Statistics Lecture

Claudio Ortelli [1]

[1]Finance Institute
Università della Svizzera italiana

Advanced Learning and Research Institute, 2011

The exponential distribution find its application in reliability theory and queuing theory. The following random variables are often modeled as exponential:

1. Time between two successive job arrivals to a file server (often called **interarrival time**).
2. Service time at a server in a queuing network; the server could be a resource such as a CPU, an I/O device, or a communication channel.
3. Time to failure (lifetime) of a component.
4. Time required to repair a component that has malfunctioned.

Remark: The choice of the exponential distribution to model the stochastic structure of the upper described variables is an assumption and not a given fact! Experimental verification of the distributional assumption will be therefore necessary before to relying on the results of the analysis.

Le $X \sim Exp(\lambda)$ be the lifetime of a component. Suppose we have observed that it has already been operating for t hours.

- What is the distribution of the remaining (residual) lifetime $Y = X - t$?

Let the conditional probability of $Y \leq y$, given that $X > t$, be denoted by $G_Y(y|t)$. For $y \geq 0$

$$
\begin{aligned}
G_Y(y|t) &= P(Y \leq y|X > t) = \frac{P(\{Y \leq y\} \text{ and } \{X > t\})}{P(X > t)} \\
&= \frac{P(\{X \leq y + t\} \text{ and } \{X > t\})}{P(X > t)} = \frac{P(t < X \leq y + t)}{P(X > t)} \\
&= \frac{exp(-\lambda t)(1 - exp(-\lambda y))}{exp(-\lambda t)} = 1 - exp(-\lambda y).
\end{aligned}
$$

Result:
The conditional distribution $G_Y(y|t)$ does not depend on t and is identical to the distribution of X, i.e. $Exp(\lambda)$.

Interpretation:
The distribution of the remaining life does not depend on how long the component has been operating, i.e. the component does not age (it is as good as new). Therefore, the exponential distribution is not suited to model components or devices that gradually deteriorate.

Let the random variable X be the lifetime (or time to failure) of a component.

Definition

The **reliability** $R(t)$ of the component is the probability that the component survives until some time t, i.e.

$$R(t) = P(X > t) = 1 - F_X(t)$$

$F_X(t)$ is often called the **unreliability** of the component.

The conditional probability that the component does not survive for an additional interval of duration x given that it has survived until time t is equal to

$$G_Y(x|t) = \frac{P(t < X \leq t + x)}{P(X > t)} = \frac{F_X(t+x) - F_X(t)}{R(t)}$$

Definition

The instantaneous failure rate $h(t)$ is defined to be

$$h(t) = \lim_{x \to 0} \frac{1}{x} G_Y(x|t) = \lim_{x \to 0} \frac{F_X(t+x) - F_X(t)}{xR(t)} ,$$

so that

$$h(t) = \frac{f_X(t)}{R(t)}.$$

Alternate terms for $h(t)$ are *hazard rate*, *force of mortality*, *intensity rate*, *conditional failure rate* or **failure rate**.

Interpretation:

- $h(t)\Delta t$ represents the conditional probability that a component having survived to age t will fail in the interval $(t, t+\Delta t]$.

- $f_X(t)\Delta t$ is the *unconditional* probability while $h(t)\Delta t$ is a conditional probability.

Next theorem shows the connection between reliability and failure rate.

Theorem

$$R(t) = \exp\left(-\int_0^t h(x)dx\right)$$

Proof.

$$\int_0^t h(x)dx = \int_0^t \frac{f_X(t)}{R(t)}dx = \int_0^t \frac{-R'(t)}{R(t)}dx = -\ln(R(t))$$

using the fact that $R'(t) = -f_X(t)$ and the boundary contition $R(0) = 1$. □

Definition

The cumulative hazard is defined to be

$$H(t) = \int_0^t h(x)dx$$

Then, reliability can also be written as $R(t) = \exp(-H(t))$.

Definition

The conditional reliability $R_t(y)$ is the probability that the component survives an additional interval of duration y given that it has survived until time t.

$$R_t(y) = \frac{R(t+y)}{R(t)} \tag{1}$$

Assume a component does not age stochastically, i.e. the survival probability over an additional time interval y is the same regardless of the age t of the component:

$$R_t(y) = R_s(y) \text{ for all } t, s \geq 0.$$

For $s = 0$

$$R_t(y) = R_0(y) = \frac{R(y)}{R(0)} = R(y),$$

so that

$$R(t+y) = R(t)R(y).$$

In particular we obtain

$$\frac{R(t+y) - R(y)}{t} = \frac{(R(t)-1)R(y)}{t} = \frac{(R(t)-R(0))R(y)}{t}.$$

Taking the limit as $t \to 0$

$$\begin{aligned} R'(y) &= R'(0)R(y) \\ R(y) &= \exp(yR'(0)) = \exp(-\lambda y) \end{aligned}$$

which shows that the lifetime $X \sim Exp(\lambda)$.
If a component has exponential lifetime distribution it follows that

1. A replacement policy of used components based on the lifetime of the components is useless.
2. In estimating mean life and reliability the age of the observed components are of no concern. The number of hours of observed live and the number of observed failures are of interest.

Definition

Increasing (decreasing) failure rate distribution
Let X be the lifetime of a component and $F_X(t)$ the corresponding distribution function. If its failure rate $h(t)$ is an increasing (decreasing) function of t for $t \geq 0$ then F_X is an Increasing (Decreasing) Failure Rate distribution: IFR (DFR) distribution.

The behavior of the failure rate $h(t)$ as a function of age is known as the *mortality curve*, *hazard function*, *life characteristic* or *lambda characteristic*.

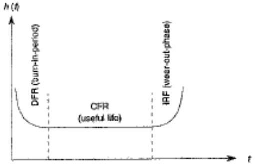


Figure 3.6. Failure rate as a function of time

The hypoexponential distribution is used to model processes that can be divided into sequential phases such that the time the process spends in each phase is independent and exponentially distributed.

- Service times for input-output operations in a computer system often follow this distribution.

A two stage hypoexponential random variable $X \sim Hypo(\lambda_1, \lambda_2)$ has pdf and distribution function equal to

$$\begin{aligned} f(t) &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(\exp(-\lambda_1 t) - \exp(\lambda_2 t)), \quad t > 0 \\ F(t) &= 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}\exp(-\lambda_1 t) + \frac{\lambda_1}{\lambda_2 - \lambda_1}\exp(-\lambda_2 t) \end{aligned}$$

When r sequential phases have identical exponential distribution the resulting density is known as $r-$stage Erlang and is given by

$$
\begin{aligned}
f(t) &= \frac{\lambda^r t^{r-1} \exp(-\lambda t)}{(r-1)!} \text{ with } t > 0,\ \lambda > 0,\ r = 1, 2, \ldots \\
F(t) &= 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} \exp(-\lambda t) \text{ with } t \geq 0,\ \lambda > 0,\ r = 1, 2, \ldots
\end{aligned}
$$

Suppose that a process consists of alternate phases, i.e. during any single experiment the process experiences one and only one of the many alternate phases, and thes phases have exponential distributions. The overall distribution is then hyperexponential with density and distribution functions given by

$$
\begin{aligned}
f(t) &= \sum_{i=1}^{k} \alpha_i \lambda_i \exp(-\lambda_i t) \text{ with } t > 0,\ \lambda_i > 0,\ \sum_{i=1}^{k} \alpha_i = 1 \\
F(t) &= \sum_{i} \alpha_i (1 - \exp(\lambda_i t)) \quad t \geq 0
\end{aligned}
$$

The Weibull distribution is the most widely used parametric family of failure distributions. It has been used to describe

- fatigue failure
- electronic component failure
- ballbearing failure

The reason is that by a proper choice of the shape parameter α we can obtain an IFR, DFR or constant failure rate distribution. The corresponding density and distribution functions are given by

$$\begin{aligned} f(t) &= \lambda \alpha t^{\alpha-1} \exp(-\lambda t^{\alpha}) \\ F(t) &= 1 - \exp(-\lambda t^{\alpha}) \end{aligned}$$

where $t \geq 0$, $\lambda > 0$ and $\alpha > 0$.

The Pareto (also knonw as double-exponential, hyperbolic or power-law) distribution has been used to model

- the amount of CPU time consumed by an arbitrary process
- the Web file size on the Internet servers
- the thinking time of the web browser
- the number of data bytes in FTP bursts
- the access frequency of Web traffic

The density and distributions functions are given by

$$
\begin{aligned}
f(x) &= \alpha k^{\alpha} x^{-\alpha-1} \quad x \geq k,\ k > 0,\ \alpha > 0 \\
F(x) &= \begin{cases} 1 - \left(\frac{k}{x}\right)^{\alpha} & x \geq k \\ 0 & x < k \end{cases}
\end{aligned}
$$

Functions of random variables

- Let X be a continuous random variable with distribution function F_X, Ψ a function and
$$Y = \Psi(X).$$
- Under regularity conditions on Ψ, Y is a random variable!
- Continuity or stepwise continuity of Ψ are sufficient conditions for Y to be a random variable.

Example: Quadratic cost function.

Let X denote a measurement error. We assume a quadratic cost function, i.e. $Y = \Psi(X) = X^2$. The random variable Y has a distribution function F_Y which depends on F_X and Ψ.

Functions of random variables

- To derive F_Y simply compute the preimage of the event $C = (-\infty, y]$. In fact by definition of F_Y

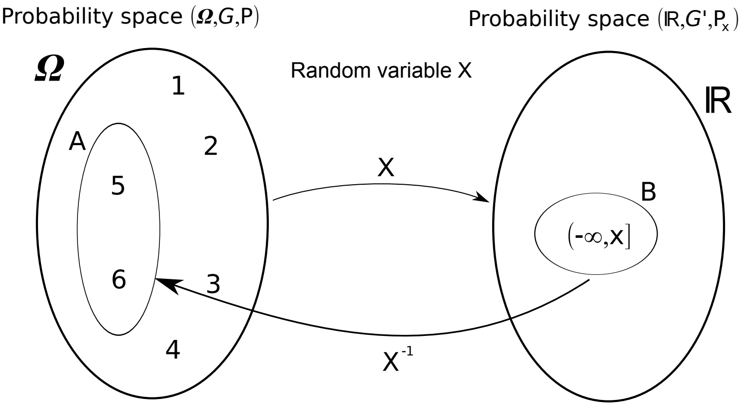
$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(Y \in C) \\ &= P(\Psi(X) \in C) \\ &= P(\Psi^{-1}(C)) \\ &= P_X(B) \end{aligned}$$

where the set B is the preimage of C, i.e.

$$B = \Psi^{-1}(C) = \{x \in \mathbb{R} \,|\Psi(x) \in C\}.$$

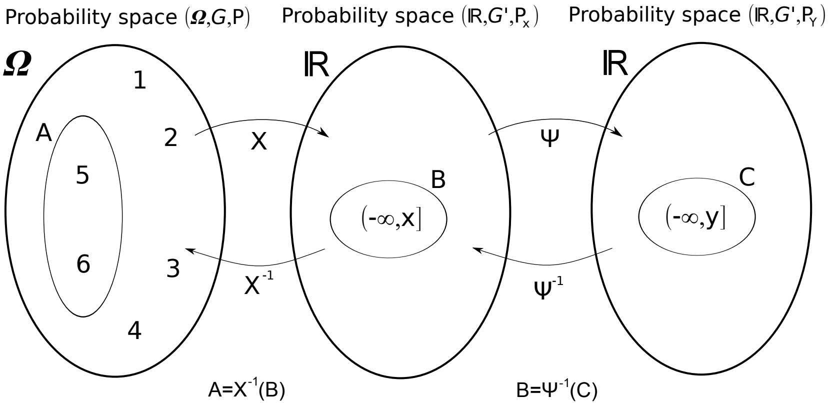
- The preimage function of Ψ is a set function (the arguments are subsets of $\mathbb{R}$) and is defined even if Ψ is not one-to-one.

Functions of random variables



Induced probability $P_X(B) := P(A)$

Functions of random variables



Induced probability $P_Y(C) = P_X(B) = P(A)$

Functions of random variables

Example continued

Because $Y = X^2$ is always positive $F_Y(y) = 0$ whenever $y \leq 0$. When $y > 0$ it follows $Y \leqslant y \Leftrightarrow -\sqrt{y} \leqslant X \leqslant \sqrt{y}$ so that $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$. If the random variable X has a density function we can differentiate the last expression to obtain the density function of Y:

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right], & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise

Let X be uniformly distributed on $(0,1)$ and define $Y = \lambda^{-1} ln(1 - X)$ where $\lambda > 0$ is a parameter. Show that Y has an exponential distribution with parameter λ.

Functions of random variables

Theorem

Let X be a continuous random variable with density function f_X satisfying

- $f_X > 0$ *for* $x \in I \subset \mathbb{R}$ *and*
- $f_X > 0$ *for* $x \notin I$

and let Φ be a differentiable and monotone real valued funtion with domain I. Then $Y = \Phi(X)$ is a continuous random variable with density function

$$f_Y(y) = \begin{cases} f_X\left[\Phi^{-1}(y)\right]\left[\left|\frac{\partial}{\partial y}\left(\Phi^{-1}\right)(y)\right|\right], & y \in \Phi(I), \\ 0, & \textit{otherwise}. \end{cases}$$

Examples

1) Let Φ be the distribution function F of the random variable X with density function f (we need to assume that F has the previous properties of continuity and differentiability) and define $Y = F(X)$. The random variable Y has density given by

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

2) Assume $Y = aX + b$, i.e. Y is a affine linear trasformation of X By Theorem 7 we have that (where I is the set over which $f_X \neq 0$)

$$f_Y(y) = \begin{cases} \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right), & y \in aI + b, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise

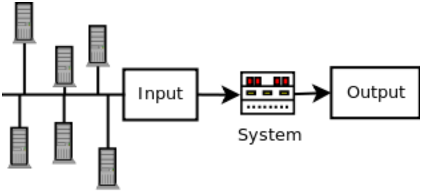
In the previous Example assume $X \sim N(\mu, sigma^2)$ and derive the density function of Y. What do you observe?

Simulation of random variables

Motivation

- We are interested in simulating a model of a real system (network, electronic device, ...).
- The output Y of the model depends on a stochastic input, i.e. a random variable X with known distribution function F_X:

$$Y = g(X).$$



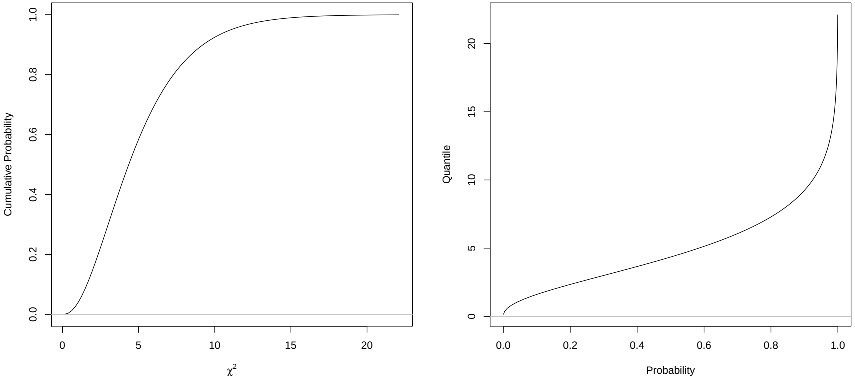
Simulation of random variables

- The model is too complex in order to analytically derive the probabilistic properties of the output, i.e. F_Y.
- Idea: simulate a possible outcome x of the input X and evaluate the corresponding outcome $y = g(x)$ of the output. Repeat the experiment N times and analyse the results.
- The simulated values of the input must be drawn from the distribution of X.
- Question: how is it possible to simulate independent realizations from a given distribution F_X?
- Answer: different methods available. The simplest of them requires simulating from the uniform distribution on the open interval (0,1), i.e. $U(0,1)$. In Matlab use the function "rand".

Simulation of random variables
Inverse Transform Method

If the distribution function F_X is continuous and strictly increasing then $F_X^{-1}: \ (0,1) \to \mathbb{R}$ exists.

Example: Chi-Squared Distribution with 5 degrees of freedom



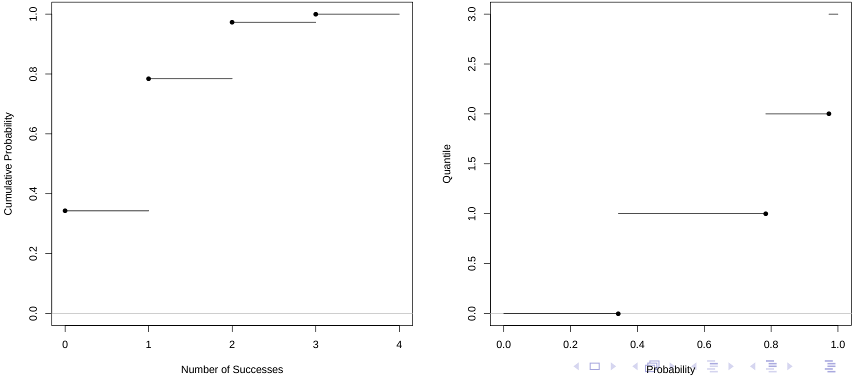
Simulation of random variables

For a general distribution function not necessarily strictly increasing define

$$F_X^{-1}(p) = \inf\{x : p \leq F_X(x)\} \quad 0 < p < 1.$$

It then follows that $F_X^{-1}(p) \leq x \iff p \leq F_X(x)$.

Example: Binomial distribution $Bin(n = 3, p = 0.3)$



Theorem

[Inverse Transform Method] Let U a continuous Unif(0,1) distributed random variable. The random variable $Y = F_X^{-1}(U)$ *has distribution function* F_X.

Proof.

By definition

$$F_Y(c) := P(Y \leq c) = P(F_X^{-1}(U) \leq c).$$

But the last equality is equivalent to (see previous slide)

$$P(U \leq F_X(c)) = F_X(c).$$

□

From the previous theorem we derive the following *two steps* simulation algorithm

1. Simulate a realization u from a Unif(0,1) random variable U.
2. Compute $x = F_X^{-1}(u)$.

Example

Simulation of $Y \sim Exp(\lambda)$

- The distribution function is $F_Y(x) = 1 - \exp(-\lambda x)$
- Compute $F_Y^{-1}(p) = -\frac{1}{\lambda}\ln(1-p)$
- Sample a random draw u from $U \sim$ Unif(0,1)
- Set $y = -\frac{1}{\lambda}\ln(1-u)$

Remark: if $U \sim$ Unif(0,1) then $1 - U$ is also Unif(0,1). Therefore we can also write $Y = -\frac{1}{\lambda}\ln(U)$.

For a descrete random variable Y with probability mass function $P(Y = x_i) = p_i, \; i = 1, \ldots, m$ consider the following algorithm:

1. Generate a Unif(0,1) random variable U
2. Compute Y as follows

$$Y = x_j \qquad \text{if} \; \sum_{i=1}^{j-1} p_i < U \leq \sum_{i=1}^{j} p_i .$$

i.e. $Y = x_j$ if $F_Y(x_{j-1}) < U \leq F_Y(x_j)$.

Example

Let Y have the following mass function

p_1	p_2	p_3	p_4
0.1	0.2	0.4	0.3

.

The simulation algorithm is the following:

1. If $u \leq p_1 = 0.1$ then $y = x_1$. Stop.
2. if $u \leq p_1 + p_2 = 0.3$ then $y = x_2$. Stop.
3. if $u \leq p_1 + p_2 + p_3 = 0.7$ then $y = x_3$. Stop.
4. $y = x_4$. Stop.

This algorithm is correct but inefficient. In fact, probabilities of one, two, ... comparisons are equal to the probabilities of x_1, x_2, ..., respectively. The expected number of comparisons is

$$p_1 + 2p_2 + 3p_3 + 4p_4 = 2.9.$$

We can improve efficiency by sorting the values of x_i by decreasing order of probabilities p_i's: x_3, x_4, x_2 and x_1.

1. If $u \leq p_3 = 0.4$ then $y = x_3$. Stop.
2. if $u \leq p_3 + p_4 = 0.7$ then $y = x_4$. Stop.
3. if $u \leq p_3 + p_4 + p_2 = 0.9$ then $y = x_2$. Stop.
4. $y = x_1$. Stop.

The expected number of comparisons is now equal to

$$p_3 + 2p_4 + 3p_2 + 4p_1 = 2.$$

The Laplace distribution is a continuous distribution with density function

$$f(x;\mu,b) = \begin{cases} \frac{1}{2b}\exp(-\frac{\mu-x}{b}) & \text{if } x < \mu \\ \frac{1}{2b}\exp(-\frac{x-\mu}{b}) & \text{if } x \geq \mu \end{cases}$$

where μ is the mean and b a scale parameter.

1. Plot the density, distribution and quantile functions of the Laplace distribution with parameter $\mu = 1$ and $b = 0.5$.
2. Using the previous values of μ and b simulate $N = 1000$ independent realizations of a Laplace distributed random variable Y.
3. Plot the histogram of the simulated random variables and compare it with the density function of Y.
4. Plot the empiric distribution function of the simulated sample and compare it with the distribution function of Y.

The theorem on *Distributions of functions of continuous random variables* allows us to generate random variables by means of ad hoc transformations of Unif(0,1) random variables. The following theorem generalizes the previous theorem to the multivariate case.

Theorem

Assume that $X = (X_1, \ldots, X_n)$ *is a random vector with joint density function* $f_X(x_1, \ldots, x_n)$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *a one-to-one and continuously differentiable function. Define* $Y = (Y_1, \ldots, Y_n) = g(X)$. *The joint density function of* Y *is then equal to*

$$f_Y(y) = f_X(g^{-1}(y)) \, |J(g^{-1}(y))|$$

where

$$J(g^{-1}(y)) = \det(M_J) = \det \left[\frac{\partial x_i(y)}{\partial y_j} \right]_{i=1,n;\; j=1,n}$$

Example

Let $X \sim N(\mu, \Sigma)$ be a 2×1 random vector and consider the affine transformation

$$Y = b + A\,X$$

where b is a deterministic 2×1 vector and A a 2×2 invertible matrix. We then have

$$X = A^{-1}(Y - b); \;\; M_J = A^{-1}$$

and $f_Y(y)$ is equal to

$$\frac{1}{\sqrt{2\pi det(\Sigma)}} \exp(-\frac{1}{2}(A^{-1}(Y-b)-\mu)'\Sigma^{-1}(A^{-1}(Y-b)-\mu))|det(A^{-1})|$$

$$\frac{1}{\sqrt{2\pi det(A\Sigma A')}} \exp(-\frac{1}{2}(Y-b-A\mu)'(A\Sigma A')^{-1}(Y-b-A\mu))$$

Example continued

Looking at the density function of Y we note that $Y \sim N(\tilde{\mu}, \tilde{\Sigma})$ whith $\tilde{\mu} = b + A\mu$ and $\tilde{\Sigma} = A\Sigma A'$.

Exercise

U_1 and U_2 are two independent Unif(0,1) distributed random variables. Define $Y = (Y_1, Y_2)$ with

$$Y_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \text{ and } Y_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$$

Show that $Y \sim N(0, I)$, i.e. Y_1 and Y_2 are independent standard normal distributed random variables.

Exercise

The joint density of the random variables X_1 and X_2 is

$$f(x_1,x_2) = 2\exp(-x_1)\exp(-x_2), \text{ for } 0 < x_1 < x_2 < \infty$$

and $f(x_1,x_2) = 0$ otherwise.
We define the transformation

$$Y_1 = 2X_1, Y_2 = X_2 - X_1.$$

Find the joint density of Y_1 and Y_2. Are Y_1 and $Y2$ independent?

Expectation

Definition

The expectation, $E[X]$, of a random variable X is defined by

$$E[X] = \begin{cases} \sum_i x_i p(x_i), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} x f(x) dx, & \text{if } X \text{ is continuous,} \end{cases}$$

provided the relevant sum or integral is absolutely convergent, i.e. $\sum_i | x_i | p(x_i) < \infty$ and $\int_{-\infty}^{\infty} | x | f(x) dx < \infty$.

Example

Assume X is Binomial di stributed with $n = 5$ and $p = 0.5$. Then

$$E[X] = \sum_i x_i p(x_i) = 0 \cdot \frac{1}{32} + 1 \cdot \frac{5}{32} + 2 \cdot \frac{10}{32} + 3 \cdot \frac{10}{32} + 4 \cdot \frac{5}{32} + 5 \cdot \frac{1}{32} = 2.5$$

The expected value need not correspond to a possible value of X!

Expectation

The expected value is a weighted average and it denotes the "center" of a probability mass or density function in the sense of a center of gravity.

Moments

Let X be a random variable, and define $Y = \phi(X)$. Suppose we want to compute $E[Y]$. In order to apply the definition of $E[Y]$ we need to derive the *pmf* (or the *pdf*) of Y. An easier method is to use the following result

$$E[Y] = E[\phi(X)] = \begin{cases} \sum_i \phi(x_i) p_X(x_i), & \text{if } X \text{ is descrete,} \\ \int_{-\infty}^{\infty} \phi(x) f_X(x) dx, & \text{if } X \text{ is continuous,} \end{cases}$$

privided the sum or the integral is absolutely convergent.

Definition

A special case is the power function $\phi(X) = X^k$, $k = 1, 2, 3, \ldots$. $E(X^k)$ is known to be the $k-$th moment of the random variable X. The first moment, i.e. $k = 1$, is the ordinary expectation of X.

Moments

Sometimes it is usefull to center the origin of measurement, i.e. to work with powers of $X - E[X]$.

Definition

The $k-$th central moment of the random variable X, μ_k, is defined as

$$\mu_k = E[(X - E[X])^k].$$

The second central moment μ_2 is called the *Variance* of the random variable X, typically denoted by σ^2, is a measure of dispersion. It measure the amount by wich the R.V. X deviates from its expected value.

Definition

The variance of a random variable X is

$$\sigma^2 = \begin{cases} \sum_i (x_i - E[X])^2 p(x_i), & \text{discrete case,} \\ \int_{-\infty}^{\infty} (x - E[X])^2 f(x) dx, & \text{continuous case.} \end{cases}$$

Moments

- The variance is a sum of squares and therefore is a nonnegative number.
- The square root of the variance is denoted by σ and is called the standard deviation.

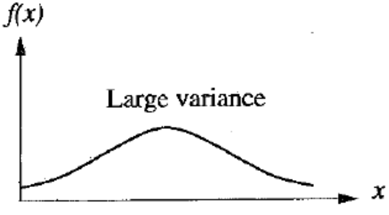


Figure 4.2. The pdf of a diffuse distribution

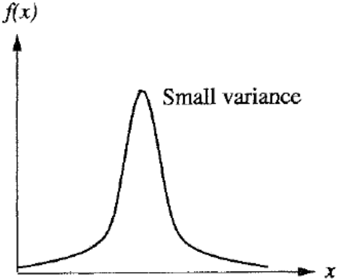


Figure 4.1. The pdf of a "concentrated" distribution

Figure: Small and large variance.

Expectation based on multiple Random Variables

Let $X_1, X_2, ..., X_n$ be n random variables defined on the same probability space and define $Y = \Phi(X_1, X_2, ..., X_n)$. Then

$$
\begin{aligned}
E[Y] &= E[\Phi(X)] \\
&= \begin{cases} \sum_{x_1} \dots \sum_{x_n} \Phi(x_1, x_2, ..., x_n) p(x_1, x_2, ..., x_n), & \text{descrete case} \\ \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \Phi(x_1, x_2, ..., x_n) p(x_1, x_2, ..., x_n) dx_1 \dots dx_n, & \text{contin. case} \end{cases}
\end{aligned}
$$

Remark: It is not necessary to derive the pmf (descrete case) or the pdf (cont. case) of the random variable Y in order to compute its expected value.

Moments

- The expected value is a linear operator. In fact let X and Y be two random variables with existing expectation and $\lambda \in \mathbb{R}$. Then

  1. $E[X+Y] = E[X] + E[Y]$.
  2. $E[\lambda X] = \lambda E[X]$.

  This property does not require that X and Y be independent and can be generalized to the case of n random variables!
- The variance satisfies the following property: $Var[a+bX] = b^2 Var[X]$.

Exercise

1) Starting from the definition of σ^2 use the property of linearity of the expected value to prove that $Var[X] = E[X^2] - (E[X])^2$.
2) Define the function $g : \mathbb{R} \to [0, \infty]$, $c \mapsto E[(X-c)^2]$. Show that $c_{min} = E[X]$ is the minimum of the function g.

Moments

Definition

Let X and Y be two random variable. The covariance between X and Y is defined to be

$$Cov(X,Y) = E[(X - E[X])(Y - E(Y)].$$

The covariance is a measure of linear dependence between two random variables. If $Cov(X,Y) = 0$ we say that X and Y are uncorrelated.

- The covariance is a *bilinear* operator. In fact let X,Y and Z be two random variables with existing expectation and $\lambda \in \mathbb{R}$. Then
  1. $Cov(X+Y,Z) = Cov(X,Z) + Cov(Y,Z)$,
  2. $Cov(\lambda X,Z) = \lambda Cov(X,Z)$,
  3. $Cov(X,Z) = Cov(Z,X)$.

  General rule: freeze the first (second) argument and consider the covariance an expectation in the second (first) argument.

Moments

examle 4.9

Moments

Exercise

1. *Starting from the definition of Cov use the property of linearity of the expected value to prove property 1. and 2. of the covariance.*
2. *Using the linearity of the expected value show that* $Cov(X,Y) = E[XY] - E[X]E[Y]$.
3. *Show that if* X *and* Y *are two independent random variables, then* $Cov(X,Y) = 0$.
4. *Show that* $Cov(X,X) = Var(X)$. *Hint: simply start from the definition of Cov.*
5. *Show that* $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.
6. *Let* $X \sim N(0,1)$ *and* Y *a random variable independent from* X *and such that* $P(Y=1) = P(Y=-1) = 0.5$. *Finally define* $Z = X \cdot Y$. *Show that* $Cov(X,Z) = 0$. *Are* X *and* Z *independent random variables?*

Moments

Definition

Correlation: the correlation coefficient $\rho(X,Y)$ between the random variables X and Y is defined by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

Remark: the correlation coefficient satisfies the following inequalities: $-1 < \rho(X,Y) < 1$.