Lecture 1 | Introduction

① Let $1=\{*\}$ be one-element set. Then for every set A, there is a unique function $A \longrightarrow 1$ given by

$$a \longmapsto *$$

$\forall a \in A$

② Let $\mathbb{Z}$ be ring of integers. Again, for every ring R there is a unique ring homomorph. $\mathbb{Z} \to R$ given by

$$n \longmapsto \begin{cases} \underbrace{1_R+\dots+1_R}_{n \text{ times}}, & n>0 \\ 0, & n=0 \\ -(\underbrace{1_R+\dots+1_R}_{-n \text{ times}}), & n<0 \end{cases}$$

where 1_R is multiplicative identity of R

① and ② are examples of universal properties

Def (Category)

A category of $\mathbb{C}$ consists of

- Collection of objects
$$Ob(\mathbb{C})$$
- Collection of maps between objects
$\forall A,B \in Ob(\mathbb{C})$
$$\mathbb{C}(A,B)$$
- Composition function
$\forall A,B,C \in Ob(\mathbb{C})$
$$\mathbb{C}(B,C)\times\mathbb{C}(A,B) \xrightarrow{\circ} \mathbb{C}(A,C)$$
$$(g,f) \longmapsto g\circ f$$
where $\begin{cases} g: B \to C \\ f: A \to B \end{cases}$
- Identity map for all objects
$\forall A \in Ob(\mathbb{C})$
$$1_A \in \mathbb{C}(A,A)$$

Subject to following axioms

1) Associativity

- $\forall A,B,C,D \in Ob(\mathbb{C})$
- $\forall f \in \mathbb{C}(A,B)$
- $\forall g \in \mathbb{C}(B,C)$
- $\forall h \in \mathbb{C}(C,D)$

$$h\circ(g\circ f)=(h\circ g)\circ f$$

2) Unit

$\forall A,B \in Ob(\mathbb{C})$, $\forall f \in \mathbb{C}(A,B)$

$$f\circ 1_A = f \ \& \ 1_B\circ f = f$$

Commutative diagrams

Let $\mathbb{C}$ be a category.

(I) For each string of maps

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_n$$

there is in general a unique map

$$A_0 \xrightarrow{f_n f_{n-1} \dots f_2 f_1} A_n$$

(II) Pictures like

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"'] & & B \arrow[d, "g"] \\ D \arrow[r, "j"] & E \arrow[r, "k"] & C \end{tikzcd}$$

are commutative if $gf = kjh$

Other examples

Note: Objects of category need not be sets and maps need not be functi

Example $\Sigma_1(G)$, G-group

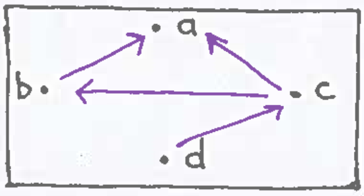
Let G be a group. Define a category $\Sigma_1(G)$ as follo

- Objects
$$Ob(\Sigma_1(G))=\{*\}$$
- Maps
$$\underbrace{\Sigma_1(G)(*,*)}_{\text{maps are elements of group } G} = G$$
- Composition func.
$$\Sigma_1(G)(*,*)\times\Sigma_1(G)(*,*) \longrightarrow \underbrace{G}_{=\Sigma_1(G)(*,*)}$$
$$G \times G \longrightarrow G$$
$$(g,h) \longmapsto gh$$
- Identity map
$$1 \in \Sigma_1(G)(*,*)$$
Here 1 is the identity of G (i.e $1 \in G$)

Lecture 2 | Examples

Let $(P, \leq)$ be a poset

1) • $a \leq a$
2) • $a \leq b$ & $b \leq c \Rightarrow a \leq c$
3) • $a \leq b$ & $b \leq a \Rightarrow a = b$

P =


$a \geq b, a \geq c, b \geq c, c \geq d$

Def (Category of posets)

Let $\underline{P}$ be a category

• Objects

$$Ob(\underline{P}) = P$$

• Maps

$\forall a, b \in P$

$$\underline{P}(a,b) = \begin{cases} \{*\}, & \text{if } a \leq b \\ \emptyset, & \text{otherwise} \end{cases}$$

• Composition func

$$\underline{P}(b,c) \times \underline{P}(a,b) \to \underline{P}(a,c)$$

This corresponds to (2) above
i.e transitivity

• Identity map

$$I_a \in P(a,a)$$

I_a always exists by (1) above, i.e reflexivity

Size issue: By Russel's paradox, there is no set of all sets. So we need to phrase the definition of category carefully.

Note: For us, a category has a class of objects, and for any two objects, a class of maps between them.

Note: There is class of all sets and class of all groups.

Def (Locally small categories)

Let $\mathbb{C}$ be a category. We say $\mathbb{C}$ is locally small if

$\forall A, B \in Ob(\mathbb{C})$

$\mathbb{C}(A,B)$ is a set

Def (Small categories)

Let $\mathbb{C}$ be a category. We say $\mathbb{C}$ is small if it's locally small and

$Ob(\mathbb{C})$ is a set.

Examples

• Locally small, but not small

Set, Grp, Top

• Small

$\Sigma_1(G)$, $\underline{P}$

Isomorphism

Def (Isomorphic map)

Let $\mathbb{C}$ be a category. We say a map $f \in \mathbb{C}(A,B)$ is an isomorphism if there is unique map $g \in \mathbb{C}(B,A)$ s.t

$$f \circ g = 1_B \ \& \ g \circ f = 1_A$$

We call g as inverse of f

Diagramatically, we say

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commute

Examples

• In Set:

iso. = bijection

• In Grp:

iso = group iso.

• In Top:

iso = homeomorphism

• In $\Sigma_1(G)$:

all maps are iso

• In $\underline{P}$

iso = equality

$$\begin{tikzcd} a \arrow[r, bend left, "\leq"] & b \arrow[l, bend left, "\leq"] \end{tikzcd} \iff a = b$$

• In $\mathbb{C}$

all identity maps are iso

Prop: Let $f \in \mathbb{C}(A,B)$.
If inverse of f exists, it's unique

Proof: Let $g_1, g_2 \in \mathbb{C}(B,A)$ be inverses of f.

WTS: $g_1 = g_2$

Since both are inverses we get

$g_1 f = 1_A \qquad f g_1 = 1_B$

$g_2 f = 1_A \qquad f g_2 = 1_B$

We also have

$g_1 = g_1 1_B$ and $g_2 = 1_A g_2$

Replacing 1_B with $f g_2$ we get

$$g_1 = g_1 f g_2$$

$$= 1_A g_2$$

$$= g_2$$

Hence $g_1 = g_2$ $\square$

<u>Note</u>: If $f: A \to B$ has inverse, we write

$$f^{-1}: B \to A$$

for it

Terminal & Initial objects

<u>Def</u> (Terminal)

Let $\mathbb{C}$ be a category. We say $T \in Ob(\mathbb{C})$ is terminal if $\forall A \in Ob(\mathbb{C})$ there is <u>a unique map</u>

$$A \longrightarrow T$$

<u>Def</u> (Initial)

let $\mathbb{C}$ be a category. We say $I \in Ob(\mathbb{C})$ is initial if $\forall A \in Ob(\mathbb{C})$ there is a <u>unique map</u>

$$I \longrightarrow A$$

<u>Examples</u>

- In <u>Set</u>

  $\{*\}$ is terminal

- In <u>Grp</u>

  $\{*\}$ is terminal

  group with just identity element

- In <u>P</u>

  maximal element is terminal

<u>Note</u> (Maximal vs Greatest)

- Maximal element: no element is bigger than this element
- Greatest element: all other elements are smaller than this element

<u>Prop</u>: let $\mathbb{C}$ be a category. If T and T' are both terminal, then they are isomorphic.

<u>Proof</u>

<u>WTS</u>: $T \xrightarrow{f} T'$ and $T' \xrightarrow{g} T$ s.t $f \circ g = 1_{T'}$ and $g \circ f = 1_T$

Since both T and T' are terminal there exist unique maps

$$f: T \longrightarrow T'$$
$$g: T' \longrightarrow T$$

Now $f \circ g$ and $1_{T'}$ are both maps $T' \longrightarrow T'$ and T' is terminal, hence

$$f \circ g = 1_{T'}$$

Similarly $g \circ f = 1_T$ $\square$

lecture 3 | **Def** (Functor)

Let $\mathbb{C}, \mathbb{D}$ be categories. A functor $F : \mathbb{C} \to \mathbb{D}$ consists of

- A function between objects:

$$\forall A \in \mathbb{C} \quad \begin{aligned} Ob(\mathbb{C}) &\longrightarrow Ob(\mathbb{D}) \\ A &\longmapsto FA \end{aligned}$$

- A function between maps:

$\forall A, B \in Ob(\mathbb{C})$

$$\begin{aligned} \mathbb{C}(A,B) &\longrightarrow \mathbb{D}(FA,FB) \\ f &\longmapsto F(f) \end{aligned}$$

subject to functoriality axioms:

1) $\forall A \in \mathbb{C}$

$$F(1_A) = 1_{FA}$$

2) $\forall f \in \mathbb{C}(A,B)$
$\forall g \in \mathbb{C}(B,C)$

$$F(g \circ f) = F(g) \circ F(f)$$

composition in $\mathbb{C}$ composition in $\mathbb{D}$

Observation

Let $\mathbb{C} \xrightarrow{F} \mathbb{D}$ be a functor:

1) The action of F any string of composable maps is well defined. i.e

for all

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \quad \text{in } \mathbb{C}$$

we have a unique map

$$FA_1 \xrightarrow{Ff_1} FA_2 \xrightarrow{Ff_2} \cdots \xrightarrow{Ff_n} F_{n+1}$$

This observation implies that:

F preserves the commutativity of diagrams

i.e

If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \quad (gf = hk)$$

commutes in $\mathbb{C}$, then

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fh"] & FB \arrow[d, "Fg"] \\ FC \arrow[r, "Fk"'] & FD \end{tikzcd}$$

commutes in $\mathbb{D}$

Remark: Let $\mathbb{C} \xrightarrow{F} \mathbb{D}$ be a functor.

If

$$f : A \to B$$

is an isomorphism in $\mathbb{C}$, then

$$Ff : FA \longrightarrow FB$$

is an isomorphism in $\mathbb{D}$

Proof:

We have $f^{-1} : B \to A$ in $\mathbb{C}$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, bend right, "1_A"'] & B \arrow[d, "f^{-1}"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^{-1}"] \arrow[dr, bend right] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

$$f^{-1} \circ f = 1_A \qquad f \circ f^{-1} = 1_B$$

WTS $Ff : FA \to FB$ is an isomorphism.

Claim: $(Ff)^{-1} : FB \to FA$ is given by

$$F(f^{-1}) : FB \to FA$$

we need to show

- $F(f)F(f^{-1}) = 1_{FB}$ $(*)$
- $F(f^{-1})F(f) = 1_{FA}$ $(+)$

To expand $(*)$

$$\begin{aligned} F(f)F(f^{-1}) &= F(ff^{-1}) \\ &= F(1_B) \\ &= 1_{FB} \end{aligned}$$

To expand $(+)$

$$\begin{aligned} F(f^{-1})F(f) &= F(f^{-1}f) \\ &= F(1_A) \\ &= 1_{FA} \end{aligned}$$

Hence, claim. $\square$

Example (Forgetful functor)

1) $U : \underline{Grp} \to \underline{Set}$

$$(G, *) \longmapsto G$$

$$\begin{tikzcd} (G, *) \arrow[r, mapsto, "U"] \arrow[d, "f"] & G \arrow[d, "f = U(f)"] \\ (H, +) \arrow[r, mapsto, "U"] & H \end{tikzcd}$$

i.e U forgets all structure

2) $U : \underline{Ring} \longrightarrow \underline{Grp}$

$$\begin{tikzcd} (R,+,\cdot,0,1) \arrow[r, mapsto] \arrow[d, "f"] & (R,+,0) \arrow[d, "f = U("] \\ (S,+,\cdot,0,1) \arrow[r, mapsto] & (S,+,0) \end{tikzcd}$$

3) U: $\underline{\text{Abelian Grp}} \longrightarrow \underline{\text{Grp}}$.

i.e Group forgets it's abelian

$(G,+,1) \longmapsto (G,+,1)$

U forgets property, structure doesn't change

Example (Free functor)

1) $F: \underline{\text{Set}} \longrightarrow \underline{\text{Grp}}$

$S \longmapsto F(S)$

Here $F(S)$ is a free group on S.

2) $F: \underline{\text{Set}} \longrightarrow \underline{\text{Ring}}$

$S \longmapsto F(S)$

Here $F(S)$ is a free ring on S.

Note (Free group on S)

- Elements: words x^2yz^{-1}, y^3x where $x,y,z \in S$
- Operation: Concatenation of words
- Identity: Empty word

Diagramatically,

$$\begin{tikzcd} x \arrow[d] & S \arrow[r, mapsto, "F"] \arrow[d, "f"] & FS \arrow[d, "Ff"] & x^2yz^{-1} \arrow[d, mapsto] \\ f(x) & T \arrow[r, mapsto, "F"] & FT & f(x)^2f(y)f(z)^{-1} \end{tikzcd}$$

Example

$$\mathrm{Top}_* \xrightarrow{\pi_1} \underline{\mathrm{Grp}}$$

$$X \longmapsto \pi_1(X)$$

The fundamental group of X

Details omitted as it's not examinable

Exercise

1) Let G,H be groups. What is

$F: \Sigma(G) \longrightarrow \Sigma(H)$

2) Let P, Q be posets. What is

$F: P \to Q$

Prop: Small categories and functors form category $\underline{\text{Cat}}$

Proof: Plugging the definition

- Objects: $\mathrm{Ob}(\underline{\mathrm{Cat}}) =$ small categories
- Morphisms: $\forall \mathbb{C}, \mathbb{D} \in \underline{\mathrm{Cat}}$

$\underline{\mathrm{Cat}}(\mathbb{C},\mathbb{D}) = \{F \mid F:\mathbb{C}\to\mathbb{D} \text{ is a functor}\}$

- Composition

$\underline{\mathrm{Cat}}(\mathbb{D},\mathbb{E}) \times \underline{\mathrm{Cat}}(\mathbb{C},\mathbb{D}) \to \underline{\mathrm{Cat}}(\mathbb{C},\mathbb{E})$

$(G,F) \longmapsto G \circ F$

Diagramatically,

$$\mathbb{C} \qquad\qquad \mathbb{E}$$

$$\begin{tikzcd} A \arrow[r, mapsto, "G\circ F"] \arrow[d, "f"'] & GFA \arrow[d, "GF(f)"] \\ B \arrow[r, mapsto, "G\circ F"] & GFB \end{tikzcd}$$

We need to check

$H(GF) = (HG)F$

$F \circ 1_{\mathbb{C}} = F$

$1_{\mathbb{D}} \circ F = F$

[left as an exercise]

Lecture 4 | Notation

Isomorpism: $A \cong B$

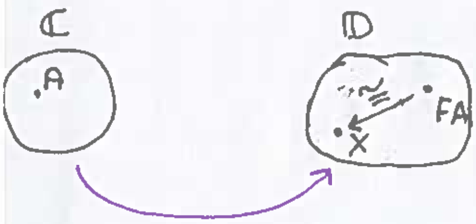
Def (Surjective functor)

Let $\mathbb{C} \xrightarrow{F} \mathbb{D}$ be a functor.

F is essentially surjective if

$\forall X \in \mathbb{D}, \exists A \in \mathbb{C}$ s.t

$$F(A) \cong \mathbb{D}$$



$F: \mathbb{C} \to \mathbb{D}$

Example

let $\mathbb{C}, \mathbb{D}$ be such that

$\mathbb{C}$ = category of $[n]$

$\mathbb{D}$ = Finset

Here

$$[n] = \begin{cases} \cdot\, [n] = \{1, \dots, n\},\ n \in \mathbb{N} \\ \cdot\, [n] \xrightarrow{f} [m] \end{cases}$$

and

$$\text{Finset} = \begin{cases} \cdot \text{ finite sets} \\ \cdot \text{ functions} \end{cases}$$

$F: \mathbb{C} \longrightarrow \mathbb{D}$

$$\begin{tikzcd} {[n]} \arrow[r, mapsto, "F"] \arrow[d, "f"'] & \{1,2,\dots,n\} \arrow[d, "f"] \\ {[m]} \arrow[r, mapsto, "F"] & \{1,\dots,m\} \end{tikzcd}$$

F maps $[n]$ to a set $\{1,2,\dots,n\}$.

Is $\{1,3\} \in \mathrm{im}(F)$?

No, but we have $F([2])$ s.t

$$F([2]) = \{1,2\}$$

which is in bijection with $\{1,3\}$. i.e

$$F([2]) \cong \{1,3\}$$

In particular, we say $\mathbb{D}$ is equal to essential image of F.

Def (Full, faithful)

let $\mathbb{C} \xrightarrow{F} \mathbb{D}$ be a functor. we say

- F is full if $\forall A, B \in \mathbb{C}$
  $$F_{A,B}: \mathbb{C}(A,B) \longrightarrow \mathbb{D}(FA,FB)$$
  is surjective
- F is faithful if $\forall A, B \in \mathbb{C}$
  $$F_{A,B}: \mathbb{C}(A,B) \longrightarrow \mathbb{D}(FA,FB)$$
  is injective

Note:

- full = surjective
- faithful = injective

Note:

- Full: $\forall u: FA \to FB$ in $\mathbb{D}$ there is $f: A \to B$ in $\mathbb{C}$ s.t $F(f) = u$
- Faithful: $\forall f_1, f_2: A \to B$ $F(f_1) = F(f_2) \Rightarrow f_1 = f_2$

Example

$$\begin{tikzcd} \text{Grp} \arrow[r] & \text{Set} \\ (G,\circ) \arrow[r, mapsto] \arrow[d, "f"'] & G \arrow[d, "f"] \\ (H,*) \arrow[r, mapsto] & H \end{tikzcd}$$

Consider,

$$\text{Grp}((G,\circ),(H,*)) \to \text{Set}(G,H)$$

- not full, since there can be functions in Set(G,H) that is not group homomorph. in Grp$((G),(H))$
- faithful

Example

$$\begin{tikzcd} \text{Ab} \arrow[r] & \text{Grp} \\ (G) \arrow[r, mapsto] \arrow[d, "f"'] & (G) \arrow[d, "f"] \\ (H) \arrow[r, mapsto] & (H) \end{tikzcd}$$

- full
- faithful

Exercise:

Find a functor
$$F: \mathbb{C} \to \mathbb{D}$$
that is full, but not faithful.

Hint · think small

$\cdot\ P \longrightarrow Q$

Answer (not confirmed)

$$\mathbb{C} \qquad \begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & b \end{tikzcd} \qquad\qquad \mathbb{D} \qquad \begin{tikzcd} * \arrow[loop above, "1_*"] \end{tikzcd}$$

$F : \mathbb{C} \to \mathbb{D}$ is a functor given by full, but not faithful

$$F(a) = F(b) = *$$

$$F(f) = F(g) = 1_*$$

Subcategories:

Def: Let $\mathbb{C}$ be a category. A subcategory $\mathbb{D}$ of $\mathbb{C}$ consists of

- Subclass of objects

$$Ob(\mathbb{D}) \subseteq Ob(\mathbb{C})$$

- Subclass of maps

$\forall A, B \in Ob(\mathbb{D})$

$$\mathbb{D}(A,B) \subseteq \mathbb{C}(A,B)$$

which is closed under composition and identities i.e

- Closed under composition

$\forall A, B, C \in Ob(\mathbb{D})$,

$$\left.\begin{array}{l} f \in \mathbb{D}(A,B) \\ \& \\ g \in \mathbb{D}(B,C) \end{array}\right. \Rightarrow g \circ f \in \mathbb{D}(A,C)$$

- Identities are included

$\forall A \in Ob(\mathbb{D})$

$$1_A \in \mathbb{D}(A,A)$$

Def (Full subcategory)

We say a subcategory $\mathbb{D}$ of $\mathbb{C}$ is full if

$\forall A, B \in Ob(\mathbb{D})$

$$\mathbb{D}(A,B) = \mathbb{C}(A,B)$$

i.e subcategory "inherits" all maps between its objects

Example

- $\underline{\text{FinSet}} \subseteq \underline{\text{Set}}$
  - full subcategory
- $\underline{\text{FinSet}}_{bij} \subseteq \text{Set}$
  - not full

category of finite functions and bijections

Exercise

1) Let G be a group. Describe, in terms of G, what is the subcategory of $\Sigma(G)$

2) Let $\underline{P}$ be a poset. Describe, in terms of $\underline{P}$, what is subcategory of $\underline{P}$.

Lecture 5 | Exercise:

Let $\mathbb{C}$ be a category and $f, g \in \mathbb{C}(A,B)$. Let
$$u : A' \to A$$
be an isomorphism. Then
$$f = g \iff fu = gu$$

Proof: ($\Longrightarrow$) is clear.

($\Longleftarrow$): If $u : A' \to A$ is an isomorphism, then there exists $u^{-1} : A \to A'$. So
$$fu = gu \Rightarrow fuu^{-1} = guu^{-1}$$
$$\Rightarrow f = g \qquad \square$$

Def (Natural transform.)

Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors. A natural transformation
$$\phi : F \Longrightarrow G$$
is a family of maps
$$(\phi_A : FA \longrightarrow GA)_{A \in \mathbb{C}}$$
such that
$$\forall f \in \mathbb{C}(A,B), \forall A, B \in \mathbb{C}$$

(Naturality condition)

$$\begin{tikzcd} \mathbb{C} & & \mathbb{D} & \\ A \arrow[d, "f"] & & FA \arrow[r, "\phi_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ B & & FB \arrow[r, "\phi_B"] & GB \end{tikzcd}$$

(i.e. $Gf \circ \phi_A = \phi_B \circ Ff$)

Example: Fix $n \in \mathbb{N}$.

- $\mathbb{C}$ = CRing (category of commutative rings)
- $\mathbb{D}$ = Mon (···· of monoids)

- $F : \mathbb{C} \to \mathbb{D}$

$$M_n : \text{CRing} \longrightarrow \text{Mon}$$

$$\begin{tikzcd} R \arrow[r, mapsto, "M_n"] \arrow[d, "f"'] & M_n(R) \arrow[d, "M_n(f)"] \\ S \arrow[r, mapsto, "M_n"] & M_n(S) \end{tikzcd}$$

$M_n(R)$ is monoid of $n \times n$ matrices with entries in R, under multiplication

- $G : \mathbb{D} \to \mathbb{C}$

$$U : \text{CRing} \longrightarrow \text{Mon}$$

$$\begin{tikzcd} (R,+,\cdot,1,0) \arrow[r, mapsto] \arrow[d] & (R,\cdot,1) \arrow[d] \\ (S,+,\cdot,1,0) \arrow[r, mapsto] & (S,\cdot,1) \end{tikzcd}$$

U is forgetful functor that forgets everything except monoid structure of $\cdot$.

Define a natural transform.
$$\det : M_n \Longrightarrow U$$
For $R = (R,+,\cdot,0,1) \in$ CRing, we need
$$\det_R : M_n(R) \longrightarrow R$$
$$M \longmapsto \det_R(M)$$
$\det_R$ is a monoid homomorphism, because
$$\det_R(M \cdot N) = \det_R(M) \cdot \det_R(N)$$
$$\det_R(I_n) = 1$$

To check naturality

$$\begin{tikzcd} \text{CRing} & & \text{Mon} & \\ (R,+,\cdot,1,0) \arrow[d, "f"] & & M_n(R) \arrow[r, "\det_R"] \arrow[d, "M_n(f)"'] & (R,\cdot,1) \arrow[d, "f"] \\ (S,+,\cdot,1,0) & & M_n(S) \arrow[r, "\det_S"] & (S,\cdot,1) \end{tikzcd}$$

when $n=2$

$$\begin{tikzcd} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \arrow[r, mapsto, "\det_R"] \arrow[d, "f"'] & ad-bc \arrow[d, "f"] \\ \begin{pmatrix} fa & fb \\ fc & fd \end{pmatrix} \arrow[r, mapsto, "\det_S"] & f(ad-bc) \end{tikzcd}$$

Since f is homomorphism of rings
$$f(a)f(d) - f(b)f(c) =$$
$$= f(ad-bc)$$
So naturality holds

Notation: For $F, G : \mathbb{C} \to \mathbb{D}$ and $\phi : F \Rightarrow G$ we often write

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathbb{D} \arrow[from=U, to=D, Rightarrow, "\phi"] \end{tikzcd}$$

Def (Composition)

Given

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M}] \arrow[r, bend right=60, "H"'{name=D}] & \mathbb{D} \arrow[from=U, to=M, Rightarrow, "\phi"] \arrow[from=M, to=D, Rightarrow, "\psi"] \end{tikzcd}$$

We define

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "H"'{name=D}] & \mathbb{D} \arrow[from=U, to=D, Rightarrow, "\psi\circ\phi"] \end{tikzcd}$$

with family of maps given by

Hence we can write (1) as:

$$G\varepsilon_B \circ (GFGf \circ y_{GA}) =$$

$$= G\varepsilon_B \circ (y_{GB} \circ Gf) = Gf$$

Since by definition of ε_B

$$\begin{tikzcd} GB \arrow[r, "y_{GB}"] \arrow[dr, "1_{GB}"'] & GFGB \arrow[d, "G\varepsilon_B"] & FGB \arrow[d, "\varepsilon_B"] \\ & GB & B \end{tikzcd}$$

we have

$$G\varepsilon_B \circ y_{GB} = 1_{GB}$$

Hence

$$1_{GB} \circ Gf = Gf$$

Similarly, to show (2) we use functoriality of G as

$$G(f \circ \varepsilon_A) \circ y_{GA} = Gf$$

$$Gf \circ G\varepsilon_A \circ y_{GA} = Gf$$

By definition of ε_A we have

$$G\varepsilon_A \circ y_{GA} = 1_{GA}$$

$$\begin{tikzcd} GA \arrow[r, "y_{GA}"] \arrow[dr, "1_{GA}"'] & GFGA \arrow[d, "G\varepsilon_A"] & FGA \arrow[d, "\varepsilon_A"] \\ & GA & A \end{tikzcd}$$

Hence

$$Gf \circ (G\varepsilon_A \circ y_{GA}) = Gf \circ 1_{GA}$$

$$= Gf$$

Since both (1) and (2) satisfy

$$Gu \circ y_{GA} = Gf$$

and such $u : FGA \to B$ is unique, we get

(1) = (2)

Step 4 : Triangle laws

The following triangular law is already provided by the definition of ε_A. Namely

$$\forall A \in \mathbb{D} \quad \begin{tikzcd} GA \arrow[r, "y_{GA}"] \arrow[dr, "1_{GA}"'] & GFGA \arrow[d, "G\varepsilon_A"] \\ & GA \end{tikzcd}$$

We need to show the other one

$$\forall X \in \mathbb{C} \quad \begin{tikzcd} FX \arrow[r, "Fy_X"] \arrow[dr, "1_{FX}"'] & FGFX \arrow[d, "\varepsilon_{FX}"] \\ & FX \end{tikzcd}$$

i.e $\varepsilon_{FX} \circ Fy_X = 1_{FX}$

Again, we will consider an object in $X \Rightarrow G$ and the unique map from the initial object to it. Namely

$\mathbb{C}$ $\qquad$ $\mathbb{D}$

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[dr, "y_X"'] & GFX \arrow[d, "Gu"] & FX \arrow[d, "\exists! u"] \\ & GFX & FX \end{tikzcd}$$

We know $u : FX \to FX$ is unique and satisfies

$$Gu \circ y_X = y_X$$

To show the second Δ law hold, i.e

$$\varepsilon_{FX} \circ Fy_X = 1_{FX}$$

we will show both LHS and RHS satisfies

$$Gu \circ y_X = y_X$$

and since u is unique, then LHS and RHS must be equal

$u = \varepsilon_{FX} \circ Fy_X$

$$G(\varepsilon_{FX} \circ Fy_X) \circ y_X$$

$$= G\varepsilon_{FX} \circ GFy_X \circ y_X = y_X$$

Using naturality of y

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "y_X"'] & GFX \arrow[d, "GFy_X"] \\ GFX \arrow[r, "y_{GFX}"] & GFGFX \end{tikzcd}$$

we get

$$G\varepsilon_{FX} \circ y_{GFX} \circ y_X = y_X$$

We can also use the first Δ law by letting $A = FX$

$$\begin{tikzcd} GFX \arrow[r, "y_{GFX}"] \arrow[dr, "1_{GFX}"'] & GFGFX \arrow[d, "G\varepsilon_{FX}"] \\ & GFX \end{tikzcd}$$

So

$$1_{GFX} \circ y_X = y_X . \qquad \square$$

$u = 1_{FX}$

Lecture 11

Motivation:

Unify several constructions that occur in mathematics. e.g

- $\underline{\mathrm{Set}}$: $A, B \in \underline{\mathrm{Set}}$

  $\to A \times B \in \underline{\mathrm{Set}}$

- $\underline{\mathrm{Top}}$: $X, Y \in \underline{\mathrm{Top}}$

  $\to X \times Y \in \underline{\mathrm{Top}}$

- $\underline{\mathrm{Grp}}$: $G, H \in \underline{\mathrm{Grp}}$

  $\to G \times H \in \underline{\mathrm{Grp}}$

These are instances of binary product in a category

Products

Def: Let $\mathbb{C}$ be a category and $X, Y \in \mathbb{C}$.

A product of X and Y consist of

- an object:

$$P \in \mathbb{C}$$

- two maps:

projections $\hookrightarrow$ $\begin{cases} p_1 : P \to X \\ p_2 : P \to Y \end{cases}$

which satisfies the following universal property:

For all $A \in \mathbb{C}$ and maps

$$\begin{cases} f_1 : A \to X \\ f_2 : A \to Y \end{cases}$$

there exists a unique map

$$u : A \to P$$

s.t

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, dashed, "u"] \arrow[dr, "f_2"] & \\ X & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

Commutes. i.e

$$\begin{cases} p_1 \circ u = f_1 \\ p_2 \circ u = f_2 \end{cases}$$

Example: In $\underline{\mathrm{Set}}$, for every $X, Y \in \underline{\mathrm{Set}}$ we can consider

- $X \times Y \in \underline{\mathrm{Set}}$
- $\begin{cases} p_1 : X \times Y \longrightarrow X, \ p_1(x,y) = x \\ p_2 : X \times Y \longrightarrow Y, \ p_2(x,y) = y \end{cases}$

and check this is a product of X and Y.

Given

- $A \in \underline{\mathrm{Set}}$
- $\begin{cases} f_1 : A \to X \\ f_2 : A \to Y \end{cases}$

We define

$$u : A \longrightarrow X \times Y$$
$$a \longmapsto (f_1(a), f_2(a))$$

Check

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "u"] \arrow[dr, "f_2"] & \\ X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

commutes since

$$(p_1 \circ u)(a) = p_1((f_1(a), f_2(a))) = f_1(a)$$

$$(p_2 \circ u)(a) = p_2((f_1(a), f_2(a))) = f_2(a)$$

To show uniqueness, let

$$v : A \longrightarrow X \times Y$$

s.t

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "v"] \arrow[dr, "f_2"] & \\ X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

WTS $v(a) = (f_1(a), f_2(a))$

$$f_1(a) = p_1(v(a)) = p_1(x,y) = x$$

$$f_2(a) = p_2(v(a)) = p_2(x,y) = y$$

Hence,

$$v(a) = (x,y) = (f_1(a), f_2(a))$$

So $v = u$

Examples:

- In $\underline{\mathrm{Top}}$: For $X, Y \in \underline{\mathrm{Top}}$, their product is given by

$$X \times Y$$

with so-called "product topology", i.e the smallest topology s.t

$$\begin{cases} p_1 : X \times Y \to X \\ p_2 : X \times Y \to Y \end{cases}$$

are continuous.

- In $\underline{\mathrm{Grp}}$: For $G, H \in \underline{\mathrm{Grp}}$, their product is given by

$$G \times H$$

considered as a group by letting

$$(g,h) \cdot (g', h') = (gg', hh')$$

and

$$1_{G \times H} = (1_G, 1_H)$$

In this way,

$$\begin{cases} p_1 : G \times H \to G \\ p_2 : G \times H \to H \end{cases}$$

become group homomorph and the universal property is satisfied

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists ! u"] \arrow[dr, "f_2"] & \\ G & G \times H \arrow[l, "p_1"] \arrow[r, "p_2"'] & H \end{tikzcd}$$

Remarks:

1) In a category $\mathbb{C}$, product need not exist

e.g.

$\mathbb{C}$ $\boxed{\overset{\bullet}{x} \qquad \overset{\bullet}{y}}$

2) In a category $\mathbb{C}$, if a product of $X, Y \in \mathbb{C}$ exists, it's unique up to isomorphism

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\cong"] & \\ X & & Y \\ & P' \arrow[ul, "p_1'"] \arrow[ur, "p_2'"'] & \end{tikzcd}$$

Equaliser

Def: let $\mathbb{C}$ be a category and $s, t \in \mathbb{C}(X, Y)$ be maps.

An equaliser of s and t consists of

- an object
  $E \in \mathbb{C}$
- a map:
  $i : E \to X$

s.t
$$si = ti$$

i.e
$$E \xrightarrow{i} X \overset{s}{\underset{t}{\rightrightarrows}} Y$$

which satisfies the following universal property:

For all $A \in \mathbb{C}$ and a map
$$f : A \to X$$
s.t
$$sf = tf$$

$$A \xrightarrow{f} X \overset{s}{\underset{t}{\rightrightarrows}} Y$$

there exists a unique map
$$u : A \to E$$
s.t

$$\begin{tikzcd} A \arrow[d, dashed, "u"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"'] & X \end{tikzcd} \qquad i \circ u = f$$

Notation: Overall, we have

$$\begin{tikzcd} \forall A \arrow[d, dashed, "\exists ! u"'] \arrow[dr, "\forall f"] & & \\ E \arrow[r, "i"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$$

Example:

- In Set, let
$$X \overset{s}{\underset{t}{\rightrightarrows}} Y$$
be maps. We can consider
$$E = \{ x \in X \mid s(x) = t(x) \}$$
and inclusion map
$$i : E \to X$$

Given $A \in \mathbb{C}$ and $A \xrightarrow{f} X$

s.t
$\forall a \in A$
$$s(f(a)) = t(f(a))$$

So $f(a) \in E$

So we define
$$u : A \to E$$
$$a \longmapsto u(a) = f(a)$$

i.e
$$u(a) = f(a), \ \forall a$$

Lecture 12 | Pullbacks:

Def: let $\mathbb{C}$ be a category.

let
$$\begin{cases} s: X \to Z \\ t: Y \to Z \end{cases}$$
be maps in $\mathbb{C}$.

A pullback of s and t consists of

- an object:
$$P \in \mathbb{C}$$
- two maps:
$$\begin{cases} p_1: P \to X \\ p_2: P \to Y \end{cases}$$

s.t
$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

commutes, i.e
$$t \circ p_2 = s \circ p_1$$

which satisfies the following universal property:

For all $A \in \mathbb{C}$ and maps
$$\begin{cases} f_1: A \to X \\ f_2: A \to Y \end{cases}$$

Such that
$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

commutes, there exists a unique map
$$u: A \to P$$

s.t
$$\begin{tikzcd} A \arrow[dr, dashed, "u"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

i.e
$$\begin{cases} p_1 \circ u = f_1 \\ p_2 \circ u = f_2 \end{cases}$$

Commutes.

Notation: We summarise this by
$$\begin{tikzcd} A \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "\forall f_2"] \arrow[ddr, bend right, "\forall f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

Example: In Set, pullbacks exists
$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

- $P = \{(x,y) \in X \times Y \mid s(x) = t(x)\}$

$p_1(x,y) = x$

$p_2(x,y) = y$

Instances of pullbacks include

- Intersection:
$$\begin{tikzcd} X \cap Y \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

- Inverse image:

Given $s: X \to Z$ and $z \in Z$
$$\begin{tikzcd} s^{-1}(z) \arrow[r] \arrow[d] & 1 \arrow[d, "z"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

$s^{-1}(z) = \{x \in X \mid s(x) = z\}$ is a pullback

Remark: We often identify

- elements: $z \in Z$
- maps: $1 \xrightarrow{z} Z$

(in Set), because there is a one-to-one correspondence between them

Diagram

Def: let $\mathbb{C}$ be a category and $\mathbb{I}$ be a small category.

A diagram of shape $\mathbb{I}$ in $\mathbb{C}$ is a functor
$$D: \mathbb{I} \to \mathbb{C}$$

Def: Let $\mathbb{I}$ be a small category and
$$D: \mathbb{I} \to \mathbb{C}$$
be a diagram.

A cone over D consists of

- an object:
$$A \in \mathbb{C}$$

• maps:

$$p_I : A \longrightarrow D(I)$$

for $I \in \mathbb{I}$

such that

for all $s : I \to J$ in $\mathbb{I}$

$$\begin{tikzcd} & A \arrow[dl, "p_I"'] \arrow[dr, "p_J"] & \\ D(I) \arrow[rr, "D(s)"'] & & D(J) \end{tikzcd}$$

commutes. i.e

$$D(s) \circ p_I = p_J$$

Example

① • $\mathbb{I}$

$$\boxed{\bullet \quad \bullet}$$

• $D : \mathbb{I} \longrightarrow \mathbb{C}$

$$X \qquad Y$$

• Cone over D

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}$$

② • $\mathbb{I}$

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

• $D : \mathbb{I} \longrightarrow \mathbb{C}$

$$\begin{tikzcd} X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & Y \end{tikzcd}$$

• Cone over D

$$\begin{tikzcd} & A \arrow[dl, bend right, "f"'] \\ X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & Y \end{tikzcd}$$

$sf = tf$

③ • $\mathbb{I}$

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

• $D : \mathbb{I} \longrightarrow \mathbb{C}$

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

• Cone over D

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

$tf_2 = sf_1$

Limits

Def: Let $\mathbb{I}$ be a small category and $D : \mathbb{I} \to \mathbb{C}$ be a diagram.

A limit of D consist of

• a cone over D: i.e

• $L \in \mathbb{C}$

• maps $p_I : L \to D(I)$

subject to the axiom for a cone

which satisfies the following universal property:

For any cone over D, say

• $A \in \mathbb{C}$

• $f_I : A \longrightarrow D(I)$

there exists a unique map

$$u : A \longrightarrow L$$

such that

$$\begin{tikzcd} & A \arrow[dl, bend right, "f_I"'] \arrow[d, dashed, "u"] \\ D(I) & L \arrow[l, "p_I"] \end{tikzcd}$$

commutes $\forall I \in \mathbb{I}$

i.e $p_I \circ u = f_I$

Remark:

• $\mathbb{I}$ is small $\Rightarrow$ small limit

• $\mathbb{I}$ is finite $\Rightarrow$ finite limit

• $\mathbb{I}$ is $\boxed{\bullet \quad \bullet}$ $\Rightarrow$ binary product

Example (continues):

① • $\mathbb{I}$

$$\boxed{\bullet \quad \bullet}$$

• Limit of D

$$\begin{tikzcd} & A \arrow[dl, bend right, "f_1"'] \arrow[d, dashed, "\exists! u"] \arrow[dr, bend left, "f_2"] & \\ X & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

② • $\mathbb{I}$

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

• Limit of D

$$\begin{tikzcd} A \arrow[d, dashed, "\exists!"'] \arrow[dr, bend left, "\forall f"] & & \\ E \arrow[r, "i"'] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$$

③ • $\mathbb{I}$

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

• Limit of D

$$\begin{tikzcd} A \arrow[dr, dashed] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "t"] \\ & X \arrow[r] & Z \end{tikzcd}$$

Lecture 13

Recall:

limit: Let $\mathbb{C}$ be a category, $\mathbb{I}$ be a small category, $\mathbb{D}: \mathbb{I} \to \mathbb{C}$ be a diagram.

A limit of D:

$$\begin{tikzcd}
 & A \arrow[ddl, bend right, "\forall f_I"'] \arrow[d, dashed, "\exists! u"] \arrow[ddr, bend left, "\forall f_J"] & \\
 & L \arrow[dl, "p_I"'] \arrow[dr, "p_J"] & \\
D(I) \arrow[rr] & & D(J)
\end{tikzcd}$$

Recall that, given

$F: \mathbb{C} \longrightarrow \mathbb{D}$

and any $D \in \mathbb{D}$, we have the comma category

$$F \Rightarrow D$$

A terminal object in $F \Rightarrow D$

$$\mathbb{C} \longrightarrow \mathbb{D}$$

$$\begin{tikzcd}
A \arrow[d, dashed, "\exists! u"'] & FA \arrow[r, "\forall f"] \arrow[d, "Fu"'] & D \arrow[d, "1_D"] \\
L & FL \arrow[r, "p"] & D
\end{tikzcd}$$

is $(L, p: FL \to D)$

Goal: To find category $\mathbb{D}$ and $F: \mathbb{C} \to \mathbb{D}$ s.t initial object in $F \Rightarrow D$ is the same thing as limit of D.

Strategy: We will take $\mathbb{D}$ to be $\mathbb{C}^{\mathbb{I}}$ (or $[\mathbb{I}, \mathbb{C}]$)

Note

- $[\mathbb{I}, \mathbb{C}]$ or $\mathbb{C}^{\mathbb{I}}$

  The functor category
  - **Object**: functors $\mathbb{I} \longrightarrow \mathbb{C}$
  - **Maps**: nat. tran of $F, G: \mathbb{I} \to \mathbb{C}$

    $\varphi: F \Rightarrow G$

Construction

functor from $\mathbb{C}$ to $\mathbb{C}^{\mathbb{I}}$

$$\mathbb{C} \xrightarrow{\Delta} \mathbb{C}^{\mathbb{I}}$$

$$\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, "f"'] & \Delta A \arrow[d, "\Delta f"] \\
B \arrow[r, mapsto] & \Delta B
\end{tikzcd}$$

where $\Delta A: \mathbb{I} \to \mathbb{C}$ is a functor and object of $\mathbb{C}^{\mathbb{I}}$

$$\Delta A: \mathbb{I} \longrightarrow \mathbb{C}$$

$$\begin{tikzcd}
I \arrow[r, mapsto] \arrow[d, "m"'] & \Delta A(I) \overset{\text{def}}{=} A \arrow[d, "\Delta A(m)"'] \arrow[d, shift left=8, "1_A"] \\
J \arrow[r, mapsto] & \Delta A(J) \overset{\text{def}}{=} A
\end{tikzcd}$$

and $\Delta f: \Delta A \to \Delta B$ is the natural transformation with components

$$\Delta A(I) \xrightarrow{(\Delta f)_I} \Delta B(I)$$

given by

$$A \xrightarrow{f} B$$

Naturality

$$\begin{tikzcd}
\mathbb{I} & \mathbb{C} & \\
I \arrow[d, "m"'] & \Delta A(I) \arrow[r, "\Delta f_I"] \arrow[d, "\Delta A(m)"'] & \Delta B(I) \arrow[d, "\Delta B(m)"] \\
J & \Delta A(J) \arrow[r, "\Delta f_J"'] & \Delta B(J)
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[d, "1_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Lemma: Let $D: \mathbb{I} \to \mathbb{C}$ be a diagram. TFAE:

(i) A cone

$$(A, (f_I: A \longrightarrow D(I))_{I \in \mathbb{I}})$$

(ii) A natural transformation $f: \Delta A \Rightarrow D$

Proof: Unfolding (ii) we get:

- A family of map

$$(f_I: \underbrace{\Delta A(I)}_{= A} \longrightarrow D(I))_{I \in \mathbb{I}}$$

Subject to naturality

$$\begin{tikzcd}
\mathbb{I} & \mathbb{C} & \\
I \arrow[d, "m"'] & \Delta A(I) \arrow[r, "f_I"] \arrow[d, "\Delta A(m)"'] & D(I) \arrow[d, "D(m)"] \\
J & \Delta A(J) \arrow[r, "f_J"] & D(J)
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
A \arrow[r, "f_I"] \arrow[d, "1"'] & D(I) \arrow[d, "D(m)"] \\
A \arrow[r, "f_J"] & D(J)
\end{tikzcd}$$

$\square$

Prop: Let $D:\mathbb{I}\to\mathbb{C}$ be a diagram. TFAE

(I) a limit of D

(II) a terminal object of $\Delta\Rightarrow D$

Note: With

$$\Delta:\mathbb{C}\longrightarrow\mathbb{C}^{\mathbb{I}}$$

and $D\in\mathbb{C}^{\mathbb{I}}$, we have the comma category $\Delta\Rightarrow D$.

A terminal object in $\Delta\Rightarrow D$

$$\mathbb{C}\longrightarrow\mathbb{C}^{\mathbb{I}}$$

$$\begin{tikzcd} A \arrow[d, dashed, "\exists! u"] \\ L \end{tikzcd} \qquad \begin{tikzcd} \Delta A \arrow[d, "\Delta(u)"'] \arrow[dr, bend left, "\forall f"] & \\ \Delta L \arrow[r, "p"] & D \end{tikzcd}$$

Remark: The data of the cone

$$(A,(f_I:A\longrightarrow D(I))_{I\in\mathbb{I}})$$

is the same as $A\in\mathbb{C}$ and $f:\Delta A\Rightarrow D$ by the previous lemma.

We need to check

$$\begin{tikzcd} & A \arrow[d, "u"] \arrow[dl, bend right, "\forall f"'] \arrow[dl, phantom, "(*)"] \\ D(I) & L \arrow[l, "p_I"] \end{tikzcd}$$

commutes $\forall I\in\mathbb{I}$ in $\mathbb{C}$

iff

$$\begin{tikzcd} \Delta A \arrow[r, "f"] \arrow[d, "\Delta(u)"'] \arrow[dr, phantom, "(\dagger)"] & D \arrow[d, "1_D"] \\ \Delta L \arrow[r, "p"'] & D \end{tikzcd}$$

commutes in $\mathbb{C}^{\mathbb{I}}$

But $(\dagger)$ commutes iff for all $I\in\mathbb{I}$

$$\begin{tikzcd} \Delta A(I) \arrow[r, "f_I"] \arrow[d, "\Delta u(I)"'] & D(I) \arrow[d, "1_{DI}"] \\ \Delta L(I) \arrow[r, "p_I"'] & D(I) \end{tikzcd}$$

commutes in $\mathbb{C}$. The last diagram is the same as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "(*)"] & D(I) \arrow[d, "1_{D(I)}"] \\ L \arrow[r, "p_I"'] & D(I) \end{tikzcd}$$

Proof: The data of limit

$$(L,(p_I:L\to D(I))_{I\in\mathbb{I}})$$

is the same as $L\in\mathbb{C}$ and $p:\Delta L\Rightarrow D$ by lemma $\square$

Recall: Given $F:\mathbb{C}\to\mathbb{D}$, in order to have a right adjoint to F, it's sufficient to have, for each $D\in\mathbb{D}$, a terminal object of $F\Rightarrow D$

Justify this

Theorem: Let $\mathbb{I}$ be a small category.

TFAE:

(i) We have, for each diagram $D:\mathbb{I}\to\mathbb{C}$, a limit

$$(\lim D,(p_I:\lim D\to D(I))_{I\in\mathbb{I}})$$

(ii) We have a right adjoint to $\Delta:\mathbb{C}\longrightarrow\mathbb{C}^{\mathbb{I}}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathbb{C}^{\mathbb{I}} \arrow[l, bend left, "\lim D"] \end{tikzcd}$$

Lecture 14 | RAPL:

Right Adjoint Preserves Limit

Obs: let $\mathbb{I}$ be a small category. and $G : \mathbb{D} \to \mathbb{C}$ be a functor

- If $D : \mathbb{I} \to \mathbb{D}$ is a diagram in $\mathbb{D}$, then
$$G \circ D : \mathbb{I} \to \mathbb{D} \to \mathbb{C}$$
is a diagram in $\mathbb{C}$

- If $(A, (p_I : A \to D(I))_I)$ is a cone over D in $\mathbb{D}$, then
$$(GA, (Gp_I : GA \to GD(I))_I)$$
is a cone over $G \circ D$ in $\mathbb{C}$

Example:

$$\mathbb{I} = \boxed{\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}}$$

In $\mathbb{D}$: $D : \mathbb{I} \to \mathbb{D}$ is a diagram

$$\begin{tikzcd} & B \arrow[d, "t"] \\ A \arrow[r, "s"'] & C \end{tikzcd}$$

A cone over D

$$\begin{tikzcd} X \arrow[r] \arrow[d] & B \arrow[d, "t"] \\ A \arrow[r, "s"'] & C \end{tikzcd}$$

In $\mathbb{C}$: $GD : \mathbb{I} \to \mathbb{C}$ is a diagram

$$\begin{tikzcd} & GB \arrow[d, "Gt"] \\ GA \arrow[r, "Gs"'] & GC \end{tikzcd}$$

A cone over GD

$$\begin{tikzcd} GX \arrow[r] \arrow[d] & GB \arrow[d, "Gt"] \\ GA \arrow[r, "Gs"'] & GC \end{tikzcd}$$

Def: let $\mathbb{I}$ be a small category. We say tha a functor
$$G : \mathbb{D} \to \mathbb{C}$$
preserves limits of shape $\mathbb{I}$ if for all diagrams
$$D : \mathbb{I} \to \mathbb{D},$$
if
$$(L, (p_I : L \to D(I))_I)$$
is a limit cone for D in $\mathbb{D}$, then
$$(GL, (Gp_I : GL \to GD(I))_I)$$
is a limit cone for GD in $\mathbb{C}$

Thm (RAPL): Right adjoint preserve limits.

Proof: We will only show right adjoints preserve pullbacks.

So let
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
be an adjunction.

Consider a diagram

$$\begin{tikzcd} & B \arrow[d, "t"] \\ A \arrow[r, "s"'] & C \end{tikzcd}$$
in $\mathbb{D}$

and let

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "t"] \\ A \arrow[r, "s"'] & C \end{tikzcd}$$
be a pullback in $\mathbb{D}$

WTS:

$$\begin{tikzcd} GP \arrow[r, "Gp_2"] \arrow[d, "Gp_1"'] & GB \arrow[d, "Gt"] \\ GA \arrow[r, "Gs"'] & GC \end{tikzcd}$$

is a pullback in $\mathbb{C}$, i.e

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists! v"] \arrow[drr, bend left, "\forall g_2"] \arrow[ddr, bend right, "\forall g_1"'] & & \\ & GP \arrow[r, "Gp_2"] \arrow[d, "Gp_1"'] & GB \arrow[d, "Gt"] \\ & GA \arrow[r, "Gs"'] & GC \end{tikzcd}$$

for all $X \in \mathbb{C}$, and for al
$$\begin{cases} g_1 : X \to GA \\ g_2 : X \to GB \end{cases}$$

Note, we are assuming
$$G(s) \circ g_1 = G(t) \circ g_2$$

Recall that we have

$$\begin{tikzcd} \mathbb{D}(FX, A) \arrow[r, shift left, "\varphi"] & \mathbb{C}(X, GA) \arrow[l, shift left, "\varphi^{-1}"] \end{tikzcd}$$

s.t

i) $\varphi(FX \xrightarrow{f} A \xrightarrow{u} A') =$

$= X \xrightarrow{\varphi(f)} GA \xrightarrow{Gu} GA'$

ii) $\varphi(FX' \xrightarrow{Fv} FX \xrightarrow{f} A) =$

$= X' \xrightarrow{v} X \xrightarrow{\varphi(f)} GA$

and

(I) $\varphi^{-1}(X \xrightarrow{f} GA \xrightarrow{Gu} GA') =$

$= FX \xrightarrow{\varphi^{-1}(f)} A \xrightarrow{u} A'$

(II) $\varphi^{-1}(X' \xrightarrow{v} X \xrightarrow{f} GA)$

$= FX' \xrightarrow{Fv} FX \xrightarrow{\varphi^{-1}(f)} A$

Consider, the adjoint transposes of

$$\begin{cases} g_1 : X \longrightarrow GA \\ g_2 : X \longrightarrow GB \end{cases}$$

Let

$$\begin{cases} f_1 : FX \longrightarrow A \\ f_2 : FX \longrightarrow B \end{cases}$$

be

$\varphi^{-1}(g_1) = f_1$ and $\varphi^{-1}(g_2) = f_2$

Sub
WTS:

$$\begin{tikzcd} FX \arrow[rr, bend left, "f_2"] \arrow[dr, bend right, "f_1"'] & & B \arrow[d, "t"] \\ & A \arrow[r, "s"'] & C \end{tikzcd}$$

Commutes, i.e

$$s f_1 = t f_2$$

To show this, it's enough to show

$$\varphi(s f_1) = \varphi(t f_2)$$

Since φ is a bijection.

$\varphi(FX \xrightarrow{f_1} A \xrightarrow{s} C) =$

$= \varphi(FX \xrightarrow{f_2} B \xrightarrow{t} C)$

Equivalently,

$X \xrightarrow{\varphi(f_1)} GA \xrightarrow{Gs} GC =$

$= X \xrightarrow{\varphi(f_2)} GB \xrightarrow{Gt} GC$

But $\varphi(f_1) = g_1$ and $\varphi(f_2) = g_2$

So it becomes

$$Gs \circ g_1 = Gt \circ g_2$$

which holds by assumption.

So, by the universal property of pullbacks in $\mathbb{D}$, there exists a unique map

$$u : FX \longrightarrow P$$

$$\begin{tikzcd} FX \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] \arrow[dr, dashed, "\exists! u"] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[d, "t"] \\ & A \arrow[r, "s"'] & C \end{tikzcd}$$

Now, we need to "transport" $u : FX \to P$ to C

let $v : X \to GP$ be the transpose of $u : FX \to A$, i.e

$$v = \varphi(u)$$

Claim 1: $v : X \to GP$ is s.t

$$\begin{tikzcd} X \arrow[drr, bend left, "g_2"] \arrow[dr, "v"] \arrow[ddr, bend right, "g_1"'] & & \\ & GP \arrow[r, "Gp_2"] \arrow[d, "Gp_1"] & GB \\ & GA & \end{tikzcd}$$

Commutes.

To show

$$Gp_1 \circ v = g_1$$

it's sufficient to show

$\varphi^{-1}(X \xrightarrow{v} GP \xrightarrow{Gp_1} GA) =$

$= \varphi^{-1}(X \xrightarrow{g_1} GA)$

i.e

$FX \xrightarrow{\varphi^{-1}(v)} P \xrightarrow{p_1} A$

$= FX \xrightarrow{\varphi^{-1}(g_1)} A$

i.e

$$p_1 \circ \varphi^{-1}(v) = \varphi^{-1}(g_1)$$

$$p_1 \circ u = f_1$$

which holds by the assumption of $u : FX \to$

Lecture - 15 Duality:

Many of the statements we have considered admit a certain "symmetric" or "dual" statement

Examples

Initial object I s.t $\forall X$ $\exists!\ I \to X$	$\leftrightarrow$	**Terminal object:** T s.t $\forall X$ $\exists!\ X \to T$
Comma category $X \Rightarrow G$; $\mathbb{C} \xleftarrow{G} \mathbb{D}$; $X \xrightarrow{f} GA$, A	$\leftrightarrow$	**Comma category** $F \Rightarrow A$; $\mathbb{C} \xrightarrow{F} \mathbb{D}$; X, $FX \xrightarrow{f} A$
left adjoint	$\leftrightarrow$	right adjoint
Colimits	$\leftrightarrow$	Limit

These are all instances of the so called Duality Principle

Note: Every statement in Category Theory has a dual, obtained by replacing domain with codomain...

A statement holds iff its dual holds.

Example:

Statement	Dual
$f: A \to B$	$f: B \to A$
$A = \mathrm{dom}(f)$	$A = \mathrm{cod}(f)$
$h = gf$	$h = fg$
1_A	1_A
u is a right inverse of f	u is a left inverse of f
X is initial object	X is terminal object

Def (The opposite of category)

Let $\mathbb{C}$ be a category. We define its opposite, $\mathbb{C}^{op}$, as follows:

- Objects:

$$Ob(\mathbb{C}^{op}) = Ob(\mathbb{C})$$

- Maps:

for $A, B \in Ob(\mathbb{C}^{op})$ we let

$$\mathbb{C}^{op}(A,B) = \mathbb{C}(B,A)$$

i.e

$$\begin{array}{ccc} \mathbb{C} & & \mathbb{C}^{op} \\ A & & A \\ \uparrow f & & \downarrow f^{op} \\ B & & B \end{array}$$

- Composition func.

$$\mathbb{C}^{op}(B,C) \times \mathbb{C}^{op}(A,B) \to \mathbb{C}^{op}(A,C)$$

Note that,

$$\mathbb{C}(C,B) \times \mathbb{C}(B,A) \cong$$

$$\mathbb{C}(B,A) \times \mathbb{C}(C,B)$$

Hence, composition func can be given as

$$\mathbb{C}(C,B) \times \mathbb{C}(B,A) \to \mathbb{C}(C,A)$$

For $\begin{cases} g^{op}: B \to C \\ f^{op}: A \to B \end{cases}$

their composite in $\mathbb{C}^{op}$

$$A \xrightarrow{(f \circ g)^{op}} C$$

i.e

$$\underbrace{g^{op} \circ f^{op}}_{\text{in } \mathbb{C}^{op}} := (\underbrace{f \circ g}_{\text{in } \mathbb{C}})^{op}$$

- Identity map: Same as those of $\mathbb{C}$.

Note: The axioms for a category can be shown to hold easily (exercise!)

Remark

- In general, $\mathbb{C}$ and $\mathbb{C}^{op}$ are very different
- We are not adding maps to $\mathbb{C}^{op}$

Example:

- An initial object of $\mathbb{C}^{op}$ is a terminal object of $\mathbb{C}$.

A is initial in $\mathbb{C}^{op}$ $\iff$ $\forall X \in \mathbb{C}^{op}$ $\exists!\ A \to X$ in $\mathbb{C}^{op}$

$\iff$ $\forall X \in \mathbb{C}$ $\exists!\ X \to A$ in $\mathbb{C}$

$\iff$ A is terminal in $\mathbb{C}$

Remark: The Duality principle holds when we consider statements that hold for all categories.

Consider such a statement Σ_1

We want to show the dual of Σ_1:

- Take a category $\mathbb{C}$
  - We then have the opposite category $\mathbb{C}^{op}$
  - Our statement holds for $\mathbb{C}^{op}$ as well, since it holds for all categories
    - But dual of Σ_1 is just Σ_1 applied to $\mathbb{C}^{op}$
    - Therefore, dual of Σ_1 holds

Example: For all $\mathbb{C}$ category, if $A, A' \in \mathbb{C}$ are both initial, then $\leftarrow \Sigma_1$

$$A \cong A$$

Dual of Σ_1

Show: For all $\mathbb{C}$, if $A, A' \in \mathbb{C}$ are both terminal then

$$A \cong A'$$

Σ_1 applied to $\mathbb{C}^{op}$

Remark: The notion of isomorphism is self-dual.

i.e.

$f: A \to B$ is an iso in $\mathbb{C}$ $\iff$ $f^{op}: B \to A$ is an iso in $\mathbb{C}^{op}$

Other uses of $\mathbb{C}^{op}$

Def (Contravariant functor)

Let $\mathbb{C}, \mathbb{D}$ be categories. A contravariant functor from $\mathbb{C}$ to $\mathbb{D}$ is a functor from $\mathbb{C}^{op}$ to $\mathbb{D}$

$$\begin{tikzcd} \mathbb{C} & \mathbb{C}^{op} \arrow[rr, "F"] & & \mathbb{D} \\ A \arrow[d, "f"] & A \arrow[rr, mapsto] & & FA \\ B & B \arrow[u, "f"] \arrow[rr, mapsto] & & FB \arrow[u, "Ff"'] \end{tikzcd}$$

Examples: Let $\mathbb{C}$ be a category. Fix an object $A \in \mathbb{C}$. Define a functor

$$\begin{tikzcd} \mathbb{C} & \mathbb{C}^{op} \arrow[rr, "F^A"] & & \underline{Set} \\ X \arrow[d, "u"] & X \arrow[rr, mapsto] & & F^A(X) \\ Y & Y \arrow[u] \arrow[rr, mapsto] & & F^A(Y) \arrow[u, "F^A(u)"'] \end{tikzcd}$$

where

$$F^A(X) = \mathbb{C}(X, A)$$
$$F^A(Y) = \mathbb{C}(Y, A)$$

So

$$\begin{tikzcd} F^A(X) & X \xrightarrow{u} Y \xrightarrow{g} A \\ F^A(Y) \arrow[u, "F^A(u)"'] & Y \xrightarrow[g]{} A \arrow[u, mapsto] \end{tikzcd}$$

Note

$$\mathbb{C}(-, -)$$

Contravariant

Covariant

Lecture 16 Colimits

Def: Let $\mathbb{C}$ be a category. Let $\mathbb{I}$ be a small category and D be a diagram of shape $\mathbb{I}$ in $\mathbb{C}$, i.e.

$$D : \mathbb{I} \to \mathbb{C}.$$

- A *cocone* on D consists of
  - an object $A \in \mathbb{C}$
  - a family of maps $(f_I : D(I) \to A)_{I \in \mathbb{I}}$

such that

$$\begin{tikzcd} D(I) \arrow[rr, "D(m)"] & & D(J) \\ & A \arrow[ul, "f_I"] \arrow[ur, "f_J"'] & \end{tikzcd}$$

commutes for every $m \in \mathbb{I}(I,J)$ in $\mathbb{I}$.

- A *colimit* of D is a cocone

$$(C, (p_I : D(I) \to C)_{I \in \mathbb{I}})$$

which is universal in the sense that for every cocone

$$(A, (f_I : D(I) \to A)_I)$$

there exists a unique map $u : C \to A$ s.t.

$$\begin{tikzcd} D(I) \arrow[r, "p_I"] \arrow[dr, bend right, "f_I"'] & C \arrow[d, "\exists! u"] \\ & A \end{tikzcd}$$

Commutes $\forall I \in \mathbb{I}$

Remark: As usual, we sum up the above with a diagram

$$\begin{tikzcd} D(I) \arrow[rr, "D(m)"] \arrow[dr, "p_I"] \arrow[ddr, bend right, "f_I"'] & & D(J) \arrow[dl, "p_J"'] \arrow[ddl, bend left, "f_J"] \\ & C \arrow[d, dashed, "\exists! u"] & \\ & A & \end{tikzcd}$$

Terminology: The maps

$$p_I : D(I) \to C$$

are sometimes called *co-projections*.

Special cases of colimits

Binary Coproducts

$$\mathbb{I} = \boxed{\overset{\bullet}{1} \qquad \overset{\bullet}{2}}$$

For a category $\mathbb{C}$, a diagram of shape $\mathbb{I}$ in $\mathbb{C}$ is just

$$X_1 \bullet \qquad \bullet X_2$$

A colimit of such diagram is

$$\begin{tikzcd} X_1 \arrow[dr, "p_1"] \arrow[ddr, bend right, "\forall f_1"'] & & X_2 \arrow[dl, "p_2"'] \arrow[ddl, bend left, "\forall f_2"] \\ & C \arrow[d, dashed, "\exists! u"] & \\ & A & \end{tikzcd}$$

Example: In **Set**, the coproduct of two sets X_1 and X_2 is the disjoint union, $X_1 + X_2$, defined as

$$X_1 + X_2 = \{(1,x) : x \in X_1\} \cup \{(2,x) : x \in X_2\}$$

$$|X_1 + X_2| = |X_1| + |X_2|$$

for X_1, X_2 finite.

$$x \longmapsto (1,x) \quad (2,x) \longmapsfrom x$$

$$\begin{tikzcd} X_1 \arrow[r, "p_1"] \arrow[dr, bend right, "f_1"'] & X_1 + X_2 \arrow[d, dashed, "\exists! u"] & X_2 \arrow[l, "p_2"'] \arrow[dl, bend left, "f_2"] \\ & A & \end{tikzcd}$$

$$u(z) = \begin{cases} f_1(x) & \text{if } z = (1,x) \text{ for some } x \in X_1 \\ f_2(x) & \text{if } z = (2,x) \text{ for some } x \in X_2 \end{cases}$$

Coequaliser:

$$\mathbb{I} = \boxed{\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}}$$

A diagram of shape $\mathbb{I}$ in $\mathbb{C}$ is

$$\begin{tikzcd} X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$$

Coequaliser of this diagram is

$$\begin{tikzcd} X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \arrow[r] \arrow[dr] & C \arrow[d, dashed, "\exists! u"] \\ & & A \end{tikzcd}$$

Example: In **Set**, the coequaliser of two functions

$$X \underset{t}{\overset{s}{\rightrightarrows}} Y$$

is the quotient of Y by the equivalence relation generated by the relation $\sim$ on Y given by

$$y \sim y' \iff \exists x \in X \text{ s.t } \begin{array}{l} s(x) = y \\ t(x) = y' \end{array}$$

Pushouts :

$$\mathbb{I} = \left(\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}\right)^{op} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

A diagram of shape $\mathbb{I}$ in $\mathbb{C}$ is

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \\ Z & \end{tikzcd}$$

A pushout of this diagram is

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "P_1"] \arrow[ddr, bend left, "\forall f_1"] & \\ Z \arrow[r, "P_2"] \arrow[drr, bend right, "\forall f_2"'] & P \arrow[dr, dashed, "\exists! u"] & \\ & & A \end{tikzcd}$$

Example : In $\underline{Set}$, pushouts

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d, "P_1"] \\ Z \arrow[r, "P_2"'] & P \end{tikzcd}$$

can be constructed by taking a quotient of

$Y + Z$

Remark :

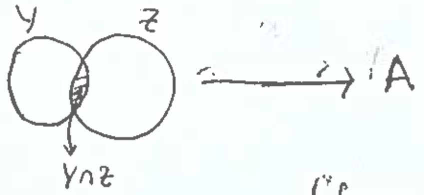
- Very special case of pushouts:

The diagram

$$\begin{tikzcd} Y \cap Z \arrow[r, hook] \arrow[d, hook] & Y \arrow[d] \\ Z \arrow[r] & Y \cup Z \end{tikzcd}$$

is a pushout in $\underline{Set}$

$$\begin{tikzcd} Y \cap Z \arrow[r, hook] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left, "f_1"] & \\ Z \arrow[r] \arrow[drr, bend right, "f_2"'] & Y \cup Z \arrow[dr, dashed, "\exists! u"] & \\ & & A \end{tikzcd}$$



$$x \longmapsto \begin{cases} f_1(x) & \text{if } x \in Y \setminus Z \\ f_2(x) & \text{if } x \in Z \setminus Y \\ f_1(x) \text{ or } f_2(x) & \text{if } x \in Y \cap Z \end{cases}$$

Thm : left adjoints preserve colimit

Proof : By duality $\square$

Application :

- $(a_1 + a_2)b = a_1 b + a_2 b$ (*)

for all $a_1, a_2, b \in \mathbb{N}$

- In $\underline{Set}$, for all $A_1, A_2, B \in \underline{Set}$, we have

$$(A_1 + A_2) \times B \cong (A_1 \times B) + (A_2 \times B)$$ (* *)

because

$$\begin{tikzcd} \underline{Set} \arrow[r, bend left, "(-)\times B"] \arrow[r, phantom, "\perp"] & \underline{Set} \arrow[l, bend left, "(-)^B"] \end{tikzcd}$$

where $(-)^B : C \longmapsto C^B$

When A_1, A_2, B are finite, we get (*) by taking cardinalities in (* *)

Lecture 18

Motivation

Our goal is to relate Boolean algebras to topological spaces, via a functor

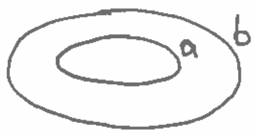
$$\underline{BoolAlg}^{op} \longrightarrow \underline{Top}$$

Given a Bool. Alg. A, we construct a top. space so that A will be isomorphic to a class of the open set of the space.

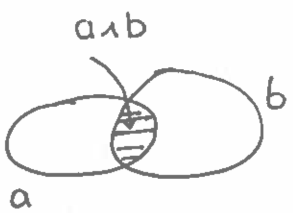
Remark: let A be a Boolean Algebra.

$a \in A$

$a \leq b$



$a \wedge b$



Where are we going to find the points of this space? ← Question

Idea: let $(X, \mathcal{O}(X))$ be a topological space and try to understand the points of X in a different way

$$x \in X = \mathcal{U}(X, \mathcal{O}(X))$$

$$1 \longrightarrow \mathcal{U}(X, \mathcal{O}(X)) = X \text{ in } \underline{Set}$$

$$\mathcal{L}(1) = (1, \mathcal{O}(1)) \xrightarrow{x} (X, \mathcal{O}(X)) \text{ in } \underline{Top}$$

$$\mathcal{O}(X) \xrightarrow{x^{-1}} \mathcal{O}(1)$$

$$u \longmapsto x^{-1}(u) = \begin{cases} 1 & \text{if } x \in u \\ \emptyset & \text{otherwise} \end{cases}$$

Remark:

$(\mathcal{O}(1), \subseteq)$ as a partially ordered set is isomorphic to $2 = \{0, 1\}$

Def: Let A be a Boolean algebra. A point of A is a Boolean algebra homomorphism

$$x : A \longrightarrow 2$$

Remark:

- 2 is initial in $\underline{BoolAlg}$
- 2 is terminal $\underline{BoolAlg}^{op}$

So points of A are maps

$$2 \longrightarrow A$$

in $\underline{BoolAlg}^{op}$, just as elements of a set X are maps

$$1 \longrightarrow X$$

Next goal:

How to understand

$$A \xrightarrow{x} 2$$

in $\underline{BoolAlg}$.

Some defs: let $F \subseteq A$

- F is filter if

$\boxed{(F1)\ F \neq \emptyset}$

$\boxed{(F2)\ a, b \in F \Rightarrow a \wedge b \in F}$

$\boxed{(F3)\ a \in F,\ b \in A,\ a \leq b \Rightarrow b \in F}$

- F is prime filter if it's filter and

$\boxed{(P1)\ F \neq A}$

$\boxed{(P2)\ a \vee b \in F \Rightarrow a \in F \text{ or } b \in F}$

- F is ultrafilter if it's filter and

$\boxed{(U1)\ F \neq A}$

$(U2)\ F \subseteq G,\ G \text{ is filter} \Rightarrow F = G \text{ or } G = A$

Prop: let $F \subseteq A$. TFAE

(i) F is ultrafilter

(ii) F is primefilter

(iii) F is a proper filter and $\forall a \in A$, we have

$$a \in F \quad \text{or} \quad \neg a \in F$$

(iv) $F \neq \emptyset$, $F \neq A$ and

$$a \wedge b \in F \iff a \in F \text{ and } b \in F$$

$$a \vee b \in F \iff a \in F \text{ or } b \in F$$

(v) There is

$$x : A \longrightarrow 2$$

in Bool Alg s.t

$$F = F_x$$

where $F_x = \{a \in A : x(a) = 1\}$

Remark: There is one-to-one correspondence between

$$\frac{x : A \longrightarrow 2}{F \subseteq A \text{ prime filter}}$$

Given $x : A \to 2$, let

$$F_x = \{a \in A : x(a) = 1\}$$

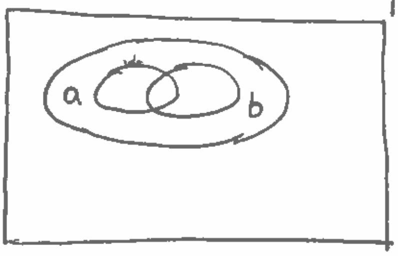
Given $F \subseteq A$ prime filter, let

$$x_F : A \longrightarrow 2$$
$$a \longmapsto \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{otherw.} \end{cases}$$

Remark:

let $pt(A)$ be the set of points of A

$pt(A)$



We now want to put topology on $pt(A)$ so as to obtain topological space

We do this by specifying a basis for the topology:

we take it to be the collection of subsets of $pt(A)$ of the form

$$V(a) = \{x \in pt(A) : \underbrace{x(a) = 1}_{a \in F_x}\}$$

"the set of points which $a \in A$ approximates"

To check the basis is closed under binary intersections, i.e

$$\forall a, b \in A, \ \exists c \in A \text{ s.t}$$

$$V(a) \cap V(b) = V(c)$$

Claim:

$\forall a, b \in A$

$$V(a) \cap V(b) = V(a \wedge b)$$

Proof:

This amounts to

$x(a) = 1$ and $x(b) = 1$ $\iff x(a \wedge b) = 1$

$\iff x(a) \wedge x(b) = 1$

The topology on $pt(A)$ has as open sets, unions of $V(a)$'s

let's write

$$\mathcal{O}(pt(A))$$

for this topology.

Goal:

$$\begin{array}{ccccc} \text{Bool Alg} & & \text{Bool Alg}^{op} & \longrightarrow & \text{Top} \\ A & & A & \longmapsto & (pt(A), \mathcal{O}(pt(A))) \\ \downarrow f & & \uparrow f & & \uparrow pt(f) \\ B & & B & \longmapsto & (pt(B), \mathcal{O}(pt(B))) \end{array}$$

where

$$(pt(A), \mathcal{O}(pt(A))) =$$

$$= (\text{Bool Alg}[A, 2], \mathcal{O}(pt(A)))$$

and

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{y} & 2 \\ & & \uparrow pt(f) & & \\ & & B & \xrightarrow{y} & 2 \end{array}$$

To check:

- $pt(f)$ is continuous

(functoriality is clear from def.)

Lecture 17(2)

Def:

A Boolean Algebra is a distributive lattice

$$(A, \leq, \top, \wedge, \bot, \vee)$$

such that every element has a complement, i.e an element

$$\neg a \in A$$

such that

$$\begin{cases} a \wedge \neg a = \bot \\ a \vee \neg a = \top \end{cases}$$

Remark: If $a \in A$ has a complement, it's unique

Examples:

- For $n \geq 2$

$$\{0, 1, \dots, n\}$$

is not a Boolean Algebra.

- For $n = 1$,

$$2 = \{0, 1\}$$

is a Boolean Algebra.

Examples

- For any set X

$$\mathcal{P}(X)$$

is a Boolean Algebra, where

$$\neg S = \{x \in X : x \notin S\}$$

Def: let $(A, \leq), (B, \leq)$ be distributive lattices. A homomorphism

$$f : (A, \leq) \longrightarrow (B, \leq)$$

of distributive latices is a function $f : A \to B$, which preserves

- order:

$$\forall x, y \in A$$

$$x \leq y \Rightarrow f(x) \leq f(y)$$

- top element

$$f(\top) = \top$$

- binary meets

$$f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2)$$

- bottom element

$$f(\bot) = \bot$$

- binary joins

$$f(a_1 \vee a_2) = f(a_1) \vee f(a_2)$$

Def: let $(A, \leq), (B, \leq)$ be Boolean algebras. A homomorphism

$$f : (A, \leq) \longrightarrow (B, \leq)$$

of Boolean Algebras i[s] just a homomorphism of distributive lattic[es] $f : (A, \leq) \to (B, \leq)$.

Example:

If $f : (A, \leq) \to (B, \leq)$ is a Bool. alg. hom., then

$$f(\neg a) = \neg f(a)$$

for all $a \in A$

Note: We write

Bool Alg

for the category of Boolean algebras and Boolean algebras homomorphisms.

Prop:

2 is an initial object of **Bool Alg**.

Proof:

$$2 \longrightarrow A$$

$$0 \longmapsto \bot$$

$$1 \longmapsto \top$$

$\square$

Lecture 17 | Remark

Let $(A, \leq)$ be partially ordered set.

$\boxed{\text{As poset}}$ $(A, \leq)$

- Top element

A has a top element i.e $T \in A$ s.t $\forall a \in A$

$$a \leq T$$

$\boxed{\text{As category}}$ $\underline{A}$

- Terminal object

$\underline{A}$ has a terminal object:

$$a \overset{\exists!}{\dashrightarrow} T$$

- Binary meets

A has binary meets, i.e $\forall a_1, a_2 \in A$, we have

$$a_1 \wedge a_2$$

s.t

$$\begin{cases} a_1 \wedge a_2 \leq a_1 \\ a_1 \wedge a_2 \leq a_2 \end{cases}$$

and $\forall b \in A$

$$\begin{matrix} b \leq a_1 \\ b \leq a_2 \end{matrix} \Rightarrow b \leq a_1 \wedge a_2$$

- Binary product

A has binary products, i.e $\forall a_1, a_2 \in A$, we have

$$a_1 \wedge a_2$$

s.t

$$\begin{cases} a_1 \wedge a_2 \dashrightarrow a_1 \\ a_1 \wedge a_2 \dashrightarrow a_2 \end{cases}$$

and $\forall b \in A$

$$\begin{matrix} b \dashrightarrow a_1 \\ b \dashrightarrow a_2 \end{matrix} \Rightarrow b \dashrightarrow a_1 \wedge a_2$$

Bottom element

A has bottom element

$$\perp \in A$$

- Initial object

$\underline{A}$ has initial object

$$\perp$$

- Binary joins

A has binary joins, i.e $\forall a_1, a_2 \in A$, we have

$$a_1 \vee a_2$$

- Binary coproducts

$\underline{A}$ has binary coproduct, $\forall a_1, a_2 \in A$, we have

$$a_1 \vee a_2$$

s.t

$$\begin{cases} a_1 \leq a_1 \vee a_2 \\ a_2 \leq a_1 \vee a_2 \end{cases}$$

and $\forall b \in A$

$$\begin{matrix} a_1 \leq b \\ a_2 \leq b \end{matrix} \Rightarrow a_1 \vee a_2 \leq b$$

s.t

$$\begin{cases} a_1 \dashrightarrow a_1 \vee a_2 \\ a_2 \dashrightarrow a_1 \vee a_2 \end{cases}$$

and $\forall b \in A$

$$\begin{matrix} a_1 \dashrightarrow b \\ a_2 \dashrightarrow b \end{matrix} \Rightarrow a_1 \vee a_2 \dashrightarrow b$$

Def: A distributive lattice is a partially ordered set $(A, \leq)$ equipped with

- A top element $\top$
- Binary meets $a_1 \wedge a_2$ for all $a_1, a_2 \in A$
- A bottom element $\bot$
- Binary joins $a_1 \vee a_2$ for all $a_1, a_2 \in A$

such that the distributive law

$$(a_1 \vee a_2) \wedge b = (a_1 \wedge b) \vee (a_2 \wedge b)$$

holds for all $a_1, a_2, b \in A$

Examples (Chain)

For any $n \geq 1$, the set $\{1, \dots, n\}$ partially ordered by

$$0 \leq 1 \leq \dots \leq n$$

is a distributive lattice, where

- top element : n
- binary meets $a_1 \wedge a_2 = \min\{a_1, a_2\}$
- bottom element : 0
- binary joins $a_1 \vee a_2 = \max\{a_1, a_2\}$

Claim: Distributive law holds.

Proof: Exercise, complete this

$a_1 \leq a_2 \leq b$

$0 \bullet\!\!-\!\!\bullet\, a_1 \quad a_2 \quad b\, \bullet\!\!-\!\!\bullet\, n$

$$\underbrace{(a_1 \vee a_2)}\wedge b \overset{?}{=} \underbrace{(a_1 \wedge b)} \vee \underbrace{(a_2 \wedge b)}$$

$a_2 \leq b \iff b \leq b$

Example (Powerset)

For any set X, its powerset

$$P(X) = \{S : S \subseteq X\}$$

has partial order given by

$$\subseteq$$

where

- top element X
- binary meets $S_1 \cap S_2$
- bottom element $\emptyset$
- binary joins $S_1 \cup S_2$

The dist. law is

$$(S_1 \cap S_2) \cap T = (S_1 \cap T) \cup (S_2 \cap T)$$

Tutorial 9

From Wed's

$$\begin{array}{ccc} \underline{\text{BoolAlg}} & \underline{\text{BoolAlg}}^{op} & \xrightarrow{pt(\cdot)} \text{Top} \\ A & A & \longmapsto (pt(A), \mathcal{O}(pt(A))) \\ \downarrow f & \uparrow f & \uparrow pt(f) \\ B & B & \longmapsto (pt(B), \mathcal{O}(pt(B))) \end{array}$$

Note: $pt(A)$ can be described as

(i) Set of Boolean alg. hom.
$$A \longrightarrow 2$$

(ii) Set of prime filters
$$F \subseteq A$$

Remark:

- Using (i) we can describe
$$pt(f) : pt(B) \longrightarrow pt(A)$$
$$\underbrace{B \xrightarrow{y} 2}_{\in pt(B)} \mapsto \underbrace{A \xrightarrow{f} B \xrightarrow{y} 2}_{\in pt(A)}$$

- Using (ii) we can describe
$$pt(f) : pt(B) \longrightarrow pt(A)$$
as mapping a prime filter $G \subseteq B$ to
$$pt(f)(G) = \{a \in A : f(a) \in G\}$$

Exercise: $pt(f)(G)$ is prime filter on A

Claim: $pt(f) : pt(B) \to pt(A)$ is continuous

Proof:

The claim follows once we show that for every
$$V(a) \subseteq pt(A),$$
its inverse image
$$pt(f)^{-1}(V(a)) = V(b)$$
for some $b \in B$

WTS: $pt(f)^{-1}(V(a)) = V(f(a))$

$$pt(f)^{-1}(V(a)) =$$
$$= \{G \in pt(B) : pt(f)(G) \in V(a)\}$$
$$= \{G \in pt(B) : f^{-1}(G) \in V(a)\}$$
$$= \{G \in pt(B) : a \in f^{-1}(G)\}$$
$$= \{G \in pt(B) : f(a) \in G\}$$
$$= V(f(a))$$

Here, $V(a) \subseteq pt(A)$ is given by
$$V(a) = \{x \in pt(A) : x(a) = 1\}$$
$$\cong \left\{ F \subseteq A : \begin{array}{l} \cdot F \text{ is prime filter} \\ \cdot a \in F \end{array} \right\}$$

Lecture 19 | **Recap:**

We constructed

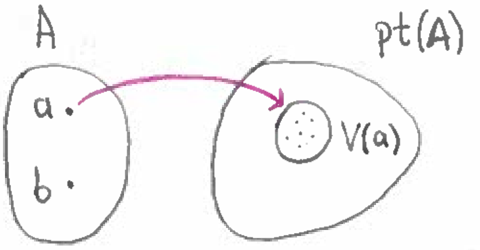
$$\mathbf{BoolAlg}^{op} \xrightarrow{pt} \mathbf{Top}$$

$$A \longmapsto pt(A)$$

- $x \in pt(A) \iff x : A \to 2$
  $\iff F \subseteq A$ is prime filter.

- Topology on $pt(A)$ is generated by

$$V(a) = \{x \in pt(A) \mid x(a) = 1\}$$



Motivation:

We know want to go back from $\mathbf{Top}$ to $\mathbf{BoolAlg}^{op}$

Let $(X, O(X))$ be a top. space. Then, the set $O(X)$ of open subsets of X is a partial order, under $\subseteq$

This is not Boolean Algebra, as $U \in O(X)$ need not have a complement.

Let's take instead

$$Clop(X) = \{U \subseteq X \mid U \text{ is open and closed}\}$$

Prop: $Clop(X)$, partially ordered by $\subseteq$, is a Boolean algebra

Proof: Exercise

Remark: We will extend

$$\mathbf{Top} \xrightarrow{Clop} \mathbf{BoolAlg}^{op}$$

to a functor

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & Clop(X) \\
Y \arrow[r, mapsto] & Clop(Y) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

where, for $V \in Clop(Y)$,

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

Claim: $f^{-1}(V)$ is closed

$$\begin{aligned}
X \setminus f^{-1}(V) &= \{x \in X \mid x \notin f^{-1}(V)\} \\
&= \{x \in X \mid f(x) \notin V\} \\
&= \{x \in X \mid f(x) \in Y \setminus V\} \\
&= f^{-1}(Y \setminus V)
\end{aligned}$$

$\square$

Summary: We have

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, bend left, "Clop"] & \mathbf{BoolAlg}^{op} \arrow[l, bend left, "pt"]
\end{tikzcd}$$

This is not equivalence.

However, we have

$$\eta : 1_{\mathbf{Top}} \Rightarrow pt \circ Clop$$

$$\varepsilon : 1_{\mathbf{BoolAlg}} \Rightarrow Clop \circ pt$$

with components

$$\eta_X : X \longrightarrow pt(Clop(X))$$

$$\varepsilon_A : A \longrightarrow Clop(pt(A))$$

The η_X's are not in general isomorphisms.

We can observe that not every $X \in \mathbf{Top}$ is in the essential image of $pt : \mathbf{BoolAlg}^{op} \to \mathbf{Top}$

The spaces of the form $pt(A)$, for some

$$A \in \mathbf{BoolAlg}^{op}$$

are quite special.

Prop:

$A \in \mathbf{BoolAlg}^{op} \Rightarrow pt(A)$ is compact

Prop: For every $A \in \mathbf{BoolAlg}^{op}$ the space $pt(A)$ has a basis made of clopen sets.

Proof:

We have the basis

$$\{V(a) \mid a \in A\}$$

So we show each $V(a)$, for $a \in A$, is also closed.

WTS:

$$pt(A) \setminus V(a) = V(\neg a)$$

let $x \in pt(A) \setminus V(a)$

$x \in pt(A) \setminus V(a) \iff$

$\iff x : A \longrightarrow 2$ and $x(a) \neq 1$

$\iff x : A \longrightarrow 2$ and $x(\neg a) = \neg x(a) = 1$

$\iff x \in V(\neg a)$ $\square$

Note: We now describe

$$\varepsilon_A, \; y_X$$

in more detail.

Lemma: For $A \in \underline{\text{BoolAlg}}^{op}$,

$$\text{Clop}(pt(A)) = \{V(a) \mid a \in A\}$$

Proof

"$\supseteq$": It's clear.

"$\subseteq$": let $U \in \text{Clop}(pt(A))$.

Since U is open,

$$U = \bigcup_{i \in I} V(a_i)$$

But U is closed and $pt(A)$ is compact.

So U is compact.

So there are

$$i_1, \dots, i_n \in I$$

s.t

$$U = V(a_1) \cup \dots \cup V(a_n)$$

$$= V(a_1 \vee \dots \vee a_n)$$

Hence,

$$U = V(a)$$

for $a = a_1 \vee \dots \vee a_n$ as requested. $\square$

Remark:

Using the lemma

$$\varepsilon_A : A \longrightarrow \text{Clop}(pt(A))$$

is just

$$\varepsilon_A : A \longrightarrow \{V(a) \mid a \in A\}$$
$$a \longmapsto V(a)$$

This is clearly surjective.

Also injective, in fact, an isomorphism of Boolean Algebras.

For $X \in \underline{\text{Top}}$, we have

$$y_X : X \longrightarrow pt(\text{Clop}(X))$$
$$x \longmapsto \{U \in \text{Clop}(X) \mid x \in U\}$$

↑ as a prime filter

It's possible to chacterise the spaces X s.t y_X is an isomorphism.

These are known as Stone Spaces (or Boolean spaces)

Notation:

let us write $\underline{\text{Stone}}$ for the full subcategory of $\underline{\text{Top}}$, whose objects are Stone spaces

---

Thm (Stone duality)

$$\begin{tikzcd} \underline{\text{Stone}} \arrow[rr, bend left, "\text{clop}"] & & \underline{\text{BoolAlg}}^{op} \arrow[ll, bend left, "pt"] \end{tikzcd}$$

is an equivalence of categories.

lecture - 20 | Recall

Stone duality

$$\begin{tikzcd} \text{Stone Spaces} \arrow[r, bend left, "\mathrm{Clop}"] \arrow[r, phantom, "\simeq"] & \mathrm{BoolAlg}^{op} \arrow[l, bend left, "\mathrm{pt}"] \end{tikzcd}$$

For $A \in \underline{\mathrm{BoolAlg}}$:

$$A \xrightarrow{\varepsilon_A} \mathrm{Clop}(\mathrm{pt}(A))$$

where

$$\mathrm{Clop}(\mathrm{pt}(A)) = \{V(a) \mid a \in A\}$$

Motivation

Application to logic

We want to be able to decide whether a certain statement (e.g postulate of parallel lines) follows or not from given "axioms"

Remark:

Let P be a set.

The set $\underline{\mathrm{Sen}}(P)$ of propositional sentences generated by P is the smallest set X s.t

- If $p \in P$ then $p \in X$. atomic sentence
- $\overset{\text{false}}{\bot}, \overset{\text{true}}{\top} \in X$
- If $s, t \in X$, then $s \overset{\text{and}}{\wedge} t,\ s \overset{\text{or}}{\vee} t \in X$
- If $s \in X$, then $\overset{\text{not}}{\neg} s \in X$

Example: Let $P = \{a, b, c\}$

- $\neg a \in \underline{\mathrm{Sen}}(P)$
- $\neg a \vee (b \wedge c) \in \underline{\mathrm{Sen}}(P)$
- $\neg\neg a \in \underline{\mathrm{Sen}}(P)$

Remark:

$\underline{\mathrm{Sen}}(P)$ is not a Boolean algebra:

- It has no partial order
- equations holding in a Boolean algebra don't necessarily hold in $\underline{\mathrm{Sen}}(P)$ e.g
$$a \wedge a = a$$

Def: A propositional theory T over P is a subset

$$T \subseteq \underline{\mathrm{Sen}}(P)$$

We call the elements of T the axioms of the theory.

Remark: Let P be a set. Let $\mathbb{T}$ be a prop. theory over P. We want to identify sentences that are "logically equivalent" w.r.t $\mathbb{T}$.

For example,

$$s \wedge s \longleftrightarrow s$$

$$\neg s \vee \neg t \longleftrightarrow \neg(s \wedge t)$$

$$p \in \mathbb{T},\ p \longleftrightarrow \underset{\text{true}}{\top}$$

This can be done by writing rules to define logical implication, written

$$s_1, \dots, s_n \underbrace{\vdash_{\mathbb{T}}}_{\text{entail}} t$$

Notation: Define an equivalence relation on $\underline{\mathrm{Sen}}(P)$, written

$$s \equiv_{\mathbb{T}} t$$

by

$$\left.\begin{array}{l} s \vdash t \text{ and} \\ t \vdash s \end{array}\right. \Rightarrow s \equiv t$$

Remark: We have an equivalence relation because

- $s \equiv s$
- $s \equiv t \Rightarrow t \equiv s$
- $\left.\begin{array}{l} s \equiv t \text{ and} \\ t \equiv r \end{array}\right. \Rightarrow s \equiv r$

Remark:

The Lindenbaum-Tarski algebra $\mathbb{B}(T)$ of $\mathbb{T}$ is the quotient of $\underline{Sen}(P)$ by $\equiv_T$

$$\mathbb{B}(\mathbb{T}) =_{def} \underline{Sen}(P) / \equiv_T$$

For a sentence s, write

$$[s]_T$$

for its equivalence class

$$[s]_T = \{ s' \mid s \equiv_T s' \}$$

Prop: $\mathbb{B}(\mathbb{T})$ is a Boolean algebra.

Proof:

- $[s] \leq [t]$ if $s \vdash t$
- $[s] \wedge [t] = [s \wedge t]$
- $[\top]$
- $[s] \vee [t] = [s \vee t]$
- $[\bot]$
- $\neg [s] = [\neg s]$

For example, we get

$$[s] \vee \neg [s] = \top$$

i.e

$$[s \vee \neg s] = \top$$

This comes from "law of excluded middle". $\square$

Def: Fix $P, \mathbb{T}$ as above. A valuation w of P is a function

$$w : P \longrightarrow \{true, false\}$$

Remark:

Given a valuation

$$w : P \to \{true, false\}$$

we can extend it to $\underline{Sen}(P)$ by

- $w(s \wedge t) = true \iff$
  $\iff w(s) = true$ and $w(t) = true$
- $w(s \vee t) = true \iff$
  $\iff w(s) = true$ or $w(t) = true$
- $w(\neg s) = true \iff$
  $\iff w(s) = false$

We can extend w also to $\mathbb{B}(\mathbb{T})$

$$\begin{tikzcd}
P \arrow[r, hook] \arrow[drr, bend right, "w"'] & \underline{Sen}(P) \arrow[r] \arrow[dr, "w"'] & \underline{Sen}(P)/\equiv_T \arrow[d, "w"] \arrow[dl, phantom, "(*)"] \\
& & \{true, false\}
\end{tikzcd}$$

We can do $(*)$ when $w(s) = true$ for any $s \in T$.

Def: A model of $\mathbb{T}$ is a valuation w such that

$$w(s) = true$$

for all $s \in \mathbb{T}$.

We then get

$$\mathbb{B}(\mathbb{T}) = \underline{Sen}(P)/\equiv_T \xrightarrow{w} \{true, false\} = 2$$

e.g

$$w([s] \wedge [t]) = w([s]) \wedge w([t])$$

Prop:

Models of $\mathbb{T}$ are the same things as maps

$$\mathbb{B}(\mathbb{T}) \longrightarrow 2$$

in $\underline{BoolAlg}$.

Tutorial - 10

Recall:

Stone duality

$$\begin{tikzcd} \text{Stone Space} \arrow[rr, bend left, "\mathrm{Clop}"] & \simeq & \mathrm{BoolAlg}^{op} \arrow[ll, bend left, "\mathrm{pt}"] \end{tikzcd}$$

$$\varepsilon_A : A \longrightarrow \mathrm{Clop}(\mathrm{pt}(A))$$

where

$$\mathrm{Clop}(\mathrm{pt}(A)) = \{ V(a) \mid a \in A \}$$

and

$$V(a) = \{ x \in \mathrm{pt}(A) \mid x(a) = 1 \}$$

Remark

Let P be a set. Let $\mathbb{T}$ be a propositional theory over P, i.e.

$$\mathbb{T} \subseteq \underline{\mathrm{Sen}}(P)$$

We defined a Boolean algebra $\mathbb{B}(\mathbb{T})$, called the Lindenbaum - Tarski algebra of $\mathbb{T}$

Observation:

- Models of $\mathbb{T}$:

$$w : P \longrightarrow 2$$

such that

$$w(s) = 1$$

$\forall s \in \mathbb{T}$

-

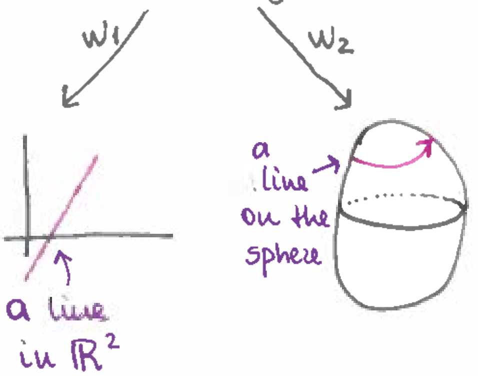
$$w : \mathbb{B}(\mathbb{T}) \longrightarrow \{\text{true}, \text{false}\} = 2$$

Idea:

$$\mathbb{T}$$

$$\underbrace{ax_1, \dots, ax_n}$$

basic axioms of Euclid's geometry



Thm (Soundness and completeness of Prop. logic)

Let P a set, $\mathbb{T}$ be a prop. theory over P. Then

$$s_1, \dots, s_n \vdash t \iff$$

$\iff$ for every model w of $\mathbb{T}$, if $w(s_1) = \text{true}$ and ... and $w(s_n) = \text{true}$ then

$$w(t) = \text{true}.$$

Proof: Let's rewrite the claim in terms of Boolean algebras:

$$s_1, \dots, s_n \vdash t \iff$$

$\iff$ for every model w of $\mathbb{T}$, if $w(s_1) = \dots = w(s_n) = \text{true}$ then $w(t) = \text{true}$

LHS $\iff$

$\iff$ $[s_1] \wedge \dots \wedge [s_n] \leq [t]$ in $\mathbb{B}(\mathbb{T})$

$\iff$ $[s_1 \wedge \dots \wedge s_n] \leq [t]$ in $\mathbb{B}(\mathbb{T})$

RHS $\iff$

$\iff$ For every $w : \mathbb{B}(\mathbb{T}) \to 2$ if $w([s_1]) = \dots = w([s_n]) = \text{true}$ then $w([t]) = \text{true}$

$\iff$ For all $w : \mathbb{B}(\mathbb{T}) \to 2$ if $w \in V([s_1]) \cap \dots \cap V([s_n])$ then $w \in V([t])$

$\iff$

$$V([s_1]) \cap \dots \cap V([s_n]) \subseteq$$
$$\subseteq V([t])$$

$\iff$ $V([s_1 \wedge \dots \wedge s_n]) \subseteq V([t])$ in $\mathrm{Clop}(\mathrm{pt}(\mathbb{B}(\mathbb{T})))$

So we are left to show

$[s_1 \wedge \dots \wedge s_n] \leq [t]$ in $\mathbb{B}(\mathbb{T})$ $\iff$

$\iff$ $V([s_1 \wedge \dots \wedge s_n]) \subseteq V([t])$ in $\mathrm{Clop}(\mathrm{pt}(\mathbb{B}(\mathbb{T})))$

But

$$\mathbb{B}(\mathbb{T}) \cong \mathrm{Clop}(\mathrm{pt}(\mathbb{B}(\mathbb{T})))$$

by Stone duality $\square$

<u>Remark</u>:

- Models of a theory can now be seen as points of a space.
- The view of Stone duality as a strong version of completeness is known as "conceptual completeness". (Makkai)
- The algebra $\mathbb{B}(\mathbb{T})$ is invariant under equivalent presentations of the theory.

Topology Cheat Sheet

MATH43031 & MATH63031 - Category Theory

For the lectures of Week 11

Preliminaries

Let X and Y be sets and $f\colon X \to Y$ be a function. For a subset $V \subseteq Y$, we define the subset $f^{-1}(V) \subseteq X$ by letting $f^{-1}(V) =_{\text{def}} \{x \in X \mid f(x) \in V\}$. We call $f^{-1}(V)$ the *inverse image* of V along f.

Topological spaces

Definition. A *topological space* is a pair $(X, \mathcal{O}(X))$, where

- X is a set, whose the elements are called the *points* of the space
- $\mathcal{O}(X)$ is a collection of subsets of X, called the *topology* of the space, whose elements are called the *open sets* of the space,

satisfying the following properties:

- X and $\varnothing$ are open sets;
- if $(U_i)_{i \in I}$ is a family of open sets, then their union $\bigcup_{i \in I} U_i$ is an open set;
- if U and V are open sets, then their intersection $U \cap V$ is an open set.

We frequently refer to a topological space simply by the set of its points, leaving the topology implicit.

Example. Consider the set of real numbers $\mathbb{R}$. We can define a topology on $\mathbb{R}$ by saying that a subset $U \subseteq \mathbb{R}$ is open if and only if every $x \in U$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$.

Definition. Let $(X, \mathcal{O}(X))$ and $(Y, \mathcal{O}(Y))$ be topological spaces. A *continuous function*

$$f\colon (X, \mathcal{O}(X)) \to (Y, \mathcal{O}(Y))$$

is a function $f\colon X \to Y$ such that $f^{-1}(V) \in \mathcal{O}(X)$ for every $V \in \mathcal{O}(Y)$.

Topological spaces and continuous functions form a category, written **Top**.

Bases for a topology

Let X be a set. In order to equip X with the structure of a topological space, i.e. with a collection of subsets $\mathcal{O}(X)$ satisfying the properties above, it is often convenient to specify what we call a *basis* for the topology, namely a collection of subsets $\mathcal{B}(X)$ satisfying the property that if $U, V \in \mathcal{B}(X)$, then their intersection $U \cap V \in \mathcal{B}(X)$. The elements of a basis are called *basic open sets*. Given a basis $\mathcal{B}(X)$, we define $\mathcal{O}(X)$ to be the family of sets that are unions (not necessarily finite) of basic open sets, i.e. sets of the form $\bigcup_{i \in I} U_i$, where $U_i \in \mathcal{B}(X)$, for all $i \in I$, as well as X and $\varnothing$. We call this the topology *generated* by the basis.

Example. The topology on $\mathbb{R}$ defined above can be described equivalently as the a topology generated by open intervals in $\mathbb{R}$, i.e. the subsets of $\mathbb{R}$ of the form

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

where $-\infty \leq a < b \leq \infty$. This is because $U \subseteq X$ is open if and only if it the union of the basic opens $(x-\varepsilon, x+\varepsilon)$, for x and ε such that $(x-\varepsilon, x+\varepsilon) \subseteq U$.

Remark. Let X and Y be topological spaces and $f\colon X \to Y$ be a function. If $\mathcal{B}(Y)$ is a basis for Y, in order to check that it is continuous, it is enough to check that $f^{-1}(V) \in \mathcal{O}(X)$ for every $V \in \mathcal{B}(Y)$, i.e. only when V is an element of the basis. This is because the inverse image of the union of a family of subsets is the union of the inverse images of the subsets in the family, i.e

$$f^{-1}(\bigcup_{i\in I} V_i) = \bigcup_{i\in I} f^{-1}(V_i).$$

Closed sets, clopen sets

Definition. Let $(X, \mathcal{O}(X))$ be a topological space.

- We say that a subset $C \subseteq X$ is *closed* if its complement $X \setminus C = \{x \in X \mid x \notin C\}$ is open.
- We say that a subset $U \subseteq X$ is *clopen* if it is both open and closed.

Example. For example, for $a, b \in \mathbb{R}$ with $a < b$, the subset

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

is a closed subset of $\mathbb{R}$, because its complement is $(-\infty, a) \cup (b, \infty)$, which is a union of basic open sets and hence it is open. The whole space $\mathbb{R}$ and $\varnothing$ are clopen.

Compact subsets

Let $(X, \mathcal{O}(X))$ be a topological space. Given a subset $S \subseteq X$, an *open cover* of S is a family of open subsets $(U_i)_{i\in I}$ such that $S \subseteq \bigcup_{i\in I} U_i$.

Definition. Let $(X, \mathcal{O}(X))$ be a topological space.

- A subset $K \subseteq X$ is *compact* if for every open cover $(U_i)_{i\in I}$ of S there is a finite subset $J \subseteq I$ such that $S \subseteq \bigcup_{j\in J} U_j$.
- We say $(X, \mathcal{O}(X))$ is *compact space* if X is compact as a subset of itself.

Proposition. Let $(X, \mathcal{O}(X))$ be a topological space. Let $C \subseteq X$ be a subset of X. Assume that X is compact and C is closed. Then C is compact.

The proof of this proposition is not difficult, but it is not part of this unit.