Random Numbers

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:

```
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
```

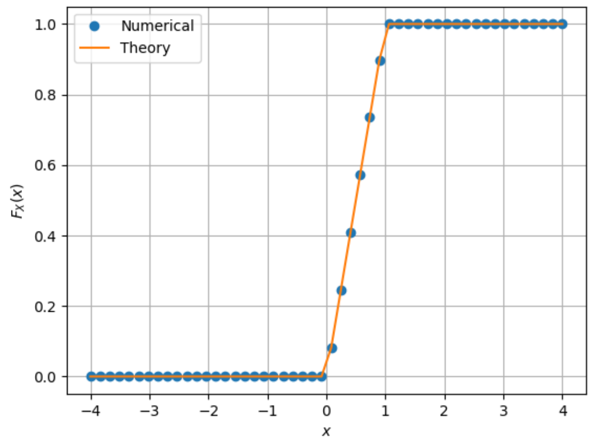

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$f_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{1.3}$$

Hence, If $x \leq 0$,

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{1.4}$$

$$F_U(x) = \int_{-\infty}^{x} 0\, dx \tag{1.5}$$

$$= 0 \tag{1.6}$$

If $0 < x < 1$,

$$F_U(x) = \int_{0}^{x} f_U(x)\, dx \tag{1.7}$$

$$F_U(x) = \int_{0}^{x} 1\, dx \tag{1.8}$$

$$= x \tag{1.9}$$

If $x \geq 1$,

$$F_U(x) = \int_{1}^{x} f_U(x)\, dx \tag{1.10}$$

$$F_U(x) = \int_{1}^{x} 0\, dx \tag{1.11}$$

$$= 0 \tag{1.12}$$

Hence,

$$F_U(x) = \begin{cases} x, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.13}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.14}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.15}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/mean_variance.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

$$\text{Mean obtained} = 0.500007 \tag{1.16}$$

$$\text{Variance obtained} = 0.083301 \tag{1.17}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.18}$$

Solution:

$$E\left[U^k\right] = 0 + \int_0^1 x^k\, dF_U(x) + 0 \tag{1.19}$$

$$= \frac{1}{k+1} \tag{1.20}$$

$$\implies E[U] = \frac{1}{2} \tag{1.21}$$

$$= 0.50 \tag{1.22}$$

$$E\left[U^2\right] = \frac{1}{3} \tag{1.23}$$

Hence,

$$\text{variance} = E\left[U^2\right] - (E[U])^2 \tag{1.24}$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.25}$$

$$= \frac{1}{12} \tag{1.26}$$

$$= 0.0833 \tag{1.27}$$

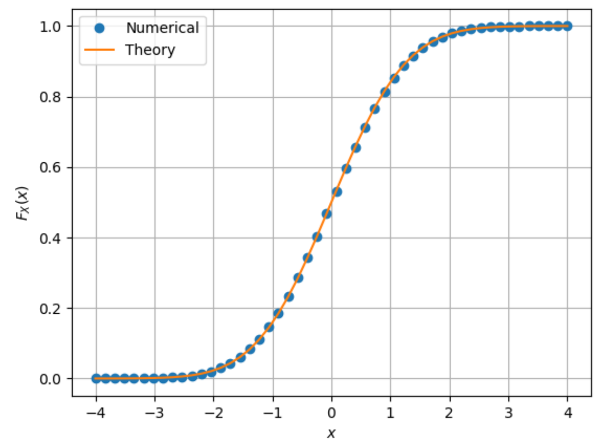

Fig. 2.2: The CDF of X

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat **Solution:**

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 using code

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
```

The CDF of a random variable U has the following properties:

a) $F_U(x)$ is a non decreasing function of x where $-\infty < x < \infty$

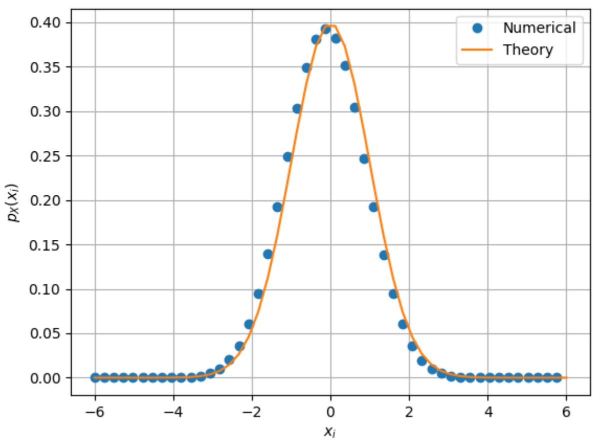

Fig. 2.3: The PDF of X

b) $F_U(x)$ ranges from 0 to 1
c) $F_U(x) = 0$ as $x \to -\infty$
d) $F_U(x) = 1$ as $x \to \infty$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/pdf_plot.py
```

The PDF of a random variable X has the following properties:

a) The probability density function is non-negative for all the possible values.

b) $\int_{-\infty}^{\infty} f(x)\, dx = 1$
c) $f(x) = 0$ as $x \to -\infty$
d) $f(x) = 0$ as $x \to \infty$

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/mean_variance.c
```

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

$$\text{Mean obtained} = 0.000326 \tag{2.3}$$

$$\text{Variance obtained} = 1.000907 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.

Solution: By definition,

$$p_X(x)\, dx = dF_U(x) \tag{2.6}$$

Hence,

$$E[U] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$

$$= 0 \tag{2.9}$$

Also,

$$\text{variance} = E\left[U^2\right] - (E[U])^2 \tag{2.10}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{2.11}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

$$\text{Using integration by parts,} \tag{2.13}$$

$$= \int_{-\infty}^{\infty} xx \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.14}$$

$$= \Big|_{-\infty}^{\infty} - x \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) \tag{2.15}$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \tag{2.16}$$

$$= 1 \implies \text{variance} = 1 \tag{2.17}$$

Also,

$$F_U(x) = \int_{-\infty}^{x} p_X(x)\, dx \tag{2.18}$$

$$= \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\, dx \tag{2.19}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

The CDF of X is plotted in Fig. 3.2 using the code below

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
```

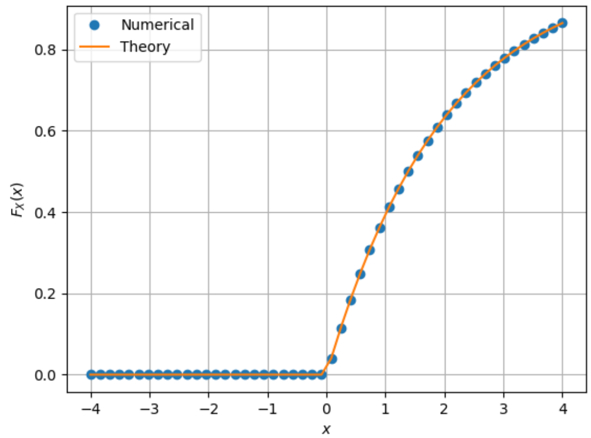

Fig. 3.2: CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Let, $V = g(U)$

$$V = -2\ln(1-U) \tag{3.2}$$

$$\implies U = 1 - e^{-\frac{V}{2}} \tag{3.3}$$

Now,

$$F_V(x) = P(g(U) \le x) \tag{3.4}$$

$$= P\left(X < g^{-1}(V)\right) \tag{3.5}$$

$$= F_U\left(g^{-1}(V)\right) \tag{3.6}$$

$$= F_U\left(1 - e^{-\frac{V}{2}}\right) \tag{3.7}$$

$$F_U\left(1 - e^{-\frac{V}{2}}\right) = \begin{cases} 1 - e^{-\frac{V}{2}}, & V \in (0, \infty) \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. 4.2

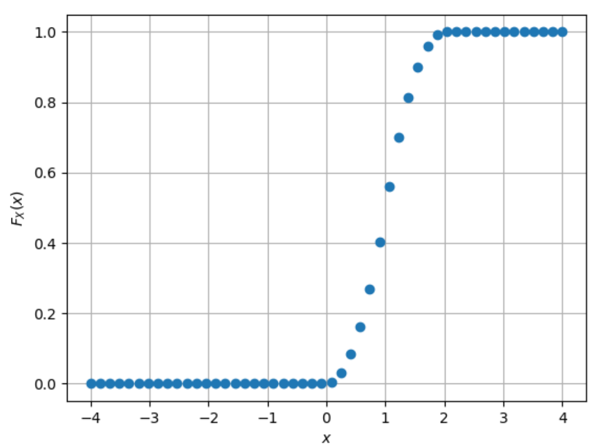

Fig. 4.2: CDF of T

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.3

4.4 Find the theoretical expressions for the PDF and CDF of T.

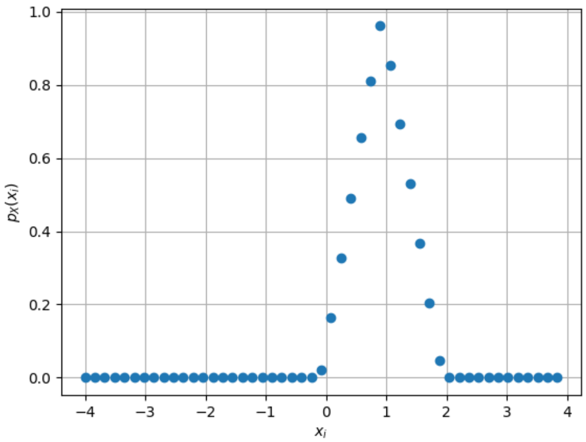

Fig. 4.3: PDF of T

Solution:

$$T = U_1 + U_2 \tag{4.2}$$
$$\implies f_T(T) = f_{U_1+U_2}(T) \tag{4.3}$$
$$= f_{U_1}(T) * f_{U_2}(T) \tag{4.4}$$
$$= \int_{-\infty}^{\infty} f_{U_1}(T-U_2)\, f_{U_2}(U_2)\; dU_2 \tag{4.5}$$

since U_1 and U_2 are independent Also,

$$f_{U_1}(U_1) = f_{U_2}(U_2) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

Hence for $0 \le U_2 \le 1$, Eq:(4.5) becomes,

$$f_T(T) = \int_0^1 f_{U_1}(T-U_2)\; dU_2 \tag{4.7}$$

If $0 \le T \le 1$,

$$f_T(T) = \int_0^T dU_2 \tag{4.8}$$
$$= T \tag{4.9}$$

If $1 < T \le 2$,

$$f_T(T) = \int_{T-1}^1 dU_2 \tag{4.10}$$
$$= 2 - T \tag{4.11}$$

$$f_T(T) = \begin{cases} T, & 0 \le T \le 1 \\ 2-T, & 1 < T \le 2 \\ 0, & \text{otherwise} \end{cases} \tag{4.12}$$

Now,

$$F_T(T) = \int_{-\infty}^T f_T(T)\; dT \tag{4.13}$$

Hence, if $T < 0$,

$$F_T(T) = \int_{-\infty}^T f_T(T)\; dT \tag{4.14}$$
$$= \int_{-\infty}^T 0\, dT \tag{4.15}$$
$$= 0 \tag{4.16}$$

If $0 \le T \le 1$,

$$F_T(T) = \int_0^T f_T(T)\; dT \tag{4.17}$$
$$= \int_0^T T\, dT \tag{4.18}$$
$$= \frac{T^2}{2} \tag{4.19}$$

If $1 < T \le 2$,

$$F_T(T) = \int_0^1 f_T(T)\; dT + \int_1^T f_T(T)\; dT \tag{4.20}$$
$$= \int_0^1 T\, dT + \int_1^T (2-T)\, dT \tag{4.21}$$
$$= \frac{1}{2} + \frac{4T - T^2 - 3}{2} \tag{4.22}$$
$$= 1 - \frac{(2-T)^2}{2} \tag{4.23}$$

If $T > 2$,

$$F_T(T) = \int_2^T f_T(T)\; dT \tag{4.24}$$
$$= \int_2^T 0\, dT \tag{4.25}$$
$$= 0 \tag{4.26}$$

$$F_T(T) = \begin{cases} \frac{T^2}{2}, & 0 \le T \le 1 \\ 1 - \frac{(2-T)^2}{2}, & 1 < T \le 2 \\ 0, & \text{otherwise} \end{cases} \tag{4.27}$$

4.5 Verify your results through a plot.
Solution: The CDF of T is plotted in Fig. 4.5 below
The PDF of T is plotted in Fig. 4.5 below
Using codes

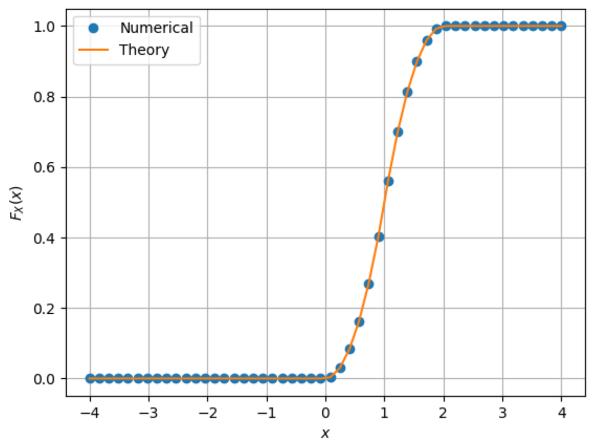

Fig. 4.5: CDF of T

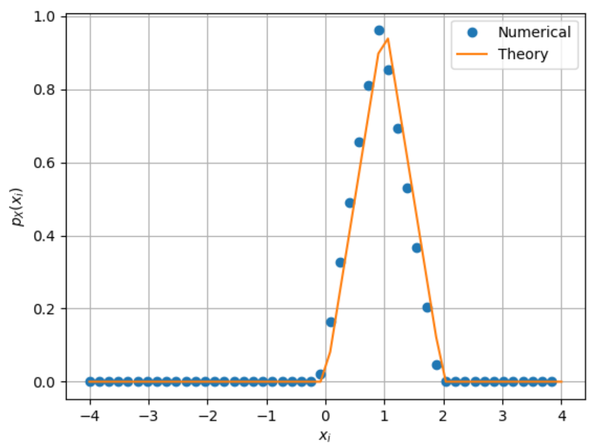

Fig. 4.5: PDF of T

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/pdf_plot.py
```

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim 01$.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

5.3 Plot Y using a scatter plot. The scatter plot of Y is plotted in Fig. 5.3 below
Solution:

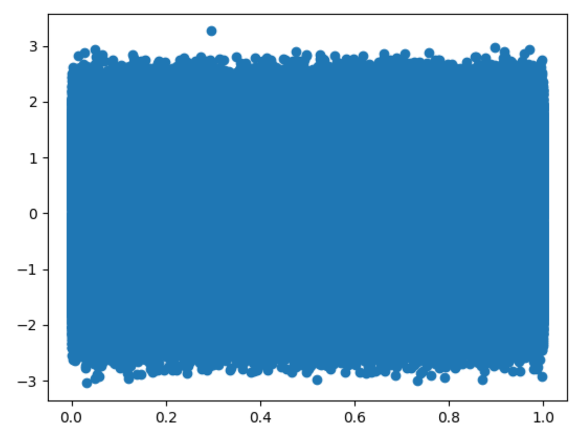

Fig. 5.3: Scatter plot of Y

5.4 Guess how to estimate X from Y.
Solution:
To estimate X from Y we define the following function,

$$g(y) = \begin{cases} 1, & y \in (0, \infty) \\ -1 & y \in (-\infty, 0] \end{cases} \tag{5.2}$$

Hence using this function we can operate on Y to find X

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.4}$$

Solution:

```
wget https://github.com/tejalkul/AI1110–
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110–
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

Values obtained

$$P_{e|0} = 0.311084 \tag{5.5}$$
$$P_{e|1} = 0.311586 \tag{5.6}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: We have,

$$\begin{aligned} P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right) && (5.7)\\ &= \Pr\left(AX + N < 0|X = 1\right) && (5.8)\\ &= \Pr\left(N < -A\right) && (5.9)\\ &= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx && (5.10)\\ &= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx && (5.11)\\ &= Q_N\left(A\right) && (5.12) \end{aligned}$$

$$\begin{aligned} P_{e|1} &= \Pr\left(\hat{X} = 1|X = -1\right) && (5.13)\\ &= \Pr\left(AX + N > 0|X = -1\right) && (5.14)\\ &= \Pr\left(N > A\right) && (5.15)\\ &= Q_N\left(A\right) && (5.16) \end{aligned}$$

$$\begin{aligned} P_e &= P_{e|0}P\left(X = 1\right) + P_{e|1}P\left(X = -1\right) && (5.17)\\ &= P_{e|0}\frac{1}{2} + P_{e|1}\frac{1}{2} && (5.18)\\ &= Q_N\left(A\right)\frac{1}{2} + Q_N\left(A\right)\frac{1}{2} && (5.19)\\ &= Q_N\left(A\right) && (5.20) \end{aligned}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

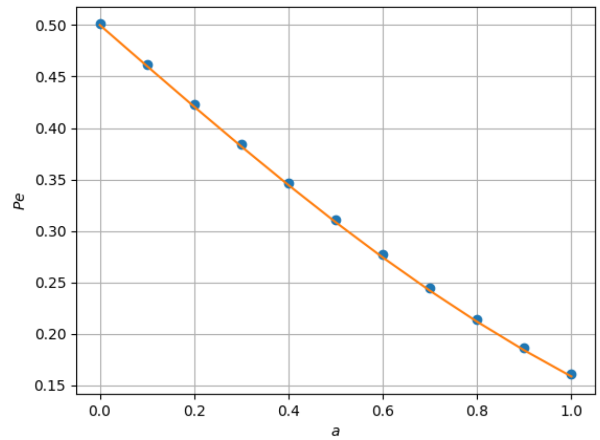

Fig. 5.7: $P_e vs A$

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimizes the theoretical P_e. To estimate X from Y we now define the following function,

$$g\left(y\right) = \begin{cases} 1, & y \in \left(\delta, \infty\right) \\ -1 & y \in \left(-\infty, \delta\right] \end{cases} \tag{5.21}$$

We have,

$$\begin{aligned} P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right) && (5.22)\\ &= \Pr\left(AX + N < \delta|X = 1\right) && (5.23)\\ &= \Pr\left(N < \delta - A\right) && (5.24)\\ &= \int_{-\infty}^{\delta-A} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx && (5.25)\\ &= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx && (5.26)\\ &= Q_N\left(A - \delta\right) && (5.27) \end{aligned}$$

$$\begin{aligned} P_{e|1} &= \Pr\left(\hat{X} = 1|X = -1\right) && (5.28)\\ &= \Pr\left(AX + N > \delta|X = -1\right) && (5.29)\\ &= \Pr\left(N > \delta + A\right) && (5.30)\\ &= Q_N\left(\delta + A\right) && (5.31) \end{aligned}$$

Hence,

$$\begin{aligned} P_e &= P_{e|0}P\left(X = 1\right) + P_{e|1}P\left(X = -1\right) && (5.32)\\ &= P_{e|0}\frac{1}{2} + P_{e|1}\frac{1}{2} && (5.33)\\ &= Q_N\left(A - \delta\right)\frac{1}{2} + Q_N\left(A + \delta\right)\frac{1}{2} && (5.34) \end{aligned}$$

To minimize P_e we differentiate P_e wrt δ,

$$0 = \frac{d}{d\delta}\left(\frac{1}{2}(Q_N(A-\delta) + Q_N(A+\delta))\right) \quad (5.35)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}}\right) \quad (5.36)$$

$$(5.37)$$

Therefore,

$$(\delta - A)^2 = (A+\delta)^2 \quad (5.38)$$

$$\implies \delta = 0 \quad (5.39)$$

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.40)$$

Solution: Let $PX = -1 =$ p, hence $PX = 1 = 1-p$ Hence,

$$P_e = P_{e|0}P(X=1) + P_{e|1}P(X=-1) \quad (5.41)$$

$$P_e = P_{e|0}(1-p) + P_{e|1}p \quad (5.42)$$

$$= Q_N(A-\delta)(1-p) + Q_N(A+\delta)p \quad (5.43)$$

Differentiating like before,

$$0 = p\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - (1-p)\frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}} \quad (5.44)$$

Taking log from both sides,

$$\ln p - \frac{(\delta-A)^2}{2} = \ln(1-p) + \frac{(\delta+A)^2}{2} \quad (5.45)$$

$$\implies \delta = \frac{1}{2A}\ln\frac{1-p}{p} \quad (5.46)$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: Using the MAP criterion we need to maximize $p_{X|Y}(x|y)$.Using Bayes Theorem,

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\,p_X(x)}{p_Y(y)} \quad (5.47)$$

There are 2 cases for $X = 1$ and $X = -1$,

$$p_{Y|X}(y|x) = \begin{cases} p_{Y|X}(y|1) & X = 1 \\ p_{Y|X}(y|-1) & X = -1 \end{cases} \quad (5.48)$$

$$p_{Y|X}(y|x) = \begin{cases} p_N(y-A) & X = 1 \\ p_N(y+A) & X = -1 \end{cases} \quad (5.49)$$

Now,

$$p_Y(y) = p_X(1)\,p_{Y|X}(y|1) + p_X(-1)\,p_{Y|X}(y|-1) \quad (5.50)$$

$$p_Y(y) = (p)\,p_N(y-A) + (1-p)\,p_N(y+A) \quad (5.51)$$

Hence,

$$p_{X|Y}(x|y) = \begin{cases} \frac{p_N(y-A)(p)}{(p)p_N(y-A)+(1-p)p_N(y+A)} & X = 1 \\ \frac{p_N(y+A)(1-p)}{(p)p_N(y-A)+(1-p)p_N(y+A)} & X = -1 \end{cases} \quad (5.52)$$

$$p_{X|Y}(x|y) = \begin{cases} \frac{(p)}{(p)+(1-p)\frac{p_N(y+A)}{p_N(y-A)}} & X = 1 \\ \frac{(1-p)}{(p)\frac{p_N(y-A)}{p_N(y+A)}+(1-p)} & X = -1 \end{cases} \quad (5.53)$$

Simplifying using $p_N(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$,

$$p_{X|Y}(x|y) = \begin{cases} \frac{(p)}{(p)+(1-p)e^{-2yA}} & X = 1 \\ \frac{(1-p)}{(p)e^{2yA}+(1-p)} & X = -1 \end{cases} \quad (5.54)$$

Therefore when $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$ we have,

$$\frac{(p)}{(p)+(1-p)e^{-2yA}} > \frac{(1-p)}{(p)e^{2yA}+(1-p)} \quad (5.55)$$

$$p^2e^{2yA} > (1-p)^2e^{-2yA} \quad (5.56)$$

$$e^{4yA} > (\frac{1-p}{p})^2 \quad (5.57)$$

$$e^{2yA} > \frac{1-p}{p} \quad (5.58)$$

$$y > \frac{1}{2A}\ln\frac{1-p}{p} \quad (5.59)$$

Hence if $y > \frac{1}{2A}\ln\frac{1-p}{p}$ we can say that $X = 1$ and if $y \le \frac{1}{2A}\ln\frac{1-p}{p}$ then $X = -1$ Now if $p = \frac{1}{2}$

$$y > \frac{1}{2A}\ln\frac{1-p}{p} \quad (5.60)$$

$$y > \frac{1}{2A}\ln 1 \quad (5.61)$$

$$y > 0 \quad (5.62)$$

Hence $\delta = 0$ for $p = \frac{1}{2}$

6 Gaussian to Other

6.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: Data for the plots is generated using

```
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

The CDF of V is plotted in Fig. 6.1

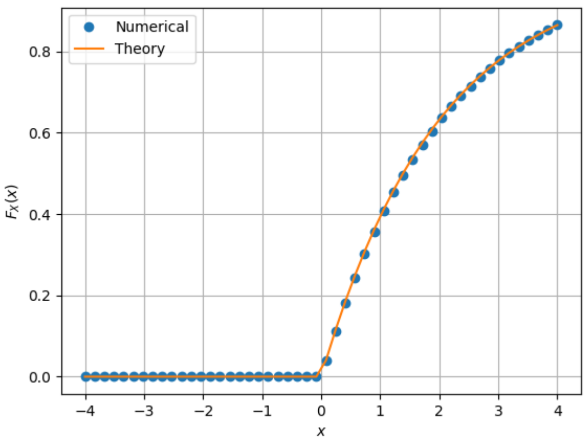

Fig. 6.1: CDF of V

The PDF of V is plotted in Fig. 6.1

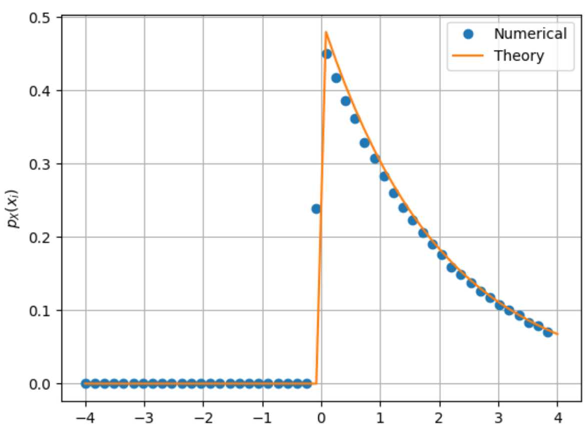

Fig. 6.1: PDF of V

Using codes

```
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/pdf_plot.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

Solution: X_1 and X_2 are i.i.d . Let,

$$X_1 = R\cos\theta \tag{6.3}$$
$$X_2 = R\sin\theta \tag{6.4}$$

Using Jacobian transform we have,

$$f_{X_1X_2}(X_1, X_2) = \frac{1}{|J(R,\theta)|} f_{R\theta}(R,\theta) \tag{6.5}$$

Now,Jacobian matrix is given as follows,

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta R} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta R} & \frac{\delta x_2}{\delta \theta} \end{pmatrix} \tag{6.6}$$
$$J = \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix} \tag{6.7}$$
$$|J| = R \tag{6.8}$$
$$\implies f_{X_1X_2}(X_1, X_2) = \frac{1}{R} f_{R\theta}(R,\theta) \tag{6.9}$$

Now,

$$f_{X_1X_2}(X_1, X_2) = f_{X_1}(X_1) f_{X_2}(X_2) \tag{6.10}$$
$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-x_1^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x_2^2}{2}} \tag{6.11}$$
$$= \frac{1}{2\pi} e^{\frac{-(x_1^2+x_2^2)}{2}} \tag{6.12}$$
$$= \frac{1}{2\pi} e^{\frac{-R^2}{2}} \tag{6.13}$$

Hence,

$$f_{R\theta}(R,\theta) = \frac{R}{2\pi} e^{\frac{-R^2}{2}} \tag{6.14}$$
$$\implies f_R(R) = \int_0^{2\pi} \frac{R}{2\pi} e^{\frac{-R^2}{2}} \, d\theta \tag{6.15}$$
$$= Re^{\frac{-R^2}{2}} \tag{6.16}$$

Hence,

$$f_R(R) = \begin{cases} Re^{\frac{-R^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.17}$$

$$F_R(R) = \begin{cases} 1 - e^{\frac{-R^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.18}$$

Now,

$$F_V(x) = P(V \leq x) \tag{6.19}$$
$$= P\left(R^2 \leq x\right) \tag{6.20}$$
$$= P\left(R \leq \sqrt{x}\right) \tag{6.21}$$
$$= F_R\left(\sqrt{x}\right) \tag{6.22}$$

Hence,

$$F_V(x) = \begin{cases} 1 - e^{\frac{-x}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.23}$$

$$\implies \alpha = \frac{1}{2} \tag{6.24}$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{6.25}$$

Solution: Data for the plots is generated using

```
wget https://github.com/tejalkul/AI1110-Assignments/blob/main/AI1110%20Random%20Variables%20Assignment/codes/exrand.c
wget https://github.com/tejalkul/AI1110-Assignments/blob/main/AI1110%20Random%20Variables%20Assignment/codes/coeffs.h
```

The CDF of A is plotted in Fig. 6.3 below
The PDF of A is plotted in Fig. 6.3 below
Using codes

```
wget https://github.com/tejalkul/AI1110-Assignments/blob/main/AI1110%20Random%20Variables%20Assignment/codes/cdf_plot.py
wget https://github.com/tejalkul/AI1110-Assignments/blob/main/AI1110%20Random%20Variables%20Assignment/codes/pdf_plot.py
```

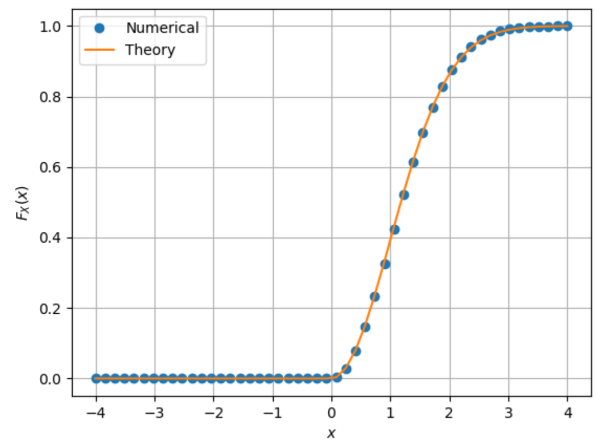

Fig. 6.3: CDF of A

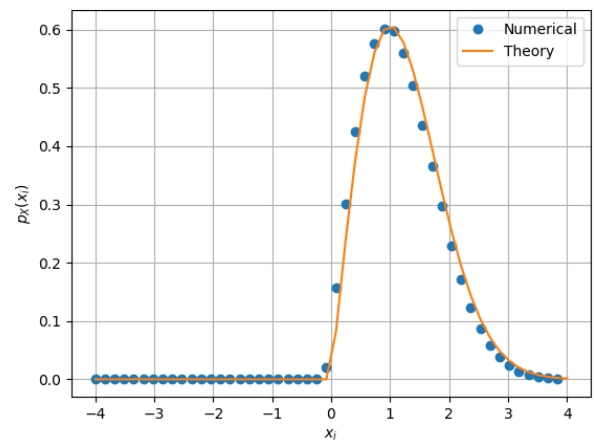

Fig. 6.3: PDF of A

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim 01, X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.