Random Numbers

Abstract—**This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:

```
wget https://github.com/tejalkul/AI1110−Assignments/blob/main/AI1110%20Random%20Variables%20Assignment/codes/exrand.c
wget https://github.com/tejalkul/AI1110−Assignments/blob/main/AI1110%20Random%20Variables%20Assignment/codes/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
https://github.com/tejalkul/AI1110−Assignments/blob/main/AI1110%20Random%20Variables%20Assignment/codes/cdf_plot.py
```

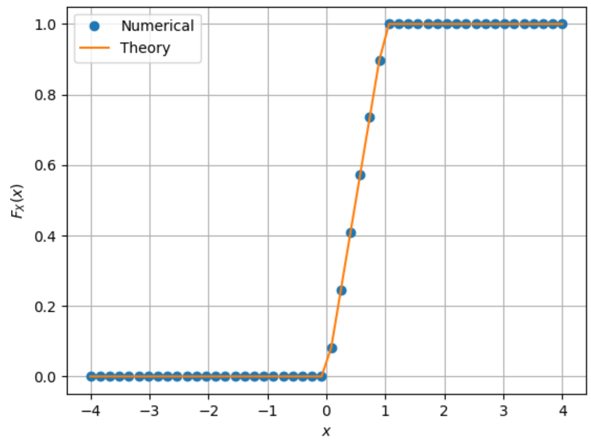

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$f_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\,dx \tag{1.3}$$

Hence, If $x \le 0$,

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\,dx \tag{1.4}$$

$$F_U(x) = \int_{-\infty}^{x} 0\,dx \tag{1.5}$$

$$= 0 \tag{1.6}$$

If $0 < x < 1$,

$$F_U(x) = \int_{0}^{x} f_U(x)\,dx \tag{1.7}$$

$$F_U(x) = \int_{0}^{x} 1\,dx \tag{1.8}$$

$$= x \tag{1.9}$$

If $x \ge 1$,

$$F_U(x) = \int_{1}^{x} f_U(x)\,dx \tag{1.10}$$

$$F_U(x) = \int_{1}^{x} 0\,dx \tag{1.11}$$

$$= 0 \tag{1.12}$$

Hence,

$$F_U(x) = \begin{cases} x, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.13}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.14}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.15}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/mean_variance.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

$$\text{Mean obtained} = 0.500007 \tag{1.16}$$

$$\text{Variance obtained} = 0.083301 \tag{1.17}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.18}$$

Solution:

$$E\left[U^k\right] = 0 + \int_0^1 x^k\, dF_U(x) + 0 \tag{1.19}$$

$$= \frac{1}{k+1} \tag{1.20}$$

$$\implies E[U] = \frac{1}{2} \tag{1.21}$$

$$= 0.50 \tag{1.22}$$

$$E\left[U^2\right] = \frac{1}{3} \tag{1.23}$$

Hence,

$$\text{variance} = E\left[U^2\right] - (E[U])^2 \tag{1.24}$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.25}$$

$$= \frac{1}{12} \tag{1.26}$$

$$= 0.0833 \tag{1.27}$$

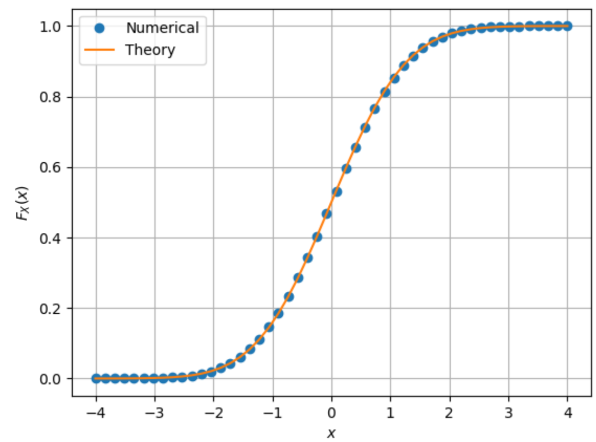

Fig. 2.2: The CDF of X

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat **Solution:**

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 using code

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
```

The CDF of a random variable U has the following properties:

a) $F_U(x)$ is a non decreasing function of x where $-\infty < x < \infty$

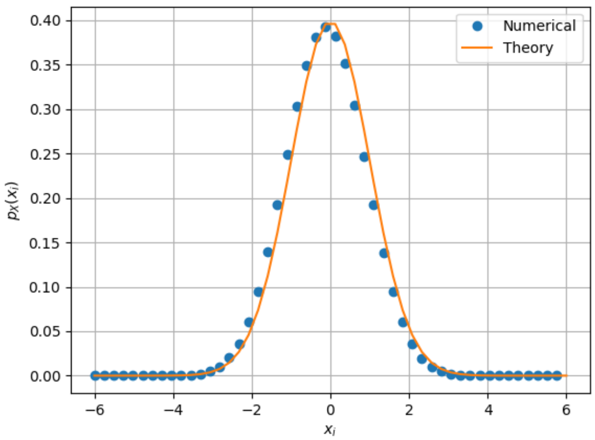

Fig. 2.3: The PDF of X

b) $F_U(x)$ ranges from 0 to 1
c) $F_U(x) = 0$ as $x \to -\infty$
d) $F_U(x) = 1$ as $x \to \infty$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/pdf_plot.py
```

The PDF of a random variable X has the following properties:

a) The probability density function is non-negative for all the possible values.
b) $\int_{-\infty}^{\infty} f(x)\, dx = 1$
c) $f(x) = 0$ as $x \to -\infty$
d) $f(x) = 0$ as $x \to \infty$

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/mean_variance.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

$$\text{Mean obtained} = 0.000326 \tag{2.3}$$

$$\text{Variance obtained} = 1.000907 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.

Solution: By definition,

$$p_X(x)\, dx = dF_U(x) \tag{2.6}$$

Hence,

$$E[U] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$

$$= 0 \tag{2.9}$$

Also,

$$\text{variance} = E\left[U^2\right] - (E[U])^2 \tag{2.10}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{2.11}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

$$\text{Using integration by parts,} \tag{2.13}$$

$$= \int_{-\infty}^{\infty} xx \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.14}$$

$$= \Big|_{-\infty}^{\infty} - x \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) \tag{2.15}$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \tag{2.16}$$

$$= 1 \implies \text{variance} = 1 \tag{2.17}$$

Also,

$$F_U(x) = \int_{-\infty}^{x} p_X(x)\, dx \tag{2.18}$$

$$= \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\, dx \tag{2.19}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

The CDF of X is plotted in Fig. 3.2 using the code below

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
```

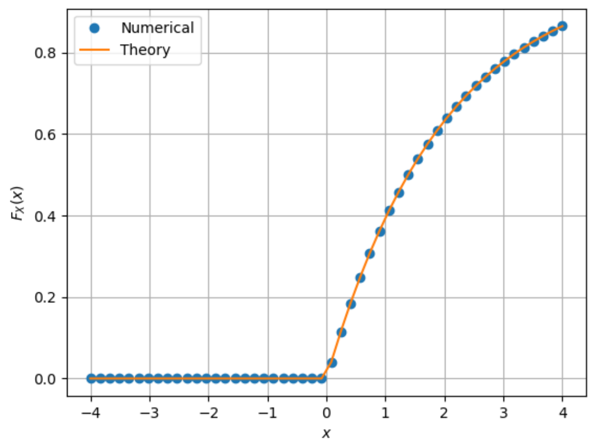

Fig. 3.2: CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Let, $V = g(U)$

$$V = -2\ln(1-U) \tag{3.2}$$

$$\implies U = 1 - e^{-\frac{V}{2}} \tag{3.3}$$

Now,

$$F_V(x) = P(g(U) \le x) \tag{3.4}$$

$$= P\left(X < g^{-1}(V)\right) \tag{3.5}$$

$$= F_U\left(g^{-1}(V)\right) \tag{3.6}$$

$$= F_U\left(1 - e^{-\frac{V}{2}}\right) \tag{3.7}$$

$$F_U\left(1-e^{-\frac{V}{2}}\right) = \begin{cases} 1 - e^{-\frac{V}{2}}, & V \in (0,\infty) \\ 0, & \text{otherwise} \end{cases} \tag{3.8}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:

```
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/exrand.c
wget https://github.com/tejalkul/AI1110-
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/coeffs.h
```

4.2 Find the CDF of T.
4.3 Find the PDF of T.
4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution:

$$T = U_1 + U_2 \tag{4.2}$$

$$\implies f_T(T) = f_{U_1+U_2}(T) \tag{4.3}$$

$$= f_{U_1}(T) * f_{U_2}(T) \tag{4.4}$$

$$= \int_{-\infty}^{\infty} f_{U_1}(T-U_2)\, f_{U_2}(U_2)\ dU_2 \tag{4.5}$$

since U_1 and U_2 are independent Also,

$$f_{U_1}(U_1) = f_{U_2}(U_2) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

Hence for $0 \le U_2 \le 1$, Eq:(4.5) becomes,

$$f_T(T) = \int_0^1 f_{U_1}(T-U_2)\ dU_2 \tag{4.7}$$

If $0 \le T \le 1$,

$$f_T(T) = \int_0^T dU_2 \tag{4.8}$$

$$= T \tag{4.9}$$

If $1 < T \leq 2$,

$$f_T(T) = \int_{T-1}^{1} dU_2 \tag{4.10}$$
$$= 2 - T \tag{4.11}$$

$$f_T(T) = \begin{cases} T, & 0 \leq T \leq 1 \\ 2-T, & 1 < T \leq 2 \\ 0, & \text{otherwise} \end{cases} \tag{4.12}$$

Now,

$$F_T(T) = \int_{-\infty}^{T} f_T(T)\, dT \tag{4.13}$$

Hence, if $T < 0$,

$$F_T(T) = \int_{-\infty}^{T} f_T(T)\, dT \tag{4.14}$$
$$= \int_{-\infty}^{T} 0\, dT \tag{4.15}$$
$$= 0 \tag{4.16}$$

If $0 \leq T \leq 1$,

$$F_T(T) = \int_{0}^{T} f_T(T)\, dT \tag{4.17}$$
$$= \int_{0}^{T} T\, dT \tag{4.18}$$
$$= \frac{T^2}{2} \tag{4.19}$$

If $1 < T \leq 2$,

$$F_T(T) = \int_{0}^{1} f_T(T)\, dT + \int_{1}^{T} f_T(T)\, dT \tag{4.20}$$
$$= \int_{0}^{1} T\, dT + \int_{1}^{T} (2-T)\, dT \tag{4.21}$$
$$= \frac{1}{2} + \frac{4T - T^2 - 3}{2} \tag{4.22}$$
$$= 1 - \frac{(2-T)^2}{2} \tag{4.23}$$

If $T > 2$,

$$F_T(T) = \int_{2}^{T} f_T(T)\, dT \tag{4.24}$$
$$= \int_{2}^{T} 0\, dT \tag{4.25}$$
$$= 0 \tag{4.26}$$

$$F_T(T) = \begin{cases} \frac{T^2}{2}, & 0 \leq T \leq 1 \\ 1 - \frac{(2-T)^2}{2}, & 1 < T \leq 2 \\ 0, & \text{otherwise} \end{cases} \tag{4.27}$$

4.5 Verify your results through a plot.
Solution: The CDF of T is plotted in Fig. 4.5 below

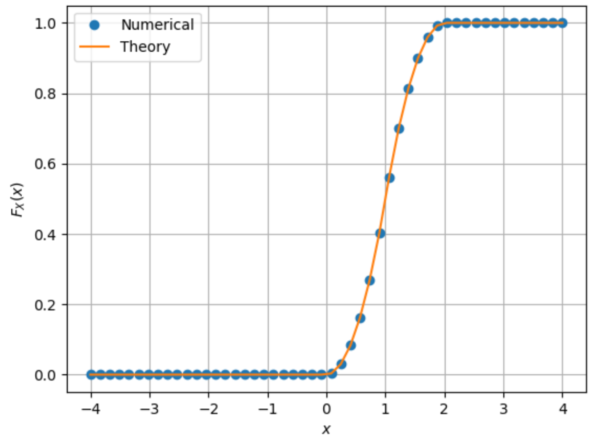

Fig. 4.5: CDF of T

The PDF of T is plotted in Fig. 4.5 below

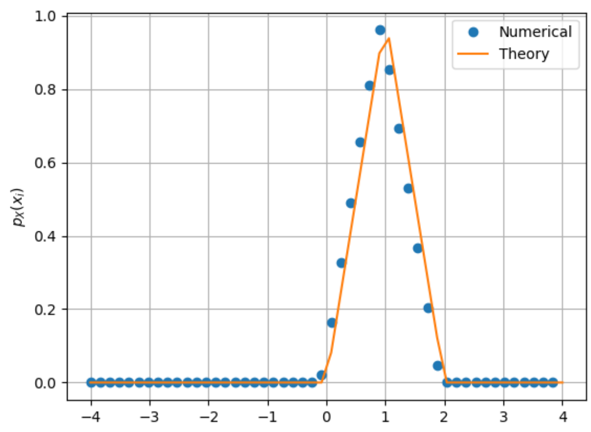

Fig. 4.5: PDF of T

Using codes

```
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
    Random%20Variables%20Assignment/
    codes/cdf_plot.py
wget https://github.com/tejalkul/AI1110−
    Assignments/blob/main/AI1110%20
```

Random%20Variables%20Assignment/codes/pdf_plot.py

5 Maximul Likelihood

5.1 Generate
$$Y = AX + N, \tag{5.1}$$
where $A = 5$ dB, $X1\{1, -1\}$, is Bernoulli and $N \sim 01$.

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find
$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$
and
$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

5.5 Find P_e.

5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of
$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If
$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$
find α.

Solution: X_1 and X_2 are i.i.d . Let,
$$X_1 = R\cos\theta \tag{6.3}$$
$$X_2 = R\sin\theta \tag{6.4}$$
Using Jacobian transform we have,
$$f_{X_1X_2}(X_1, X_2) = \frac{1}{|J(R,\theta)|} f_{R\theta}(R,\theta) \tag{6.5}$$
Now,Jacobian matrix is given as follows,
$$J = \begin{pmatrix} \frac{\delta x_1}{\delta R} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta R} & \frac{\delta x_2}{\delta \theta} \end{pmatrix} \tag{6.6}$$
$$J = \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix} \tag{6.7}$$
$$|J| = R \tag{6.8}$$
$$\implies f_{X_1X_2}(X_1, X_2) = \frac{1}{R} f_{R\theta}(R,\theta) \tag{6.9}$$
Now,
$$f_{X_1X_2}(X_1, X_2) = f_{X_1}(X_1) f_{X_2}(X_2) \tag{6.10}$$
$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-X_1^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-X_2^2}{2}} \tag{6.11}$$
$$= \frac{1}{2\pi} e^{\frac{-\left(X_1^2 + X_2^2\right)}{2}} \tag{6.12}$$
$$= \frac{1}{2\pi} e^{\frac{-R^2}{2}} \tag{6.13}$$
Hence,
$$f_{R\theta}(R,\theta) = \frac{R}{2\pi} e^{\frac{-R^2}{2}} \tag{6.14}$$
$$\implies f_R(R) = \int_0^{2\pi} \frac{R}{2\pi} e^{\frac{-R^2}{2}} \, d\theta \tag{6.15}$$
$$= Re^{\frac{-R^2}{2}} \tag{6.16}$$
Hence,
$$f_R(R) = \begin{cases} Re^{\frac{-R^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.17}$$
$$F_R(R) = \begin{cases} 1 - e^{\frac{-R^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.18}$$
Now,
$$F_V(x) = P(V \leq x) \tag{6.19}$$
$$= P\left(R^2 \leq x\right) \tag{6.20}$$
$$= P\left(R \leq \sqrt{x}\right) \tag{6.21}$$
$$= F_R\left(\sqrt{x}\right) \tag{6.22}$$
Hence,
$$F_V(x) = \begin{cases} 1 - e^{\frac{-x}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.23}$$
$$\implies \alpha = \frac{1}{2} \tag{6.24}$$

6.3 Plot the CDF and PDf of
$$A = \sqrt{V} \tag{6.25}$$

7 Conditional Probability

7.1

7.2 Plot
$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$
for
$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim 01, X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in \left(\mathbf{s}_0, \mathbf{s}_1\right), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.