Modelling of the Rotary Inverted Pendulum System

M. Roman, E. Bobasu, D. Sendrescu
University of Craiova, Faculty of Automation, Computers and Electronics,
Department of Automation, e-mail: monica@automation.ucv.ro

***Abstract*- This paper presents an approach on the Bond Graph modelling applied to a Quanser Rotary Inverted Pendulum experiment followed by the computation of the mathematical model. The system was decomposed into subsystems that were modelled separately. The obtained subsystems generated submodels and the overall model was then built up by combining these separate structures. The model was built up and simulated using BondSim environment.**

I. INTRODUCTION

The study of system dynamics resides in modelling its behaviour. Systems models are simplified, abstracted structures used to predict the behaviour of the studied systems. Our interest is pointing towards the mathematical model used to predict certain aspects of the system response to the inputs. In mathematical notations a system model is described by a set of ordinary differential equations in terms of state variables and a set of algebraic equations that relate the state variable to other system variables.

In order to model a system it is usually necessary to decompose the system into smaller parts – subsystems - that can be modelled separately. The subsystem is a part of the system that can be modelled as a system itself obtaining submodels. The overall model can then be built up by combining the separate submodels.

System models will be constructed using a uniform notation for all types of physical systems which is Bond Graph method based on energy and information flow [1]. Using this method it is possible to develop models of electrical, mechanical, magnetic, hydraulic, pneumatic, thermal, and other systems using a small set of variables. The method uses the effort-flow analogy to describe physical processes. A Bond Graph consists of subsystems linked together by lines representing power bonds. Each process is described by a pair of variables, effort (*e*) and flow (*f*), and their product is the power. The direction of power is depicted by a half arrow. In a dynamic system the effort and the flow variables, and hence the power fluctuate in time.

It is remarkable how models of various systems belonging to different engineering domains can be express using a set of only nine elements, called elementary components. These elements are sufficient to describe any physical system regardless of the energy types processed by it [2].

A classification of Bond Graph elements can be made up by the number of ports. The ports are places where interactions with other processes take place. There are one port elements represented by inertial elements (I), capacitive elements (C), resistive elements (R), effort sources (SE) and flow sources (SF), two ports elements represented by transformer elements (TF) and gyrator elements (GY), and multi ports elements - effort junctions (J0) and flow junctions (J1). These elements are sufficient to describe any physical system regardless of the energy types processed by it.

I, C, and R elements are passive elements because they convert the supplied energy into stored or dissipated energy. The half arrow is always pointing towards these elements.

Se and Sf elements are active elements because they supply power to the system and the half arrow is always pointed outwards these elements.

TF, GY, 0 and 1 junctions are junction elements that serve to connect I, C, R, and source elements and constitute the junction structure of Bond Graph model.

The concept of causality is an important concept embedded in Bond Graph theory. This refers to cause (input) and effect (output) relationship. Thus, as part of the Bond Graph modelling process, a causality assignment is implicitly introduced. Causality is graphically represented by a short stroke, called causal stroke, placed perpendicular to the bond at one of its ends indicating the direction of the effort variable. Causal stroke assignment is independent of the power flow direction. This leads to the description of bond-graphs in the form of state – space equation.

The sources (Se and Sf) have fixed causality, the element of dissipation (R) has free causality depending on the causality of the other elements of Bond Graph, and the storage elements (I, C) have preferential causality, that is integral causality or derivative causality, but it is always desirable that C and I elements to be in integral causality.

Transformers, gyrators and junction elements have constrainedly causality. Thus on a 0-junction one effort pushes inward, all others outward and on a 1-junction one flow points inwards, all others point outwards.

Besides the power variables, two other types of variables are very important in describing dynamic systems and these variables, sometimes called energy variables, are the generalized momentum (p) as time integral of effort and the generalized displacement (q) as time integral of flow.

The goal of this paper is to model the Quanser Rotary Inverted Pendulum experiment using a systematic way of building it in small steps [3].

A first step is to write a word Bond Graph which contains words instead of standard symbols for the main components

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and bonds for power and signal exchange. The component name is useful, but it is more important the connection of the components to other components through ports.

There are two types of ports, power ports characterized by power flow into or out of the component, graphically represented by a half arrow, and control ports characterized by negligible power flow and high information content, depicted by a full arrow.

The next step is to replace words by standards elements which contain precise mathematical or functional relations. Each element represents a definite effect or action in the system. When the Bond Graph model is done it is possible to formulate the state space equations starting from the constitutive relations of elements.

II. QUANSER INVERTED PENDULUM EXPERIMENT

The Quanser Inverted Pendulum experiment [4] consists of a mechanical and an electrical subsystem. The Quanser experimental set-up contains the following components: Quanser Universal Power Module UPM 2405/1503; Quanser MultiQ PCI data acquisition board; Quanser Rotpen – Rotary Pendulum Module; Quanser SRV02-E servo-plant; PC equipped with Matlab/Simulink and WinCon software.

The modeling of the mechanical subsystem consists in describing the tip deflection and the base rotation dynamics. The electrical subsystem involves modeling a DC servomotor that dynamically relates voltage to torque.

Figure 1. SRV02 plant – DC motor and gear box

The SRV02 rotary plant module serves as the base component for the rotary family of experiments. Its modularity facilitates the change from one experimental setup to another.

The SRV02 plant consists of a DC motor in a solid aluminium frame equipped with a gearbox whose output drives external gears. The basic unit is equipped with a potentiometer to measure the output/load angular position.

The external gear can be reconfigured in two configurations:

- Low Gear Ratio - this is the recommended configuration to perform the position and speed control experiments with no other module attached to the output.

The only loads that are recommended for this configuration are the bar and circular loads supplied with the system;

- High Gear Ratio - this is the recommended configuration for all other experiments that require an additional module such as the flexible beam, ball and beam, gyro, rotary inverted pendulum etc.

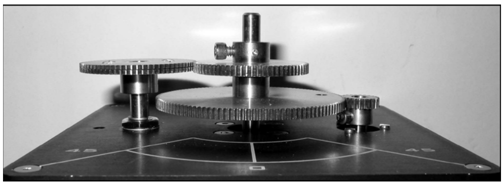

Figure 2. High gear configuration

The pendulum module is attached to the SRV02 load gear by two thumbscrews (Fig. 3). The Pendulum Arm is attached to the module body by a set screw. The Inverted Pendulum experiment is a classical example of how the use of control may be employed to stabilize an inherently unstable system. The Inverted pendulum is also an accurate model in the pitch and yaw of a rocket in flight and can be used as a benchmark for many control methodologies [5].

Figure 3. The Quanser rotary inverted pendulum experiment

III. BOND GRAPH MODELING OF THE SYSTEM

The models are built up using BondSim software (BondSim Research Pack 3.0 is a registered trademark of Vjekoslav Damic, The Polytechnic of Dubrovnik, Croatia, 2002) [6]. Modelling in BondSim consists of creating objects in the computer memory that are represented on screen as Bond Graphs. Models development is done entirely visually, without coding, using the support of the Windows system. The components models are real objects that serve as interfaces to the document that contains the model.

This is a modelling environment that supports the fundamental modelling approach, systematic problem decomposition and model creation at every level of decomposition. The number of levels of decomposition depends on the complexity of the engineering system modelled. For simple problems it is sufficient a single level, but real systems consist of many components and thus two or more levels of model decomposition may be necessary. The model can be created as multi-level structures. The modelling environment supports building the model from pieces.

Many engineering systems consist of components and simulation models of such components can be represented as objects in the computer memory and depicted on the screen by their name that is any word description chosen to describe the component. The component name is useful for reference to the model, but is more important to represent how the component is connected to other components. The places where interactions with other components take place are called ports. A component represented by its word description and its ports is taken as the most fundamental representation of a component model and is called the word model. The word model is used as the starting point of component model development.

In the model representation besides the elementary components we use some components corresponding to block diagram operations and these can be used to define control laws and to process the results of simulations. These are similar to the other components but they can have only control input and control output ports. The elementary block diagram components are: input, output, function, integrator, differentiator, summator, and node.

We proceed to the development of the model by identifying the system components and connecting them as they are in the real system [7]. The block diagram representation of the system in terms word model is presented in the figure below:

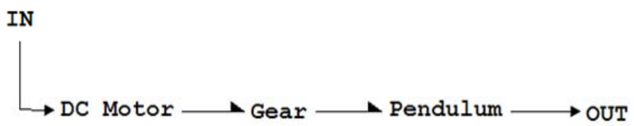

Figure 4. Block diagram representation of the system

The Bond Graph model of DC Motor component is presented in Fig.5.

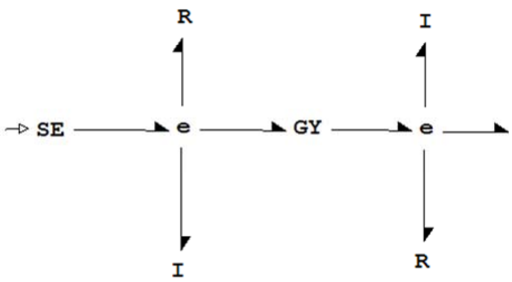

Figure 5. Bond Graph model of DC motor component

The gyrator GY describes the electromechanical conversion in the motor relating the back emf from the electrical part to the angular velocity of the rotor from the mechanical part, respectively the armature current from the electrical part to the torque acting on the rotor. For this reason, the gyrators are called overcrossed transformers.

$$\begin{cases} V_b = k_m \cdot \omega \\ T_m = k_t \cdot i_a \end{cases} \tag{1}$$

where: k_t is the motor torque constant, and k_m is the back emf constant.

The electrical process in the armature is described in Bond Graph terms by the armature resistance R_a represented using a resistive element (R), and the armature inductance L_a represented using an inertial element (I). These two elements are joined through an effort junction (1 junction). The mechanical process is also described using an inertial element that models the rotation of the rotor mass moment of inertia J_m, and a resistive element that models the linear friction coefficient B_m. These two elements are joined through an effort junction (1 junction).

The gearbox named Gear is represented by a transformer (TF) having its parameter equal to the reduction ratio of the gearbox, an inertial element, representing the equivalent high gear inertia J_g and a resistive element to model the viscous friction forces.

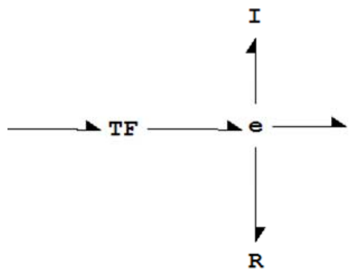

Figure 6. The Bond Graph model of Gear component

In order to model the component Pendulum it is necessary to write the equations of motion of the rotary inverted pendulum. Fig. 7 depicts the rotary inverted pendulum in motion. It is important to notice the direction in witch the arm is moving. Fig. 8 depicts the pendulum as a lump mass at half the length of the pendulum. The arm is displaced with a given α. Notice that the direction of θ is now in the x-direction of this illustration. We shall begin the derivation by examining the velocity of the pendulum centre of mass.

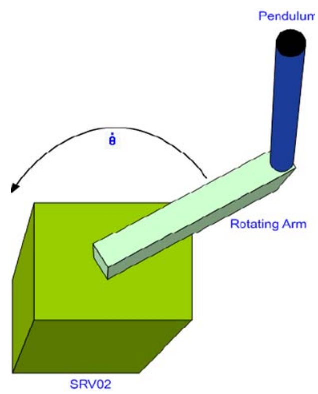

Figure 7. Top view of rotary inverted pendulum

As it can be seen in the Fig. 8 below, we notice that there are two components for the velocity of the pendulum lumped mass:

$$v_{pcm} = -L\cos\alpha(\dot{\alpha})\hat{x} - L\sin\alpha(\dot{\alpha})\hat{y} \quad (2)$$

We also know that the pendulum arm is moving with the rotating arm at a rate of:

$$v_{arm} = r\dot{\theta} \quad (3)$$

Using (2) and (3) the velocities for *x*-direction and *y*-direction are given by (4):

$$\begin{aligned} v_x &= r\dot{\theta} - L\cos\alpha(\dot{\alpha}) \\ v_y &= -L\sin\alpha(\dot{\alpha}) \end{aligned} \quad (4)$$

Equation (4) leaves us with the complete velocity of the pendulum.

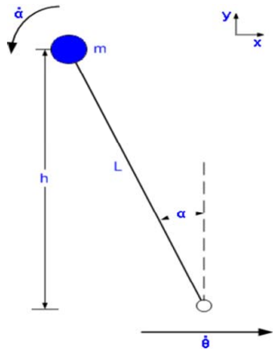

Figure 8. Side view with pendulum in motion

Using the relations presented above, the Bond Graph model of inverted pendulum has the following form given by Fig. 9. We use (2), (3) and (4) in order to model the dependencies between the velocities involved in the system $\dot{\theta}, v_{arm}$, $\dot{\alpha}$, $\dot{x}$ and $\dot{y}$.

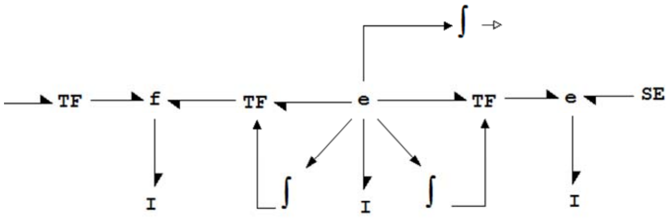

Figure 9. The Bond Graph model of Pendulum component

Thus the angular velocity with respect to rotational gear and the velocity of the pendulum arm are related using a transformer element, a modulated one, with the transformer modulus $k_1 = r$ (the pendulum arm length). Other two modulated transformer were used to relate the angular velocity of the pendulum, $\dot{\alpha}$, with the velocities on the *x*-direction, $\dot{x}$, and *y*-direction, $\dot{y}$. These elements are characterized by $k_2 = -L\cos(\alpha)$ and respectively $k_3 = -L\sin(\alpha)$. The Bond Graph model of the Pendulum component also contains an inertial element that models the moment of inertia of the pendulum and two inertial elements to model the pendulum mass over the *x* and *y*-directions. The weight force G was modeled using an effort source.

By joining together these three models, the DC motor component, the Gear component and the Pendulum component, we obtained the complete Bond Graph model of the Quanser rotary inverted pendulum system (Fig. 11).

The model of the Quanser rotary inverted pendulum system was built up and simulated using BondSim environment. Starting from this model, we shall assign the causality of elements and junctions in the Bond Graph model and we'll compute the mathematical model of the system in terms of state space equations. Before writing the constitutive equations it is desirable to name all the bonds in the graph, to assign to each bond a reference power direction and to assign causality to each bond. Causality is specified by means of the causal stroke, a short, perpendicular line made at one of the bond line ends indicating the direction in which the effort signal is directed.

It can be easily seen that we have three elements in integral causality which means that we have three state variables in terms of generalized momentums and generalized displacements (p_2, p_{10}, p_{17}).

The constitutive equations of the inertial elements in integral causality I_2, I_{10} and I_{17} are given by the following relations:

$$f_2 = \frac{1}{L_a}p_2,\ f_{10} = \frac{1}{J_g}p_{10},\ f_{17} = \frac{1}{J_p}p_{17} \quad (5)$$

For the inertial elements in derivative causality and R elements, the constitutive equations are as follows:

$$p_6 = J_m f_6,\ p_{14} = m_p f_{14},\ p_{20} = m_p f_{20} \quad (6)$$
$$e_3 = R_a f_3,\ e_7 = B_m f_7,\ e_{11} = B_g f_{11} \quad (7)$$

The effort junctions (J1) and the flow junctions (J0) are characterized by, the flows on all bonds equal to zero and the algebraic sum of the efforts equal to zero, respectively the efforts on all bonds equal to zero and the algebraic sum of the flows equal to zero. For TF and GY elements we have the following relations:

$$\text{(TF)} \begin{aligned} f_8 &= \frac{1}{n}f_9 \\ e_9 &= \frac{1}{n}e_8 \end{aligned},\quad \text{(GY)} \begin{aligned} e_4 &= k_t f_5 \\ e_5 &= k_t f_4 \end{aligned} \quad (8)$$

It can be seen that we have three more transformer elements, called modulated or controlled transformers, denoted MTF. These components satisfy the power conservation requirement. This is satisfied not only by the transformer modulus, but also by the ratios dependent on a control variable.

Thus, the modulated transformers are described by the following constitutive relations:

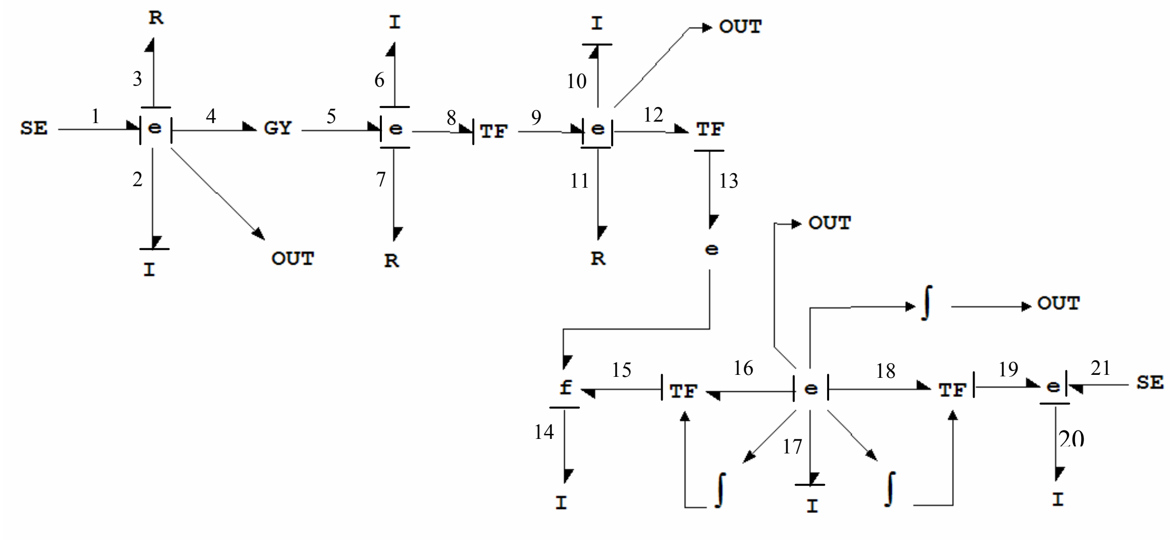

Figure 10. The complete Bond Graph model of the system

$$e_{12}=k_1e_{13},\quad e_{16}=k_2e_{15},\quad e_{18}=k_3e_{19}$$
$$f_{13}=k_1f_{12},\quad f_{15}=k_2f_{16},\quad f_{19}=k_3f_{18} \tag{9}$$

Combining all these equations and using the following notations:

$$J_e=n^2J_g+n^2k_1^2m_p+J_m,\ r_e=n^2B_g+R_a,$$
$$J_{ep}=(k_2^{\,2}+k_3^2)m_p+J_p \tag{10}$$
$$k_1=r,\ k_2=-L\cos(\alpha),\ k_3=-L\sin(\alpha),$$
$$k_4=k_1k_2m_p,\ k_5=J_eJ_{ep}-n^2k_4^2$$

we arrive at the state space equations in terms of energy variables.

$$\begin{cases}\dot{p}_2=-\dfrac{R_a}{L_a}p_2-\dfrac{k_t}{nJ_g}p_{10}+e_1\\[2mm] \dot{p}_{10}=\dfrac{nJ_gJ_{ep}k_t}{k_5L_a}p_2-\dfrac{J_{ep}r_e}{k_5}p_{10}-\dfrac{n^2k_3k_4J_gG}{k_5}\\[2mm] \dot{p}_{17}=-\dfrac{nk_tk_4J_p}{k_5L_a}p_2+\dfrac{J_pk_4r_e}{J_gk_5}p_{10}+\dfrac{J_pk_3G(k_4^2n^2+k_5)}{J_{ep}k_5}\end{cases} \tag{11}$$

Deriving the constitutive equations of inertial elements we will obtain the state space equations in terms of power variables. Taking into account the physical significance of the flow *f*:

$$f_2=i_a,\ f_{10}=\omega,\ f_{17}=\omega_l \tag{12}$$

we arrive at the final form of state space equations given by (13) where: i_a - the armature current, u_a - the armature voltage, ω - angular velocity of gear shaft, ω_p - angular velocity of the pendulum.

$$\begin{cases}\dfrac{di_a}{dt}=-\dfrac{R_a}{L_a}i_a-\dfrac{k_t}{nL_a}\omega+\dfrac{1}{L_a}u_a\\[2mm] \dfrac{d\omega}{dt}=\dfrac{nJ_{ep}k_t}{k_5}i_a-\dfrac{J_{ep}r_e}{k_5}\omega-\dfrac{n^2k_3k_4G}{k_5}\\[2mm] \dfrac{d\omega_p}{dt}=-\dfrac{nk_tk_4}{k_5}i_a+\dfrac{k_4r_e}{k_5}\omega+\dfrac{k_3G(k_4^2n^2+k_5)}{J_{ep}k_5}\end{cases} \tag{13}$$

IV. Simulation Results

Fig. 11-14 represent the variations of main model parameters.

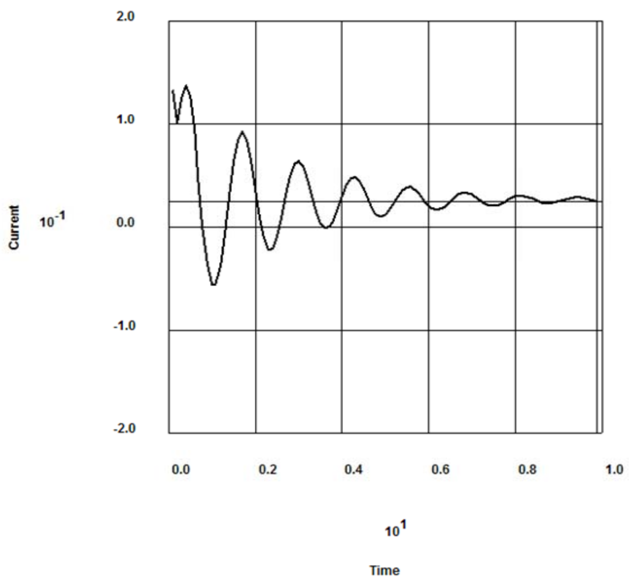

Figure 11. The current variation in the DC motor

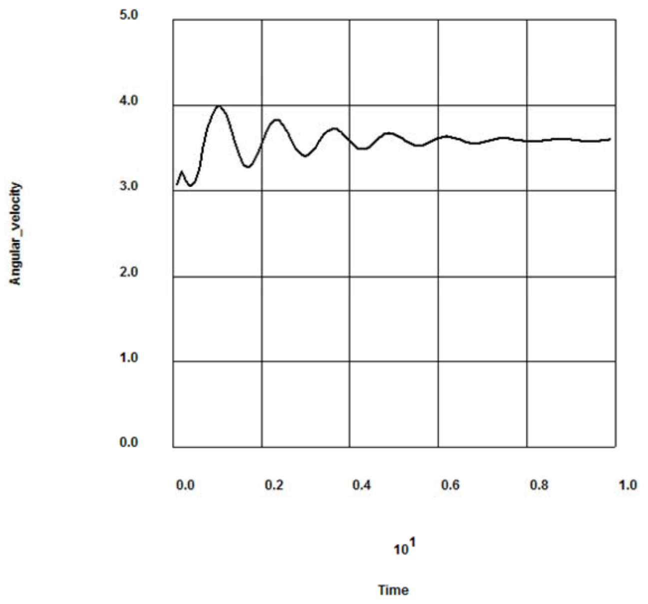

Figure 12. The gear shaft angular velocity as time function

For the simulation the following values were used: u_a=2V - motor input voltage, R_a=2.6 Ω - armature resistance, L_a=*0.18e-3H* - armature inductance, k_t=0.00767*Nm* motor torque constant, J_m=3.87e-7Kgm^2 motor inertia, $J_g = 2e-3\,Kg \cdot m^2$ equivalent high gear inertia, $B_g = 4e-3\,Nm/(rd/s)$ viscous damping coefficient, $r = 0.158m$ is the rotating arm length, $L = 0.32m$ the flexible pendulum arm length, $m_p = 0.231\,Kg$ pendulum arm mass.

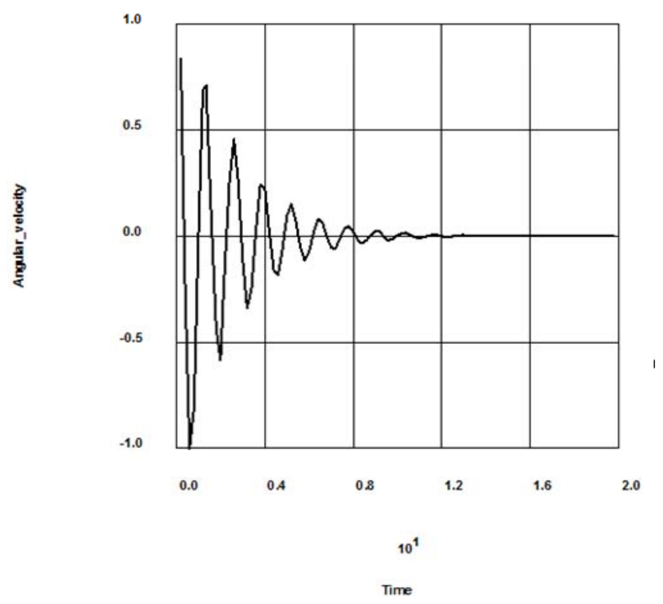

Figure 13. The angular velocity of the pendulum arm

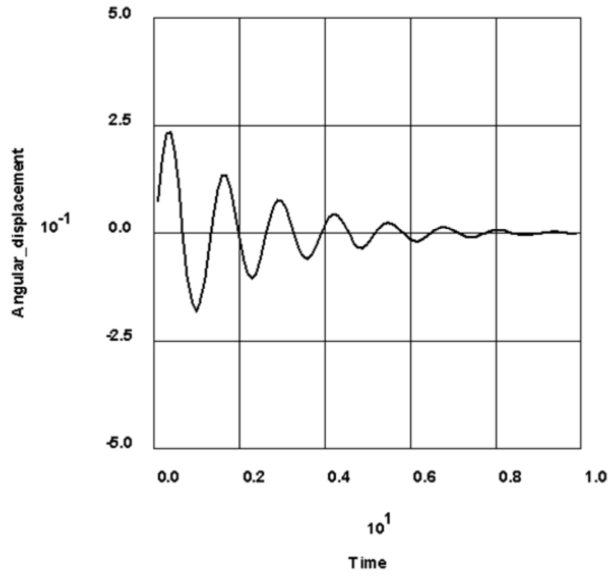

Figure 14. The angular displacement of the pendulum arm

Based on parameters evolution in the diagrams above we notice that the settling time is almost 10 s.

V. Concluding remarks

In this paper our interest was pointed towards the mathematical model used to predict certain aspects of the system response to the inputs.

The system model was built up using a uniform notation for all types of physical systems, the Bond Graph method, based on energy and information flow. Using this method, models of various systems belonging to different engineering domains can be express using a set of only nine elements. First we wrote a word Bond Graph containing words instead of standard symbols for the main components and bonds for power and signal exchange. The next step was to replace words by standards elements that contain precise mathematical or functional relations. The system was decomposed into three subsystems that were modelled separately. By joining together these three models, we obtained the complete Bond Graph model of the Quanser Rotary Inverted Pendulum system. The model was created and simulated using BondSim modelling environment.

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