AI1110 Assignment

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Probability Assignment

1.3: Find a theoretical expression for $F_U(x)$

Solution:

$$f_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{2}$$

Hence, If $x \leq 0$,

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{3}$$

$$F_U(x) = \int_{-\infty}^{x} 0\, dx \tag{4}$$

$$= 0 \tag{5}$$

If $0 < x < 1$,

$$F_U(x) = \int_{0}^{x} f_U(x)\, dx \tag{6}$$

$$F_U(x) = \int_{0}^{x} 1\, dx \tag{7}$$

$$= x \tag{8}$$

If $x \geq 1$,

$$F_U(x) = \int_{1}^{x} f_U(x)\, dx \tag{9}$$

$$F_U(x) = \int_{1}^{x} 0\, dx \tag{10}$$

$$= 0 \tag{11}$$

Hence,

$$F_U(x) = \begin{cases} x, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

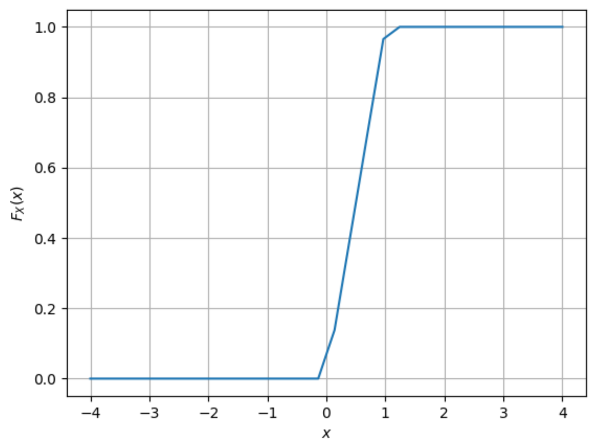

Fig. 1: CDF of U

1.5: Verify your result theoretically given that,

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k\, dF_U(x) \tag{13}$$

Solution:

$$E\left[U^k\right] = 0 + \int_{0}^{1} x^k\, dF_U(x) + 0 \tag{14}$$

$$= \frac{1}{k+1} \tag{15}$$

$$\implies E[U] = \frac{1}{2} \tag{16}$$

$$= 0.50 \tag{17}$$

$$E\left[U^2\right] = \frac{1}{3} \tag{18}$$

Hence,

$$\text{variance} = E\left[U^2\right] - \left(E\left[U\right]\right)^2 \tag{19}$$
$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{20}$$
$$= \frac{1}{12} \tag{21}$$
$$= 0.0833 \tag{22}$$

2.2: What properties does a CDF have?

Solution: The CDF of a random variable U has the following properties:

1) $F_U\left(x\right)$ is a non decreasing function of x where $-\infty < x < \infty$
2) $F_U\left(x\right)$ ranges from 0 to 1
3) $F_U\left(x\right) = 0$ as $x \to -\infty$
4) $F_U\left(x\right) = 1$ as $x \to \infty$

2.3: What properties does a PDF have?

Solution: The PDF of a random variable X has the following properties:

1) The probability density function is non-negative for all the possible values.
2) $\int_{-\infty}^{\infty} f\left(x\right)\, dx = 1$
3) $f\left(x\right) = 0$ as $x \to -\infty$
4) $f\left(x\right) = 0$ as $x \to \infty$

2.5: Given that

$$p_X\left(x\right) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{23}$$

repeat the above exercise theoretically.

Solution: By definition,

$$p_X\left(x\right) dx = dF_U\left(x\right) \tag{24}$$

Hence,

$$E\left[U\right] = \int_{-\infty}^{\infty} x p_X\left(x\right)\, dx \tag{25}$$
$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)\, dx \tag{26}$$
$$= 0 \tag{27}$$

Also,

$$\text{variance} = E\left[U^2\right] - \left(E\left[U\right]\right)^2 \tag{28}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 p_X\left(x\right)\, dx \tag{29}$$
$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)\, dx \tag{30}$$
$$= 1 \tag{31}$$
$$\implies \text{variance} = 1 \tag{32}$$

Also,

$$F_U\left(x\right) = \int_{-\infty}^{x} p_X\left(x\right)\, dx \tag{33}$$
$$= \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\, dx \tag{34}$$

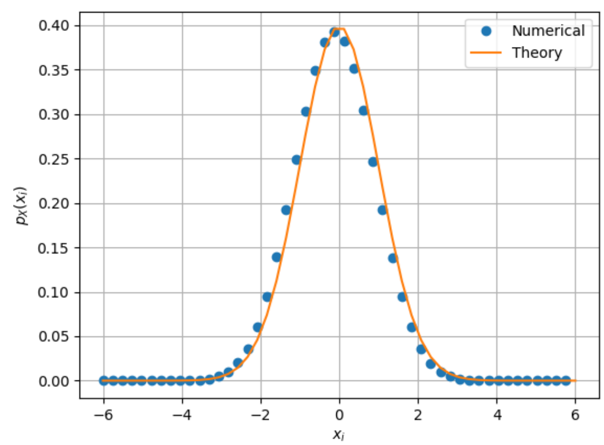

Fig. 2: PDF of X

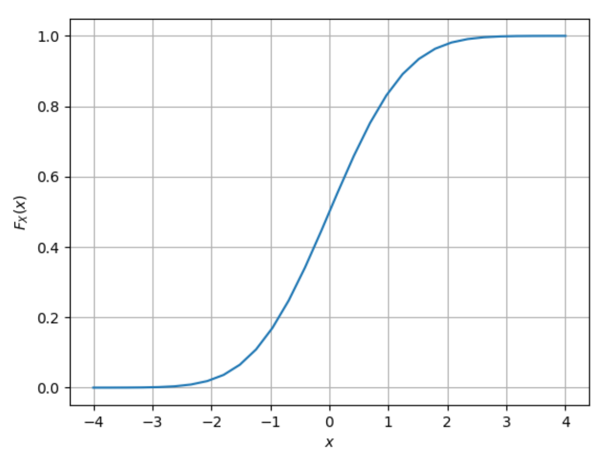

Fig. 3: CDF of X

3.2: Find a theoretical expression for $F_V\left(x\right)$

Solution: Let, $V = g\left(U\right)$

$$V = -2\ln\left(1-U\right) \tag{35}$$
$$\implies U = 1 - e^{-\frac{V}{2}} \tag{36}$$

Now,

$$F_V\left(x\right) = P\left(g\left(U\right) \le x\right) \tag{37}$$
$$= P\left(X < g^{-1}\left(V\right)\right) \tag{38}$$
$$= F_U\left(g^{-1}\left(V\right)\right) \tag{39}$$
$$= F_U\left(1 - e^{-\frac{V}{2}}\right) \tag{40}$$

$$F_U\left(1 - e^{-\frac{V}{2}}\right) = \begin{cases} 1 - e^{-\frac{V}{2}}, & V \in (0, \infty) \\ 0, & \text{otherwise} \end{cases} \tag{41}$$

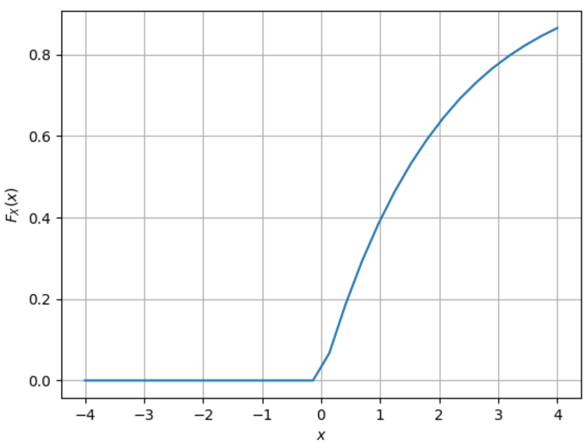

Fig. 4: CDF of V