Möbius Function

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* **An arithmetic function f(n)** : N → C is multiplicative if for any relatively prime n, m ∈ N: f(mn) = f(m)f(n).

The following are further examples of well-known multiplicative functions.

- $\mu(n)$, the Möbius function
- $e(n) = \delta_{1,n}$, the Dirichlet identity → $e(n) = 1 \to n = 1$; $e(n) = 0 \to n > 1$
- $I(n) = 1$ for all $n \in \mathbb{N}$
- $id(n) = n$ for all $n \in \mathbb{N}$

$$\mu(n) = \begin{cases} 1 & \to n = 1 \\ 0 & \to p^2 \mid n \\ (-1)^r & \to n = p_1 p_2 \ldots p_r \end{cases}$$

★ Möbius Function

→ $F(n) = \sum_{d \mid n} f(d)$

f(d)	μ	e	I	id
F(n)	e	I	τ	σ

→ Thm: $f(n) \to$ multiplicative $\iff F(n) = \sum_{d \mid n} f(d)$ is multiplicative

★ **Möbius Inversion Theorem:**

→ $F(n) = \sum_{d \mid n} f(d) \Rightarrow f(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right)$

→ Code to find Möbius func of 1 to N

```
void mobius()
{
        for (int i = 1; i <= N; i++)
        {
                if (i == 1)
                        mobi[i] = 1;
                else if (spf[i] == spf[i / spf[i]])
                        mobi[i] = 0;
                else
                        mobi[i] = -1 * mobi[i / spf[i]];
        }
}
```

→ spf: smallest prime factor

★ T.C. → O(N)

★ **General Steps to Solve:**

1. Let the problem be to find $G = \sum_{i=1}^{n} \sum_{j=i+1}^{n} h(gcd(i,j))$, here h(n) should be a multiplicative function.
   For example if the problem was to find $G = \sum_{i=1}^{n} \sum_{j=i+1}^{n} gcd(i,j)^3$, then the function h() will be $h(n) = n^3$.
2. Re-write the equation like this: $G = \sum_{g=1}^{n} h(g) * cnt[g]$ → ⊛
   → Where cnt[g] = number of pairs (i, j) such that gcd(i, j) = g, (1 <= i < j <= n).
3. Find the function f(n), such that $h(n) = \sum_{d \mid n} f(d)$. This can be done using mobius inversion formula and sieve.

— In case of an given array and to find:

	h(n)	f(n)
1) No of coprime pairs	e(n)	μ(n)
2) Sum of gcd for all pairs	id(n)	φ(n)

```
for (int i = 1; i <= N; i++) {
    for (int j = i; j <= N; j += i) {
        f[j] += h[i] * mobi[j/i];
    }
}
```

→ Once we have the values of f(n)

⊛ ⇒ $G = \sum_{g=1}^{n} \left(\sum_{d \mid g} f(d)\right) . cnt[g]$

d	cnt[1]	cnt[2]	.....	cnt[n]
1	f(1)	f(1)	....	f(1)
2		f(2)		
.				
n				

Horizontal Sum → ; Vertical sum ↓

⇒ $G = \sum_{d=1}^{n} f(d) . \left(\sum_{d \mid g} cnt[g]\right)$ → No of ordered tuples ∋ ∋ when property applied to the tuple we get a multiple of d.

Hence the summation represents

$cnt[g] \to$ No. of ordered tuples satisfying some property given in question.

★ **Typical Questions:**

$\sum \sum h(\text{property}) = G$

→ For eg → $\left(\sum_{d \mid g} cnt[g]\right)$ → where we want to find sum of all pairs of gcd is:
→ The no. of ordered pairs (i, j) ∋ d | gcd (a[i], a[j])