Statistics

ALaRI Exam — 25 June 2012

- Duration: 1 hour and 30 minutes
- Open book exam
- Solve all exercises

Problem 1

1. Let X and Y be two Bernoulli distributed random variables with parameter p and let their joint probability mass function defined as

	$Y=0$	$Y=1$
$X=0$	P_{00}	P_{01}
$X=1$	P_{10}	P_{11}

Table 1: Joint probability mass function

   (a) Compute the marginal mass function of X and Y as a function of P_{ij}. Write explicitly the relation between the P_{ij} probabilities and p in this general case.

   (b) Prove that $P_{01} = P_{10}$.

   (c) What are the values of P_{00}, P_{01}, P_{10} and P_{11} as a function of p if X and Y are independent?

   (d) We remove now the independence assumption an replace it with the assumption that $Cov(X,Y) = 0$. Show that in this case X and Y are independent, i.e. the probabilities P_{ij} are the same as those calculated in (c). Hint: Compute the covariance as a function of the unknown probabilities. Use the information derived in (a) and (b).

   You have just proved that for two Bernoulli distributed random variables with equal success parameter p

$$Cov(X,Y) = 0 \Leftrightarrow X \text{ and } Y \text{ are independent.}$$

Problem 2

1. Let X and Y two uncorrelated random variables, each having expectation μ and variance σ^2. Show that $X+Y$ is uncorrelated with $X-Y$.

2. The joint probability function of X and Y, $p(x,y)$, is given by

	$Y=1$	$Y=2$	$Y=3$
$X=1$	$p(1,1)=0$	$p(1,2)=0.1$	$p(1,3)=\beta$
$X=2$	$p(2,1)=\alpha$	$p(2,2)=0$	$p(2,3)=0.3$
$X=3$	$p(3,1)=0.1$	$p(3,2)=0.1$	$p(3,3)=0$

Table 2: Joint probability mass function

As a function of the parameters α and β:

(a) compute the marginal probabilities mass functions of X and Y.

(b) Compute the expected values of X and Y .

(c) Compute the covariance between X and Y.

(d) For which values of α and β are X and Y uncorrelated?

Problem 3

Let X an exponential distributed random variable with density function given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0,\ \lambda > 0 \\ 0 & \text{otherwise} \end{cases} .$$

and $Y = X^3$.

1. Compute the density function of Y.

2. Compute $E(Y)$ using the previously derived density.

Problem 4

Let $X_1, \dots, X_n$ be a random sample from a population distributed according to the geometric distribution:

$$p(x) = (1-p)^{x-1} p,\ \ x = 1, 2, 3, \dots$$

1. Which kind of experiment is described by the geometric distribution?

2. Compute the maximum likelihood estimator of the parameter p for a random sample of size n.