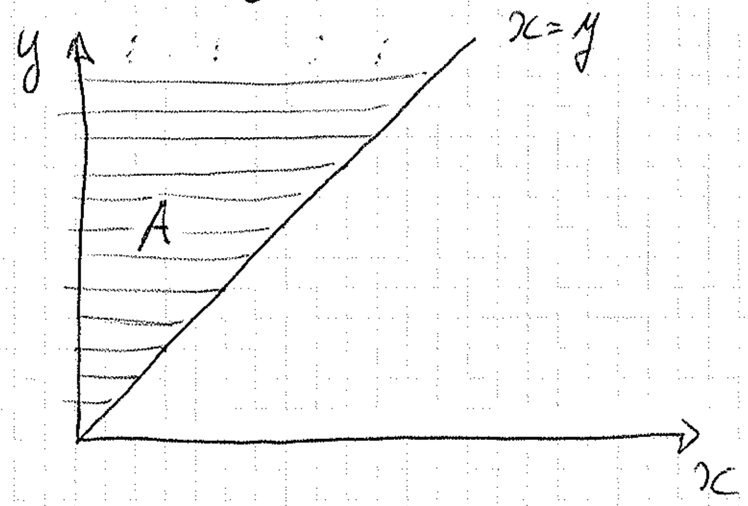
Exercise 11/43

Let the execution times X and Y of two...

We need to compute the probability of the event $A = \{X < Y\} \subset \mathbb{R}^2$



$$P(A) = \int_A f(x,y)\,dx\,dy = \int_0^{t_y}\int_0^{y} f(x,y)\,dx\,dy.$$

But $f(x,y) = f(x \mid Y=y)\cdot f(y)$ so

$$\int_0^{t_y}\int_0^{y} f(x \mid Y=y)\cdot f(y)\,dx\,dy =$$

$$\int_0^{t_x}\int_0^{y} f(x \mid y=y)\cdot f(y)\,dx\,dy + \int_{t_x}^{t_y}\int_0^{y} f(x \mid y=y)\cdot f(y)\,dx\,dy$$

Because X and Y are independent

$$f(x \mid y=y) = f(x)$$

so that

$$\int_0^{t_x}\int_0^{y} f(x)\cdot f(y)\,dx\,dy + \int_{t_x}^{t_y}\int_0^{y} f(x)\cdot f(y)\,dx\,dy$$

$$\int_0^{t_x} \underbrace{\int_0^{y} f(x)\,dx}_{F_X(y)=\frac{y}{t_x}} \cdot f(y)\,dy + \int_{t_x}^{t_y} \underbrace{\int_0^{y} f(x)\cdot dx}_{=1 \text{ (because } y \geq t_x !)} \cdot f(y)\,dy$$

$$\int_0^{t_x} \frac{y}{t_x}\cdot\frac{1}{t_y}\,dy + \int_{t_x}^{t_y} \frac{1}{t_y}\,dy = 1 - \frac{1}{2}\,\frac{t_x}{t_y}$$

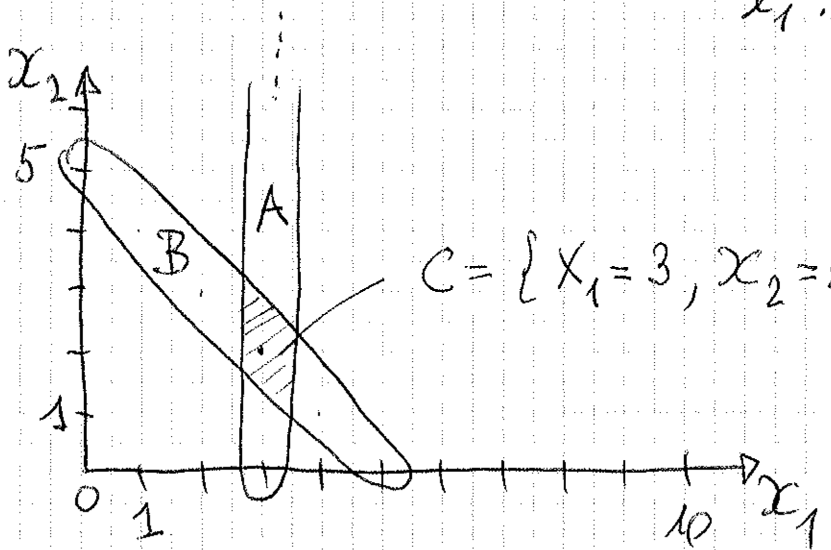
It follows that if $t_x = t_y$ we have a proba= bility of $\frac{1}{2}$ that X finisches before Y.
Otherwise ($t_y > t_x$) this probability will be $> \frac{1}{2}$!

Exercise 10/61

X_1 and X_2 are <u>independent</u> random variables with Poisson distribution ...

The joint probability mass function is given by

$$p(X_1 = x_1, X_2 = x_2) = \frac{\lambda_1^{x_1} \cdot e^{-\lambda_1}}{x_1!} \cdot \frac{\lambda_2^{x_2} \cdot e^{-\lambda_2}}{x_2!}$$



$C = \{X_1 = 3, x_2 = 5\} = A \cap B$

Let A the event " $X_1 = 3$ " and B the event " $X_1 + X_2 = 5$ "

$$p(X_1 = 3 \mid X_1 + X_2 = 5) = \frac{p(X_1 = 3, X_2 = 2)}{P(X_1 + X_2 = 5)} = \frac{P(A \cap B)}{P(B)}$$

$\Rightarrow$ in general we will obtain

$$P(X_1 = x \mid X_1 + X_2 = y) = \begin{cases} p(X_1 = x, X_2 = y - x) / P(X_1 + X_2 = y), & x \leq y \\ 0, & x > y \end{cases}$$

$$p(X_1 = x \mid X_1 + X_2 = y) = \frac{\dfrac{\lambda_1^{x} \cdot e^{-\lambda_1} \cdot \lambda_2^{y-x} \cdot e^{-\lambda_2}}{x! \cdot (y-x)!}}{\displaystyle\sum_{j=0}^{y} \frac{\lambda_1^{j} \cdot e^{-\lambda_1}}{j!} \cdot \frac{\lambda_2^{y-j} \cdot e^{-\lambda_2}}{(y-j)!}}$$

$$= \frac{\dfrac{\lambda_1^{x} \cdot \lambda_2^{y-x}}{x! \cdot (y-x)!}}{\displaystyle\sum_{j=0}^{y} \frac{y!}{y!} \cdot \frac{\lambda_1^{j}}{j!} \cdot \frac{\lambda_2^{y-j}}{(y-j)!}}$$

$$= \frac{\dfrac{y!}{x! \cdot (y-x)!} \cdot \lambda_1^{x} \cdot \lambda_2^{y-x}}{\underbrace{\displaystyle\sum_{j=0}^{y} \frac{y!}{j!(y-j)!} \cdot \lambda_1^{j} \cdot \lambda_2^{y-j}}_{(\lambda_1+\lambda_2)^y}}$$

$$= \frac{y!}{x!(y-x)!} \cdot \frac{\lambda_1^{x} \cdot \lambda_2^{y-x}}{(\lambda_1+\lambda_2)^y} = \frac{y!}{x!(y-x!)} \cdot p_1^{x} \cdot p_2^{y-x}$$

where $p_1 = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$ and $p_2 = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$

Exercise page 40/61

Assume the regression model

$$y_i = \theta + \varepsilon_i \qquad i = 1, \dots, n$$

(1)

$$\underset{(n\times 1)}{y} = \underset{(n\times 1)}{X}\,\underset{(1\times 1)}{\theta} + \underset{(n\times 1)}{\varepsilon}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}; \quad X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}; \quad \theta = [\theta_1]; \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

(2) In this case $X'X = n$ and $X'y = \sum y_i$.
From the general formula (on page 40/61)

$$\hat{\theta}(y) = (X'X)^{-1} X'y$$

we obtain

$$\hat{\theta}(y) = n^{-1} \sum y_i = \bar{y}$$

Exercise page 41/61

Assume the regression model

$$y_i = \theta_1 + \theta_2 x_i + \varepsilon_i \qquad i = 1, \dots, n$$

1

$$\underset{(n\times 1)}{y} = \underset{(n\times 2)}{X}\,\underset{(2\times 1)}{\theta} + \underset{n\times 1}{\varepsilon}$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} ; \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} ; \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

and $\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$

The <u>random</u> vector ε is not observed and the <u>deterministic</u> vector θ is unknown.

② From the first order conditions (see page 39/61)

$$\underbrace{X'X}_{2\times 2}\,\theta = \underbrace{X'y}_{(2\times 1)}$$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \cdot \theta = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \end{bmatrix}$$

$$\begin{cases} n \cdot \cancel{x}\,\theta_1 + \sum x_i \cdot \theta_2 = \sum y_i & (1) \\ \sum x_i \cdot \theta_1 + \sum x_i^2 \cdot \theta_2 = \sum y_i\, x_i & (2) \end{cases}$$

- Divide the first equation by n so that

$$\theta_1 + \bar{x} \cdot \theta_2 = \bar{y} \implies \hat{\theta}_1 = \bar{y} - \theta_2 \cdot \bar{x}$$

- Insert the previous result in (2) and note that

$$\sum (x_i - \bar{x})(y_i - \bar{y}) = \sum (x_i - \bar{x}) \cdot y_i$$
$$= \sum x_i \cdot y_i - \bar{x} \sum y_i$$
$$\sum (x_i - \bar{x})^2 = \sum (x_i - \bar{x})\, x_i = \sum x_i^2 - \bar{x} \cdot \sum x_i$$

to obtain the result

$$\hat{\theta}_2 = \frac{\sum (x_i - \bar{x})\, y_i}{\sum (x_i - \bar{x})^2}$$

Exercise 45/61

$$f(x_1 | \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}(x_1-\mu)^2}$$

where μ is known!

1) Compute the likelihood function of the sample
2) Compute the log-likelihood
3) Maximize the log-likelihood w.r.t. σ^2.

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1) $f(x_1, \dots, x_n | \sigma^2) = \prod_{i=1}^{n} f(x_i | \sigma^2)$

2) $\ln f(x_1, \dots, x_n | \sigma^2) = \sum_{i=1}^{n} \ln f(x_i | \sigma^2)$

$$\ln f(x_i | \sigma^2) = -\frac{1}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x_i-\mu)^2$$

$$\sum_{i=1}^{n} \ln f(x_i | \sigma^2) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2$$

3) $\frac{\partial}{\partial\sigma^2} \ln f(x_1, \dots, x_n | \sigma^2) =$

$$= -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \cdot \sum_{i=1}^{n} (x_i-\mu)^2 \overset{!}{=} 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i-\mu)^2$$

Exercise 48/61

In medical applications ...

The density function of a log-normal distributed R.V. is

$$f(x \mid \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0$$

and 0 otherwise.

$$E[X] = e^{\mu + \sigma^2/2} = g_1(\mu, \sigma^2)$$

$$V[X] = (e^{\sigma^2} - 1)\, e^{2\mu + \sigma^2} = g_2(\mu, \sigma^2)$$

Both moments are functions of (μ, σ^2). The invariance principle of the Maximum Likelihood estimator (see page 46/61) states that the ML estimates of $E[X]$ and $V[X]$ are equal to $E[X]$ and $V[X]$ evaluated at the ML estimates of μ and σ^2.

The log-Likelihood function of a sample of n independent observations x_i, $i = 1, \dots, n$ is

$$\ln f(x_1, \dots, x_n) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\sigma^2 - \sum_{i=1}^{n} \ln x_i + - \frac{1}{2\sigma^2} \cdot \sum_{i=1}^{n} (\ln x_i - \mu)^2$$

$$\frac{\partial}{\partial \mu} \ln f(x_1, \dots, x_n \mid \mu, \sigma^2) = \dots \stackrel{!}{=} 0$$

$$\frac{\partial}{\partial \sigma^2} \ln f(x_1, \dots, x_n \mid \mu, \sigma^2) = \dots \stackrel{!}{=} 0$$

$$\Longrightarrow \hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \ln x_i \text{ and } \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (\ln x_i - \hat{\mu})^2$$

The ML-estimator of $E[X]$ is

$$g_1(\hat{\mu}, \hat{\sigma}^2) = e^{\hat{\mu} + \hat{\sigma}^2/2}$$

The ML-estimator of $V[X]$ is

$$g_2(\hat{\mu}, \hat{\sigma}^2) = \left(e^{\hat{\sigma}^2} - 1\right) e^{2\hat{\mu} + \hat{\sigma}^2}$$

Exercise 51/61

Check that the ML-estimator of β is

$$\beta^* = \sum_{i=1}^{n} |x_i| / n$$

Steps:

1) Compute the joint density function of $X_1, \dots, X_n$ (Likelihood)
2) Compute the natural logarithm (log-likelihood)
3) Maximize the log-likelihood function with respect to β.

1) $$f(x_1, \dots, x_n \mid \beta) = \prod_{i=1}^{n} \frac{1}{2\beta} \cdot e^{-|x_i|/\beta}$$

2) $$\ln f(x_1, \dots, x_n \mid \beta) = -n \ln 2\beta - \sum_{i=1}^{n} \frac{|x_i|}{\beta}$$

3) $$\frac{\partial \ln f}{\partial \beta} = -n \cdot \frac{1}{\beta} + \frac{\sum |x_i|}{\beta^2} \overset{!}{=} 0$$

solve the equation

$$\beta^* = \frac{\sum |x_i|}{n}$$

Exercise 55/61

1) i) simple
ii) composite
iii) composite
iv) composite

2) i) composite
ii) simple
iii) composite
iv) simple