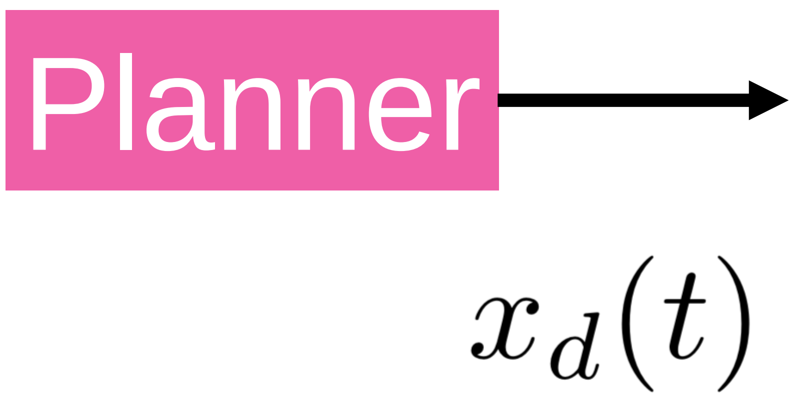
Kinematics and dynamics of the double pendulum

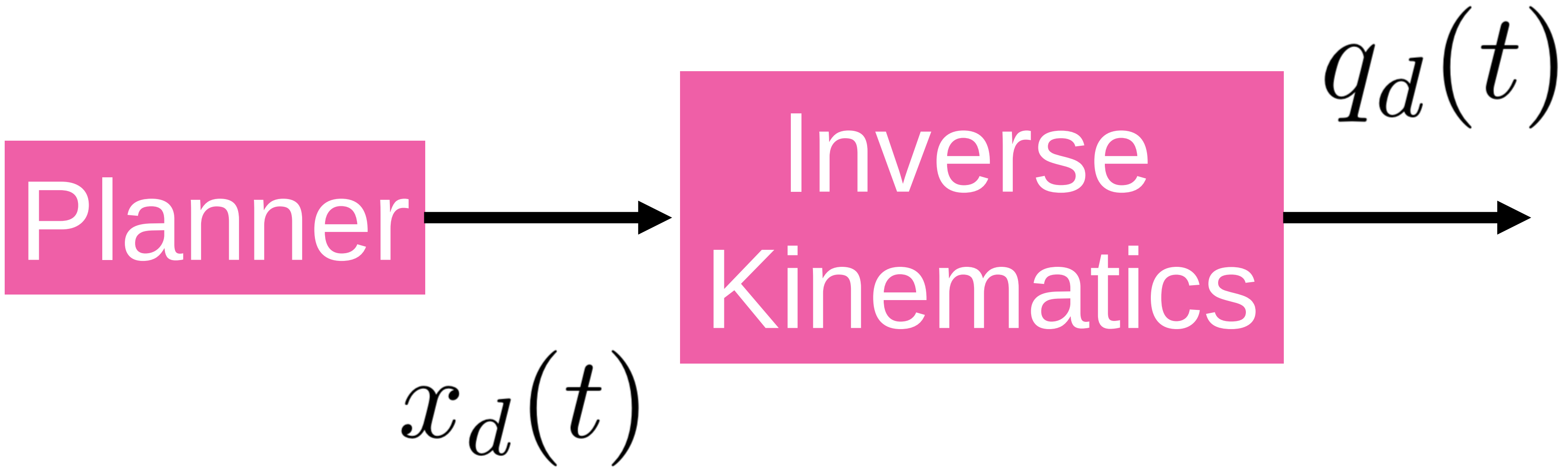
Legged Robots Course

Control design pipeline

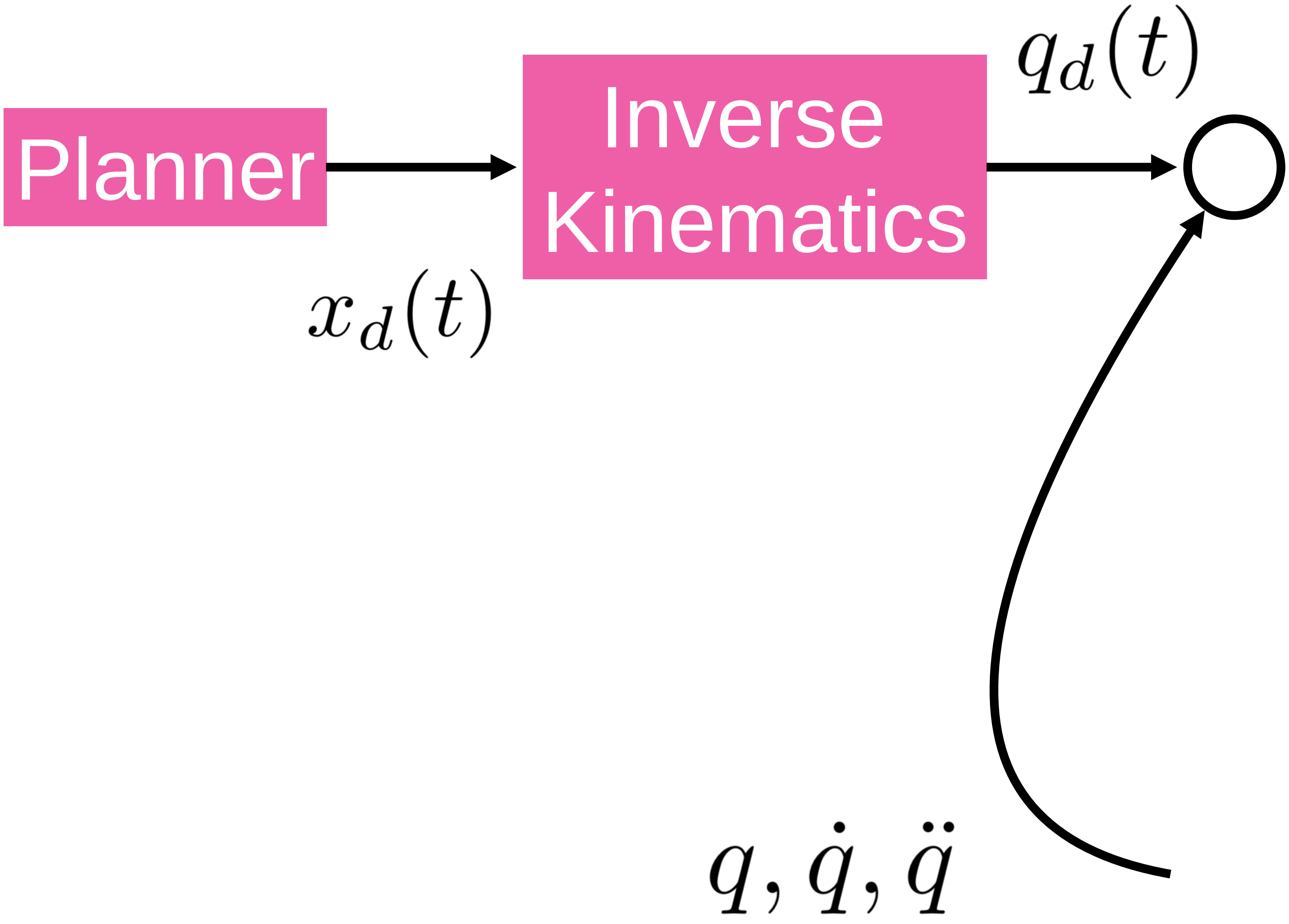
Control design pipeline



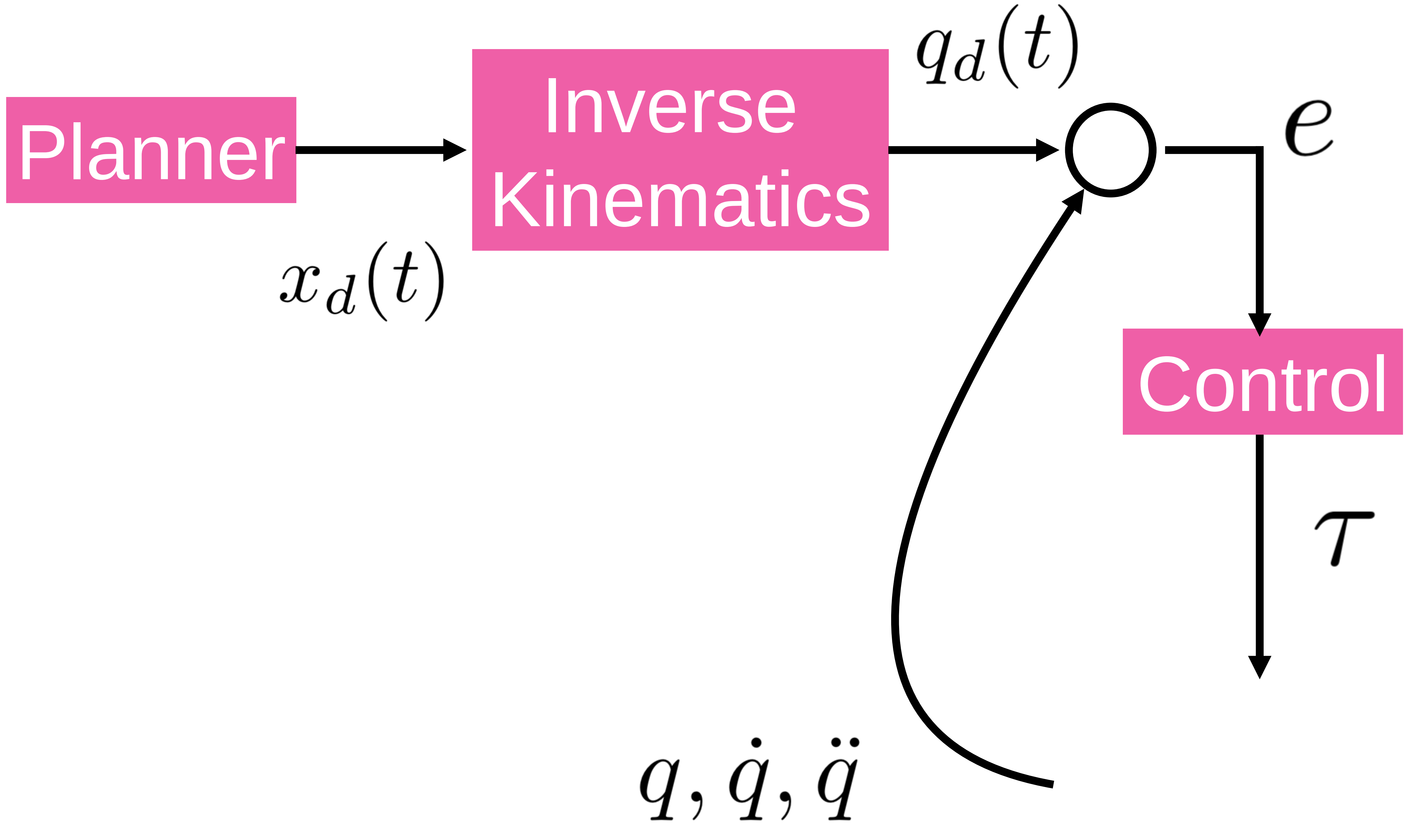
Control design pipeline



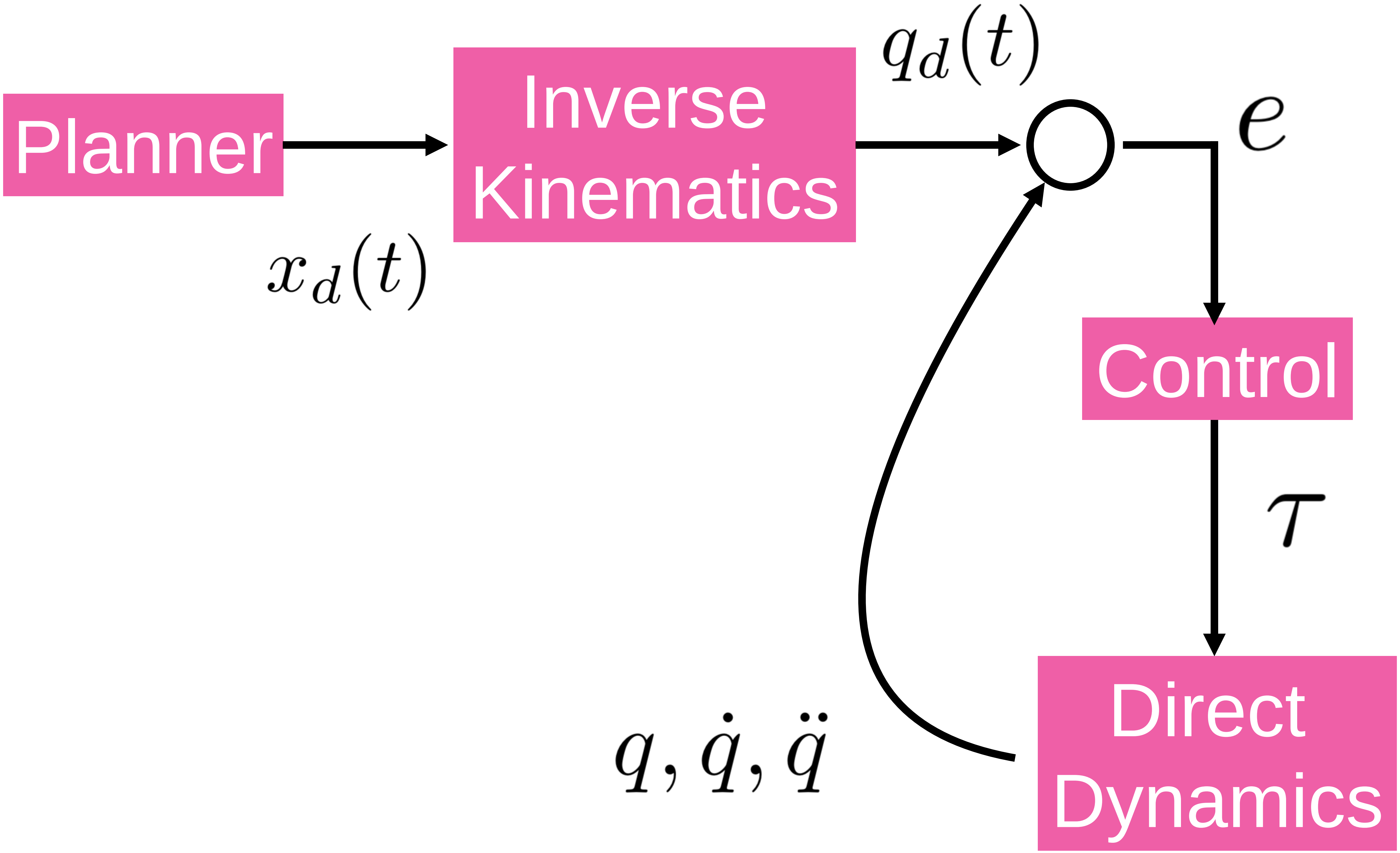
Control design pipeline



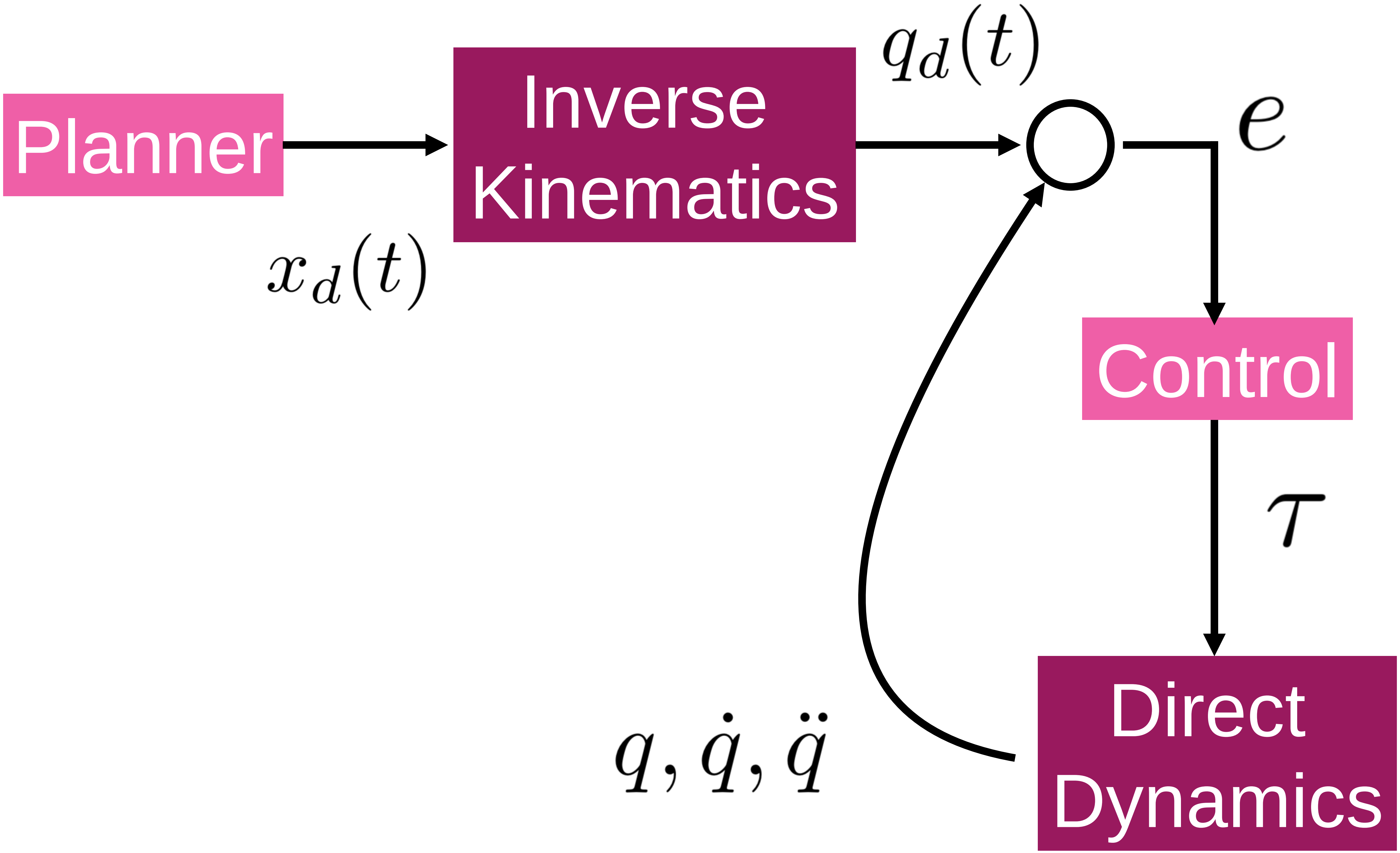
Control design pipeline



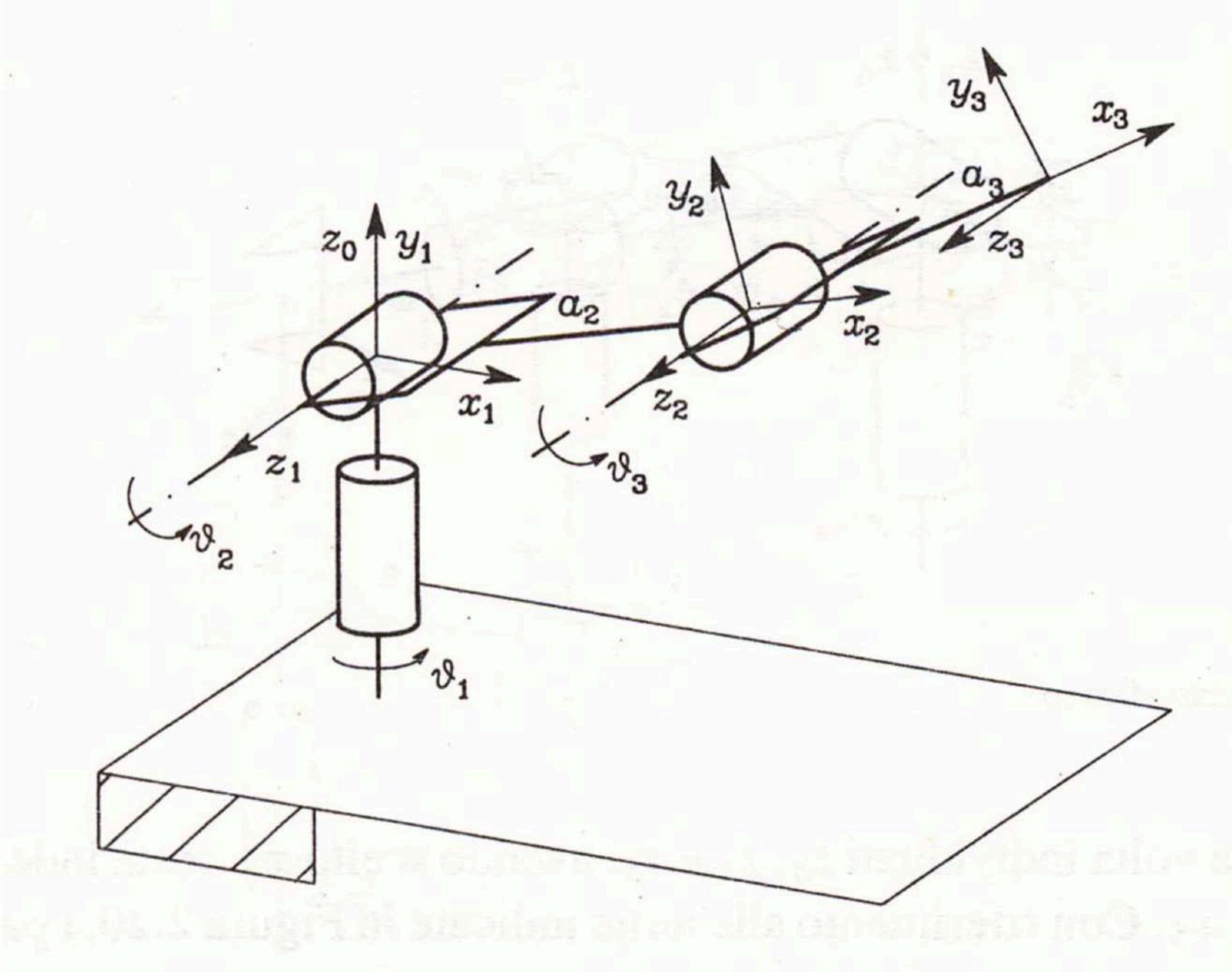
Control design pipeline



Control design pipeline



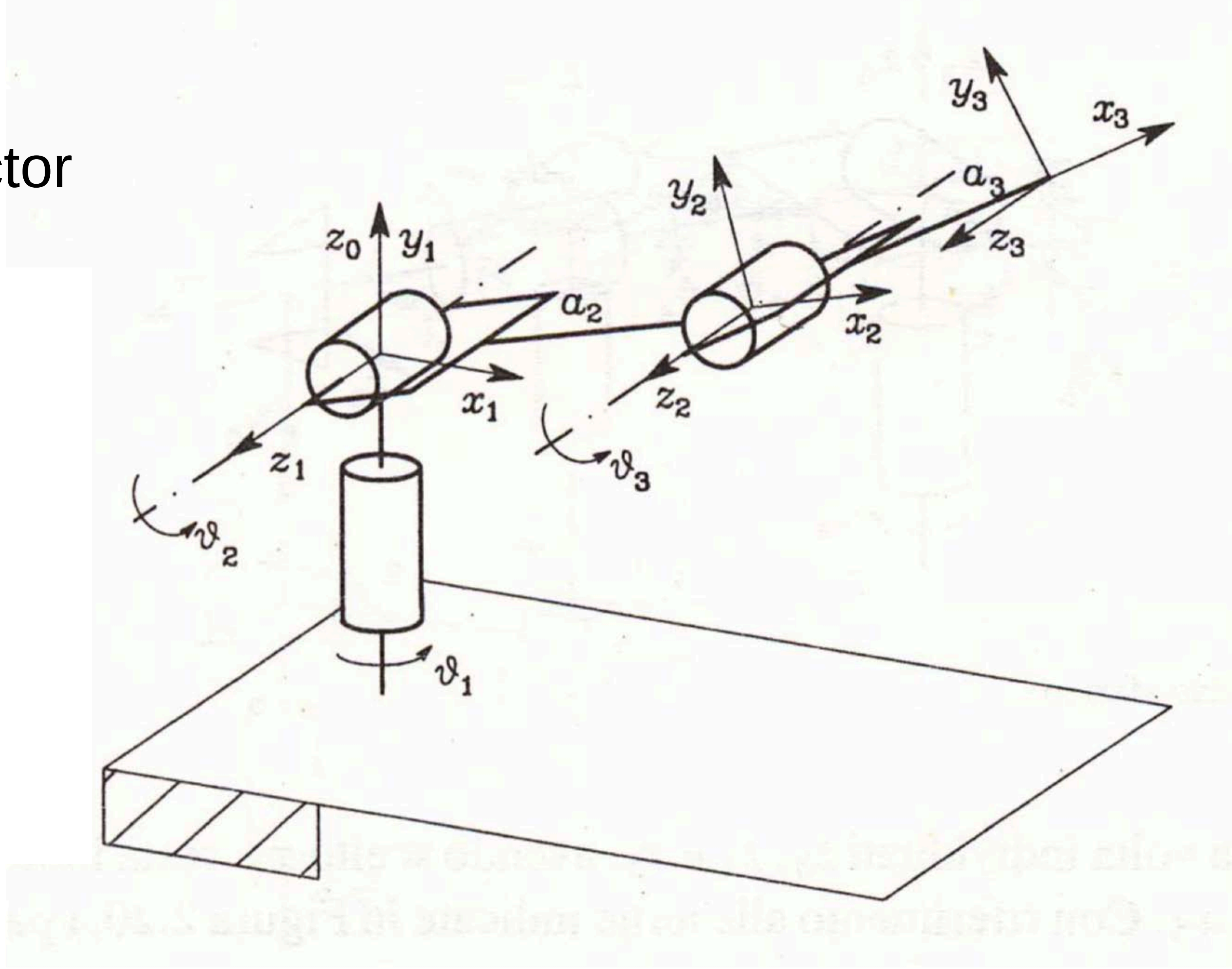
Kinematics



y_3
x_3
a_3
y_2
z_3
z_0
y_1
a_2
x_2
x_1
z_2
z_1
ϑ_3
ϑ_2
ϑ_1

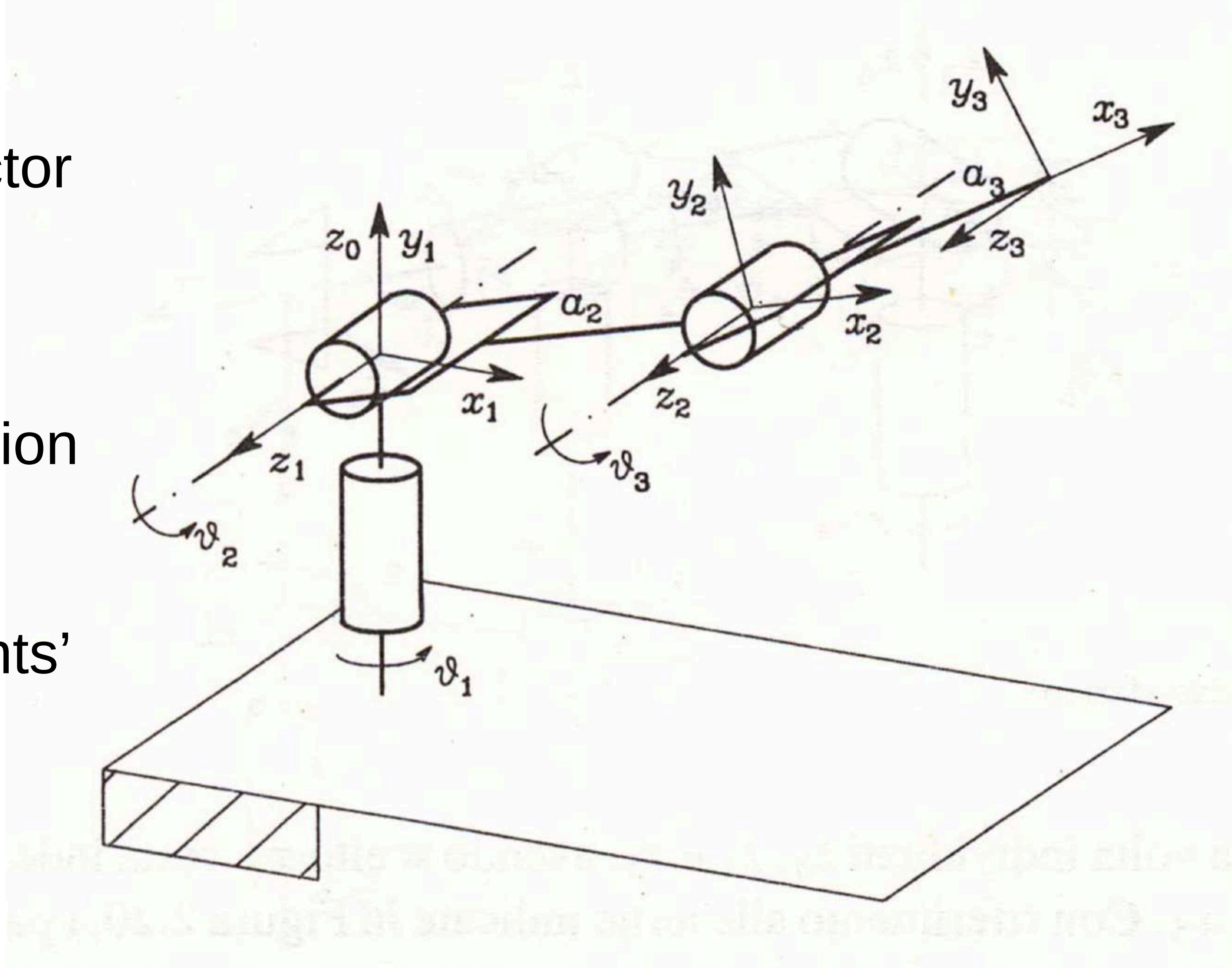
Kinematics

- **Direct Kinematics**:
  - Input: joints' angles
  - Output: position and orientation of the end effector

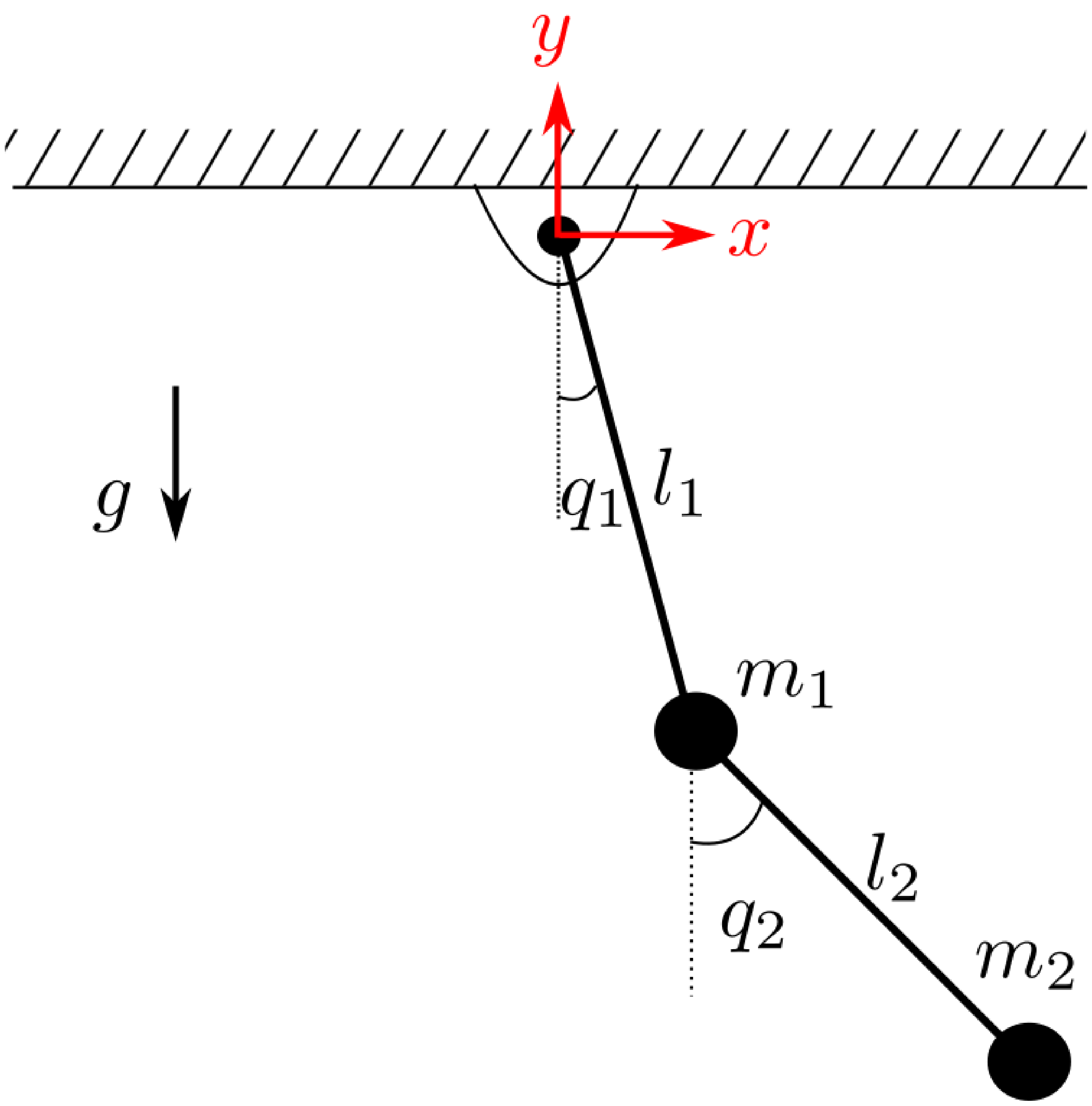


Kinematics

- **Direct Kinematics**:
  - Input: joints' angles
  - Output: position and orientation of the end effector
- **Inverse Kinematics**:
  - Input: position and orientation of the end effector
  - Output: all the possible joints' angles combinations



Direct kinematics

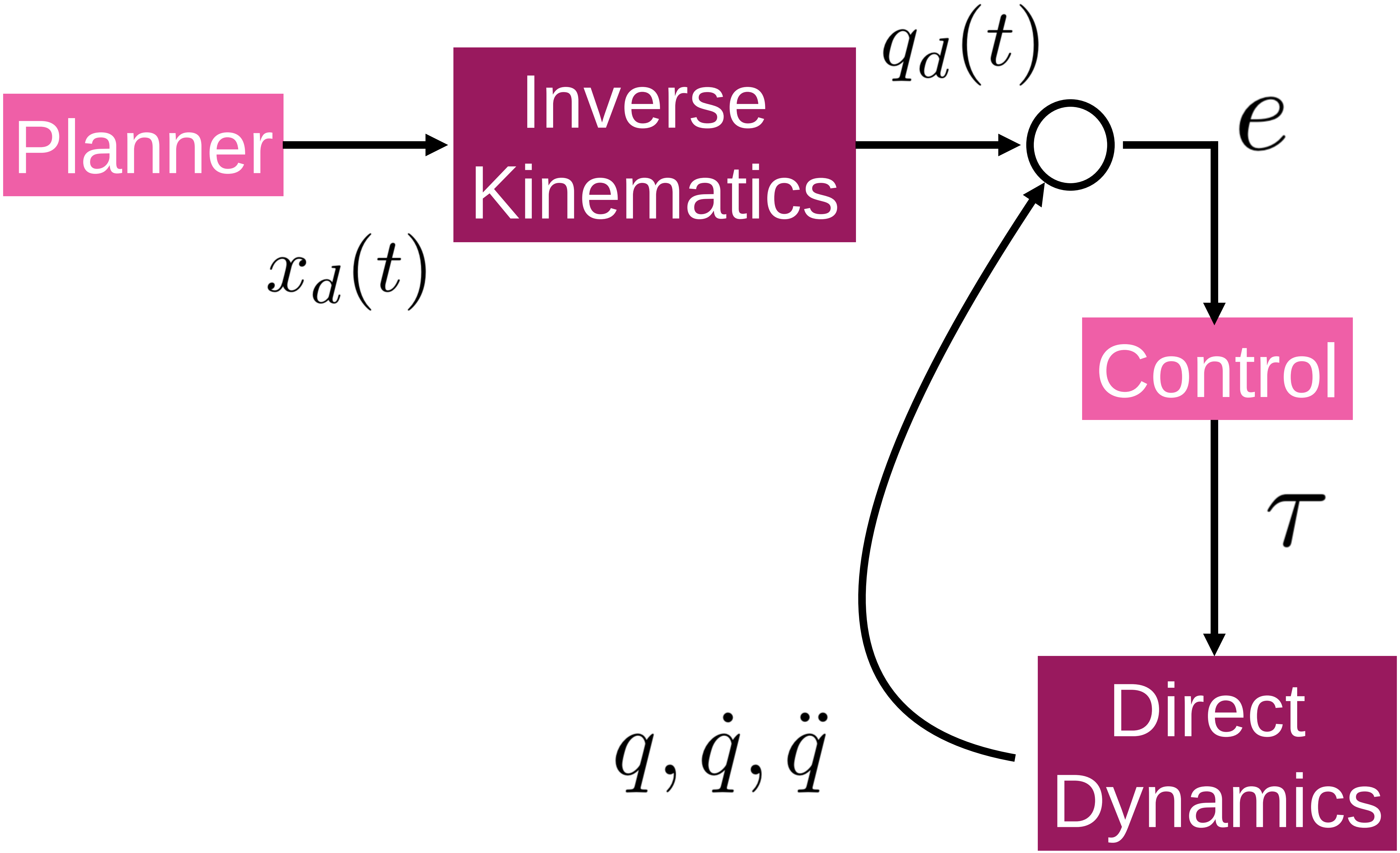


Invert these relationships to obtain the inverse kinematics model (unique?)

$$x_1(q_1) = l_1 sin(q_1(t))$$
$$y_1(q_1) = -l_1 cos(q_1(t))$$
$$x_2(q_1, q_2) = l_1 sin(q_1(t)) + l_2 sin(q_2(t))$$
$$y_2(q_1, q_2) = -l_1 cos(q_1(t)) - l_2 cos(q_2(t))$$

$$\dot{x}_1(q_1, \dot{q}_1) = l_1 cos(q_1(t))\dot{q}_1(t) \qquad \dot{x} = \frac{dx}{dt}$$
$$\dot{y}_1(q_1, \dot{q}_1) = l_1 sin(q_1(t))\dot{q}_1(t)$$
$$\dot{x}_2(\dots) = l_1 cos(q_1(t))\dot{q}_1(t) + l_2 cos(q_2(t))\dot{q}_2(t)$$
$$\dot{y}_2(\dots) = l_1 sin(q_1(t))\dot{q}_1(t) + l_2 sin(q_2(t))\dot{q}_2(t)$$

Control design pipeline



Dynamics

- Dynamics studies the relation between the **joint actuator torques** and the **resulting motion**

Dynamics

- Dynamics studies the relation between the **joint actuator torques** and the **resulting motion**
- **Inverse Dynamics (used for designing controllers):**

$$\boldsymbol{f} = \boldsymbol{m}\ddot{\boldsymbol{x}}$$

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

- **Direct (Forward) Dynamics (use for simulation)**

$$\ddot{\boldsymbol{x}} = \frac{\boldsymbol{f}}{\boldsymbol{m}}$$

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}))$$

Equations of motion

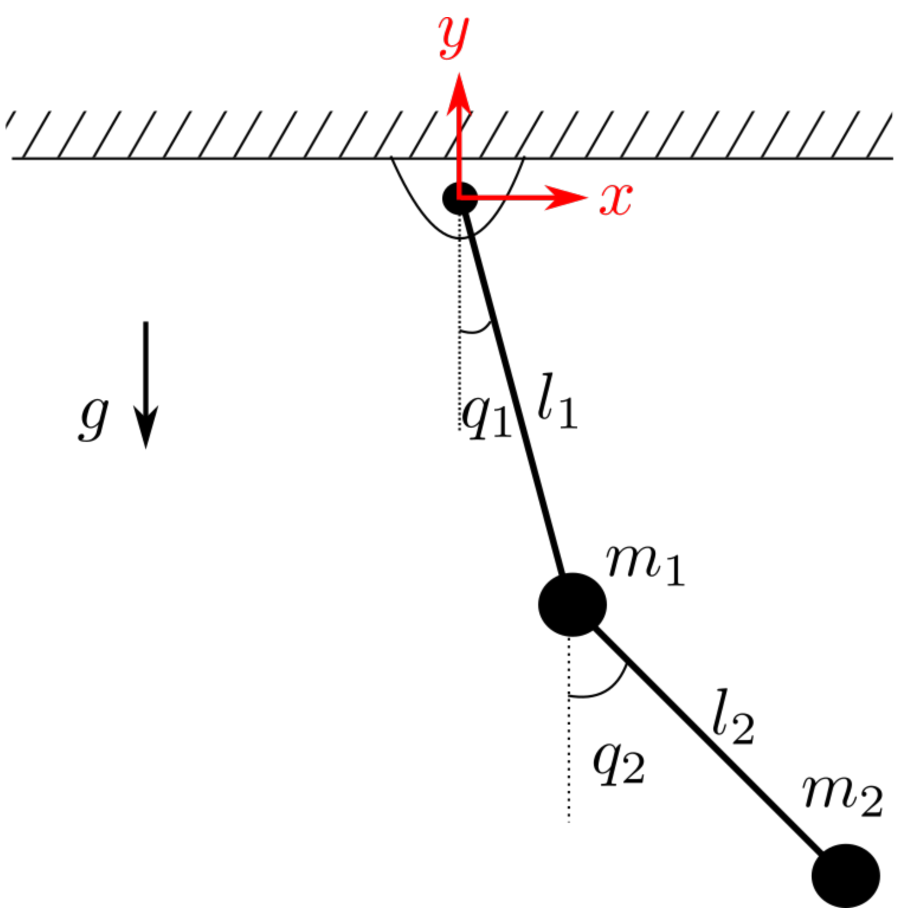
- **Lagrangian method:**
  - Variational method based on kinetics and potential energy

$$L(q,\dot{q}) = T(q,\dot{q}) - V(q)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

- **Newton-Euler recursive method**
  - Relies on $F = ma$ applied to each individual link of the robot

Dynamics



Potential Energy

$$\boldsymbol{q} = (\boldsymbol{q_1}, \boldsymbol{q_2})$$

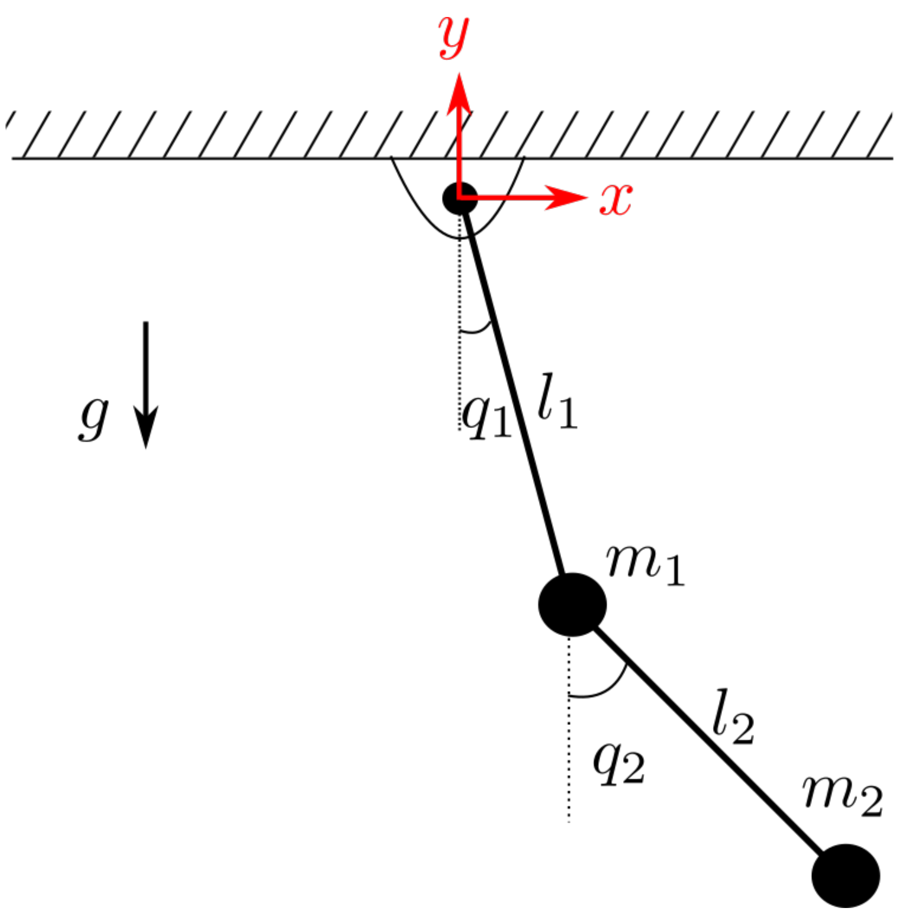
$$\boldsymbol{V}(\boldsymbol{q}) = \boldsymbol{V_1}(\boldsymbol{q}) + \boldsymbol{V_2}(\boldsymbol{q})$$

$$\boldsymbol{V_1}(\boldsymbol{q}) = \boldsymbol{m_1 g y_1} = -\boldsymbol{m_1 g l_1 cos}(\boldsymbol{q_1})$$

$$\boldsymbol{V_2}(\boldsymbol{q}) = \boldsymbol{m_2 g y_2} = \cdots$$

Potential energy is always a function of q and not $\dot{q}$!

Dynamics



Kinetic Energy

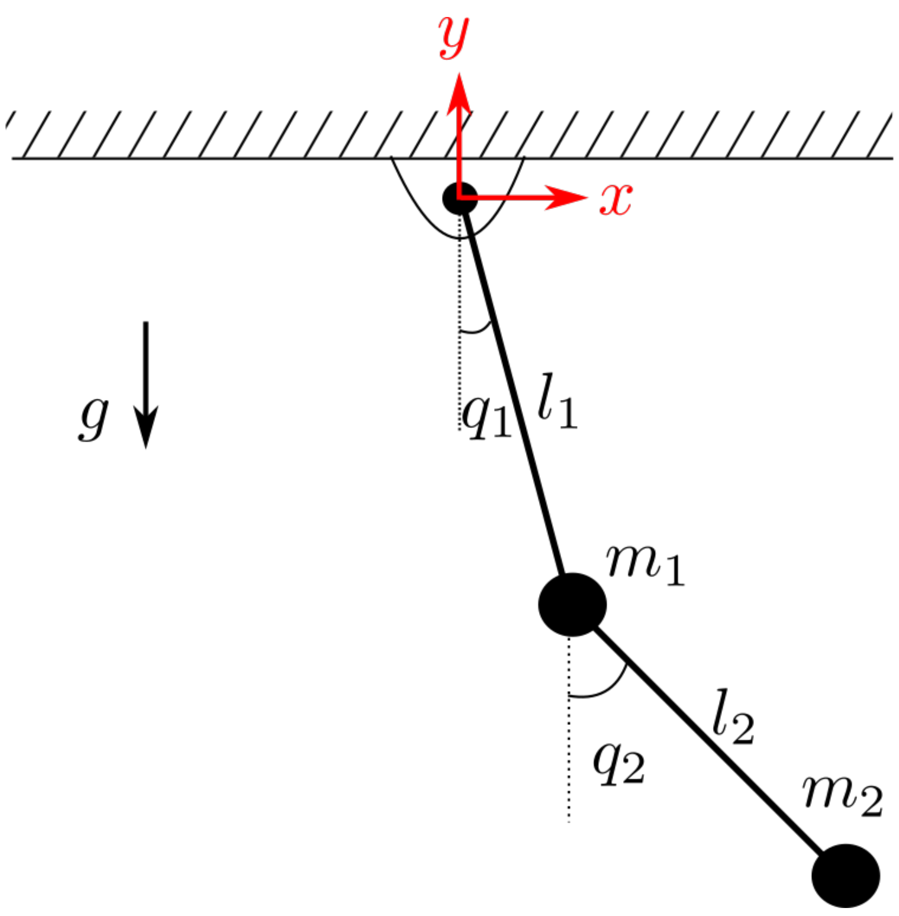
$$\boldsymbol{T(q,\dot{q}) = T_1(q,\dot{q}) + T_2(q,\dot{q})}$$

$$\boldsymbol{T_1(q,\dot{q}) = \frac{1}{2}m_1\left(\dot{x}_1{}^2 + \dot{y}_1{}^2\right) = \frac{1}{2}m_1 l_1^2\,\dot{q}_1^2}$$

$$\boldsymbol{T_2(q,\dot{q}) = \frac{1}{2}m_2\left(\dot{x}_2{}^2 + \dot{y}_2{}^2\right) = \cdots}$$

Kinetic energy is always a function of q and $\dot{q}$!

Dynamics



Equation of Motion

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}_1}\right) - \frac{\partial L(q,\dot{q})}{\partial q_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}_2}\right) - \frac{\partial L(q,\dot{q})}{\partial q_2} = 0$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\frac{\partial T(q,\dot{q})}{\partial \dot{q}_1}\right) - \frac{\partial T(q,\dot{q})}{\partial q_1} + \frac{\partial V(q)}{\partial q_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T(q,\dot{q})}{\partial \dot{q}_2}\right) - \frac{\partial T(q,\dot{q})}{\partial q_2} + \frac{\partial V(q)}{\partial q_2} = 0$$

Dynamics

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}_1}) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}_2}) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

$$\boldsymbol{q} = (\boldsymbol{\theta_1}, \boldsymbol{\theta_2})$$

$$\frac{\partial T_1}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1$$

$$\frac{\partial T_2}{\partial \dot{\theta}_1} = m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial T_1}{\partial \theta_1} = 0$$

$$\frac{\partial T_2}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial V_1}{\partial \theta_1} = m_1 g l_1 \sin(\theta_1)$$

$$\frac{\partial V_2}{\partial \theta_1} = m_2 g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

Dynamics

$$\frac{\partial T_1}{\partial \dot\theta_2} = 0$$

$$\frac{\partial T_2}{\partial \dot\theta_2} = m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial T_1}{\partial \theta_2} = 0$$

$$\frac{\partial T_2}{\partial \theta_2} = m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial V_1}{\partial \theta_2} = 0$$

$$\frac{\partial V_2}{\partial \theta_2} = m_2 g l_2 \sin(\theta_1)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
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$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q)\ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q)\ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$m_1 l_1^2 \ddot\theta_1 + m_2 l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot\theta_2 - m_2 l_1 l_2 \dot\theta_2 (\dot\theta_1 - \dot\theta_2)\sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot\theta_1 - m_2 l_1 l_2 \dot\theta_1 (\dot\theta_1 - \dot\theta_2)\sin(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$C(q, \dot q)\, \dot q$$

$$\tau = B(q)\ddot q + C(q, \dot q) + g(q)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$C(q, \dot{q})\, \dot{q}$$

$$\tau = B(q)\ddot{q} + C(q, \dot{q}) + g(q)$$