MATHEMATIKA

A JOURNAL OF PURE AND APPLIED MATHEMATICS

VOL. 15. PART 1. JUNE, 1968. No. 29.

BOMBIERI'S MEAN VALUE THEOREM

P. X. GALLAGHER

The purpose of this paper is to give a short proof of an important recent theorem of Bombieri [**2**] on the mean value of the remainder term in the prime number theorem for arithmetic progressions. Applications of the theorem have been made by Bombieri and Davenport [**3**], Rodriques [**9**], and Elliott and Halberstam [**5**]. For earlier versions of the theorem and a survey of other applications, see Barban [**1**], and Halberstam and Roth [**7**, Chapter 4].

For $(a, q) = 1$, we put

$$\psi(x, a, q) = \sum_{\substack{n \leqslant x \\ n \equiv a\ (q)}} \Lambda(n),$$

where $\Lambda(n) = \log p$ if n is a power of the prime p, and $\Lambda(n) = 0$ otherwise. We write l for $\log x$.

THEOREM. *For each positive constant A, there is a positive constant B such that if $Q = x^{\frac{1}{2}} l^{-B}$, then*

$$\sum_{q \leqslant Q} \max_{y \leqslant x} \max_{(a,q)=1} \left| \psi(y, a, q) - \frac{y}{\phi(q)} \right| \ll x l^{-A}. \tag{1}$$

Bombieri derived this result from an estimate of the total density of zeros of all Dirichlet L-functions to moduli $\leqslant Q$, which he derived from general estimates of character sums (the "large sieve"). We avoid the zeros and estimate the left side of (1) directly in terms of character sums, L-series in particular. Our estimates are good only for small t; a smoothing device is used to get around this. The unsmoothing at the end of the proof inflates B as a function of A. As a result, our value for B is $16A + 103$ compared to Bombieri's $3A + 23$.

1. *Smoothing; reduction to the large sieve.* For functions F piecewise continuous on $[1, \infty]$, we put

$$F_0 = F, \quad \text{and} \quad F_{k+1}(x) = \int_1^x F_k(y) \frac{dy}{y}.$$

then

$$\psi_k(x, a, q) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) . \psi_k(x, \chi), \tag{2}$$

where

$$\psi_k(x, \chi) = \frac{1}{k!} \sum_{n \leqslant x} \chi(n) \Lambda(n) \log^k \left(\frac{x}{n} \right).$$

If χ is induced by the primitive character χ^*, then

$$\psi_k(x,\chi) = \psi_k(x,\chi^*) + O(l^{k+1}\log q), \tag{3}$$

since the terms in the two sums agree, except possibly for $n = p^a$ with $p|q$, which contribute $\leqslant l^{k+1}$ for each $p|q$. Using $\psi_k(x,\chi_0{}^*) = \psi_k(x)$ we get from (2) and (3)

$$\max_{(a,q)=1}\left|\psi_k(x,a,q) - \frac{\psi_k(x)}{\phi(q)}\right| \ll \frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}|\psi_k(x,\chi^*)| + l^{k+1}\log q.$$

We have

$$\sum_{d|q\leqslant Q}\frac{1}{\phi(q)} \ll \frac{1}{\phi(d)}\sum_{e\leqslant Q/d}\frac{1}{\phi(e)} \ll \frac{\log(Q/d)+1}{\phi(d)} = \varepsilon_d, \quad \text{say.}$$

Hence for $Q \leqslant x^{\frac{1}{2}}$, and bounded k and A,

$$\sum_{q\leqslant Q}\max_{y\leqslant x}\max_{(a,q)=1}\left|\psi_k(y,a,q) - \frac{\psi_k(y)}{\phi(q)}\right| \ll \sum_{1<q\leqslant Q}\varepsilon_q\sum_{\chi}{}^*\max_{y\leqslant x}|\psi_k(y,\chi)| + xl^{-A}, \tag{4}$$

where the star indicates that the sum is over the primitive characters to the modulus q.

We use the Siegel-Walfisz theorem [**8**, pp. 134, 144], in the form

$$\max_{y\leqslant x}|\psi(y,\chi)| \ll xl^{-E}, \quad \text{for} \quad \chi\neq\chi_0 \quad \text{and} \quad q\leqslant l^C,$$

with arbitrarily large constants C and E. It follows easily from this that for bounded k, A and C, the terms with $q \leqslant l^C = D$ contribute $\ll xl^{-A}$ to the right side of (4). Similarly, from the prime number theorem in the form

$$\max_{y\leqslant x}|\psi(y) - y| \ll xl^{-E},$$

with arbitrarily large E, we get for bounded k and A, and $Q \leqslant x^{\frac{1}{2}}$,

$$\sum_{q\leqslant Q}\max_{y\leqslant x}\frac{|\psi_k(y) - y|}{\phi(q)} \ll xl^{-A}.$$

Hence for $Q \leqslant x^{\frac{1}{2}}$, and bounded k, A and C, and writing $D = l^C$, we have

$$\sum_{q\leqslant Q}\max_{y\leqslant x}\max_{(a,q)=1}\left|\psi_k(y,a,q) - \frac{y}{\phi(q)}\right| \ll \sum_{D<q\leqslant Q}\varepsilon_q\sum_{\chi}{}^*\max_{y\leqslant x}|\psi_k(y,\chi)| + xl^{-A}. \tag{5}$$

Put $\alpha = 1 + l^{-1}$. Then for $k \geqslant 1$,

$$\psi_k(y,\chi) = \frac{1}{2\pi i}\int_{(\alpha)}\frac{y^s}{s^{k+1}}\left(-\frac{L'}{L}(s,\chi)\right)ds, \quad \int_{(\alpha)} = \int_{\alpha-i\infty}^{\alpha+i\infty}.$$

This follows on multiplying the absolutely convergent series

$$-\frac{L'}{L}(s,\chi) = \sum_{1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^s} \tag{6}$$

by y^s/s^{k+1} and integrating term by term, using

$$\frac{1}{2\pi i}\int_{(\alpha)}\frac{x^s}{s^{k+1}}ds = \begin{cases}\dfrac{1}{k!}\log^k x & (x\geqslant 1),\\ 0 & (0<x<1).\end{cases}$$

For $S = S(s, \chi)$ bounded and analytic in $\sigma \geqslant \frac{1}{2}$, we write

$$\frac{L'}{L} = \frac{L'}{L}(1 - LS)^2 + (2L'S - L'LS^2).$$

For $\chi \neq \chi_0$, the second term is analytic and $\ll |s|$ in $\sigma \geqslant \beta = \frac{1}{2} + l^{-1}$ (the constant depending on q and S), since $L(s) \ll |s|^{\frac{1}{2}}$ for $\sigma \geqslant \frac{1}{2}$ [**8**, p. 115], and

$$L'(s) = \frac{1}{2\pi i}\int_{\gamma} \frac{L(\zeta)}{(\zeta - s)^2}\, d\zeta, \tag{7}$$

where γ is the circle of radius l^{-1} centred at s. Hence for $k \geqslant 2$,

$$\psi_k(y, \chi) = \frac{1}{2\pi i}\int_{(\alpha)} \frac{y^s}{s^{k+1}}\left(-\frac{L'}{L}\right)(1 - LS)^2 ds + \frac{1}{2\pi i}\int_{(\beta)} \frac{y^s}{s^{k+1}}(L'LS^2 - 2L'S)\, ds.$$

On $\sigma = \alpha$, we have $y^s \ll x$ for $y \leqslant x$, and, using (6),

$$\frac{L'}{L}(s, \chi) \ll \sum \frac{\Lambda(n)}{n^\alpha} = \alpha \int_0^\infty \frac{\psi(u)}{u^{1+\alpha}}\, du \ll \int_1^\infty \frac{du}{u^\alpha} = l.$$

On $\sigma = \beta$, we have $y^s \ll x^{\frac{1}{2}}$ for $y \leqslant x$. Hence

$$\max_{y \leqslant k} |\psi_k(y, \chi)| \ll xl \int_{(\alpha)} \frac{|1 - LS|^2}{|s|^{k+1}}|ds| + x^{\frac{1}{2}} \int_{(\beta)} \frac{|L'LS^2| + |L'S|}{|s|^{k+1}}|ds|.$$

It follows that the right side of (5) is

$$\ll xl \int_{(\alpha)} \frac{A(s)}{|s|^{k+1}}|ds| + x^{\frac{1}{2}} \int_{(\beta)} \frac{B(s)}{|s|^{k+1}}|ds| + xl^{-A}, \tag{8}$$

where

$$A(s) = \sum_{D<q\leqslant Q} \varepsilon_q \sum_{\chi}{}^{*} |1 - LS|^2, \quad B(s) = \sum_{D<q\leqslant Q} \varepsilon_q \sum_{\chi}{}^{*} (|L'LS^2| + |L'S|). \tag{9}$$

2. *Application of the large sieve.* To estimate $A(s)$ on $\sigma = \alpha$ and $B(s)$ on $\sigma = \beta$, we use the following " large sieve " inequalities, valid for arbitrary complex constants a_n, and for $D \leqslant Q/\log Q$:

$$\sum_{D<q\leqslant Q} \varepsilon_q \sum_{\chi}{}^{*} \left|\sum_{M+1}^{M+N} a_n \chi(n)\right|^2 \ll \left(Q + \frac{N.\log Q}{D}\right) \sum_{M+1}^{M+N} |a_n|^2, \tag{10}$$

$$\sum_{q\leqslant X} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \left|\sum_{M+1}^{M+N} a_n\, \chi(n)\right|^2 \ll (X^2 + N) \sum_{M+1}^{M+N} |a_n|^2. \tag{11}$$

The inequality (11) follows from

$$\frac{q}{\phi(q)} \sum_{\chi}{}^{*} \left|\sum_{M+1}^{M+N} a_n \chi(n)\right|^2 \ll \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\sum_{M+1}^{M+N} a_n\, e\left(\frac{na}{q}\right)\right|^2,$$

which is proved in [6, p. 16], and Bombieri's inequality [2, Theorem 2]

$$\sum_{q \leqslant X} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{M+1}^{M+N} a_n e\left(\frac{na}{q}\right) \right|^2 \ll (X^2+N) \sum_{M+1}^{M+N} |a_n|^2,$$

for which short proofs are given in [4] and [6]. The inequality (10) follows as in [6, p. 18] from (11) by partial summation.

For $\sigma \geqslant 1$ and $H \geqslant 1$, we have (as in [6, p. 20])

$$L(s,\chi) = \sum_{1}^{H} \frac{\chi(n)}{n^s} + O\left(\frac{|s|\, q^{\frac{1}{2}} \log q}{H}\right), \quad (\chi \neq \chi_0).$$

We put

$$S(s,\chi) = \sum_{1}^{H} \frac{\mu(n)\chi(n)}{n^s}.$$

Then for $\sigma \geqslant 1$ and $H \leqslant x$, we have $S(s,\chi) \ll \log H \leqslant l$, so for $q \leqslant x$,

$$1 - L(s,\chi)\, S(s,\chi) = \sum_{1}^{\infty} \frac{c(n)\chi(n)}{n^s} + O\left(\frac{|s|\, q^{\frac{1}{2}}\, l^2}{H}\right),$$

where $c(1) = 0$, and for $n > 1$, $c(n) = -\Sigma\mu(d)$, where d ranges over the divisors of n for which $d \leqslant H$ and $n/d \leqslant H$. Thus $c(n) = 0$ for $n > H^2$ and for $n \leqslant H$, and $|c(n)| \leqslant \tau(n)$ for $H < n \leqslant H^2$. Using the Schwarz inequality,

$$\left| \sum_{1}^{\infty} \frac{c(n)\chi(n)}{n^s} \right|^2 = \left| \sum_{0 \leqslant h \ll l} \left(\sum_{2^h H+1}^{2^{h+1}H} \frac{c(n)\chi(n)}{n^s} \right) \right|^2 \ll l \sum_{0 \leqslant h \ll l} \left| \sum_{2^h H+1}^{2^{h+1}H} \frac{c(n)\chi(n)}{n^s} \right|^2.$$

From (10) with $a_n = c(n)/n^s$, we get for each $N = 2^h H$ $(\leqslant x^2)$ and $Q \leqslant x$,

$$\sum_{D<q\leqslant Q} \varepsilon_q \sum_{\chi}{}^{*} \left| \sum_{N+1}^{2N} \frac{c(n)\chi(n)}{n^s} \right|^2 \ll \left(Q + \frac{Nl}{D}\right) \sum_{N+1}^{2N} \frac{\tau^2(n)}{n^2} \ll \left(\frac{Q}{N} + \frac{l}{D}\right) . l^3.$$

Here we have used $\Sigma_{n \leqslant M} \tau^2(n) \ll M . \log^3 M$ [8, p. 26]. It follows that

$$A(s) \ll \sum_{0 \leqslant h \ll l} \left(\frac{Q}{2^h H} + \frac{l}{D}\right) l^4 + \sum_{D<q\leqslant Q} (\log(Q/q) + 1) \frac{|s|^2 q l^4}{H^2}$$
$$\ll QH^{-1}l^4 + D^{-1}l^6 + |s|^2 Q^2 H^{-2} l^4.$$

Choosing $H = QDl^{-2}$ (and assuming $D \geqslant l$), we get

$$A(s) \ll D^{-1}|s|^2 l^6, \quad \text{on } \sigma = \alpha. \tag{12}$$

To estimate $B(s)$, we apply (11) to

$$S^2(s,\chi) = \sum_{1}^{H^2} \frac{d(n)\chi(n)}{n^s}, \quad (|d(n)| \leqslant \tau(n)),$$

and get for $\sigma \geqslant \frac{1}{2}$

$$\sum_{q \leqslant X} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} |S(s,\chi)|^4 \ll (X^2+H^2) \sum_{1}^{H^2} \frac{\tau^2(n)}{n} \ll (X^2+H^2)\, l^4. \tag{13}$$

By the argument of [6, Theorem 4], we also get from (11)

$$\sum_{q \leqslant X} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} |L(s,\chi)|^4 \ll X^2 |s|^2 \log^4(X|s|+2) \qquad (\sigma \geqslant \tfrac{1}{2}). \tag{14}$$

Using Hölder's inequality, (7) gives

$$|L'(s,\chi)|^4 \ll l^5 \int_{\gamma} |L(\zeta,\chi)|^4 \, |d\zeta|.$$

Hence on $\sigma = \beta$, we have

$$\sum_{q \leqslant X} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} |L'(s,\chi)|^4 \ll X^2 |s|^2 \log^4(X|s|+2)\, l^4. \tag{15}$$

Using the Schwarz inequality several times, we get from (13)–(15)

$$\sum_{q \leqslant X} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} (|L'LS^2| + |L'S|) \ll (X+H)\, l^2 . X |s|\, l . \log^2(X|s|+2).$$

By partial summation, it follows, using $H = DQl^{-2}$, that

$$B(s) \ll DQl^5 |s| \log^2(|s|+2), \qquad \text{on } \sigma = \beta. \tag{16}$$

3. *End of the proof; unsmoothing.* From (8), (12) and (16), the right side of (5) is

$$\ll xl \int_{(\alpha)} \frac{D^{-1}|s|^2 l^6}{|s|^{k+1}} |ds| + x^{\frac{1}{2}} \int_{(\beta)} \frac{DQl^5 |s| \log^2(|s|+2)}{|s|^{k+1}} |ds| + xl^{-A}.$$

For $k = 3$, this is

$$\ll xD^{-1}l^7 + x^{\frac{1}{2}} DQl^5 + xl^{-A}.$$

Choosing $D = l^{A+7}$ and $Q = x^{\frac{1}{2}} l^{-(2A+12)}$, we conclude that

$$\sum_{q \leqslant Q} \max_{y \leqslant x} \max_{(a,q)=1} \left| \psi_3(y,a,q) - \frac{y}{\phi(q)} \right| \ll xl^{-A}. \tag{17}$$

It remains to deduce (1) from (17). Since $\psi_k(y,a,q)$ is an increasing function of y, we get, for $0 < \lambda \leqslant 1$,

$$\frac{1}{\lambda} \int_{e^{-\lambda}y}^{y} \psi_{k-1}(z,a,q) \frac{dz}{z} \leqslant \psi_{k-1}(y,a,q) \leqslant \frac{1}{\lambda} \int_{y}^{e^{\lambda}y} \psi_{k-1}(z,a,q) \frac{dz}{z}.$$

The integrals are

$$\psi_k(y,a,q) - \psi_k(e^{-\lambda}y,a,q) \quad \text{and} \quad \psi_k(e^{\lambda}y,a,q) - \psi_k(y,a,q).$$

Putting

$$\psi_k(x,a,q) = \frac{x}{\phi(q)} + r_k(x,a,q),$$

the integrals are both

$$(\lambda + O(\lambda^2)) \frac{y}{\phi(q)} + O\Big(\max_{y \leqslant ex} |r_k(y,a,q)|\Big).$$

Hence

$$\max_{y \leqslant x} |r_{k-1}(y, a, q)| \ll \frac{\lambda x}{\phi(q)} + \frac{1}{\lambda} \max_{y \leqslant ex} |r_k(y, a, q)|,$$

so for $Q \leqslant x$,

$$\sum_{q \leqslant Q} \max_{y \leqslant x} \max_{(a,q)=1} |r_{k-1}(y, a, q)| \ll \lambda x l + \lambda^{-1} \sum_{q \leqslant Q} \max_{y \leqslant ex} \max_{(a,q)=1} |r_k(y, a, q)|.$$

Using a decreasing induction, starting at $k = 3$, the right side is

$$\ll \lambda x l + \lambda^{-1} x l^{-A}, \quad \text{for } Q \leqslant x^{\frac{1}{2}} l^{-B_k(A)}$$

$$\ll x l^{-\frac{1}{2}(A-1)}, \quad \text{choosing } \lambda = l^{-\frac{1}{2}(A+1)}.$$

Thus $B_{k-1}(A) = B_k(2A+1)$. Starting with $B_3(A) = 2A + 12$, this gives $B_0(A) = 16A + 103$.

References

1. M. B. Barban, " The ' large sieve ' method and its applications in the theory of numbers ", *Russian Math. Surveys*, 21 (1966), 49–103.
2. E. Bombieri, " On the large sieve ", *Mathematika*, 12 (1965), 201–225.
3. ——— and H. Davenport, " Small differences between prime numbers ", *Proc. Royal Soc. A.*, 293 (1966), 1–18.
4. H. Davenport and H. Halberstam, " The values of a trigonometric polynomial at well spaced points ", *Mathematika*, 13 (1966), 91–96; 14 (1967), 229.
5. P. D. T. A. Elliott and H. Halberstam, " Some applications of Bombieri's theorem ", *Mathematika*, 13 (1966), 196–203.
6. P. X. Gallagher, " The large sieve ", *Mathematika*, 14 (1967), 14–20.
7. H. Halberstam and K. F. Roth, *Sequences Vol.* 1, (Oxford, 1966).
8. K. Prachar, *Primzahlverteilung* (Springer, 1957).
9. G. Rodriques, " Sul problema dei divisori di Titchmarsh ", *Bollettino Unione Math. Italiana* (3), 20 (1965), 358–366.

Barnard College, Columbia University,
New York, 27, New York (U.S.A.).

(*Received on the* 18*th of September*, 1967.)