AI1110 Assignment 11

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Example 7-8: Using the equation,

$$\phi_z(w) = E\left\{e^{jw(x_1+x_2+....x_n)}\right\} = \phi_1(w)....\phi_n(w) \tag{1}$$

where x_i are independent and $\phi_i(w)$ is the characteristic function of x_i.
Prove:

(a) Bernoulli Trials Fundamental theorem
(b) Poisson Theorem

Solution:

(a) Bernoulli Trials:
Consider the random variable x_i as follows: $x_i = 1$ if head shows the at the ith trial and $x_i = 0$ otherwise. Thus,

$$\Pr(x_i = 1) = \Pr(h) = p \tag{2}$$

$$\Pr(x_i = 0) = \Pr(t) = q \tag{3}$$

$$\phi_i(w) = pe^{jw} + q \tag{4}$$

The random variable $z = x_1 + ... + x_n$ takes the values 0,1.....,n and $\{z = k\}$ is the event {k heads in n tosses}. Furthermore,

$$\phi_z(w) = E\left\{e^{jwz}\right\} = \sum_{k=0}^{n} \Pr(z = k) e^{jkw} \tag{5}$$

The random variables x_i are independent because x_i depends only on the the outcomes of the ith trial.Hence,

$$\phi_i(w) = \left(pe^{jw} + q\right)^n = \sum_{k=0}^{n} {}^nC_k p^k e^{jkw} q^{n-k} \tag{6}$$

Comparing with (5), we get,

$$\Pr(z = k) = \Pr(\text{k heads}) = {}^nC_k p^k q^{n-k} \tag{7}$$

Hence Proved.

(b) Poisson Theorem:

We will show that if $p << 1$, then

$$\Pr(z = k) = \frac{e^{-np}(np)^k}{k!}. \tag{8}$$

We will establish a more general result. Suppose the random variables x_i are independent and each takes values 1 and 0 with respective probabilities p_i and $q_i = 1 - p_i$. If $p_i << 1$ then,

$$e^{p_i\left(e^{jw}-1\right)} = 1 + p_i\left(e^{jw} - 1\right) \tag{9}$$

$$= p_i e^{jw} + q_i \tag{10}$$

$$= \phi_i(w) \tag{11}$$

With $z = x_1 + ... + x_n$, it follows from (1) that,

$$\phi_Z(w) = e^{p_1\left(e^{jw}-1\right)}...e^{p_n\left(e^{jw}-1\right)} = e^{a\left(e^{jw}-1\right)} \tag{12}$$

where $a = p_1 + ... + p_n$. This leads to the conclusion that the random variable z is approximately Poisson distributed with parameter a. It can be shown that the result is exact in the limit if
$p_i \to 0$ and $p_1 + ... + p_n \to a$ as $n \to \infty$
Hence Proved