# Statistics Lecture

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# Bernoulli Trials

Consider a random experiment with only two outcomes, “success” and “failure”, and denote the probability of the two outcomes by $p$ and $q$, respectively, with $p + q = 1$.

## Definition

The experiment consisting in observing a sequence of $n$ independent repetitions of the above described experiment is called a sequence of Bernoulli trials.

## Examples

1. Observe $n$ consecutive executions of an if statement, with success = “then clause is executed” and failure = ”else clause is executed”.
2. Examine components produced on an assembly line, with success = ”acceptable” and failure = ”defective”.

# Bernoulli Trials

Let 0 denote failure and 1 denote success. Let $S_n$ be the sample space of an experiment involving $n$ Bernoulli trials

$$\begin{aligned} S_1 &= \{0,1\}, \\ S_2 &= \{(0,0),(0,1),(1,0),(1,1)\}, \\ S_n &= \{0,1\}^n = \{2^n\ n-\text{tuples of 0s and 1s}\}. \end{aligned}$$

For all sample spaces $S_n$ we define the $\sigma-$ algebra $\mathscr{P}(S_n)$ as the relevant $\sigma-$ algebra on which to define the probability $P$. On $S_1$ we then have $P(\{0\}) = q$ and $P(\{1\}) = p$. We wish to assign probability to points in $S_n$.
Define $A_i$ =”success on trail $i$” and $\bar{A}_i$ =”failure on trial $i$”. We then have $P(A_i) = p$ and $P(\bar{A}_i) = q$. Let $s$ be an outcome of $S_n$ with $k$ “1” and $n-k$ “0”, i.e.

$$s = (1,1,\ldots,1,0,0,\ldots 0)$$

# Bernoulli Trials

The elementary event $\{s\}$ can be written

$$\{s\} = A_1 \cap A_2 \cap \cdots \cap A_k \cap \bar{A}_{k+1} \cap \cdots \cap \bar{A}_n.$$

Because events $A_i$ are independent we obtain

$$P(\{s\}) = P(A_1)P(A_2)\ldots P(A_k)P(\bar{A}_{k+1})\ldots P(\bar{A}_n)$$

so that $P(\{s\}) = p^k q^{n-k}$. Note that that we can construct $\left( \begin{array}{c} n \\ k \end{array} \right)$ different outcomes with $k$ successes and $n-k$ failures, therefore defining $A =$"we observe exactly $k$ successes in $n$ trials"

$$P(A) = \left( \begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k}.$$

Since by the binomial theorem $(p+q)^n = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k} = 1$, $P$ is a well defined probability law on $(S_n, \mathscr{P}(S_n))$.

# Bernoulli Trials

In connection with reliability theory let us assume that a particular system with $n$ components requires at least $k$ components to function in order for the entire system to work correctly. Such systems are called $k$-out-of-$n$ systems.

- If we let $k = n$ we have a series system.
- If we let $k = 1$ we have a system with parallel redundancy.

Assuming all components are statistically identical and function independently of each other, and denoting by $R$ the reliability of a component ($q = 1 - R$ gives its unreliability), then the experiment of observing the statuses of $n$ components can be thought of as a sequence of $n$ Bernoulli trials with probability $p = R$.

$$\begin{aligned} R_{k|n} &= P(\text{"at least } k\text{components functioning properly"}) \\ &= P(\bigcup_{i=k}^{n} \text{"exactly } i\text{components functioning properly"}) \end{aligned}$$

# Bernoulli Trials

$$
\begin{aligned}
R_{k|n} &= \sum_{i=k}^{n} P(\text{"exactly } i \text{components functioning properly"}) \\
&= \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i}.
\end{aligned}
$$

## Example

Triple modular redundancy (TMR).

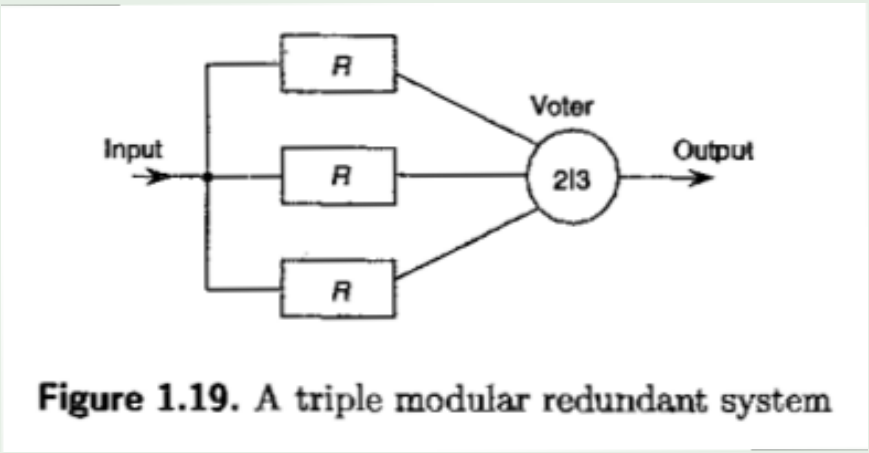
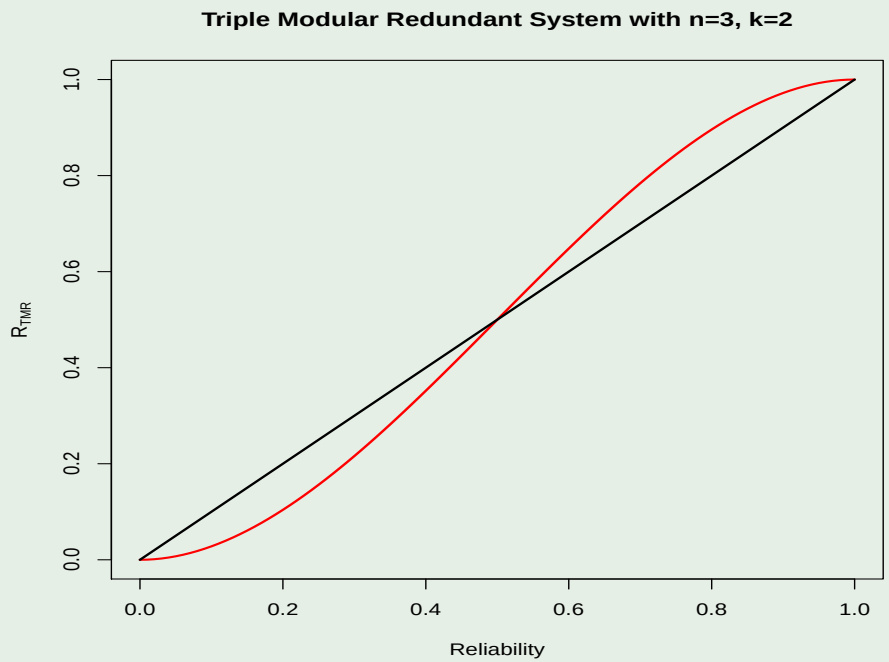



**Figure 1.19.** A triple modular redundant system

## Example

Applying the above formula with $n = 3$ and $k = 2$:

$$R_{TMR} = \sum_{i=2}^{3} \binom{3}{i} R^i (1-R)^{3-i} = 3R^2(1-R) + R^3.$$



# Non homogeneous Bernoulli Trials

When the components are nonhomegeneous w.r.t. the corresponding reliabilities, then the calculation is a bit more complicated:

$$R_{k|n} = 1 - \sum_{|I|>=k} \left( \prod_{i \in I} (1 - R_i) \right) \left( \prod_{i \notin I} R_i \right)$$

where $I$ ranges over all choises $i_1 < i_2 < \cdots < i_m$ such that $k \leq m \leq n$ and $R_i$ denotes the reliability of th $i-$th component.

## Example

Consider a non homogeneous TMR with $n = 3$ and $k = 2$.

$$\begin{aligned} R_{2|3} &= 1 - (1 - R_1)(1 - R_2)R_3 - (1 - R_1)(1 - R_3)R_2 + \\ &\quad - (1 - R_2)(1 - R_3)R_1 - (1 - R_1)(1 - R_2)(1 - R_3) \\ &= R_1 R_2 + R_2 R_3 + R_1 R_3 - 2R_1 R_2 R_3 \end{aligned}$$

# Generalized Bernoulli Trials

Next, we consider **generalized Bernoulli trials**. Here we have a sequence of $n$ independent trials, and on each trial the result is exactly one of the $k$ possibilities $b_1, b_2, \ldots, b_k$. On a given trial, let $b_i$ occur with probability $p_i, i = 1, 2, \ldots, k$ such that

$$p_i \geq 0 \text{ and } \sum_{i=1}^{k} p_i = 1.$$

The sample space $S$ consists of all $k^n$ $n$-tuples with components $b_1, b_2, \ldots, b_k$. To a point $s \in S$

$$s = (\underbrace{b_1, b_1, \ldots, b_1}_{n_1}, \underbrace{b_2, b_2, \ldots, b_2}_{n_2}, \ldots, \underbrace{b_k, \ldots, b_k}_{n_k})$$

we assign the probability of $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $\sum_{i=1}^{k} n_i = n$. This is

# Generalized Bernoulli Trials

the probability assigned to any $n$-tuple having $n_i$ occurrences of $b_i$, where $i = 1, 2, \ldots, k$. The number of such $n$-tuples are given by the multinomial coefficient [LIU 1968]:

$$\binom{n}{n_1 \quad n_2 \quad \cdots \quad n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

As before, the probability that $b_1$ will occur $n_1$ times, $b_2$ will occur $n_2$ times, $\ldots$, and $b_k$ will occur $n_k$ times is given by

$$p(n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! n_2! \cdots n_k!} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \tag{1.23}$$

and

$$\begin{aligned} \sum_{n_i \geq 0} p(n_1, n_2, \ldots, n_k) &= (p_1 + p_2 + \cdots + p_k)^n \\ &= 1 \end{aligned}$$

# Generalized Bernoulli Trials

## Example

Trivedi, page 52.

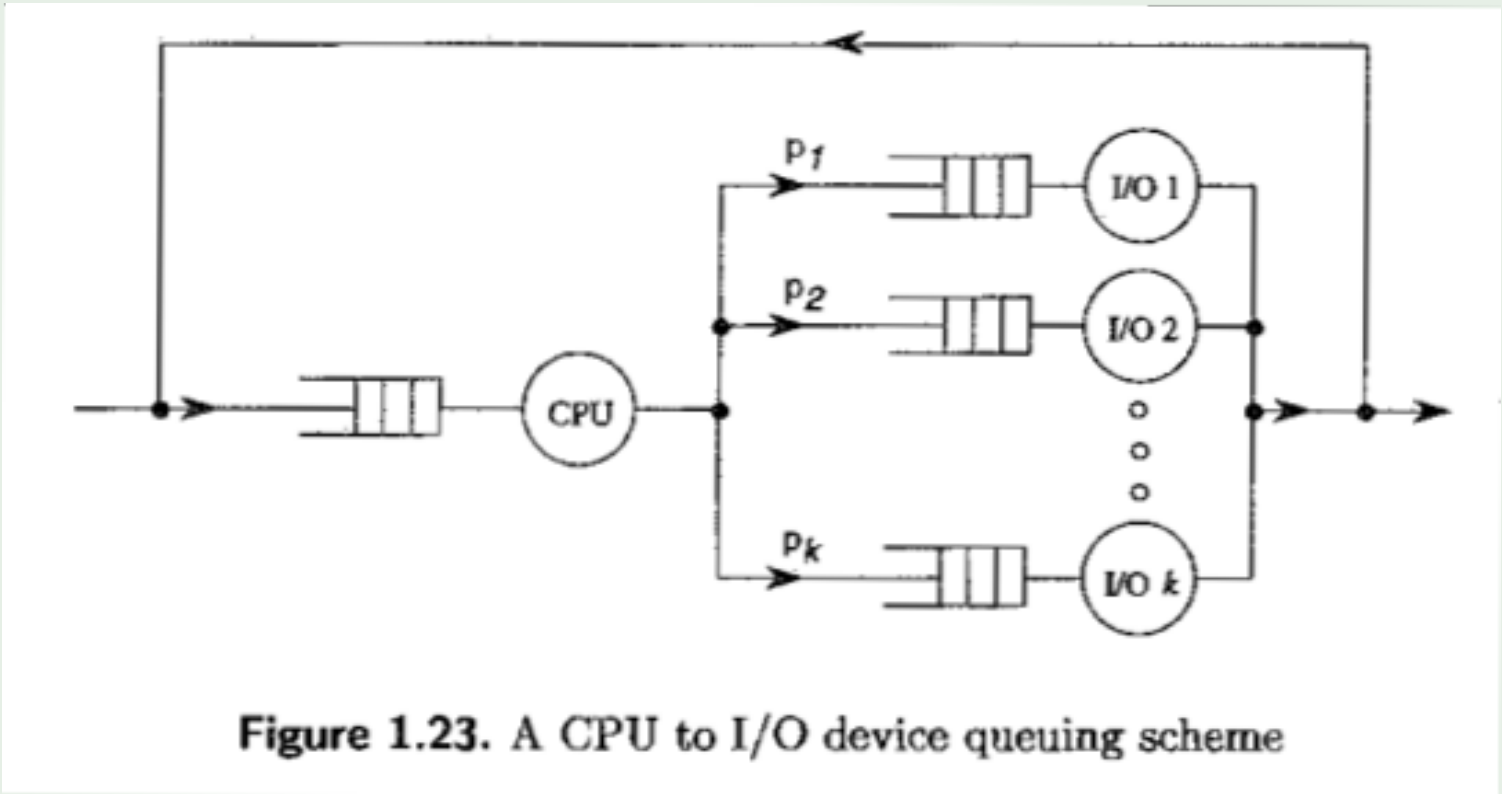



**Figure 1.23.** A CPU to I/O device queuing scheme

Solve problems 2,3,4 at page 56 and review problem 1 at page 57 of Trivedi.

# Random variables

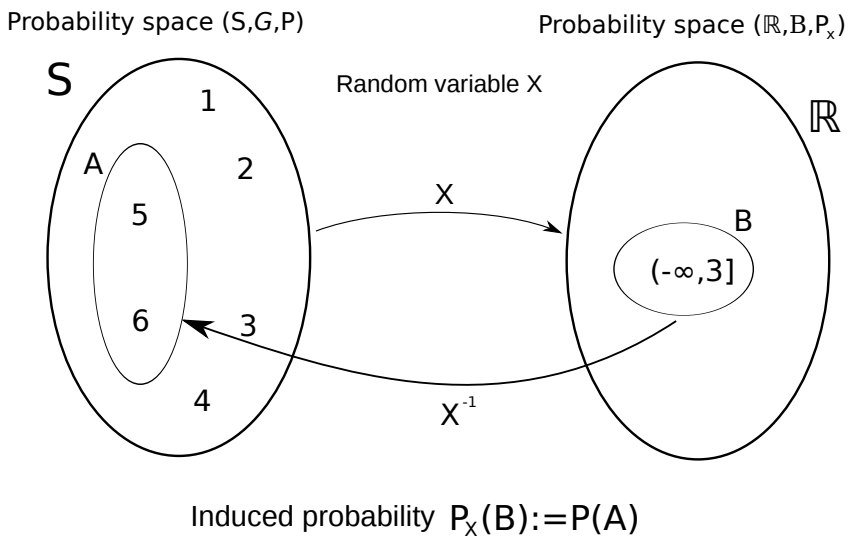
Let $(\Omega, \mathscr{G})$ denote a measurable space with sample space $\Omega$ and $\sigma-$algebra $\mathscr{G}$.

## Definition

A random variable is a real valued function $X : \Omega \to \mathbb{R}$ such that

$$X^{-1}(B) \in \mathscr{G} \text{ for all events } B \in \mathscr{B}.$$

The following picture clarifies the situation



# Random variables

*Remarks:*

1. A random variable is a function between two measurable spaces satisfying a *measurability* condition: the preimage of any event $B \subset \mathbb{R}$ must be an event of $\mathscr{G}$.
2. $S$ can be finite, countable infinite or uncountable. If the image of $X$ is discrete, i.e. finite or countable, then $X$ is a discrete random variable.

## Definition

Let $X$ be a random variable and $x \in \mathbb{R}$ a real number. The event

$$A_x = \{s \in S \mid X(s) = x\}$$

is called the inverse image of the set $\{x\}$ and represents the outcomes of the random experiment which are mapped to $x$.

# Random variables

It is clear that $A_x \cap A_y = \emptyset$ if $x \neq y$ and that

$$\bigcup_{x \in \mathbb{R}} A_x = S.$$

Attention:

1. Unions over an uncountable number of events are not, in general, events (see the definition of $\sigma-$algebra).
2. If the random variable $X$ is discrete, then $\bigcup_{x \in I} A_x$ is an event for all $I \subset \mathbb{R}$. Why?

Notation:

1. The notation $[X = x]$ will be used as an abbreviation for $A_x$.
2. Similarly, $[X \leq x]$ denotes the event $E = \{s \in S \mid X(s) \leq x\}$. Analogous definitions apply for the other inequality operators.

# Random variables

The set of random variables is closed under addition and scalar multiplication, under maximum, minimum, multiplication and division as well as under limit operation.

## Theorem

*Let $X$ and $Y$ two random variables defined on the same measurable space $(S, \mathcal{G})$. then*

1. $aX + bY$ *is a random variable for all* $a, b \in \mathbb{R}$.
2. $\max\{X, Y\}$ *and* $\min\{X, Y\}$ *are random variables.*
3. $XY$ *is random variable.*
4. *Provided that* $Y(s) \neq 0$ *for each* $s \in S$, $X/Y$ *is a random variable.*
5. *Let* $X_1, X_2, \ldots$ *be a sequence of random variables. If*

$$X(s) = \lim_{n \to \infty} X_n(s)$$

*exists for every* $s \in S$, *the* $X$ *is a random variable.*

# Probability mass function

Let $X$ be a discrete random variable and $I \subset \mathbb{R}$ the image set of $X$. Because $X$ is discrete, set $I$ is countable and

$$\sum_{x \in I} P(X = x) = \sum_{x \in I} P(A_x) = 1.$$

Furthermore, for all $x \notin I$, $P(X = x) = 0$ while for all events $B \in \mathscr{B}$ the event $A_B = [X \in B] = \{s \in S \mid X(s) \in B\}$ so that the probability

$$P(X \in B) = P(\bigcup_{x \in B} A_x) = P(\bigcup_{x \in B \cap I} A_x) = \sum_{x \in B \cap I} P(A_x).$$

The probability of any event $B \in \mathscr{B}$ can be computed as a sum over a countable number of points $x \in B \cap I$.

## Definition

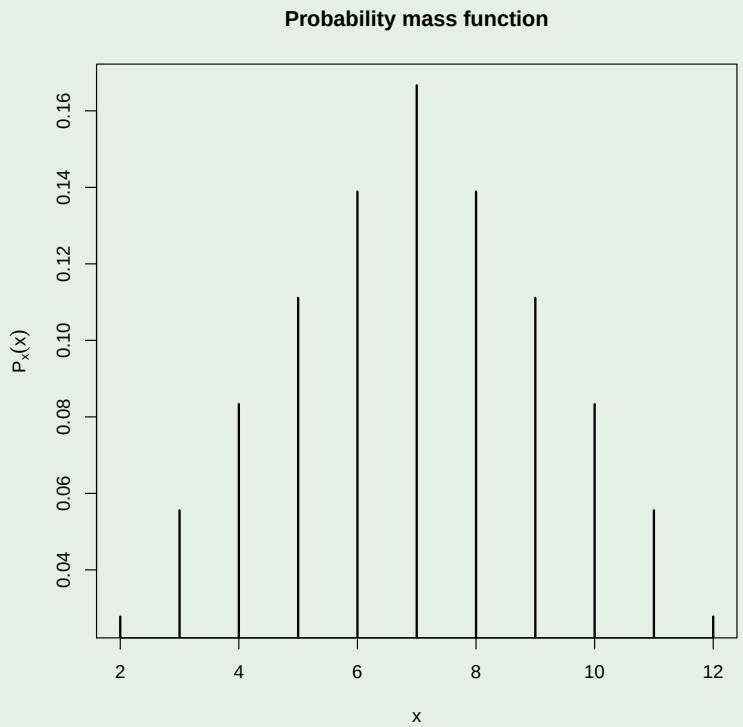
We define the *probability mass function* or *discrete density function* of the random variable $X$, denoted by $p_X$, the function

$$p_X(x) = P(A_x) \text{ for all } x \in \mathbb{R}.$$

# Probability mass function

## Example

A fair die is tossed twice. Let $S$ denote the sample space of this random experiment and define $X$ to be the sum of the outcomes of the first and second toss. The image of $X$ is the set $I = \{2, 3, \ldots, 12\}$. The corresponding probability mass function is given in the following plot:

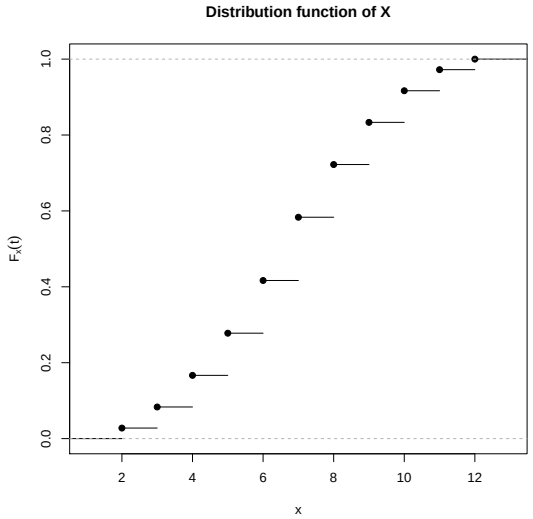


# Cumulative distribution function

## Definition

Let $X$ be a discrete random variable and let $A_{(-\infty,t]}$ be the preimage of the event $(-\infty, t]$, $t \in \mathbb{R}$. The *cumulative distribution function* (CDF) or the probability distribution function or the *distribution function* of $X$ is

$$\begin{aligned} F_X(t) &= P(X \leq t) = P(A_{(-\infty,t]}) \\ &= \sum_{x \in (-\infty,t] \cap I} P(A_x) = \sum_{x \in (-\infty,t] \cap I} p_X(x) \end{aligned}$$



# Cumulative distribution function

A cumulative distribution $F$ has the following properties:

1. $0 \leq F(x) \leq 1$ for $-\infty < x < \infty$. This follows because $F$ is a probability.
2. $F$ is a monotone increasing function of $x$, i.e. $F(x_1) \leq F(x_2)$ if $x_1 \leq x_2$ (think about the corresponding events ...).
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. If the random variable $X$ has a finite image, then $F(x) = 0$ $(1)$ for $x$ sufficiently small (large).
4. Let $x_1, x_2, \ldots$ the elements of the image $I$ of $X$. $F$ has a positive jump equal to $p_X(x_i)$ at $i = 1, 2, \ldots$ and in the interval $[x_i, x_{i+1})$ $F$ has a constant value $F(x_i)$.

It can be shown that any function $F$ satisfying properties 1-4 is the distribution function of some discrete random variable!

# Special discrete distributions

- The Bernoulli pmf is the density function of a discrete random variable $X$ having 0 or 1 as its only possible values.
- The Binomial pmf is the density function of a discrete random variable $Y_n$ which denotes the number of successes in $n$ independent Bernoulli trials where each of them has a probability of success equal to $p$.
- The geometric pmf is the density function of a random variable $Z$ describing the necessary number of independent Bernoulli trials in order to obtain the first success. The sample space of this random experiment is described by

$$S = \left\{ 0^{i-1}1 \mid i = 1, 2, 3, \dots \right\}.$$

This sample space has an infinite number of outcomes.

# Special discrete distributions

- The negative Binomial pmf is the density function of a random variable $Z$ that describes the number of trials of a Bernoulli experiment which are necessary in order to obtain the $r^{th}-$success. It generalizes the geometric distribution.
- The Hypergeometric pmf is the density function of a random variable $X$ wich computes the number $k$ of defective components in a random sample of $m$ components, chosen without replacement, from a total of $n$ components, $d$ of which are defective.
  - $n$, the number of components
  - $d$, the number of defective components
  - $m$, the number of components to sample (without replacement!)
  - $k$, the number of defective components found in the sample
- The uniform discrete pmf, the pmf of the constant random variable and the pmf of the indicator R.V.

# Special discrete distributions

The Poisson distribution is the last discrete distribution we want to study. It is a very popular and very used distribution in probability and statistics because of its peculiarity.

# Exercises

Solve problems 2,3,4,5 at page 91 of Trivedi.

# Probability generating function

The probability generating function is a convenient tool which simplifies calculations involving nonnegative, integer valued random variables.

## Definition

The probability generating function (PGF) of a nonnegative integer valued random variable $X$ is

$$G_X(z) = \sum_{i=0}^{\infty} p_i z^i$$

$G_X(z)$ is also known as the $z-$transform of $X$ and converges for all $z \in \mathbb{C}$ with $\mid z \mid < 1$. By definition of $G_X$ we have that $G_X(1) = 1$.

## Theorem

*If two discrete random variables $X$ and $Y$ have the same PGFs, then they have the same distributions and probability mass functions.*

# Probability generating function

## Exercise

*Compute the PGF of the following discrete distributions:*

1. *The binomial random variable.*
2. *The Bernoulli random variable.*
3. *The Uniform $U[1, n]$ distribution.*
4. *The constant random variable $X = 3$.*
5. *The Poisson distribution.*
6. *Solve probems 1 and 2 at page 99 of Trivedi.*

# Discrete random vector

**Motivation:** We are often interested in the relationship between two or more (discrete) random variables. Assume we have two softwares implementing the same functionality. We are interested in the execution time in seconds (rounded to the nearest integer). Let us denote by $X_1$ and $X_2$ the execution time of the first and second software, respectively. Examples of events on which we could be interested in are $\{X_1 > 4, X_2 > X_1\}$, $\{X_1 \leq 5, X_2 \leq 4\}$ and so on.

## Definition

A random vector $X = (X_1, \ldots, X_r)$ is an $r-$dimensional vector-valued function $X : S \to \mathbb{R}^r$, $s \mapsto X(s) = (X_1(s), \ldots, X_r(s))$. A discrete $r-$dimentional random vector is a function from $S$ to $\mathbb{R}^r$ taking on finite or countably infinite set of vector values $x$.

# Discrete random vector

## Definition

The compound (or joint) pmf for a random vector $X$ is defined to be

$$\begin{aligned} p_X(x) &= P(X = x) \\ &= P(X_1 = x_1, \ldots, X_r = x_r). \end{aligned}$$

## Fact

*The joint pmf of a discrete random vector has the following 4 properties:*

1. $p_X(x) \geq 0$ *for all* $x \in \mathbb{R}^r$;
2. *The set* $\{x \in \mathbb{R}^r \mid p_X(x) \geq 0\}$ *is a finite or countable infinite subset of* $\mathbb{R}^r$ *which is denoted by* $\{x_i,\ i = 1, 2, \ldots\}$ ;
3. $P(X \in A) = \sum_{x \in A} p_X(x)$;
4. $\sum_{i=1}^{\infty} p_X(x_i) = 1$.

Every real valued function defined on $\mathbb{R}^r$ satisfying properties 1-4 is the joint pmf of some discrete $r-$dim. random vector.

# Discrete random vector

## Example

Consider a program with two modules with stochastic execution times $X$ and $Y$, respectively. Assume for simplicity that the images of the discrete random variables $X$ and $Y$ are $\{1,2\}$ and $\{1,2,3,4\}$. The joint pmf of the random vector $Z = (X,Y)'$ is described by the following table:

| | $y=1$ | $y=2$ | $y=3$ | $y=4$ |
|---|---|---|---|---|
| $X=1$ | $\frac{1}{4}$ | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{8}$ |
| $X=2$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{16}$ |

The marginal pmf of the random variable $X$ is by definition $p_X(x) = P(X=x)$ and can be derived from the joint pmf by "integrating out" the unrelevant random variables ($Y$ in this case).

# Discrete random vector

## Example

(Continued).

$$\begin{aligned} p_X(x) &= P(X = X) = P(\bigcup_j \{X = x, Y = y_j\}) \\ &= \sum_j P(X = x, Y = y_j) \\ &= \sum_j p_Z(x, y_j). \end{aligned}$$

# Discrete random vector

## Examples

Let $X$ and $Y$ be two random variables, each with image $\{1,2\}$ and with a joint pmf given by

$$\begin{aligned} P_{X,Y}(1,1) &= P_{X,Y}(2,2) = a, \\ P_{X,Y}(1,2) &= P_{X,Y}(2,1) = 1/2 - a, \text{ for } 0 \leq a \leq \frac{1}{2}. \end{aligned}$$

It is easy to see that $p_X(1) = p_X(2) = p_Y(1) = p_Y(2) = \frac{1}{2}$, whatever be the value of $a$. This means we have uncountable many distinct joint pmf's associated with the same marginal pmf's!

# Discrete random vector

## Examples

The multinomial pmf. Consider a sequence of $n$ generalized Bernoulli trials where at each trial there are the same finite number of distinct outcomes having probabilities $p_1, p_2, \ldots, p_r$ where $\sum_{i=1}^{r} p_i = 1$. Define the random vector $X = (X_1, X_2, \ldots, X_r)$ such that $X_i$ is the number of trials that resulted in the $i-th$ outcome. The joint pmf of $X$ is then given by

$$\begin{aligned} p_X(n) &= P(X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r) \\ &= \underbrace{\binom{n}{n_1\, n_2 \cdots n_r}}_{\frac{n!}{n_1! n_2! \cdots n_r!}} p_1^{n_1} p_1^{n_1} \cdots p_r^{n_r} \end{aligned}$$

See Trivedi, page 102 for a practical example involving the multinomial distribution!

# Discrete random vector

## Examples

An inspection plan calls for inspecting five chips and for either accepting each chip, rejecting each chip or submitting it for reinspection, with probabilities of $p_1 = 0.70$, $p_2 = 0.20$ and $p_3 = 0.10$, respectively. Questions:

1. What is the probability that all five chips must be reinspected?
2. What is the probability that none of the chips must be reinspected?
3. What is the probability that at least one of the chips must be reinspected?

Solve problems 1 and 2 at page 104 of Trivedi.

## Definition

(Independent Random Variables). Two discrete random variables $X$ and $Y$ are defined to be independent if and only if their joint pmf is the product of their marginal pmf's, i.e.

$$P_{X,Y}(x,y) = p_X(x)p_Y(y) \text{ for all } x \text{ and } y.$$

# Independent random variables

Let assume $X$ and $Y$ be two independent random variables. Define the random variable $Z = X + Y$. The event $\{Z = t\}$ can be represented by all the outcomes on the line $X + Y = t$.

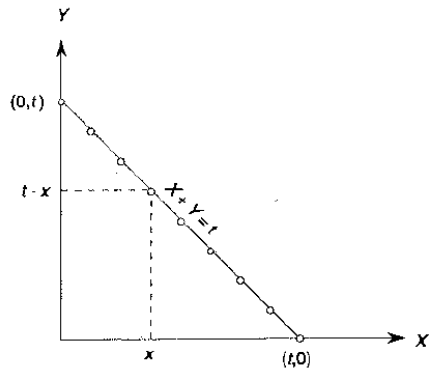



**Figure 2.20.** Computing the pmf of the random variable $Z_1 = X + Y$

$$
\begin{aligned}
P(Z = t) &= \sum_{x=0}^{t} P(X = x, Y = t - x) \\
&= \sum_{x=0}^{t} P(X = x) P(Y = t - x)
\end{aligned}
$$

# Independent random variables

From previous formula it follows that

$$p_Z(t) \quad = \quad \sum_{x=0}^{t} p_X(x) P_Y(t-x).$$

This summation is called the discrete convolution and it gives the formula for the pmf of the sum of two nonnegative, indipendent, discrete random variables.

## Definition

Let $X_1, X_2, \ldots, X_r$ be $r$ discrete random variable with pmf's $p_{X_1}, p_{X_2}, \ldots, p_{X_r}$, respectively. The R.V. are said to be *mutually independent* if and only if their joint pmf is given by

$$p_{X_1,X_2,\ldots X_r}(x_1, x_2, \ldots x_r) \quad = \quad p_{x_1}(x_1) p_{x_2}(x_2) \cdots p_{x_r}(x_r).$$

Remark: it is possible for every pair of R.V. in the set $\{X_1, \ldots, X_r\}$ to be pairwise independent without the entire set to be mutually independent.

# Independent random variables

## Example

Consider a sequence of two Bernoulli trials and define $X_1$ and $X_2$ as the number of successes on the first and second trials, respectively. Let $X_3$ define the number of matches on the two trials. Then it can be shown that the pairs $(X_1, X_2)$, $(X_1, X_3)$ and $(X_2, X_3)$ are each independent, but that the set $\{X_1, X_2, X_3\}$ is not mutally independent.

Restricting our attention to nonnegative integer-valued R.V. and recalling the definition of probability generating function PGF, the PGF of the sum of two independent random variables is the product of their PGFs:

$$\begin{aligned} G_Z(z) &= G_{X+Y}(z) \\ &= G_X(z) G_Y(z). \end{aligned}$$

# Independent random variables

*Proof:*

$$
\begin{aligned}
G_Z(z) &= \sum_{t=0}^{\infty} p_Z(t) z^t \\
&= \sum_{t=0}^{\infty} z^t \sum_{x=0}^{t} p_X(x) p_Y(t-x) \\
&= \sum_{t=0}^{\infty} \sum_{x=0}^{t} z^t p_X(x) p_Y(t-x) \\
&= \sum_{x=0}^{\infty} \sum_{t=x}^{\infty} z^t p_X(x) p_Y(t-x) \\
&= \sum_{x=0}^{\infty} z^x p_X(x) \sum_{t=x}^{\infty} z^{t-x} p_Y(t-x) \\
&= G_X(z) G_Y(z).
\end{aligned}
$$

# Independent random variables

Why is this theorem important?

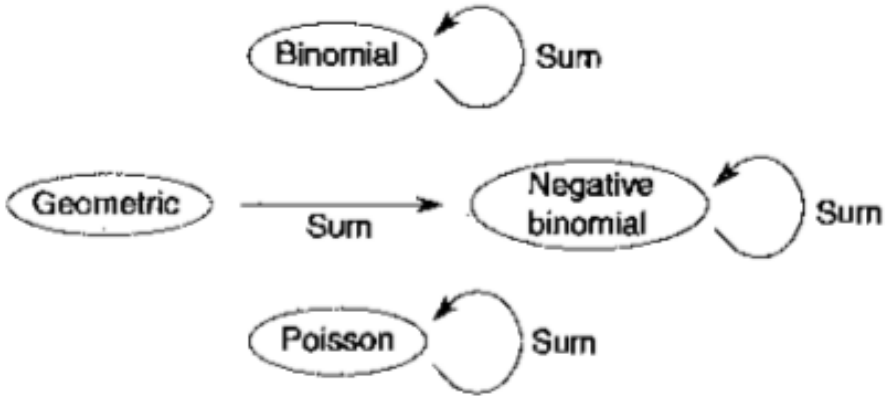



**Figure 2.21.** Theorem 2.2

Because some families of random variables are closed w.r.t. independent sum of random variables of the same family.
It follow by induction that if $X_1, X_2, \ldots, X_n$ are mutually independent R.V.

$$G_{X_1+\cdots+X_n}(z) = G_{X_1}(z)G_{X_2}(z)\cdots G_{X_n}(z)$$

# Independent random variables

## Theorem

*Let $X_1, X_2, \ldots, X_r$ be mutually independent.*

1. *If $X_i$ has the binomial distribution with parameters $n_i$ and $p_i$ then $\sum_{i=1}^{r} X_i$ has the binomial distribution with parameters $n_1 + n_2 + \cdots + n_r$ and $p$.*
2. *If $X_i$ has the (modified) negative binomial distribution with parameters $\alpha_i$ and $p_i$ then $\sum_{i=1}^{r} X_i$ has the (modified) negative binomial distribution with parameters $\alpha_1 + \alpha_2 + \cdots + \alpha_r$ and $p$.*
3. *If $X_i$ has the Poisson distribution with parameter $\alpha_i$, then $\sum_{i=1}^{r} X_i$ has the Poisson distribution with parameter $\sum_{i=1}^{r} \alpha_i$.*

## Proof.

Use the properties of the PGF (see Trivedi). □

Solve problems 4 and 5 at page 112 of Trivedi.

# Continuous random variables

When the sample space $S$ is nondenumerable (as mentioned in Section 1.7), not every subset of the sample space is an event that can be assigned a probability. As before, let $\mathcal{F}$ denote the class of measurable subsets of $S$. Now if $X$ is to be a random variable, it is natural to require that $P(X \leq x)$ be well defined for every real number $x$. In other words, if $X$ is to be a random variable defined on a probability space $(S, \mathcal{F}, P)$, we require that $\{s|X(s) \leq x\}$ be an event (i.e., a member of $\mathcal{F}$). We are, therefore, led to the following extension of our earlier definition.

***Definition (Random Variable).*** A random variable $X$ on a probability space $(S, \mathcal{F}, P)$ is a function $X : S \to \Re$ that assigns a real number $X(s)$ to each sample point $s \in S$, such that for every real number $x$, the set of sample points $\{s|X(s) \leq x\}$ is an event, that is, a member of $\mathcal{F}$.

***Definition (Distribution Function).*** The (cumulative) distribution function or CDF $F_X$ of a random variable $X$ is defined to be the function

$$F_X(x) = P(X \leq x), \qquad -\infty < x < \infty.$$

# Continuous random variables

## Definitions

1. A random variable $X$ such that its CDF $F_X$ is a continuous function is called a continous random variable.
2. A random variable $X$ such that the derivative $dF/dx$ exists everywhere (except perhaps for a finite number of points) is called an absolutely continuous random variable.

## Example

The continuous uniform random variable having CDF

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \leq 1 \end{cases}$$

is absolutely continuous having a derivative at all point except at $x = 0$ and $x = 1$.

# Continuous random variables

## Definition

(Probability Density Function). For an absolutely continuous random variable $X$, $f(x) = dF(x)/dx$ is called the probability density function (pdf or density function) of $X$.

The pdf enables us to obtain the CDF and hence to compute probabilities by integrating the pdf, i.e.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt \text{ for } -\infty < x < \infty \text{ and}$$

$$\begin{aligned} P(X \in (a,b]) &= P(a < X \leq b) \\ &= P(X \leq b) - P(X \leq a) \\ &= \int_{-\infty}^{b} f_X(t)dt - \int_{-\infty}^{a} f_X(t)dt \\ &= \int_{a}^{b} f_X(t)dt \end{aligned}$$

# Continuous random variables

The pdf satisfies the following properties:

1. $f(x) \geq 0$ for all $x$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

The CDF $F$ of a continuous random variable satisfies the following properties

1. $0 \leq F(x) \leq 1$, $-\infty < x < \infty$.
2. $F(x)$ is a monotone increasing function of $x$.
3. $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

The CDF of a continuous random variable does not have any jumps, therefore

$$P(X = c) = \int_{c}^{c} f_X(t)dt = 0.$$

This does not imply that the event $\{X = c\}$ is empty, but that the probability assigned to it is 0.

## Example

The time in years, denoted by $X$, required to complete a software project has a pdf of the form

$$f_X(x) = \begin{cases} kx(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$

Since $f_x$ must satisfy property 1, $k \geq 0$. Because of property 2 we have

$$\int_0^1 kx(1-x)dx \quad = \quad k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)|_0^1 = 1$$

from which we deduce $k = 6$. The probability that the project will be completed by less than four months is given by

$$P(X < \frac{4}{12} \quad = F_X(\tfrac{1}{3}) \quad = \int_0^{1/3} f_X(x)dx = \frac{7}{27}$$

# Mixed random variables

Sometimes the involved random variable will either be descrete or continuous, but a mixed random variable.

## Example

There may be a nonzero probability, say $p_0$, of initial failure of a component at time 0 due to manufacturing defects. In this case the time to failure $X$ of the component is neither discrete nor a continuous random variable! Assuming an exponential distribution for $x > 0$ the $CDF$ will be

$$F_X(x) = \begin{cases} 0, & x < 0, \\ p_0, & x = 0, \\ p_0 + (1 - p_0)(1 - e^{-\lambda x}), & x > 0. \end{cases}$$

In general, a mixed random variable will have a $CDF$ given by

$$F_X(x) = \alpha_d F^{(d)}(x) + (1 - \alpha_d) F^{(c)}(x)$$

where $F^{(d)}$ and $F^{(c)}$ are two discrete and continuous $CDF$, respectively.

# Mixed random variables

Solve problems 1, 2 and 3 at page 119 of Trivedi.

The exponential distribution find its application in reliability theory and queuing theory. The following random variables are often modeled as exponential:

1. Time between two successive job arrivals to a file server (often called **interarrival time**).
2. Service time at a server in a queuing network; the server could be a resource such as a CPU, an I/O device, or a communication channel.
3. Time to failure (lifetime) of a component.
4. Time required to repair a component that has malfunctioned.

Remark: The choice of the exponential distribution to model the stochastic structure of the upper described variables is an assumption and not a given fact! Experimental verification of the distributional assumption will be therefore necessary before to relying on the results of the analysis.

Let $X \sim Exp(\lambda)$ be the lifetime of a component. Suppose we have observed that it has already been operating for $t$ hours.

- What is the distribution of the remaining (residual) lifetime $Y = X - t$?

Let the conditional probability of $Y \leq y$, given that $X > t$, be denoted by $G_Y(y \mid t)$. For $y \geq 0$

$$\begin{aligned} G_Y(y \mid t) &= P(Y \leq y \mid X > t) = \frac{P(\{Y \leq y\} \text{ and } \{X > t\})}{P(X > t)} \\ &= \frac{P(\{X \leq y + t\} \text{ and } \{X > t\})}{P(X > t)} = \frac{P(t < X \leq y + t)}{P(X > t)} \\ &= \frac{exp(-\lambda t)(1 - exp(-\lambda y))}{exp(-\lambda t)} = 1 - exp(-\lambda y). \end{aligned}$$

*Result:*
The conditional distribution $G_Y(y \mid t)$ does not depend on $t$ and is identical to the distribution of $X$, i.e. $Exp(\lambda)$.

*Interpretation:*
The distribution of the remaining life does not depend on how long the component has been operating, i.e. the component does not age (it is as good as new). Therefore, the exponential distribution is not suited to model components or devices that gradually deteriorate.

Let the random variable $X$ be the lifetime (or time to failure) of a component.

## Definition

The **reliability** $R(t)$ of the component is the probability that the component survives until some time $t$, i.e.

$$R(t) = P(X > t) = 1 - F_X(t)$$

$F_X(t)$ is often called the **unreliability** of the component.

The conditional probability that the component does not survive for an additional interval of duration $x$ given that it has survived until time $t$ is equal to

$$G_Y(x \mid t) = \frac{P(t < X \leq t + x)}{P(X > t)} = \frac{F_X(t+x) - F_X(t)}{R(t)}$$

## Definition

The instantaneous failure rate $h(t)$ is defined to be

$$h(t) = \lim_{x \to 0} \frac{1}{x} G_Y(x \mid t) = \lim_{x \to 0} \frac{F_X(t+x) - F_X(t)}{xR(t)},$$

so that

$$h(t) = \frac{f_X(t)}{R(t)}.$$

Alternate terms for $h(t)$ are *hazard rate*, *force of mortality*, *intensity rate*, *conditional failure rate* or **failure rate**.
Interpretation:

- $h(t)\Delta t$ represents the conditional probability that a component having survived to age $t$ will fail in the interval $(t, t+\Delta t]$.

- $f_X(t)\Delta t$ is the *unconditional* probability while $h(t)\Delta t$ is a conditional probability.

Next theorem shows the connection between reliability and failure rate.

**Theorem**

$$R(t) = \exp\left(-\int_0^t h(x)dx\right)$$

**Proof.**

$$\int_0^t h(x)dx = \int_0^t \frac{f_X(t)}{R(t)}dx = \int_0^t \frac{-R'(t)}{R(t)}dx = -\ln(R(t))$$

using the fact that $R'(t) = -f_X(t)$ and the boundary contition $R(0) = 1$. □

## Definition

The cumulative hazard is defined to be

$$H(t) = \int_0^t h(x)dx$$

Then, reliability can also be written as $R(t) = \exp(-H(t))$.

## Definition

The conditional reliability $R_t(y)$ is the probability that the component survives an additional interval of duration $y$ given that it has survived until time $t$.

$$R_t(y) = \frac{R(t+y)}{R(t)} \tag{1}$$

Assume a component does not age stochastically, i.e. the survival probability over an additional time interval $y$ is the same regardless of the age $t$ of the component:

$$R_t(y) \quad = \quad R_s(y) \text{ for all } t, s \geq 0.$$

For $s = 0$

$$R_t(y) \quad = \quad R_0(y) = \frac{R(y)}{R(0)} = R(y),$$

so that

$$R(t+y) \quad = \quad R(t)R(y).$$

In particular we obtain

$$\frac{R(t+y)-R(y)}{t} \quad = \quad \frac{(R(t)-1)R(y)}{t} = \frac{(R(t)-R(0))R(y)}{t}.$$

Taking the limit as $t \to 0$

$$\begin{aligned} R'(y) &= R'(0)R(y) \\ R(y) &= \exp(yR'(0)) = \exp(-\lambda y) \end{aligned}$$

which shows that the lifetime $X \sim Exp(\lambda)$.
If a component has exponential lifetime distribution it follows that

1. A replacement policy of used components based on the lifetime of the components is useless.
2. In estimating mean life and reliability the age of the observed components are of no concern. The number of hours of observed live and the number of observed failures are of interest.

## Definition

Increasing (decreasing) failure rate distribution
Let $X$ be the lifetime of a component and $F_X(t)$ the corresponding distribution function. If its failure rate $h(t)$ is an increasing (decreasing) function of $t$ for $t \geq 0$ then $F_X$ is an Increasing (Decreasing) Failure Rate distribution: IFR (DFR) distribution.

The behavior of the failure rate $h(t)$ as a function of age is known as the *mortality curve*, *hazard function*, *life characteristic* or *lambda characteristic*.

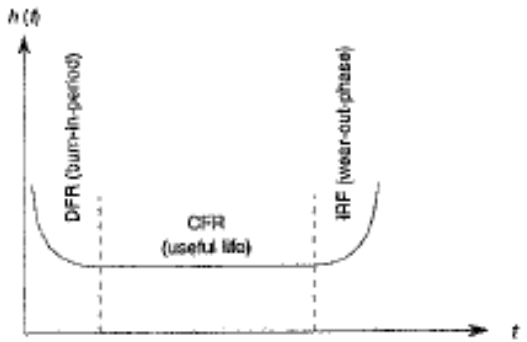



**Figure 3.6.** Failure rate as a function of time

The hypoexponential distribution is used to model processes that can be divided into sequential phases such that the time the process spends in each phase is independent and exponentially distributed.

- Service times for input-output operations in a computer system often follow this distribution.

A two stage hypoexponential random variable $X \sim Hypo(\lambda_1, \lambda_2)$ has pdf and distribution function equal to

$$\begin{aligned} f(t) &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(\exp(-\lambda_1 t) - \exp(\lambda_2 t)), \quad t > 0 \\ F(t) &= 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}\exp(-\lambda_1 t) + \frac{\lambda_1}{\lambda_2 - \lambda_1}\exp(-\lambda_2 t) \end{aligned}$$

When $r$ sequential phases have identical exponential distribution the resulting density is known as $r-$stage Erlang and is given by

$$f(t) = \frac{\lambda^r t^{r-1} \exp(-\lambda t)}{(r-1)!} \text{ with } t > 0,\ \lambda > 0,\ r = 1,2,\ldots$$

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} \exp(-\lambda t) \text{ with } t \geq 0,\ \lambda > 0,\ r = 1,2,\ldots$$

Suppose that a process consists of alternate phases, i.e. during any single experiment the process experiences one and only one of the many alternate phases, and thes phases have exponential distributions. The overall distribution is then hyperexponential with density and distribution functions given by

$$\begin{aligned} f(t) &= \sum_{i=1}^{k} \alpha_i \lambda_i \exp(-\lambda_i t) \text{ with } t > 0,\ \lambda_i > 0,\ \sum_{i=1}^{k} \alpha_i = 1 \\ F(t) &= \sum_{i} \alpha_i (1 - \exp(\lambda_i t)) \quad t \geq 0 \end{aligned}$$

The Weibull distribuiton is the most widely used parametric family of failure distributions. It has been used to describe

- fatigue failure
- electronic component failure
- ballbearing failure

The reason is that by a proper choice of the shape parameter $\alpha$ we can obtain an IFR, DFR or constant failure rate distribution. The corresponding density and distribution functions are given by

$$\begin{aligned} f(t) &= \lambda \alpha t^{\alpha-1} \exp(-\lambda t^{\alpha}) \\ F(t) &= 1 - \exp(-\lambda t^{\alpha}) \end{aligned}$$

where $t \geq 0$, $\lambda > 0$ and $\alpha > 0$