Simulation of Random Variables

Exercise page 15

Solution

Let U_1 and U_2 two uniform $(0,1)$ distributed random variable and define

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = g(U_1, U_2) = \begin{pmatrix} \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \\ \sqrt{-2\ln(U_1)}\sin(2\pi U_2) \end{pmatrix}.$$

We show that Y_1 and Y_2 are two independent standard normally distributed random variables.

1. Compute g^{-1}

   $$\begin{cases} Y_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \\ Y_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2) \end{cases}$$

   Divide the second equation by the first one.

   $$\begin{cases} Y_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \\ Y_2/Y_1 = \tan(2\pi U_2) \end{cases} \quad \text{i.e.}$$

   $$\begin{cases} Y_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \\ U_2 = \frac{1}{2\pi}\arctan(Y_2/Y_1) \end{cases}$$

   Square the first equation and insert U_2 from the second:

   $$\begin{cases} Y_1^2 = -2\ln(U_1)\left(\cos(\arctan(Y_2/Y_1))\right)^2 \\ U_2 = \frac{1}{2\pi}\arctan(Y_2/Y_1) \end{cases}$$

   Recall that $\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$ so that we can rewrite the previous system of equations as

   $$\begin{cases} Y_1^2 = -2\ln(U_1)\frac{Y_1^2}{Y_1^2+Y_2^2} \\ U_2 = \frac{1}{2\pi}\arctan(Y_2/Y_1) \end{cases} \quad \text{i.e.}$$

   $$\begin{cases} U_1 = \exp\left(-\frac{1}{2}\left(Y_1^2+Y_2^2\right)\right) \\ U_2 = \frac{1}{2\pi}\arctan(Y_2/Y_1) \end{cases} = \begin{cases} U_1 = g_1^{-1}(Y_1, Y_2) \\ U_2 = g_2^{-1}(Y_1, Y_2) \end{cases} = g^{-1}(Y_1, Y_2).$$

2. Compute $J(g^{-1}(Y))$

   Recall that $\frac{\partial \arctan(x)}{\partial x} = \frac{1}{1+x^2}$

   $$\frac{\partial g_1^{-1}}{\partial y_1} = -y_1 \exp(-\tfrac{1}{2}\left(Y_1^2+Y_2^2\right))$$

   $$\frac{\partial g_1^{-1}}{\partial y_2} = -y_2 \exp(-\tfrac{1}{2}\left(Y_1^2+Y_2^2\right))$$

   $$\frac{\partial g_2^{-1}}{\partial y_1} = -\frac{1}{2\pi}\frac{1}{1+y_2^2/y_1^2}\frac{y_2}{y_1^2} = -\frac{y_2}{2\pi}\frac{1}{y_1^2+y_2^2}$$

   $$\frac{\partial g_2^{-1}}{\partial y_2} = \frac{y_1}{2\pi}\frac{1}{y_1^2+y_2^2}$$

$$M_J = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial y_1} & \frac{\partial g_1^{-1}}{\partial y_2} \\ \frac{\partial g_2^{-1}}{\partial y_1} & \frac{\partial g_2^{-1}}{\partial y_2} \end{bmatrix}.$$

$\det(M_J) = -\frac{1}{2\pi}\exp(-\frac{1}{2}\left(Y_1^2 + Y_2^2\right))$.

3. Compute the joint density function of Y_1 and Y_2

   Recall that $f_U(u_{1,}u_2) = 1$ for $0 < u_1 < 1$, $0 < u_2 < 1$ and 0 otherwise so that

$$\begin{aligned} f_Y(y_1, y_2) &= f_U(g^{-1}(y_1, y_2)) \mid J(g^{-1}(y_1, y_2)) \mid \\ &= 1\ \frac{1}{2\pi}\exp(-\frac{1}{2}\left(y_1^2 + y_2^2\right)) \\ &= \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y_1^2)\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y_2^2). \end{aligned}$$