Statistics Lecture

Claudio Ortelli [1]

[1]Finance Institute
USI

Advanced Learning and Research Institute, 2012

Bernoulli Trials

Consider a random experiment with only two outcomes, “success” and “failure”, and denote the probability of the two outcomes by p and q, respectively, with $p+q=1$.

Definition

The experiment consisting in observing a sequence of n independent repetitions of the above described experiment is called a sequence of Bernoulli trials.

Examples

1. Observe n consecutive executions of an if statement, with success = “then clause is executed” and failure = ”else clause is executed”.
2. Examine components produced on an assembly line, with success = ”acceptable” and failure = ”defective”.

Bernoulli Trials

Let 0 denote failure and 1 denote success. Let S_n be the sample space of an experiment involving n Bernoulli trials

$$
\begin{aligned}
S_1 &= \{0,1\}, \\
S_2 &= \{(0,0),(0,1),(1,0),(1,1)\}, \\
S_3 &= \{0,1\}^n = \{2^n\, n-\text{tuples of 0s and 1s}\}.
\end{aligned}
$$

For all sample spaces S_n we define the $\sigma-$ algebra $\mathscr{P}(S_n)$ as the relevant $\sigma-$ algebra on which to define the probability P. On S_1 we then have $P(\{0\}) = q$ and $P(\{1\}) = p$. We wish to assign probability to points in S_n.
Define $A_i =$"success on trail i" and $\bar{A}_i =$"failure on trial i". We then have $P(A_i) = p$ and $P(\bar{A}_i) = q$. Let s be an outcome of S_n with k "1" and $n-k$ "0", i.e.

$$
s = (1,1,\ldots,1,0,0,\ldots 0)
$$

Bernoulli Trials

The elementary event $\{s\}$ can be written

$$\{s\} = A_1 \cap A_2 \cap \cdots \cap A_k \cap \overline{A}_{k+1} \cap \cdots \cap \bar{A}_n.$$

Because events A_i are independent we obtain

$$P(\{s\}) = P(A_1)P(A_2)\ldots P(A_k)P(\bar{A}_{k+1})\ldots P(\bar{A}_n)$$

so that $P(\{s\}) = p^k q^{n-k}$. Note that that we can construct $\left(\begin{array}{c} n \\ k \end{array}\right)$ different outcomes with k successes and $n-k$ failures, therefore defining $A =$"we observe exactly k successes in n trials"

$$P(A) = \left(\begin{array}{c} n \\ k \end{array}\right) p^k q^{n-k}.$$

Since by the binomial theorem $(p+q)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) p^k q^{n-k} = 1$, P is a well defined probability law on $(S_n, \mathscr{P}(S_n))$.

Bernoulli Trials

In connection with reliability theory let us assume that a particular system with n components requires at least k components to function in order for the entire system to work correctly. Such systems are called k-out-of-n systems.

- If we let $k = n$ we have a series system.
- If we let $k = 1$ we have a system with parallel redundancy.

Assuming all components are statistically identical and function independently of each other, and denoting by R the reliability of a component ($q = 1 - R$ gives its unreliability), then the experiment of observing the statuses of n components can be thought of as a sequence of n Bernoulli trials with probability $p = R$.

$$
\begin{aligned}
R_{k|n} &= P(\text{"at least } k \text{components functioning properly"}) \\
&= P\Big(\bigcup_{i=k}^{n} \text{"exactly } i \text{components functioning properly"}\Big)
\end{aligned}
$$

Bernoulli Trials

$$R_{k|n} = \sum_{i=k}^{n} P(\text{"exactly } i \text{components functioning properly"}) = \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i}.$$

Example

Triple modular redundancy (TMR).

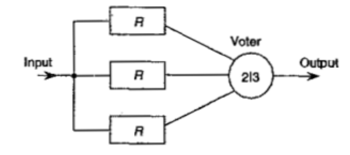


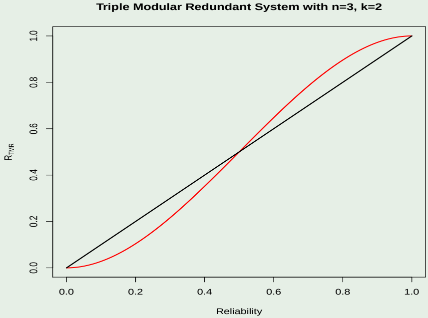
Figure 1.19. A triple modular redundant system

Bernoulli Trials

Example

Applying the above formula with $n = 3$ and $k = 2$:

$$R_{TMR} = \sum_{i=2}^{3} \binom{3}{i} R^i (1-R)^{3-i} = 3R^2(1-R) + R^3.$$



Non homogeneous Bernoulli Trials

When the components are nonhomegeneous w.r.t. the corresponding reliabilities, then the calculation is a bit more complicated:

$$R_{k|n} \quad = \quad 1 - \sum_{|I|>=k} \left(\prod_{i \in I} (1 - R_i) \right) \left(\prod_{i \notin I} R_i \right)$$

where I ranges over all choises $i_1 < i_2 < \cdots < i_m$ such that $k \leq m \leq n$ and R_i denotes the reliability of th $i-$th component.

Example

Consider a non homogeneous TMR with $n = 3$ and $k = 2$.

$$\begin{aligned} R_{2|3} &= 1-(1-R_1)(1-R_2)R_3-(1-R_1)(1-R_3)R_2+ \\ &\quad -(1-R_2)(1-R_3)R_1-(1-R_1)(1-R_2)(1-R_3) \\ &= R_1R_2+R_2R_3+R_1R_3-2R_1R_2R_3 \end{aligned}$$

Generalized Bernoulli Trials

Next, we consider **generalized Bernoulli trials**. Here we have a sequence of n independent trials, and on each trial the result is exactly one of the k possibilities $b_1, b_2, \ldots, b_k$. On a given trial, let b_i occur with probability $p_i, i = 1, 2, \ldots, k$ such that

$$p_i \geq 0 \text{ and } \sum_{i=1}^{k} p_i = 1.$$

The sample space S consists of all k^n n-tuples with components $b_1, b_2, \ldots, b_k$. To a point $s \in S$

$$s = (\underbrace{b_1, b_1, \ldots, b_1}_{n_1}, \underbrace{b_2, b_2, \ldots, b_2}_{n_2}, \ldots, \underbrace{b_k, \ldots, b_k}_{n_k})$$

we assign the probability of $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $\sum_{i=1}^{k} n_i = n$. This is the probability assigned to any n-tuple having n_i occurrences of b_i, where $i = 1, 2, \ldots, k$. The number of such n-tuples are given by the multinomial coefficient [LIU 1968]:

$$\binom{n}{n_1 \; n_2 \; \cdots \; n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

As before, the probability that b_1 will occur n_1 times, b_2 will occur n_2 times, $\ldots$, and b_k will occur n_k times is given by

$$p(n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! n_2! \cdots n_k!} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \qquad (1.23)$$

and

$$\begin{aligned} \sum_{n_i \geq 0} p(n_1, n_2, \ldots, n_k) &= (p_1 + p_2 + \cdots + p_k)^n \\ &= 1 \end{aligned}$$

Generalized Bernoulli Trials

Example

Trivedi, page 52.

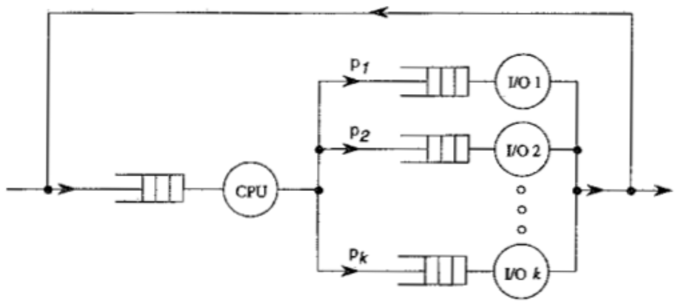


Figure 1.23. A CPU to I/O device queuing scheme

Solve problems 2,3,4 at page 56 and review problem 1 at page 57 of Trivedi.

Random variables

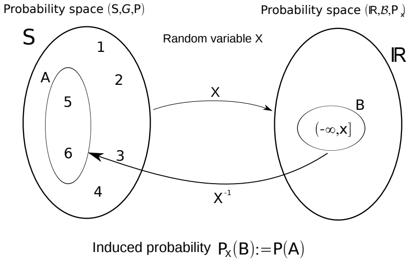
Let $(\Omega, \mathscr{G})$ denote a measurable space with sample space Ω and $\sigma-$algebra $\mathscr{G}$.

Definition

A random variable is a real valued function $X : \Omega \to \mathbb{R}$ such that

$$X^{-1}(B) \in \mathscr{G} \text{ for all events } B \in \mathscr{B}.$$

The following picture clarifies the situation



Random variables

Remarks:

1. A random variable is a function between two measurable spaces satisfying a *measurability* condition: the preimage of any event $B \subset \mathbb{R}$ must be an event of $\mathscr{G}$.
2. S can be finite, countable infinite or uncountable. If the image of X is discrete, i.e. finite or countable, then X is a discrete random variable.

Definition

Let X be a random variable and $x \in \mathbb{R}$ a real number. The event

$$A_x = \{s \in S \mid X(s) = x\}$$

is called the inverse image of the set $\{x\}$ and represents the outcomes of the random experiment which are mapped to x.

Random variables

It is clear that $A_x \cap A_y = \emptyset$ if $x \neq y$ and that

$$\bigcup_{x \in \mathbb{R}} A_x = S.$$

Attention:

1. Unions over an uncountable number of events are not, in general, events (see the definition of $\sigma-$algebra).
2. If the random variable X is discrete, then $\bigcup_{x \in I} A_x$ is an event for all $I \subset \mathbb{R}$. Why?

Notation:

1. The notation $[X = x]$ will be used as an abbreviation for A_x.
2. Similarly, $[X \leq x]$ denotes the event $E = \{s \in S \mid X(s) \leq x\}$. Analogous definitions apply for the other inequality operators.

Random variables

The set of random variables is closed under addition and scalar multiplication, under maximum, minimum, multiplication and division as well as under limit operation.

Theorem

Let X and Y two random variables defined on the same measurable space $(S, \mathscr{G})$. then

1. $aX + bY$ *is a random variable for all* $a, b \in \mathbb{R}$.
2. $\max\{X, Y\}$ *and* $\min\{X, Y\}$ *are random variables.*
3. XY *is random variable.*
4. *Provided that* $Y(s) \neq 0$ *for each* $s \in S$, X/Y *is a random variable.*
5. *Let* $X_1, X_2, \ldots$ *be a sequence of random variables. If*

$$X(s) = \lim_{n \to \infty} X_n(s)$$

exists for every $s \in S$, *the* X *is a random variable.*

Probability mass function

Let X be a discrete random variable and $I \subset \mathbb{R}$ the image set of X. Because X is discrete, set I is countable and

$$\sum_{x \in I} P(X = x) = \sum_{x \in I} P(A_x) = 1.$$

Furthermore, for all $x \notin I$, $P(X = x) = 0$ while for all events $B \in \mathscr{B}$ the event $A_B = [X \in B] = \{s \in S \mid X(s) \in B\}$ so that the probability

$$P(X \in B) = P(\bigcup_{x \in B} A_x) = P(\bigcup_{x \in B \cap I} A_x) = \sum_{x \in B \cap I} P(A_x).$$

The probability of any event $B \in \mathscr{B}$ can be computed as a sum over a countable number of points $x \in B \cap I$.

Definition

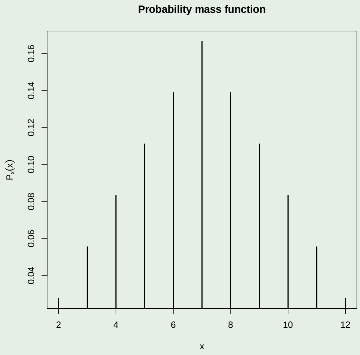
We define the *probability mass function* or *discrete density function* of the random variable X, denoted by p_X, the function

$$p_X(x) = P(A_x) \text{ for all } x \in \mathbb{R}.$$

Probability mass function

Example

A fair die is tossed twice. Let S denote the sample space of this random experiment and define X to be the sum of the outcomes of the first and second toss. The image of X is the set $I = \{2, 3, \ldots, 12\}$. The corresponding probability mass function is given in the following plot:

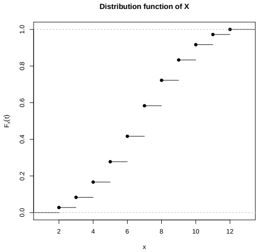


Cumulative distribution function

Definition

Let X be a discrete random variable and let $A_{(-\infty,t]}$ be the preimage of the event $(-\infty, t]$, $t \in \mathbb{R}$. The *cumulative distribution function* (CDF) or the probability distribution function or the *distribution function* of X is

$$\begin{aligned} F_X(t) &= P(X \leq t) = P(A_{(-\infty,t]}) \\ &= \sum_{x \in (-\infty,t] \cap I} P(A_x) = \sum_{x \in (-\infty,t] \cap I} p_X(x) \end{aligned}$$



Cumulative distribution function

A cumulative distribution F has the following properties:

1. $0 \leq F(x) \leq 1$ for $-\infty < x < \infty$. This follows because F is a probability.
2. F is a monotone increasing function of x, i.e. $F(x_1) \leq F(x_2)$ if $x_1 \leq x_2$ (think about the corresponding events ...).
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. If the random variable X has a finite image, then $F(x) = 0$ (1) for x sufficiently small (large).
4. Let $x_1, x_2, \ldots$ the elements of the image I of X. F has a positive jump equal to $p_X(x_i)$ at $i = 1, 2, \ldots$ and in the interval $[x_i, x_{i+1})$ F has a constant value $F(x_i)$.

It can be shown that any function F satisfying properties 1-4 is the distribution function of some discrete random variable!

Special discrete distributions

- The Bernoulli pmf is the density function of a discrete random variable X having 0 or 1 as its only possible values.
- The Binomial pmf is the density function of a discrete random variable Y_n which denotes the number of successes in n independent Bernoulli trials where each of them has a probability of success equal to p.
- The geometric pmf is the density function of a random variable Z describing the necessary number of independent Bernoulli trials in order to obtain the first success. The sample space of this random experiment is described by

  $$S = \left\{0^{i-1}1 \mid i = 1, 2, 3, \ldots\right\}.$$

  This sample space has an infinite number of outcomes.

Special discrete distributions

- The negative Binomial pmf is the density function of a random variable Z that describes the number of trials of a Bernoulli experiment which are necessary in order to obtain the $r^{th}-$success. It generalizes the geometric distribution.
- The Hypergeometric pmf is the density function of a random variable X wich computes the number k of defective components in a random sample of m components, chosen without replacement, from a total of n components, d of which are defective.
  - n, the number of components
  - d, the number of defective components
  - m, the number of components to sample (without replacement!)
  - k, the number of defective components found in the sample
- The uniform discrete pmf, the pmf of the constant random variable and the pmf of the indicator R.V.