

$$x^y = y^x$$

take natural log on both sides.

$$\Rightarrow y \ln(x) = x \ln(y)$$

$$\Rightarrow \ln(x)/x = \ln(y)/y$$

now differentiate $\ln(x)/x$ wrt x and compare it with zero.

$$\Rightarrow d/dx(\ln(x)/x)$$

$$\Rightarrow 1/x^2 - \ln(x)/x^2$$

$$\Rightarrow (1 - \ln(x))/x^2$$

for all real x , $x^2 \geq 0$

therefore : $(1 - \ln(x)) \geq 0$ for $x \leq e$ (~ 2.71)

$1 - \ln(x) < 0$ for $x > e$

So $\ln(x)/x$ is increasing in range $\leq e$, i.e. for integers, its increasing for 1,2 and decreasing else where.

Following are the cases :

Case 1 : $x < 3$ and $y < 3$

As in this range function $\ln(x)/x$ is increasing

so $f(x) > f(y)$, for $x > y$

$$\Rightarrow \ln(x)/x > \ln(y)/y$$

$$\Rightarrow x^y > y^x, \text{ for } x > y \text{ and } x < 3 \text{ and } y < 3$$

Case 2 : $x > 3$ and $y > 3$

As in this range function is decreasing, so

$f(x) < f(y)$, for $x > y$

$$\Rightarrow \ln(x)/x < \ln(y)/y$$

$$\Rightarrow x^y < y^x, \text{ for } x > y \text{ and } x > 3 \text{ and } y > 3$$

Case 3 : $x < 3$ and $y \geq 3$

Here $x = 1, 2$

Case 3.1 : $x = 1$

this is trivial case $1^n = 1$

so $x^y < y^x$

$\Rightarrow 1 < y$ case 3.2 : $x = 2$ here $x^y = 2^y$ and $y^x = y^2$ comparing the graph of exponential and parabola, following are results as $y \geq 3$,

$$2^3 < 3^2 \Rightarrow 8 < 9$$

$$2^4 = 4^2 \Rightarrow 16 = 16$$

$$2^5 > 5^2 \Rightarrow 32 > 25$$

$$2^6 > 6^2 \Rightarrow 64 > 36$$

$$2^7 > 7^2 \Rightarrow 128 > 49$$

$$2^8 > 8^2 \Rightarrow 256 > 64$$

..... and so on.

So, $x^y > y^x$, for $x = 2$, and $y > 3$

$x^y < y^x$, for $x=1,2$ and $y \geq 3$

Note :

I have taken natural, you can take log for any base, answer will remain same because :

$$d/dx(\ln x) = 1/x$$

$$\text{and } d/dx(\log(x)) = 1/x \cdot \log(b)$$

where b is base, for which log is taken.

Hope this will help.

If any this is incorrect, let me know.