Statistics Lecture

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Advanced Learning and Research Institute, 2012

Why is this theorem important?

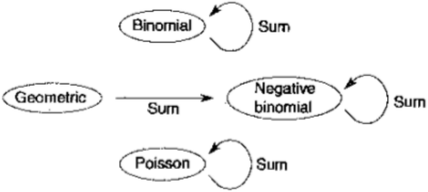


Figure 2.21. Theorem 2.2

Because some families of random variables are closed w.r.t. independent sum of random variables of the same family.
It follow by induction that if $X_1, X_2, \dots, X_n$ are mutually independent R.V.

$$G_{X_1+\cdots+X_n}(z) = G_{X_1}(z)G_{X_2}(z)\cdots G_{X_n}(z)$$

Independent random variables

Theorem

Let $X_1, X_2, \ldots, X_r$ be mutually independent.

1. *If X_i has the binomial distribution with parameters n_i and p_i then $\sum_{i=1}^{r} X_i$ has the binomial distribution with parameters $n_1 + n_2 + \cdots + n_r$ and p.*
2. *If X_i has the (modified) negative binomial distribution with parameters α_i and p_i then $\sum_{i=1}^{r} X_i$ has the (modified) negative binomial distribution with parameters $\alpha_1 + \alpha_2 + \cdots + \alpha_r$ and p.*
3. *If X_i has the Poisson distribution with parameter α_i, then $\sum_{i=1}^{r} X_i$ has the Poisson distribution with parameter $\sum_{i=1}^{r} \alpha_i$.*

Proof.

Use the properties of the PGF (see Trivedi). $\square$

Solve problems 4 and 5 at page 112 of Trivedi.

Continuous random variables

When the sample space S is nondenumerable (as mentioned in Section 1.7), not every subset of the sample space is an event that can be assigned a probability. As before, let $\mathcal{F}$ denote the class of measurable subsets of S. Now if X is to be a random variable, it is natural to require that $P(X \leq x)$ be well defined for every real number x. In other words, if X is to be a random variable defined on a probability space $(S, \mathcal{F}, P)$, we require that $\{s|X(s) \leq x\}$ be an event (i.e., a member of $\mathcal{F}$). We are, therefore, led to the following extension of our earlier definition.

Definition (Random Variable). A random variable X on a probability space $(S, \mathcal{F}, P)$ is a function $X : S \to \Re$ that assigns a real number $X(s)$ to each sample point $s \in S$, such that for every real number x, the set of sample points $\{s|X(s) \leq x\}$ is an event, that is, a member of $\mathcal{F}$.

Definition (Distribution Function). The (cumulative) distribution function or CDF F_X of a random variable X is defined to be the function

$$F_X(x) = P(X \leq x), \qquad -\infty < x < \infty.$$

Continuous random variables

Definitions

1. A random variable X such that its CDF F_X is a continuous function is called a continous random variable.
2. A random variable X such that the derivative dF/dx exists everywhere (except perhaps for a finite number of points) is called an absolutely continuous random variable.

Example

The continuous uniform random variable having CDF

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \leq 1 \end{cases}$$

is absolutely continuous having a derivative at all point except at $x = 0$ and $x = 1$.

Definition

(Probability Density Function). For an absolutely continuous random variable X, $f(x) = dF(x)/dx$ is called the probability density function (pdf or density function) of X.

The pdf enables us to obtain the CDF and hence to compute probabilities by integrating the pdf, i.e.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt \text{ for } -\infty < x < \infty \text{ and}$$

$$\begin{aligned} P(X \in (a,b]) &= P(a < X \leq b) \\ &= P(X \leq b) - P(X \leq a) \\ &= \int_{-\infty}^{b} f_X(t)dt - \int_{-\infty}^{a} f_X(t)dt \\ &= \int_{a}^{b} f_X(t)dt \end{aligned}$$

Continuous random variables

The pdf satisfies the following properties:

1. $f(x) \geq 0$ for all x.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

The CDF F of a continuous random variable satisfies the following properties

1. $0 \leq F(x) \leq 1$, $-\infty < x < \infty$.
2. $F(x)$ is a monotone increasing function of x.
3. $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

The CDF of a continuous random variable does not have any jumps, therefore

$$P(X = c) = \int_c^c f_X(t)dt = 0.$$

This does not imply that the event $\{X = c\}$ is empty, but that the probability assigned to it is 0.

Continuous random variables

Example

The time in years, denoted by X, required to complete a software project has a pdf of the form

$$f_X(x) = \begin{cases} kx(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} .$$

Since f_x must satisfy property 1, $k \geq 0$. Because of property 2 we have

$$\int_0^1 kx(1-x)dx \quad = \quad k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 1$$

from which we deduce $k = 6$. The probability that the project will be completed by less than four months is given by

$$P(X < \frac{4}{12} \quad = F_X(\tfrac{1}{3}) \quad = \int_0^{1/3} f_X(x)dx = \frac{7}{27}$$

Mixed random variables

Sometimes the involved random variable will either be descrete or continuous, but a mixed random variable.

Example

There may be a nonzero probability, say p_0, of initial failure of a component at time 0 due to manufacturing defects. In this case the time to failure X of the component is neither discrete nor a continuous random variable! Assuming an exponential distribution for $x > 0$ the CDF will be

$$F_X(x) = \begin{cases} 0, & x < 0, \\ p_0, & x = 0, \\ p_0 + (1 - p_0)(1 - e^{-\lambda x}), & x > 0. \end{cases}$$

In general, a mixed random variable will have a CDF given by

$$F_X(x) = \alpha_d F^{(d)}(x) + (1 - \alpha_d) F^{(c)}(x)$$

where $F^{(d)}$ and $F^{(c)}$ are two discrete and continuous CDF, respectively.

The exponential distribution find its application in reliability theory and queuing theory. The following random variables are often modeled as exponential:

1. Time between two successive job arrivals to a file server (often called **interarrival time**).
2. Service time at a server in a queuing network; the server could be a resource such as a CPU, an I/O device, or a communication channel.
3. Time to failure (lifetime) of a component.
4. Time required to repair a component that has malfunctioned.

Remark: The choice of the exponential distribution to model the stochastic structure of the upper described variables is an assumption and not a given fact! Experimental verification of the distributional assumption will be therefore necessary before to relying on the results of the analysis.

The exponential distribution

Solve problems 1, 2 and 3 at page 119 of Trivedi.
The CDF of the exponential distribution is

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and the density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} P(X \geq t) &= \int_t^{\infty} f(x)dx = e^{-\lambda t} \\ P(a \leq X \leq b) &= e^{-\lambda a} - e^{-\lambda b} \end{aligned}$$

Let $X \sim Exp(\lambda)$ be the lifetime of a component. Suppose we have observed that it has already been operating for t hours.

- What is the distribution of the remaining (residual) lifetime $Y = X - t$?

Let the conditional probability of $Y \leq y$, given that $X > t$, be denoted by $G_Y(y \mid t)$. For $y \geq 0$

$$\begin{aligned}
G_Y(y \mid t) &= P(Y \leq y \mid X > t) = \frac{P(\{Y \leq y\} \text{ and } \{X > t\})}{P(X > t)} \\
&= \frac{P(\{X \leq y + t\} \text{ and } \{X > t\})}{P(X > t)} = \frac{P(t < X \leq y + t)}{P(X > t)} \\
&= \frac{exp(-\lambda t)(1 - exp(-\lambda y))}{exp(-\lambda t)} = 1 - exp(-\lambda y).
\end{aligned}$$

Result:
The conditional distribution $G_Y(y \mid t)$ does not depend on t and is identical to the distribution of X, i.e. $Exp(\lambda)$.

Interpretation:
The distribution of the remaining life does not depend on how long the component has been operating, i.e. the component does not age (it is as good as new). Therefore, the exponential distribution is not suited to model components or devices that gradually deteriorate.

Let the random variable X be the lifetime (or time to failure) of a component.

Definition

The **reliability** $R(t)$ of the component is the probability that the component survives until some time t, i.e.

$$R(t) = P(X > t) = 1 - F_X(t)$$

$F_X(t)$ is often called the **unreliability** of the component.

The conditional probability that the component does not survive for an additional interval of duration x given that it has survived until time t is equal to

$$G_Y(x \mid t) = \frac{P(t < X \leq t + x)}{P(X > t)} = \frac{F_X(t+x) - F_X(t)}{R(t)}$$

Definition

The instantaneous failure rate $h(t)$ is defined to be

$$h(t) = \lim_{x \to 0} \frac{1}{x} G_Y(x \mid t) = \lim_{x \to 0} \frac{F_X(t+x) - F_X(t)}{xR(t)} ,$$

so that

$$h(t) = \frac{f_X(t)}{R(t)} .$$

Alternate terms for $h(t)$ are *hazard rate*, *force of mortality*, *intensity rate*, *conditional failure rate* or **failure rate**.

Interpretation:

- $h(t)\Delta t$ represents the conditional probability that a component having survived to age t will fail in the interval $(t, t+\Delta t]$.

- $f_X(t)\Delta t$ is the *unconditional* probability while $h(t)\Delta t$ is a conditional probability.

Next theorem shows the connection between reliability and failure rate.

Theorem

$$R(t) = \exp\left(-\int_0^t h(x)dx\right)$$

Proof.

$$\int_0^t h(x)dx = \int_0^t \frac{f_X(x)}{R(x)}dx = \int_0^t \frac{-R'(x)}{R(x)}dx = -\ln(R(t))$$

using the fact that $R'(t) = -f_X(t)$ and the boundary contition $R(0) = 1$. □

Definition

The cumulative hazard is defined to be

$$H(t) = \int_0^t h(x)dx$$

Then, reliability can also be written as $R(t) = \exp(-H(t))$.

Definition

The conditional reliability $R_t(y)$ is the probability that the component survives an additional interval of duration y given that it has survived until time t.

$$R_t(y) = \frac{R(t+y)}{R(t)} \tag{1}$$

Assume a component does not age stochastically, i.e. the survival probability over an additional time interval y is the same regardless of the age t of the component:

$$R_t(y) \quad = \quad R_s(y) \text{ for all } t, s \geq 0.$$

For $s = 0$

$$R_t(y) \quad = \quad R_0(y) = \frac{R(y)}{R(0)} = R(y),$$

so that

$$R(t+y) \quad = \quad R(t)R(y).$$

In particular we obtain

$$\frac{R(t+y) - R(y)}{t} \quad = \quad \frac{(R(t)-1)R(y)}{t} = \frac{(R(t)-R(0))R(y)}{t}.$$

Taking the limit as $t \to 0$

$$\begin{aligned} R'(y) &= R'(0)R(y) \\ R(y) &= \exp(yR'(0)) = \exp(-\lambda y) \end{aligned}$$

which shows that the lifetime $X \sim Exp(\lambda)$.
If a component has exponential lifetime distribution it follows that

1. A replacement policy of used components based on the lifetime of the components is useless.
2. In estimating mean life and reliability the age of the observed components are of no concern. The number of hours of observed live and the number of observed failures are of interest.

Definition

Increasing (decreasing) failure rate distribution
Let X be the lifetime of a component and $F_X(t)$ the corresponding distribution function. If its failure rate $h(t)$ is an increasing (decreasing) function of t for $t \geq 0$ then F_X is an Increasing (Decreasing) Failure Rate distribution: IFR (DFR) distribution.

The behavior of the failure rate $h(t)$ as a function of age is known as the *mortality curve*, *hazard function*, *life characteristic* or *lambda characteristic*.

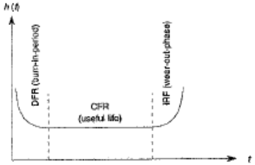


Figure 3.6. Failure rate as a function of time

The hypoexponential distribution is used to model processes that can be divided into sequential phases such that the time the process spends in each phase is independent and exponentially distributed.

- Service times for input-output operations in a computer system often follow this distribution.

A two stage hypoexponential random variable $X \sim Hypo(\lambda_1, \lambda_2)$ has pdf and distribution function equal to

$$
\begin{aligned}
f(t) &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(\exp(-\lambda_1 t) - \exp(\lambda_2 t)), \quad t > 0 \\
F(t) &= 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}\exp(-\lambda_1 t) + \frac{\lambda_1}{\lambda_2 - \lambda_1}\exp(-\lambda_2 t)
\end{aligned}
$$

When r sequential phases have identical exponential distribution the resulting density is known as $r-$stage Erlang and is given by

$$
\begin{aligned}
f(t) &= \frac{\lambda^r t^{r-1} \exp(-\lambda t)}{(r-1)!} \text{ with } t > 0,\ \lambda > 0,\ r = 1, 2, \ldots \\
F(t) &= 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} \exp(-\lambda t) \text{ with } t \geq 0,\ \lambda > 0,\ r = 1, 2, \ldots
\end{aligned}
$$

Suppose that a process consists of k alternate phases, i.e. during any single experiment the process experiences one and only one of the many alternate phases, and thes phases have exponential distributions. The overall distribution is then hyperexponential with density and distribution functions given by

$$\begin{aligned} f(t) &= \sum_{i=1}^{k} \alpha_i \lambda_i \exp(-\lambda_i t) \text{ with } t > 0,\ \lambda_i > 0,\ \sum_{i=1}^{k} \alpha_i = 1 \\ F(t) &= \sum_{i} \alpha_i (1 - \exp(\lambda_i t)) \ \ t \geq 0 \end{aligned}$$

The Weibull distribuiton is the most widely used parametric family of failure distributions. It has been used to describe

- fatigue failure
- electronic component failure
- ballbearing failure

The reason is that by a proper choice of the shape parameter α we can obtain an IFR, DFR or constant failure rate distribution. The corresponding density and distribution functions are given by

$$\begin{aligned} f(t) &= \lambda \alpha t^{\alpha-1} \exp(-\lambda t^{\alpha}) \\ F(t) &= 1 - \exp(-\lambda t^{\alpha}) \end{aligned}$$

where $t \geq 0$, $\lambda > 0$ and $\alpha > 0$

The Normal distribuiton is the most known continuous distribution. It is very important from a theoretic point of view because the asymptotic distribution of many common estimators and statistics is Normal even if the underlying observations are sampled from a non normal distribution. It is characterized by a density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty.$$

The mean, the variance, and the characteristic function are

$$E[X] = \mu, \quad Var[X] = \sigma^2, \quad N_X(\tau) = e^{i\tau\mu - \tau^2\sigma^2/2}$$

respectively.

Functions of random variables

- Let X be a continuous random variable with distribution function F_X, Ψ a function and
$$Y = \Psi(X).$$
- Under regularity conditions on Ψ, Y is a random variable!
- Continuity or stepwise continuity of Ψ are sufficient conditions for Y to be a random variable.

Example: Quadratic cost function.

Let X denote a measurement error. We assume a quadratic cost function, i.e. $Y = \Psi(X) = X^2$. The random variable Y has a distribution function F_Y which depends on F_X and Ψ.

Functions of random variables

- To derive F_Y simply compute the preimage of the event $C = (-\infty, y]$. In fact by definition of F_Y

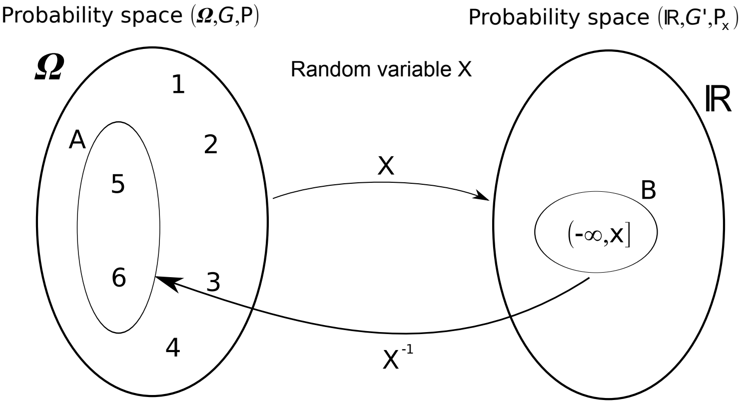
$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(Y \in C) \\ &= P(\Psi(X) \in C) \\ &= P(X \in \Psi^{-1}(C)) \\ &= P_X(B) \end{aligned}$$

  where the set B is the preimage of C, i.e.

$$B = \Psi^{-1}(C) = \{x \in \mathbb{R} \,|\Psi(x) \in C\}.$$

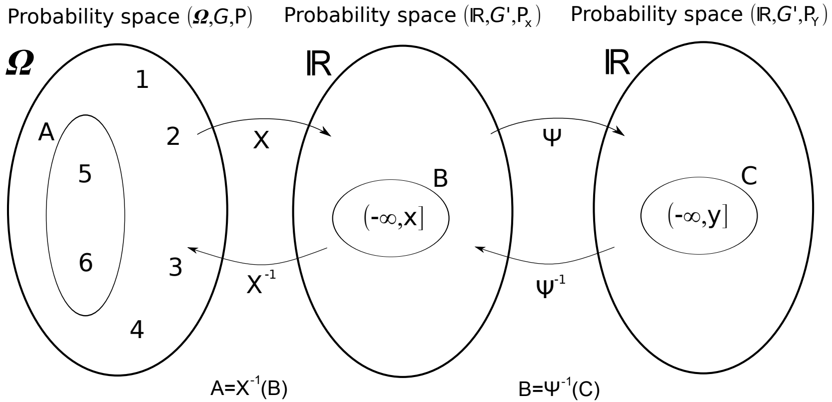
- The preimage function of Ψ is a set function (the arguments are subsets of $\mathbb{R}$) and is defined even if Ψ is not one-to-one.

Functions of random variables



Induced probability $P_X(B) := P(A)$

Functions of random variables



Induced probability $P_Y(C) = P_X(B) = P(A)$

Example continued

Because $Y = X^2$ is always positive $F_Y(y) = 0$ whenever $y \leq 0$. When $y > 0$ it follows $Y \leqslant y \Leftrightarrow -\sqrt{y} \leqslant X \leqslant \sqrt{y}$ so that $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$. If the random variable X has a density function we can differentiate the last expression to obtain the density function of Y:

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right], & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise

Let X *be uniformly distributed on* $(0,1)$ *and define* $Y = -\lambda^{-1} ln(1-X)$ *where* $\lambda > 0$ *is a parameter. Show that* Y *has an exponential distribution with parameter* λ.

Functions of random variables

Theorem

Let X be a continuous random variable with density function f_X satisfying

- $f_X > 0$ *for* $x \in I \subset \mathbb{R}$ *and*
- $f_X = 0$ *for* $x \notin I$

and let Φ be a differentiable and monotone real valued function with domain I. Then $Y = \Phi(X)$ is a continuous random variable with density function

$$f_Y(y) = \begin{cases} f_X\left[\Phi^{-1}(y)\right]\left[\left|\frac{\partial}{\partial y}\left(\Phi^{-1}\right)(y)\right|\right], & y \in \Phi(I), \\ 0, & \textit{otherwise.} \end{cases}$$

Functions of random variables

Examples

1) Let Φ be the distribution function F of the random variable X with density function f (we need to assume that F has the previous properties of continuity and differentiability) and define $Y = F(X)$. The random variable Y has density given by

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

2) Assume $Y = aX + b$, i.e. Y is a affine linear trasformation of X. By the previous Theorem we have that (I is the set over which $f_X \neq 0$)

$$f_Y(y) = \begin{cases} \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right), & y \in aI + b, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise

1. *In the second part of the previous Example assume that* $X \sim N(\mu, sigma^2)$ *and derive the density function of* Y. *What do you observe?*
2. *Let as before* X *be* $N(\mu, sigma^2)$ *and assume* $Y = e^X$. *Derive the density and distribution function of* Y.
3. *Show that if* X *has the* $k-$*stage Erlang distribution with parameter* λ, *then* $Y = 2\lambda X$ *has the* $chi-$*square distribution with* $2k$ *degrees of freedom.*

Jointly distributed random variables

Definition. *If* $\mathbf{X} = (X_1, \ldots, X_n)$ *is a* n*-dimensional random variable* ($n \geq 1$), *its joint distribution function is defined by*

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = P[X_1 \leq x_1, \ldots, X_n \leq x_n], \text{ for } x_i \in \mathbb{R}, i = 1, \ldots, n.$$

In the case that $\mathbf{X}$ *is a multivariate continuous random variable, then*

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f_{\mathbf{X}}(y_1, \ldots, y_n) dy_1 \ldots dy_n,$$

where $f_{\mathbf{X}}(x_1, \ldots, x_n)$ *is the joint probability density function of* $\mathbf{X}$, *that is, by definition,*

(i) $f_{\mathbf{X}}(x_1, \ldots, x_n) \geq 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *and* (ii) $\int_{\mathbb{R}^n} f_{\mathbf{X}}(y_1, \ldots, y_n) dy_1 \ldots dy_n = 1.$

In most cases we are interested only in the distribution of a subset of the n variables. For this reason, we introduce the marginal density function.

Jointly distributed random variables

Definition. *If* $\mathbf{X} = (X_1, \ldots, X_n)$ *is a continuous* n*-dimensional random vector, the distribution function of any subset of* $X_1, \ldots, X_n$ *is continuous and its probability density function is called a marginal density function. In particular, the marginal density of* $X_1, \ldots, X_i$ *for* $i \in \{1, \ldots, n\}$ *equals*

$$f_{X_1,\ldots,X_i}(x_1, \ldots, x_i) = \int_{\mathbb{R}^{n-i}} f_{\mathbf{X}}(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n) dy_{i+1} \ldots dy_n.$$

In the bivariate case $n = 2$ we have that

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, y) dy \quad \text{and}$$
$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y, x_2) dy.$$

In the case $n = 3$ we have that, for example,

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, y_2, y_3) dy_2 dy_3 \quad \text{or}$$
$$f_{X_1,X_3}(x_1, x_3) = \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, y, x_3) dy.$$

Example. Let we consider a three-dimensional vector $\mathbf{X} = (X_1, X_2, X_3)$ having the following joint density function

$$f_{\mathbf{X}}(x_1, x_2, x_3) = \begin{cases} 6x_1x_2^2x_3, & \text{if } 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq \sqrt{2} \\ 0, & \text{otherwise.} \end{cases}$$

We show that $f_{\mathbf{X}}(x_1, x_2, x_3)$ is a probability density function. First note that $f_{\mathbf{X}}(x_1, x_2, x_3) \geq 0$ for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. Next,since

$$\int_{\mathbb{R}^3} f_{\mathbf{X}}(x_1, x_2, x_3)dx_1dx_2dx_3 = \int_0^{\sqrt{2}}\int_0^1\int_0^1 6x_1x_2^2x_3dx_1dx_2dx_3 = 1,$$

by definition $f_{\mathbf{X}}(x_1, x_2, x_3)$ is a probability density function.
We then compute the marginal densities of X_1, X_2, X_3 and (X_1, X_3).

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, y_2, y_3)dy_2dy_3 \\ &= \begin{cases} \int_0^{\sqrt{2}}\int_0^1 6x_1y_2^2y_3dy_2dy_3 = 2x_1, & \text{if } 0 \leq x_1 \leq 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Jointly distributed random variables

Similarly,

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{X}}(y_1, x_2, y_3)dy_1dy_3$$
$$= \begin{cases} \int_0^{\sqrt{2}}\int_0^1 6y_1x_2^2y_3dy_1dy_3 = 3x_2^2, & \text{if } 0 \leq x_2 \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$
$$f_{X_3}(x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{X}}(y_1, y_2, x_3)dy_1dy_2$$
$$= \begin{cases} \int_0^1\int_0^1 6y_1x_2^2y_3dy_1dy_2 = x_3, & \text{if } 0 \leq x_3 \leq \sqrt{2} \\ 0, & \text{otherwise,} \end{cases}$$

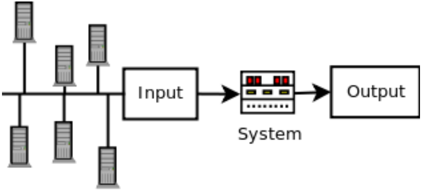
and

$$f_{X_1,X_3}(x_1, X_3) = \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, y_2, x_3)dy_2$$
$$= \begin{cases} \int_0^1 6x_1y_2^2x_3dy_2 = 2x_1x_3, & \text{if } 0 \leq x_1 \leq 1, 0 \leq x_3 \leq \sqrt{2} \\ 0, & \text{otherwise.} \end{cases}$$

Note that we can test whether the calculations are correct verifying that $\int_{\mathbb{R}} f_{X_i}(x_i)dx_i = 1$ for all $i = 1, 2, 3$ and $\int_{\mathbb{R}^2} f_{X_1,X_3}(x_1, x_3)dx_1dx_3 = 1$ since all these density functions must be probability density functions.

- We are interested in simulating a model of a real system (network, electronic device, ...).
- The output Y of the model depends on a stochastic input, i.e. a random variable X with known distribution function F_X:

$$Y = g(X).$$

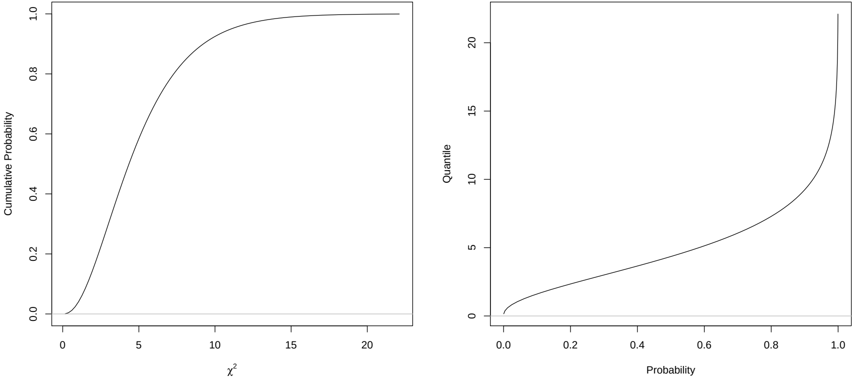


Simulation of random variables

- The model is too complex in order to analytically derive the probabilistic properties of the output, i.e. F_Y.
- Idea: simulate a possible outcome x of the input X and evaluate the corresponding outcome $y = g(x)$ of the output. Repeat the experiment N times and analyse the results.
- The simulated values of the input must be drawn from the distribution of X.
- Question: how is it possible to simulate independent realizations from a given distribution F_X?
- Answer: different methods available. The simplest of them requires simulating from the uniform distribution on the open interval (0,1), i.e. $U(0,1)$. In Matlab use the function “rand”.

If the distribution function F_X is continuous and strictly increasing then $F_X^{-1}: (0,1) \to \mathbb{R}$ exists.

Example: Chi-Squared Distribution with 5 degrees of freedom



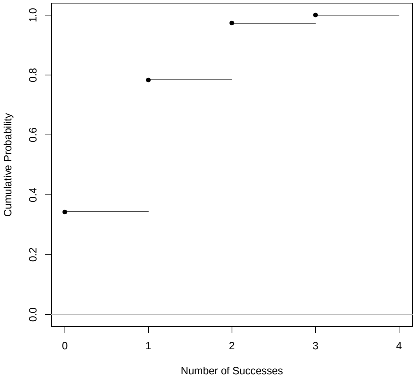
Simulation of random variables

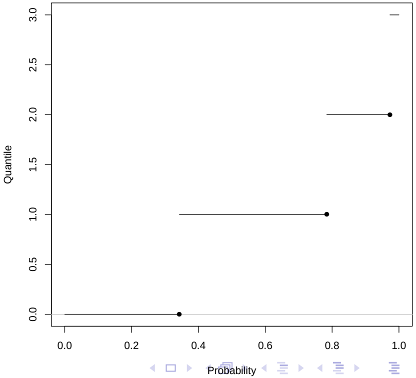
For a general distribution function not necessarily strictly increasing define

$$F_X^{-1}(p) = \inf\{x : p \leq F_X(x)\} \quad 0 < p < 1.$$

It then follows that $F_X^{-1}(p) \leq x \Longleftrightarrow p \leq F_X(x)$.

Example: Binomial distribution $Bin(n = 3, p = 0.3)$





Theorem

[Inverse Transform Method] Let U a continuous Unif(0,1) distributed random variable. The random variable $Y = F_X^{-1}(U)$ has distribution function F_X.

Proof.

By definition

$$F_Y(c) := P(Y \leq c) = P(F_X^{-1}(U) \leq c).$$

But the last equality is equivalent to (see previous slide)

$$P(U \leq F_X(c)) = F_X(c).$$

□

From the previous theorem we derive the following *two steps* simulation algorithm

1. Simulate a realization u from a Unif(0,1) random variable U.
2. Compute $x = F_X^{-1}(u)$.

Example

Simulation of $Y \sim Exp(\lambda)$

- The distribution function is $F_Y(x) = 1 - \exp(-\lambda x)$
- Compute $F_Y^{-1}(p) = -\frac{1}{\lambda}\ln(1-p)$
- Sample a random draw u from $U \sim$ Unif(0,1)
- Set $y = -\frac{1}{\lambda}\ln(1-u)$

Remark: if $U \sim$ Unif(0,1) then $1 - U$ is also Unif(0,1). Therefore we can also write $Y = -\frac{1}{\lambda} \ln(U)$.

For a descrete random variable Y with probability mass function $P(Y = x_i) = p_i$, $i = 1, ..., m$ consider the following algorithm:

1. Generate a Unif(0,1) random variable U
2. Compute Y as follows

$$Y = x_j \quad \text{if} \sum_{i=1}^{j-1} p_i < U \leq \sum_{i=1}^{j} p_i .$$

i.e. $Y = x_j$ if $F_Y(x_{j-1}) < U \leq F_Y(x_j)$.

Example

Let Y have the following mass function

p_1	p_2	p_3	p_4
0.1	0.2	0.4	0.3

.

The simulation algorithm is the following:

1. If $u \leq p_1 = 0.1$ then $y = x_1$. Stop.
2. if $u \leq p_1 + p_2 = 0.3$ then $y = x_2$. Stop.
3. if $u \leq p_1 + p_2 + p_3 = 0.7$ then $y = x_3$. Stop.
4. $y = x_4$. Stop.

This algorithm is correct but inefficient. In fact, probabilities of one, two, ... comparisons are equal to the probabilities of x_1, x_2, ..., respectively. The expected number of comparisons is

$$p_1 + 2p_2 + 3p_3 + 4p_4 = 2.9.$$

We can improve efficiency by sorting the values of x_i by decreasing order of probabilities p_i's: x_3, x_4, x_2 and x_1.

1. If $u \leq p_3 = 0.4$ then $y = x_3$. Stop.
2. if $u \leq p_3 + p_4 = 0.7$ then $y = x_4$. Stop.
3. if $u \leq p_3 + p_4 + p_2 = 0.9$ then $y = x_2$. Stop.
4. $y = x_1$. Stop.

The expected number of comparisons is now equal to

$$p_3 + 2p_4 + 3p_2 + 4p_1 = 2.$$

The Laplace distribution is a continuous distribution with density function

$$f(x;\mu,b)=\begin{cases}\frac{1}{2b}\exp(-\frac{\mu-x}{b}) & \text{if } x<\mu\\ \frac{1}{2b}\exp(-\frac{x-\mu}{b}) & \text{if } x\geq\mu\end{cases}$$

where μ is the mean and b a scale parameter.

1. Plot the density, distribution and quantile functions of the Laplace distribution with parameter $\mu=1$ and $b=0.5$.
2. Using the previous values of μ and b simulate $N=1000$ independent realizations of a Laplace distributed random variable Y.
3. Plot the histogram of the simulated random variables and compare it with the density function of Y.
4. Plot the empiric distribution function of the simulated sample and compare it with the distribution function of Y.

The theorem on *Distributions of functions of continuous random variables* allows us to generate random variables by means of ad hoc transformations of Unif(0,1) random variables. The following theorem generalizes the previous theorem to the multivariate case.

Theorem

Assume that $X = (X_1, \ldots, X_n)$ is a random vector with joint density function $f_X(x_1, \ldots, x_n)$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ a one-to-one and continuously differentiable function. Define $Y = (Y_1, \ldots, Y_n) = g(X)$. The joint density function of Y is then equal to

$$f_Y(y) = f_X(g^{-1}(y))\,|J(g^{-1}(y))|$$

where

$$J(g^{-1}(y)) = \det(M_J) = \det\left[\frac{\partial x_i(y)}{\partial y_j}\right]_{i=1,n;\ j=1,n}$$

Example

Let $X \sim N(\mu, \Sigma)$ be a 2×1 random vector and consider the affine transformation

$$Y = b + A\,X$$

where b is a deterministic 2×1 vector and A a 2×2 invertible matrix. We then have

$$X = A^{-1}(Y - b);\; M_J = A^{-1}$$

and $f_Y(y)$ is equal to

$$\frac{1}{\sqrt{2\pi det(\Sigma)}} \exp(-\frac{1}{2}(A^{-1}(Y-b)-\mu)'\Sigma^{-1}(A^{-1}(Y-b)-\mu))|det(A^{-1})|$$

$$\frac{1}{\sqrt{2\pi det(A\Sigma A')}} \exp(-\frac{1}{2}(Y-b-A\mu)'(A\Sigma A')^{-1}(Y-b-A\mu))$$

Example continued

Looking at the density function of Y we note that $Y \sim N(\tilde{\mu}, \tilde{\Sigma})$ with $\tilde{\mu} = b + A\mu$ and $\tilde{\Sigma} = A\Sigma A'$.

Exercise

U_1 and U_2 are two independent Unif(0,1) distributed random variables. Define $Y = (Y_1, Y_2)$ with

$$Y_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \text{ and } Y_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$$

Show that $Y \sim N(0, I)$, i.e. Y_1 and Y_2 are independent standard normal distributed random variables.

Exercise

The joint density of the random variables X_1 *and* X_2 *is*

$$f(x_1, x_2) = 2\exp(-x_1)\exp(-x_2), \text{ for } 0 < x_1 < x_2 < \infty$$

and $f(x_1, x_2) = 0$ *otherwise.*
We define the transformation

$$Y_1 = 2X_1, Y_2 = X_2 - X_1.$$

Find the joint density of Y_1 *and* Y_2*. Are* Y_1 *and* Y_2 *independent?*

Expectation

Definition

The expectation, $E[X]$, of a random variable X is defined by

$$E[X] = \begin{cases} \sum_i x_i p(x_i), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} x f(x) dx, & \text{if } X \text{ is continuous,} \end{cases}$$

provided the relevant sum or integral is absolutely convergent, i.e. $\sum_i \mid x_i \mid p(x_i) < \infty$ and $\int_{-\infty}^{\infty} \mid x \mid f(x) dx < \infty$.

Example

Assume X is Binomial di stributed with $n = 5$ and $p = 0.5$. Then

$$E[X] = \sum_i x_i p(x_i) = 0 \cdot \frac{1}{32} + 1 \cdot \frac{5}{32} + 2 \cdot \frac{10}{32} + 3 \cdot \frac{10}{32} + 4 \cdot \frac{5}{32} + 5 \cdot \frac{1}{32} = 2.5$$

The expected value need not correspond to a possible value of X!

Expectation

The expected value is a weighted average and it denotes the “center” of a probability mass or density function in the sense of a center of gravity.

Expectation

Let X be a random variable, and define $Y = g(X)$. Suppose we want to compute $E[Y]$. In order to apply the definition of $E[Y]$ we need to derive the *pmf* (or the *pdf*) of Y. An easier method is to use the following result

$$E[Y] = E[g(X)] = \begin{cases} \sum_i g(x_i) p_X(x_i), & \text{if } X \text{ is descrete,} \\ \int_{-\infty}^{\infty} g(x) f_X(x) dx, & \text{if } X \text{ is continuous,} \end{cases}$$

privided the sum or the integral is absolutely convergent, i.e.

$$\sum_i |\, g(x_i) \,|\, p_X(x_i) < \infty \text{ or } \int_{-\infty}^{\infty} |\, g(x) \,|\, f_X(x) dx < \infty.$$

Expectation

Theorem. Properties of Expectation. *If c is a constant and $g(X), g_1(X)$ and $g_2(X)$ are functions whose expectations exist, then*

(i) $E[c] = c$;

(ii) $E[cg(X)] = cE[g(X)]$;

(iii) $E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$;

(iv) $E[g_1(X)] \leq E[g_2(X)]$ *if* $g_1(x) \leq g_2(x)\ \forall x$;

(v) $\mid E[g(X)] \mid \leq E[\mid g(X) \mid]$;

(vi) $E[\mid g(X) \mid] = 0$ *then it follows that* $g(x) = 0$ *for all* x *with positive probability.*

(vii) *If X is a random variable with distribution function $F(x)$ then $E[X]$ exists if and only if $\int_0^\infty (1 - F(x))dx$ and $\int_{-\infty}^0 F(x)dx$ are finite, in which case*

$$E[X] = \int_0^\infty (1 - F(x))dx - \int_{-\infty}^0 F(x)dx.$$

Example. Suppose that X has the following distribution (called Laplace distribution)

$$f_X(x) = \frac{1}{2}\exp(- \mid x \mid /2), \; -\infty < x < +\infty.$$

Then the average value $E[X]$ exists since

$$\int_0^\infty x/2 \exp(- \mid x \mid /2)dx = \int_0^\infty x/2 \exp(-x/2)dx = 2.$$

Similarly, $\int_{-\infty}^0 x/2 \exp(- \mid x \mid /2)dx = -2$ and

$$E[X] = \int_{\mathbb{R}} x f_X(x)dx = \int_{-\infty}^0 f_X(x)dx + \int_0^\infty x f_X(x)dx = -2 + 2 = 0.$$

Moments

Definition

A special case is the power function $g(X) = X^k$, $k = 1, 2, 3, \dots$. $E(X^k)$ is known to be the $k-$th moment of the random variable X. The first moment, i.e. $k = 1$, is the ordinary expectation of X.

Sometimes it is usefull to center the origin of measurement, i.e. to work with powers of $X - E[X]$.

Definition

The $k-$th central moment of the random variable X, μ_k, is defined as

$$\mu_k = E[(X - E[X])^k].$$

Moments

The second central moment μ_2 is called the *Variance* of the random variable X, typically denoted by σ^2, is a measure of dispersion. It measure the amount by wich the R.V. X deviates from its expected value.

Definition

The variance of a random variable X is

$$\sigma^2 = \begin{cases} \sum_i (x_i - E[X])^2 p(x_i), & \text{discrete case,} \\ \int_{-\infty}^{\infty} (x - E[X])^2 f(x) dx, & \text{continuous case.} \end{cases}$$

- The variance is a sum of squares and therefore is a nonnegative number.
- The square root of the variance is denoted by σ and is called the standard deviation.

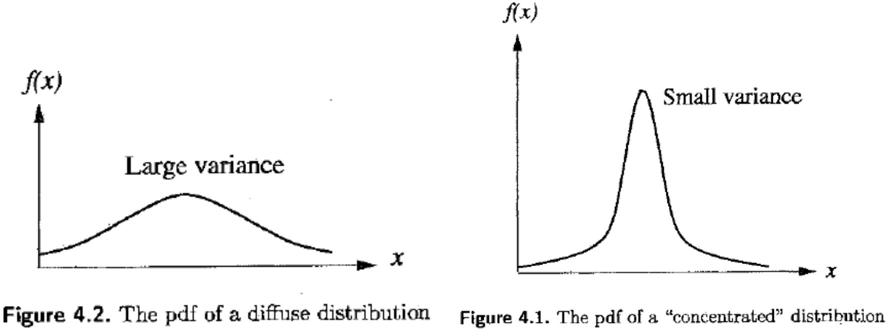


Figure: Small and large variance.

Theorem. Properties of the Variance. *Let $X, X_i, i = 1, \ldots, n$, be independent discrete random variables with finite second moment, and a, b two constants. Then*

(i) $Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$;

(ii) $Var(aX + b) = a^2 Var(X)$;

(iii) $Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} Var(X_i)$.

Note that relaxing the assumption of independence property (iii) does not hold any more (see the following sections).

We define two special measures based on μ_3 and μ_4:

- $n = 3$, μ_3 divided by the standard deviation cubed is called *skewness* of the distribution of X and is often denoted

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = \frac{E[(X - E[X])^3}{\sigma(X)^3}.$$

If the distribution of X is *symmetric about* μ, then $\alpha_3(X) = 0$.

- $n = 4$, μ_4 divided by $\sigma(X)^4$ is called *kurtosis*, is denoted by α_4 and is used as a measure of how "*heavy*" the tails of a distribution are.

Exercise

1) Starting from the definition of σ^2 *use the property of linearity of the expected value to prove that* $Var[X] = E[X^2] - (E[X])^2$.
2) Define the function $g : \mathbb{R} \to [0, \infty]$, $c \mapsto E[(X-c)^2]$. *Show that* $c_{min} = E[X]$ *is the minimum of the function* g.

Examples and Exercises.

Example. Let X be a random variable having a "semitriangular" distribution given by (for some $a > 0$)

$$f_X(x) = \begin{cases} 2(a-x)/a^2, & \text{if } 0 \leq x \leq a \\ 0, & \text{otherwise.} \end{cases}$$

First of all, note that this is a probability density function since

$$(i)\ f_X(x) \geq 0\ \forall x, \text{ and } (ii)\ \int_{\mathbb{R}} f_X(x)dx = \int_0^a \left(\frac{2}{a} - \frac{2x}{a^2}\right)dx = 2 - 1 = 1.$$

We compute $E[X]$ and $Var[X]$.

$$E[X] = \int_{\mathbb{R}} x f_X(x)dx = \int_0^a \left(\frac{2x}{a} - \frac{2x^2}{a^2}\right)dx = \frac{x^2}{a}\Big|_0^a - \frac{2x^3}{3a^2}\Big|_0^a = a - \frac{2}{3}a = \frac{1}{3}a,$$

$$E[X^2] = \int_{\mathbb{R}} x^2 f_X(x)dx = \int_0^a \left(\frac{2x^2}{a} - \frac{2x^3}{a^2}\right)dx = \frac{2x^3}{3a}\Big|_0^a - \frac{x^4}{2a^2}\Big|_0^a = \frac{2}{3}a^2 - \frac{1}{2}a^2 = \frac{1}{6}a^2.$$

Therefore $Var(X) = E[X^2] - E[X]^2 = a^2/6 - a^2/9 = a^2/18$.

As an exercise, compute the median, skewness α_3 and kurtosis α_4 of X.

Example. Mean and Variance of the uniform distribution. Let we consider X uniformly distributed on (b,c), i.e. $f_X(x) = 1/(c-b)$ if $x \in (b,c)$ and 0 otherwise. Then

$$E[X] = \int_b^c \frac{x}{c-b}dx = \frac{x^2}{2(c-b)}\Big|_b^c = \frac{c^2-b^2}{2(c-b)} = \frac{b+c}{2};$$
$$E[X^2] = \int_b^c \frac{x^2}{c-b}dx = \frac{x^3}{3(c-b)}\Big|_b^c = \frac{c^3-b^3}{3(c-b)} = \frac{c^2+bc+b^2}{3}.$$

Hence $Var(X) = E[X^2] - E[X]^2 = (c-b)^2/12$.

Let we also compute the median m of X. m must satisfy

$$P[X \leq m] = \int_b^m \frac{1}{c-b}dx = \frac{m-b}{c-b} = 0.5.$$

Thus, $m = (b+c)/2 = E[X]$.

Exercise. Compute higher order moments of X. In particular, compute skewness and kurtosis of X.

Moments

Example. An urn contains 4 red balls, 1 yellow ball, and 3 green balls. Balls are drawn successively at random *without replacement*, until a green ball is drawn. Let X be the number of draws required. We compute $E[X]$ and $\sigma^2(X)$.

The possible values of X are $1, 2, 3, 4, 5, 6$ with corresponding probabilities $p_X(1) = 3/8, p_X(2) = 5/8 \cdot 3/7, p_X(3) = 5/8 \cdot 4/7 \cdot 3/6, p_X(4) = 5/8 \cdot 4/7 \cdot 3/6 \cdot 3/5, p_X(5) = 5/8 \cdot 4/7 \cdot 3/6 \cdot 2/5 \cdot 3/4, p_X(6) = 5/8 \cdot 4/7 \cdot 3/6 \cdot 2/5 \cdot 1/4$. Hence

$$E[X] = 1 \cdot 3/8 + 2 \cdot 15/56 + 3 \cdot 5/28 + 4 \cdot 3/28 + 5 \cdot 3/56 + 6 \cdot 1/56 = 9/4$$

and

$$E[X^2] = 1 \cdot 3/8 + 4 \cdot 15/56 + 9 \cdot 5/28 + 16 \cdot 3/28 + 25 \cdot 3/56 + 36 \cdot 1/56 = 27/4.$$

Therefore

$$Var(X) = \sigma^2(X) = E[X^2] - E[X]^2 = 27/4 - (9/4)^2 = 27/4 - 81/16 = 27/16 = 1.6875.$$

Example. Mean and Variance of the binomial distribution. To begin, let X be Bernoulli distributed with success parameter p. Then

$$E[X] = 0 \cdot (1-p) + 1 \cdot p = p \text{ and } E[X^2] = 0^2 \cdot (1-p) + 1^2 \cdot p = p.$$

Consequently, $Var(X) = p - p^2 = p(1-p)$.
Now, we know that $Y = X_1 + \ldots + X_n \sim B(n,p)$ for n independent Bernoulli variables X_i with same success parameter p. Thus, the expectation and variance of the binomial variable Y equal

$$E[Y] = \sum_{i=1}^{n} E[X_i] = np \text{ and } Var[Y] = \sum_{i=1}^{n} Var(X_i) = np(1-p).$$

Exercise. 1. Derive the results for the expectation and variance of the binomial distribution *directly* from its definition.

2. Find $E[X]$ and $Var(X)$ for the following distribution

$$p_X(x) = \binom{5}{x} 0.4^x 0.6^{5-x} \text{ if } x \in \{0,1,2,3,4,5\}, \text{ and 0 otherwise.}$$

Moments

Example. Mean and Variance of the Poisson distribution. If X has the Poisson distribution with parameter λ. Then

$$E[X] = \sum_{x=0}^{\infty} x \exp(-\lambda)\lambda^x/x! = \exp(-\lambda) \sum_{x=1}^{\infty} \lambda^x/(x-1)!$$
$$= \exp(-\lambda) \cdot \lambda \sum_{x=1}^{\infty} \lambda^{x-1}/(x-1)! = \exp(-\lambda)\lambda \exp(\lambda) = \lambda;$$
$$E[X^2] = \sum_{x=0}^{\infty} x^2 \exp(-\lambda)\lambda^x/x! = \sum_{x=0}^{\infty} (x(x-1)+x) \exp(-\lambda)\lambda^x/x!$$
$$= \sum_{x=2}^{\infty} \exp(-\lambda)\lambda^x/(x-2)! + \sum_{x=0}^{\infty} x \exp(-\lambda)\lambda^x/x! = \lambda^2 + \lambda.$$

Thus $Var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Exercise. Compute skewness and kurtosis of the Poisson distribution with parameter λ. What is the skewness of the Poisson distribution with expectation equal to 2?
Trick: Compute $E[X^3]$ as $E[X(X-1)(X-2)] + 3E[X(X-1)] + E[X]$.

Example: Mean and Variance of the normal distribution.

Let we consider $X \sim N(\mu, \sigma^2)$. We will show that $E[X] = \mu$ and $Var(X) = \sigma^2$. In fact

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{\sigma y + \mu}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}y^2} \sigma dy \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{1}{2}y^2} dy + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \frac{\sigma}{\sqrt{2\pi}} \left(-e^{-\frac{1}{2}y^2}\right) \Big|_{-\infty}^{\infty} + \mu \cdot 1 = \mu
\end{aligned}
$$

and

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{(\sigma y + \mu)^2}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}y^2} \sigma dy$$
$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-\frac{1}{2}y^2} dy + \frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{1}{2}y^2} dy + \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$
$$= \frac{\sigma^2}{\sqrt{2\pi}} \left(\left(-y e^{-\frac{1}{2}y^2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \right) + 0 + \mu^2 \cdot 1 = \sigma^2 + \mu^2.$$

Thus, $Var(X) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$.

Exercise: Let $X \sim N(\mu, \sigma^2)$. Find the skewness and kurtosis of X. In particular, compute these values when $\mu = 0$ and $\sigma = 1$.

Moments

Exercise: Let $Y = aX^2 + b$ for a,b two constants. Compute the mean and variance of Y for the case

1. $X \sim N(0,1)$;
2. $X \sim N(\mu, \sigma^2)$;
3. X exponentially distributed with parameter λ.
4. A particular container of one brand of a certain food is labeled as containing 500 grams of food. Suppose it is known that the weight of the food in a container selected at random is distributed $N(500, 25)$. A container is considered to be underweight if its net food weight is less than 98% of the label claim, in this case less than 490 grams. If 1000 containers are chosen at random, how many should we expect to be underweight?
5. Let $X \sim N(\mu, \sigma^2)$. Let $Y = exp(X)$. Y is called a lognormal random variable. Find $E[Y]$ and $Var(Y)$.

Expectation based on multiple Random Variables

Let $X_1, X_2, ..., X_n$ be n random variables defined on the same probability space and define $Y = \Phi(X_1, X_2, ..., X_n)$. Then

$$
\begin{aligned}
E[Y] &= E[\Phi(X)] \\
&= \begin{cases} \sum_{x_1} \ldots \sum_{x_n} \Phi(x_1, x_2, ..., x_n) p(x_1, x_2, ..., x_n), & \text{descrete case} \\ \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \Phi(x_1, x_2, ..., x_n) f(x_1, x_2, ..., x_n) dx_1 \ldots dx_n, & \text{contin. case} \end{cases}
\end{aligned}
$$

Remark: It is not necessary to derive the pmf (descrete case) or the pdf (cont. case) of the random variable Y in order to compute its expected value.

Expectation based on multiple Random Variables

Recall the linearity property of expectation:

1. $E[\lambda X] = \lambda E[X]$, where $\lambda \in \mathbb{R}$,
2. $E[X+Y] = E[X] + E[Y]$.

If function Ψ is linear, we than obtain that

$$E[Y] = E[\Phi(X)] = E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i].$$

Theorem

If X and Y are independent random variables, then $E[XY] = E[X]E[Y]$. The converse does not hold, i.e. it is possible that two random variables X and Y satisfy $E[XY] = E[X]E[Y]$ without being indendent.

Expectation based on multiple Random Variables

Definition

Let X and Y be two random variable. The covariance between X and Y is defined to be

$$Cov(X,Y) = E[(X - E[X])(Y - E(Y)].$$

The covariance is a measure of linear dependence between two random variables. If $Cov(X,Y) = 0$ we say that X and Y are uncorrelated.

- The covariance is a *bilinear* operator. In fact let X,Y and Z be two random variables with existing expectation and $\lambda \in \mathbb{R}$. Then
  1. $Cov(X+Y,Z) = Cov(X,Z) + Cov(Y,Z)$,
  2. $Cov(\lambda X,Z) = \lambda Cov(X,Z)$,
  3. $Cov(X,Z) = Cov(Z,X)$.

  General rule: freeze the first (second) argument and consider the covariance an expectation in the second (first) argument.

Expectation based on multiple Random Variables

Remark: Zero covariance does not imply that the two random variables are independent!

Example

Let X be uniformly distributed over the interval $(-1, 1)$ and let $Y = X^2$ so that Y is completely dependent on X. Noting that for all odd values of $k > 0$, the $k-$th moment $E[X^k] = 0$, we have (see part 2 of next Exercise for the equality $Cov(X, Y) = E[XY] - E[X]E[Y]$)

$$Cov(X, Y) = E[XY] - E[X]E[Y] = E[X^3] - 0E[Y] = 0.$$

Expectation based on multiple Random Variables

Exercise

1. *Starting from the definition of* Cov *use the property of linearity of the expected value to prove property 1. and 2. of the covariance.*
2. *Using the linearity of the expected value show that* $Cov(X,Y) = E[XY] - E[X]E[Y]$.
3. *Show that if* X *and* Y *are two independent random variables, then* $Cov(X,Y) = 0$.
4. *Show that* $Cov(X,X) = Var(X)$. *Hint: simply start from the definition of* Cov.
5. *Show that* $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.
6. *Let* $X \sim N(0,1)$ *and* Y *a random variable independent from* X *and such that* $P(Y=1) = P(Y=-1) = 0.5$. *Finally define* $Z = X \cdot Y$. *Show that* $Cov(X,Z) = 0$. *Are* X *and* Z *independent random variables?*

Expectation based on multiple Random Variables

Definition

Correlation: the correlation coefficient $\rho(X, Y)$ between the random variables X and Y is defined by

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\, Var(Y)}}$$

Remark: the correlation coefficient satisfies the following inequalities: $-1 < \rho(X, Y) < 1$.

- Transform methods are transformations of the probability mass function (discrete case) or the density function (continuous case).
- They are particular useful to compute moments of a distribution and in problems involving sums of independent random variables.

Definition

The moment generating function (MGF) $M_X(\theta)$, abbreviated $M(\theta)$, of the random variable X is defined by

$$M(\theta) = E\left[\exp(X\theta)\right]$$

provided the expectation exists ($M(\theta)$ may not exist for all $\theta \in \mathbb{R}$).

Definition

The characteristic function of a random variable X is given by

$$N_X(\tau) = N(\tau) = E\left[\exp(iX\tau)\right] = M_X(i\tau) \text{ where } i = \sqrt{-1}.$$

Note that $N_X(\tau)$ is always defined for any X and all τ.

Definition

Let X be a nonnegative continuous random variable. The Laplace - Stieltjes transform of X is

$$L_X(s) = L(s) = M_X(-s) = \int_0^\infty \exp(-sx)f(x)dx.$$

Definition

Let X be a discrete nonnegative integer-valued random variable. The z transform (or probability generating function) of X is defined as

$$G_X(z) = G(z) = E\left[z^X\right] = M_X(ln(z)) = \sum_{i=0}^{\infty} p_X(i) z^i.$$

Theorem

Affine transformation. Let $Y = aX + b$. *Then*

$$M_Y(\theta) = \exp(b\theta) M_X(a\theta)$$

Theorem

[The Convolution Theorem] Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables. Define $Y = \sum_{i=1}^{n} X_i$. If $M_{X_i}(\theta)$ exists for all i, then $M_Y(\theta)$ exists, and

$$M_Y(\theta) = \prod_{i=1}^{n} M_{X_i}(\theta).$$

Theorem

[Uniqueness Theorem] If $M_X(\theta) = M_Y(\theta)$ for all θ, then $F_X = F_Y$, i.e. X and Y have the same distribution.

Theorem

[Moment generating property of the MGF] Let X be a random variable such that all moments exist. Then

$$E\left[X^k\right] = \frac{\partial^k M_X}{\partial \theta^k}\Big|_{\theta=0} \quad k = 1, 2, \ldots$$

Proof.

$$\exp(X\theta) = 1 + X\theta + \frac{X^2\theta^2}{2!} + \cdots + \frac{X^k\theta^k}{k!} + \ldots$$

Taking expectation on both sides

$$M_X(\theta) = E\left[\exp(X\theta)\right] = 1 + E\left[X\right]\theta + \frac{E\left[X^2\right]\theta^2}{2!} + \cdots + \frac{E\left[X^k\right]\theta^k}{k!} + \ldots$$

$\square$

The corresponding properties for the characteristic function N_X, the Laplace - Stieltjes transform L_X and the z transform G_X are

$$E\left[X^k\right] = (-i)^k \frac{\partial^k N_X}{\partial \tau^k}|_{\tau=0} \quad k = 0, 1, \ldots$$

$$E\left[X^k\right] = (-1)^k \frac{\partial^k L_X}{\partial s^k}|_{s=0} \quad k = 0, 1, \ldots$$

$$E\left[\frac{X!}{(X-k)!}\right] = \lim_{\tilde{z}\uparrow 1} \frac{\partial^k G_X}{\partial z^k}|_{z=\tilde{z}} \quad k = 0, 1, \ldots$$

respectively, where $\left[\frac{X!}{(X-k)!}\right] = X(X-1)\ldots(X-k+1)$.

Finally, let X be a discrete nonnegative integer-valued random variable with z transform G_X. The probability mass function of X can be recovered by taking derivatives of G_X:

$$p_k = P(X = k) = \frac{1}{k!}\frac{\partial^k G_X}{\partial z^k}|_{z=0}$$

Examples: Let X be exponentially distributed with parameter λ. Then

$$f_X(x) = \lambda \exp(-\lambda x),\ x > 0.$$

$$\begin{aligned} L_X(s) &= \int_0^\infty exp(-sx)\lambda exp(-\lambda x)dx \\ &= \frac{\lambda}{s+\lambda}\int_0^\infty (\lambda + s)exp(-(\lambda + s)x)dx \\ &= \frac{\lambda}{s+\lambda}. \end{aligned}$$

Example (continued):

$$E[X] = (-1)\frac{\partial L_X}{\partial s}|_{s=0} = (-1)\frac{-\lambda}{(\lambda+s)^2}|_{s=0} = \frac{1}{\lambda}.$$

$$E\left[X^2\right] = \frac{\partial^2 L_X}{\partial s^2}|_{s=0} = \frac{2\lambda}{(\lambda+s)^3}|_{s=0} = \frac{2}{\lambda^2}.$$

Example: Let X be a n trials Binomial distributed random variable with probability of success p. The z transform of X is by definition

$$\begin{aligned} G_X(z) &= E(z^X) = \sum_{k=0}^{n} z^k \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k} \\ &= (pz+1-p)^n \end{aligned}$$

Exercise:
Let X be a Bernoulli distributed random variable with probability of success p.

1. Compute the MGF M_X.
2. Compute skewness and kurtosis of X.

Exercise:
Let $X_1, X_2, \ldots, X_n$ a sequence of independent Bernoulli distributed random variables.

1. Compute the moment generating function of $Y = \sum_{i=1}^{n} X_i$.
2. Show that M_Y is the MGF of a Bernoulli (n, p) distributed random variable.

Exercise:
Let X be a standard normally distributed random variable.

1. Compute the MGF of X.
2. Compute the Kurtosis of X.
3. Define $Y = \sigma X + \mu$. Derive the MGF of Y and compute its expected value and variance.

Exercise:
Let X be a geometric distributed random variable with probability mass function $p_X(i) = p(1-p)^i$, $i = 1, 2, ...$

1. Compute the z transform of X.
2. Compute the skewness of X.

Exercise:
Let X be a continuous $\text{Unif}(a, b)$ distributed random variable with $0 \leq a < b$.

1. Compute the MGF and the Laplace - Stieltjes transform of X.
2. Compute skewness and kurtosis of X.

Mean time to failure

Let X denote the lifetime of a component. The reliability of the component is defined to be $R(t) = P(X > t) = 1 - F(t)$ so that $R'(t) = -f(t)$.

Definition

The expected life or the mean time to failure (MTTF) of the component is given by

$$E[X] = \int_0^\infty tf(t)dt = -\int_0^\infty tR'(t)dt.$$

Because of Property vii) of the Expectation

$$E[X] = \int_0^\infty (1 - F(t))dt - \int_{-\infty}^0 F(t)dt$$

we obtain the alternative expression

$$E[X] = \int_0^\infty R(t)dt.$$

Mean time to failure

More generally, using partial integration, we derive the following general result.

$$
\begin{aligned}
E[X^k] &= \int_0^\infty t^k f(t)dt \\
&= -\int_0^\infty t^k R'(t)dt \\
&= -t^k R(t)\,|_0^\infty + \int_0^\infty kt^{k-1} R(t)dt.
\end{aligned}
$$

Because $lim_{t\to\infty} t^k R(t) = 0$ we obtain the final formula

$$E[X^k] = \int_0^\infty kt^{k-1} R(t)dt.$$

In particular

$$Var(X) = \int_0^\infty 2tR(t)dt - \left[\int_0^\infty R(t)dt\right]^2.$$

Mean time to failure: Series System

The system reliability of a series system with n independent components is

$$R(t) = \prod_{i=1}^{n} R_i(t)$$

where $R_i(t)$ denotes the reliability of the ith component. The MTTF of a series system is much smaller that the MTTF of its components. If X_i denotes the lifetime of component i and X the system lifetime, since $0 \leq R_i(t) \leq 1$,

$$R_X(t) = \prod_{j=1}^{n} R_{X_j}(t) \leq R_{X_i}(t) \, \forall i$$

so that

$$\begin{aligned} E[X] &= \int_0^{\infty} R_X(t) dt \leq \int_0^{\infty} R_{X_i}(t) dt \, \forall i \\ &\leq E[X_i] \, \forall i \\ &\leq \min_i \{E[X_i]\} \end{aligned}$$

Mean time to failure: Parallel System

Because $X = \max\{X_1, X_2, \ldots, X_n\}$ and the components are independent

$$F_X(t) = \prod_{i=1}^{n} F_{X_i}(t) = \prod_{i=1}^{n} [1 - R_{X_i}(t)].$$

The system reliability of a parallel system with n independent components is

$$R_X(t) = 1 - F_X(t) = 1 - \prod_{i=1}^{n} [1 - R_{X_i}(t)] \geq 1 - [1 - R_{X_i}(t)] \; \forall i.$$

This implies that the reliability of a parallel redundant system is larger than that of any of its components. Therefore

$$E[X] \quad = \quad \int_0^\infty R_X(t)dt \geq \max_i\{E[X_i]\}.$$

Mean time to failure: Standby Redundancy

- The system has one component operating and $(n-1)$ unpowered spares.
- The failure rate of an operating component is λ. A cold spare does not fail.
- The switching equipment is failure free.
- Define X_i to be the lefetime of the ith component from the point it is put into operation until its failure.

The system lifetime X is

$$X = \sum_{i=1}^{n} X_i,$$

X has an $n-$stage Erlang distribution and

$$E[X] = \frac{n}{\lambda} \text{ and } Var[X] = \frac{n}{\lambda^2}.$$