Exercise 10/61

Consider the three-dimensional vector $X = (X_1, X_2, X_3)$ *having the following joint density function*

$$f_X(x_1, x_2, x_3) = \begin{cases} 6x_1x_2^2x_3, & \text{if } 0 \leq x_1 \leq 1, 0 \leq x_x \leq 1, 0 \leq x_3 \leq \sqrt{2}. \\ 0, & \text{otherwise.} \end{cases}$$

Part 1)

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, y_2, y_3)dy_2dy_3 \\ &= \begin{cases} \int_0^{\sqrt{2}}\int_0^1 6x_1y_2^2y_3dy_2dy_3 = 2x_1, & \text{if } 0 \leq x_1 \leq 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} f_{X_2}(x_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{X}}(y_1, x_2, y_3)dy_1dy_3 \\ &= \begin{cases} \int_0^{\sqrt{2}}\int_0^1 6y_1x_2^2y_3dy_1dy_3 = 3x_2^2, & \text{if } 0 \leq x_2 \leq 1 \\ 0, & \text{otherwise,} \end{cases} \\ f_{X_3}(x_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{X}}(y_1, y_2, x_3)dy_1dy_2 \\ &= \begin{cases} \int_0^1\int_0^1 6y_1x_2^2y_3dy_1dy_2 = x_3, & \text{if } 0 \leq x_3 \leq \sqrt{2} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

$$f_{X_1,X_2|X_3=k}(x_1, x_2) = \begin{cases} \frac{f_{\mathbf{X}}(x_1,x_2,k)}{f_{X_3}(k)}, & \text{if } f_{X_3}(k) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6x_1x_2^2k}{k} = 6x_1x_2^2, & \text{if } 0 \leq k \leq \sqrt{2}, 0 \leq x_1, x_2 \leq 1 \\ 0, & \text{otherwise;} \end{cases}$$

Part 2)

Definition. *The continuous random variables $X_1, \ldots, X_n$ are independent if and only if for all $x_1, \ldots, x_n$*

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i),$$

that is the joint density function of $\mathbf{X}$ equals the product of the marginal densities of $X_i,\ i = 1, \ldots, n$.

This means that we check whether $f_{X_1,X_2,X_3}(x_1, x_2, x_3)$ equals the product $f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3)$ for all (x_1, x_2, x_3):

$$f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3) = \begin{cases} (2x_1)\cdot(3x_2^2)\cdot(x_3) = 6x_1x_2^2x_3, & \text{if } 0 \leq x_3 \leq \sqrt{2}, 0 \leq x_1, x_2 \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

which, for all (x_1, x_2, x_3), equals $f_{X_1,X_2,X_3}(x_1, x_2, x_3)$. Hence, X_1, X_2 and X_3 are independent random variables.