n = {p_1}^{m_1}{p_2}^{m_2}\cdots{p_N}^{m_N}$

$$\sigma(n) = \sum_{d|n} d = \left(\frac{{p_1}^{m_1+1}-1}{p_1-1}\right)\left(\frac{{p_2}^{m_2+1}-1}{p_2-1}\right)\cdots\left(\frac{{p_N}^{m_N+1}-1}{p_N-1}\right)$$

The Erdos-Sierpinski Problem is to find solutions to the equations $\sigma(n) = \sigma(n+1)$. The first few solutions are:

$$\sigma(206) = \sigma(207) = 312$$
$$\sigma(957) = \sigma(958) = 1440$$
$$\sigma(1334) = \sigma(1335) = 2160$$
$$\sigma(1364) = \sigma(1365) = 2688$$
$$\sigma(1634) = \sigma(1635) = 2688$$
$$\sigma(2685) = \sigma(2686) = 4320$$
$$\sigma(2974) = \sigma(2975) = 4464$$
$$\sigma(4364) = \sigma(4365) = 7644$$

Guy and Shanks noted that some solutions have the form

$$n = 2p, \quad n+1 = 3^m q$$

when q and p are both prime with

$$q = 3^{m+1} - 4, \quad p = \frac{3^m q - 1}{2}$$

Yields the solution $n = 14, 206, 19358$ for $m = 1, 2$, and 4 Similarly,

$$n = 3^m q, \quad n + 1 = 2p$$

when q and p are both prime for

$$q = 3^{m+1} - 10, \quad p = \frac{3^m q + 1}{2}$$

yield the solutions when $m = 4$ and 5. However this does not solve the problem since it only produces the 3 and 2 addtional solutions respectively.

2. The Sigma Fucntion in the Gaussian Integers

The ring $\mathbb{Z}[\sqrt{-1}]$ is usually called the Gaussian Integers. In Spira's Paper, Spira defined a mutiplicative sum-of-divisors function on the Gaussian Integers which we will denote $g\sigma(z)$. As there are four units in the Gaussian Integers, each prime π Gaussian Sigma has 4 associates: π, $i\pi$, $-\pi$, and $-i\pi$. Therefore in the prime factorization of a Gaussian Integer, $z = \epsilon\pi^{m_1} \cdots \pi^{m_k}$, we choose all the primes to be in the first quadrant, and not on the imaginary axis by factoring out various units. (Then ϵ is the product of those units). Assuming the π, are all in the 1st quandrant and not on the Imaginary Axis, Spria defined:

$$g\sigma(z) = g\sigma(\pi^{m_1} \cdots \pi^{m_k}) = \prod_{j=1}^{k} \left(\frac{\pi_j^{m_j - 1}}{\pi_j - 1} \right) = \prod_{j=1}^{k} \left(1 + \pi_j + \cdots + \pi_j^{m_j}\right)$$

Which is evidently multpilicative.

For example, in $\mathbb{Z}[\sqrt{-1}]$, $5 = (2+i)(2-i)$. Since $2-i$ is not in the first quadrant, we factor out $-i$

$$5 = (2 + i)(2 - i) = (2 + i)(-i)(1 + 2i) = -i(2 + i)(1 + 2i)$$

Then both the primes $2 + i$ and $1 + 2i$ are in quadrant one, and

$$g\sigma(5) = (1 + (2 + i))(1 + (1 + 2i)) = (3 + i)(2 + 2i) = 4 + 8i$$

We now seek the solutions to the equations:

$$g\sigma(z) = g\sigma(z + 1)$$

$$g\sigma(z) = g\sigma(z + i)$$

$$g\sigma(z + 1 + i)$$

We call this the "Erdos-Sierpinski Problem in the Gaussian Integers".

3. Erdos-Sierpinski Problem in $\mathbb{Z}[\sqrt{-1}]$

Let n be prime, and $z+1$ be the product of two distinct primes p and q, then $z=(z+1)-1=pq-1$. If p and q are of odd parity, then $n=(1+i)$. This would force $pq=2+i$, a contradiction since $2+i$ is prime and can't be the product of two primes. So either p or q must be of even parity, namely $(1+i)$, WLOG let $p=1+i$ and $q=a+bi$, then $z=(1+i)q-1$. Assume $g\sigma(z)=z+1=g\sigma(z+1)=(1+i)=(2+i)(a+bi+1)$, then we have the equations:

$$1=2a+2$$

and

$$i(1)=i(a+2b+1)$$

By Spira's definition of Complex Divisor Function we are unable to have negative numbers in the domain, therefore leads to a contradiction, so there exists no solutions in this form.

This time let $z+1$ be prime, and z be the product of two distinct primes p and q, then $z=(z+1)-1=pq$. For $z+1$ to be of even parity, then $z=i$, and i is not the product of two primes. That leaves z to be of even parity, so either p or q is $1+i$. WLOG let $p=1+i$, $q=a+bi$, and $z=c+di$, then $(1+i)(a+bi)=(c+di+1)-1=c+di$. Assume $g\sigma((1+i)(a+bi))=(2+i)(a+1+bi)=g\sigma(z+1)=n+2=c+di+1$, then we have the four equations:

$$a-b=c$$
$$i(a+b)=i(d)$$
$$2a+2-b=c+1$$
$$i(a+1+2b)=i(d)$$

Taking $d=a+1+2b=a+b$ we get $b=-1$, however the input may not be negative leading to our contradiction. Thus there are no consecutive integers in this form or any form.

4. Solutions to $g\sigma(pq)=g\sigma(rs)$

Furthermore, using Guy and Shanks method with Gaussian Integers we have z and $z+1$ the form of two distinct primes. For the distinct primes q and p (either not equal to $1+i$ or $2+i$), let $z=(1+i)p$ and $z+1=(2+i)q$. If we let $p=a+bi$ and $q=c+di$, then $z=(z+1)-1=(1+i)(a+bi)=(2+i)(c+di)-1$. Assume $g\sigma(z)=g(z+1)$, then $g\sigma(1+i)g\sigma(p)=(2+i)(a+bi+1)=g\sigma(2+i)g\sigma(q)=(3+i)(c+di+1)$. Splitting the real and imaginary parts we have the four equations:

$$a-b=2c-d-1$$
$$(a+b)i=i(c+2d)$$

$$2a + 2 - b = 3c + 3 - d$$
$$i(a + 2b + 1) = i(c + 3d + 1)$$

From the four equations we have: $a = 5$, $b = 2$, $c = 3$, and $d = 2$. Since $a + bi$ is prime $(5^2 + 2^2 = 29)$, and same holds for for $2 + 3$.

Suppose we know a, b, e, f, then we can solve for c, d, g, h. Let

$$n = (a + bi)(c + di), \quad n + 1 = (e + fi)(g + hi)$$

and by $g\sigma$ of primes

$$g\sigma(z) = (a + bi + 1)(c + di + 1) = (ac + a + c - bd) + i(ad + d + bc + b)$$
$$g\sigma(z + 1) = (e + fi + 1)(g + hi + 1) = (eg + e + g - fh) + i(eh + fg + h + f)$$

and we have the four equations

$$ac - bd = eg - fh - 1$$
$$bc + ad = fg + eh$$
$$ac + a + c - bd = eg + e + g - fh$$
$$ad + d + bc + b = eh + fg + h + f$$

Thus our solutions:

$$c = \frac{ea^2 - 2e^2a - ea - a + eb^2 - bf - 2ebf + ef^2 + f^2 + e^3 + e^2 + e}{a - e^2 + b - f^2}$$
$$d = \frac{a^2b - 2eab - af + b^3 - 2b^2f + bf^2 + e^2b + eb + b - f}{a - e^2 + b - f^2} - b + f$$
$$g = \frac{a^3 - 2ea^2 - a^2 + ab^2 - 2abf + af^2 + e^2a + ea - a - b^2 + bf + e}{a - e^2 + b - f^2}$$
$$h = \frac{a^2b - 2eab - af + b^3 - 2b^2f + e^2b + eb + b - f}{a - e^2 + b - f^2}$$

Examples of where $g\sigma(z) = g\sigma(z + 1)$

$$g\sigma(3 + 7i) = 10 + 10i = g\sigma(4 + 7i)$$
$$g\sigma(19 + 25i) = -20 + 60i = g\sigma(20 + 25i)$$
$$g\sigma(19 + 75i) = -100 + 100i = g\sigma(20 + 75i)$$
$$g\sigma(40 + 85i) = -100 + 140i = g\sigma(41 + 85i)$$
$$g\sigma(90 + 245i) = -320 + 480i = g\sigma(91 + 245i))$$

5. Non-Consecutive Gaussian Integers Containg Two Primes

For $g\sigma(z) = g\sigma(z+k+qi)$ can also be solved using the previous formula. It is easy to show using the previous equations.

$$n = (a+bi)(c+di), \quad z+k+qi = (e+fi)(g+hi)$$

and by $g\sigma$ of primes

$$g\sigma(z) = (a+bi+1)(c+di+1) = (ac+a+c-bd) + i(ad+d+bc+b)$$

$$g\sigma(z+k+qi) = (e+fi+1)(g+hi+1) = (eg+e+g-fh) + i(eh+fg+h+f)$$

and we have the four equations

$$ac - bd = eg - fh - k$$
$$bc + ad = fg + eh - q$$
$$ac + a + c - bd = eg + e + g - fh$$
$$ad + d + bc + b = eh + fg + h + f$$

Giving the solutions:

$$c = \frac{-ea^2-afq+eak+ak+2e^2a-eb^2+bfk+2ebf+bq+ebq-f^2k-ef^2-fq-e^2k-ek-e^3}{a^2-2ea+b^2-2bf+f^2+e^2}$$
$$d = \frac{-a^2b+a^2q+2eab+afk+aq-eaq-b^3+2b^2f+b^2q-bf^2-bfq-ebk-bk-e^2k+fk-ea}{a^2-2ea+b^2-2bf+f^2+e^2} + b - f - q$$
$$g = \frac{-a^3+a^2k+2ea^2-ab^2+2abf-af^2-afq-eak+ak-e^2a+b^2k-bfk+bq+ebq-fq-ek}{a^2-2ea+b^2-2bf+f^2+e^2}$$
$$h = \frac{-a^2b+a^2q+2eab+afk+aq-eaq-b^3+2b^2f+b^2q-bf^2-bfq-ebk-bk-e^2b+fk-ea}{a^2-2ea+b^2-2bf+f^2+e^2}$$

A few solutions for $g\sigma(z) = g\sigma(z+i)$

$$g\sigma(15+9i) = 40i = g\sigma(15+10i)$$
$$g\sigma(25+39i) = -40+80i = g\sigma(25+40i)$$
$$g\sigma(25+90i) = -120+120i = g\sigma(25+91i)$$
$$g\sigma(65+60i) = -40+160i = g\sigma(65+61i)$$

Lastly, solutions for $g\sigma(z) = g\sigma(z+1+i)$

$$g\sigma(8+16i) = 10+10i = g\sigma(3+12i)$$
$$g\sigma(5+5i) = 20i = g\sigma(6+6i)$$
$$g\sigma(13+4i) = 20i = g\sigma(14+5i)$$
$$g\sigma(84+5i) = 8+96i = g\sigma(85+6i)$$
$$g\sigma(90+6i) = -160+160i = g\sigma(91+7i))$$
$$g\sigma(138+102i) = -400+80i = g\sigma(139+103i)$$