On Generating the Ordered Mixed Quadratic Surds over Nonnegative Integers

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Abstract

The set of mixed quadratic surds over nonnegative integers is well-ordered. Finding an iterative formula in order to generate them from smallest to largest is not easy, and likely uncomputably large. Unfortunately, the known method for generating them algorithmically is also very slow. We abstract the pattern by restricting it to a subset of allowed integral parts. We conjecture a pattern for splitting up the discrete signal dependent on the highest allowed integer of the mixed quadratic surd. That discrete signal is then to be converted into a formula for finding the success values for the integer aspect for the special integral cases: 0, 1, 2. Then we make conjectures about the form.

I. Introduction

There is a set of numbers called the constructible numbers. They are precisely the set of numbers: $\{r + \sqrt{s},\ with\, r, s \in \mathbb{Q}_0\}$.
The set of mixed quadratic surds over nonnegative integers may be written as:

$$\{\alpha + \sqrt{\beta},\ with\, \alpha, \beta \in \aleph_{+0}\}.$$

This subset can be ordered from the smallest to the next smallest and is thus well-ordered unlike the set of constructible numbers.

The first 10 ordered values of our set are: $0,\ 1,\ \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, 1 + \sqrt{2},\ \sqrt{6},\ \sqrt{7}, 1 + \sqrt{3}$.

II. Experimental Procedure

We generated the sequence for a restricted $\alpha \leq 4$. Then we broke up the signals by surds and integers. They were broken up with identical length in units. Then we attempted Mathematica's FindSequenceFunction on the starts and ends of breaks.

```
p[a_Integer] = {a, 0};
p[a_. + u_. Sqrt[b_Integer]] = {a, u^2 b};
GetIntegerPart[z_] := p[z][[1]];
GetSurdPart[z_] := Inactive[Sqrt][p[z][[2]]];

sortedMixedQuadraticSurds=
Sort[Flatten[Table[Table[a+Sqrt[n+Floor[0.5 + Sqrt[n]]],{n,0,40000}],{a,0,α}]],Less];

orderedIntegerParts=Activate[Table[GetIntegerPart[sortedMixedQuadraticSurds[[i]]],{i,1,1200}]];
orderedSurdParts=Activate[Table[GetIntegerPart[sortedMixedQuadraticSurds[[i]]],{i,1,1200}]];
```

III. Results

Table 1:

Rasterize[Partition[Table[ListLinePlot[

Take[orderedIntegerParts,{$2n^2 + 3, 4 + 4n + 2n^2$}], Axes->False,Filling->Axis],{n,1,10}],3]//Grid]

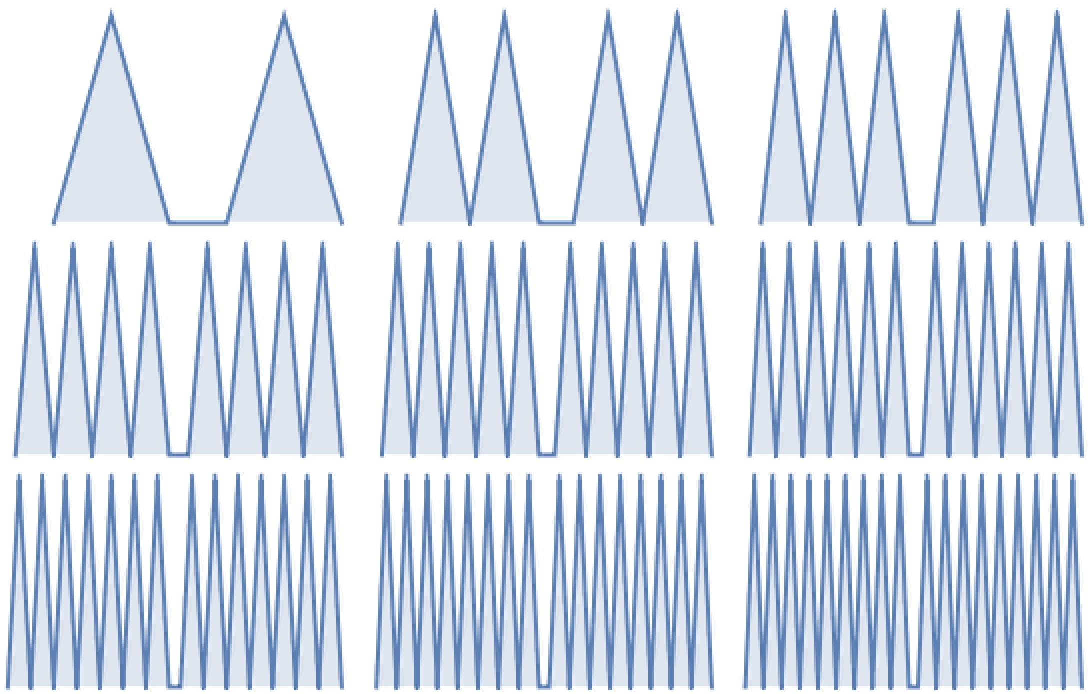

Figure 1: This is the figure created by Table 1 with $\alpha \in \{0, 1\}$. It only shows the pattern of the integer part.

Table 2:

Rasterize[Partition[Table[ListLinePlot[

Take[orderedIntegerParts,{$6 - 3n + 3n^2, 5 + 3n + 3n^2$}], Axes->False,Filling->Axis],{n,1,10}],3]//Grid]

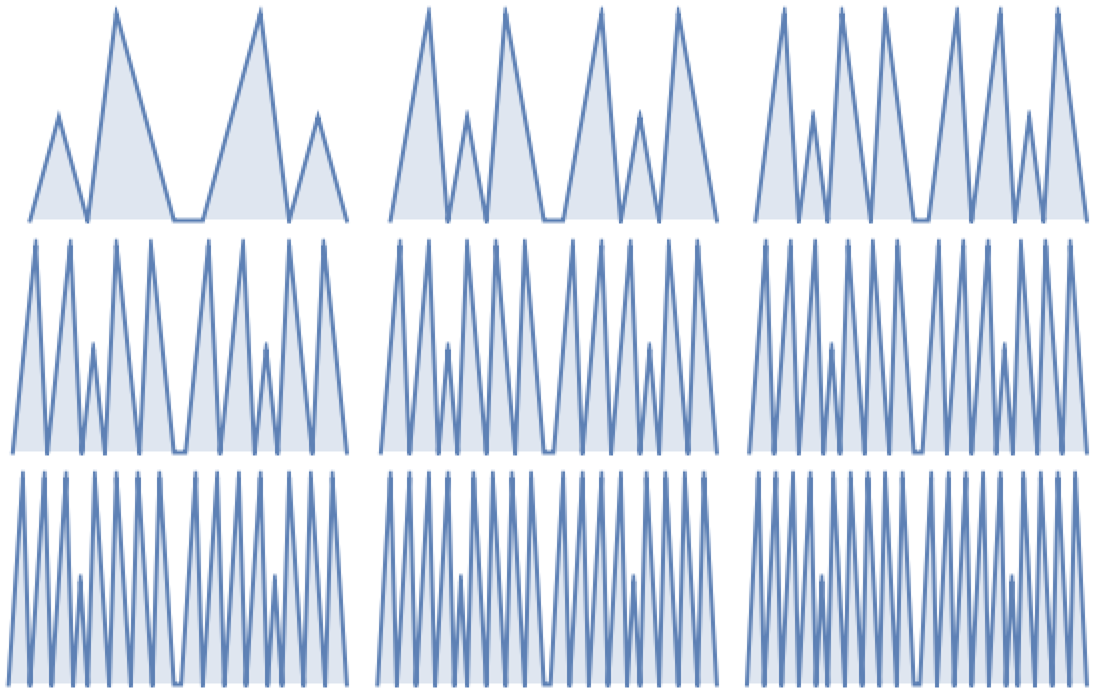

Figure 2: This is the figure created by Table 2 with $\alpha \in \{0, 1, 2\}$. It only shows the pattern of the integer part.

Table 3:

Rasterize[Partition[Table[ListLinePlot[

Take[orderedIntegerParts,{$13 - 8n + 4n^2$, $8 + 4n^2$ }], Axes->False,Filling->Axis],{n,4,12}],3]//Grid]

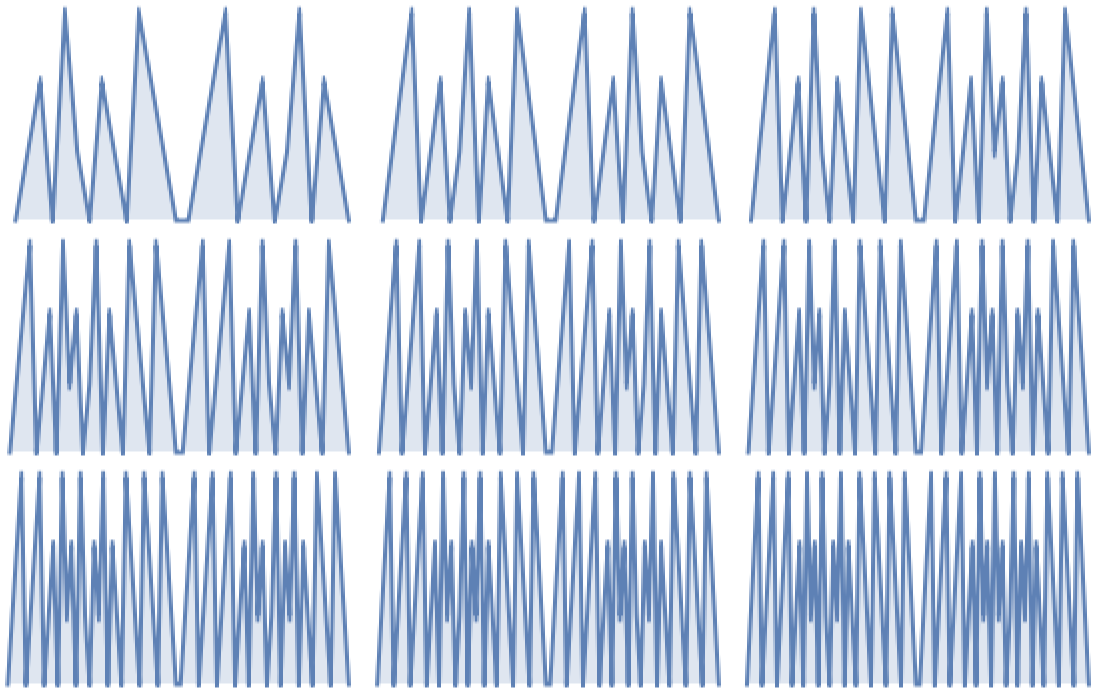

Figure 3: This is the figure created by Table 3 with $\alpha \in \{0, 1, 2, 3\}$. It only shows the pattern of the integer part.

The next Tables, Tables 4 and 5, generate the sequences of integers from the spliced digit signal when $max(\alpha) = 1, 2$.

Table 4 (binary):

FindSequenceFunction[Table[FromDigits[Take[orderedIntegerParts,{$2n^2 + 3, 4 + 4n + 2n^2$}],2],{n,0,15}],t];

Table 5 (ternary):

FindSequenceFunction[Table[FromDigits[Take[orderedIntegerParts,{$6 - 3n + 3n^2, 5 + 3n + 3n^2$}],3],{n,1,15}],t];

IV. Conjectures

The conjectured formula for the endpoints of the discrete interval that breaks up the polynomial that is dependent on $max(\alpha)$, which we simply call α, is:

$$\{f_{\alpha_1} + f_{\alpha_2}t + f_{\alpha_3}t^2, f_{\alpha_1} + f_{\alpha_2}(t + 1) + f_{\alpha_3}(t + 1)^2 - 1\},\ t > \alpha\},\ \text{(Form. 1)},$$

with,

$$f_{\alpha_1} := \frac{(-1)^{-\alpha}(126(-1+(-1)^{2\alpha})+6(-16+45(-1)^{\alpha}+19(-1)^{2\alpha})\alpha+(292+375(-1)^{\alpha}-259(-1)^{2\alpha})\alpha^2+(344+206(-1)^{\alpha}-326(-1)^{2\alpha})\alpha^3+(128+67(-1)^{\alpha}-125(-1)^{2\alpha})\alpha^4-16(-1-(-1)^{2\alpha})\alpha^5+2(-1)^{\alpha}\alpha^6}{24\alpha(2+\alpha)(3+\alpha)}$$

$$f_{\alpha_2} := \frac{(1+(-1)^{\alpha})(1+\alpha)}{2}$$

$$f_{\alpha_3} := \alpha + 1;$$

Converting the separate signals using Table 4(1) and Table 5(2) from binary and ternary gave us the respective conjectured Formulas 2 and 3. We could not calculate it for Table 3.

$$\frac{-8 - 2^{2t+1} + 2^{4t}}{12},\ t \geq 0.\ \text{(Form. 2)}$$

Gives

$$\frac{-63-2\cdot 3^{\frac{3t}{2}}-2(-1)^t 3^{\frac{3t}{2}}-2\cdot 3^{\frac{3+3t}{2}}+2(-1)^t 3^{\frac{3+3t}{2}}-16\cdot 3^{1+3t}-2\cdot 3^{\frac{3+9t}{2}}+2(-1)^{\frac{6+9t}{2}}-2(-1)^t 3^{\frac{3+9t}{2}}+5\cdot 3^{1+6t}}{78},\ t \geq 1.\ \text{(Form. 3)}$$

If a generative formula that can be used for all of them is discovered generated from base $\alpha + 1$, then a digit extraction formula can be used to generate the sequence of mixed quadratic surds over nonnegative integers provided that one makes sure that the α is high enough depending on which n-th value is to be calculated.