Burnside Lemma

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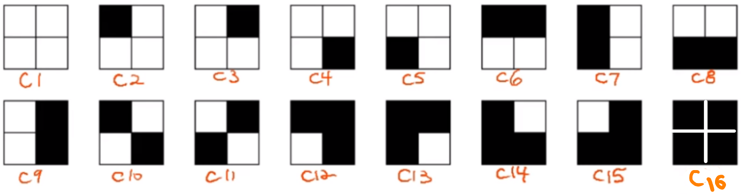
★ How many different ways (patterns) are there to color the squares of a 2x2 chessboard using only the black and white colors?



$G = \{e, a, b, c, p, q, r, s\}$

$a = \frac{\pi}{2}$, $b = \pi$, $c = \frac{3\pi}{2}$; $p \to$ Horizontal, $q \to$ Vertical, $r, s \to$ Diagonal

★ Definition :

① Orbit $\longrightarrow$ $x, y \in X$

$O_x = O_y \iff x \underset{G}{\sim} y$

★ $O_x = \{g.x \mid g \in G\}$

② Stabilizer $\longrightarrow$ Stabilizer of $x \in X$

$S_x = \{g \in G : g.x = x\}$

★ $S_{c_1} = \{e, a, b, c, p, q, r, s\} \longrightarrow$ example

Element of X	Orbit	Stabilizers
C1	{C1}	{e, a, b, c, p, q, r, s}
C2	{C2, C3, C4, C5}	{e, r}
C3		{e, s}
C4		{e, r}
C5		{e, s}
C6	{C6, C7, C8, C9}	{e, q}
C7		{e, p}
C8		{e, q}
C9		{e, p}
C10	{C10, C11}	{e, b, r, s}
C11		{e, b, r, s}
C12	{C12, C13, C14, C15}	{e, s}
C13		{e, r}
C14		{e, s}
C15		{e, r}
C16	{C16}	{e, a, b, c, p, q, r, s}

$\longrightarrow$ apply stabilizer to colouring to get the same colouring

★ Orbit – Stabilizer Theorem:

Let G be a group acting on set X:

$\forall x \in X$

$|O_x| . |S_x| = |G|$

$\Rightarrow \frac{|S_x|}{|G|} = |O_x|$

$\Rightarrow \frac{1}{|G|} \sum_{x \in O_x} |S_x| = \frac{|O_x|}{|O_x|} = 1$

$\Rightarrow$ No. of orbits / Colourings $= \frac{1}{|G|} \sum_{x \in X} |S_x|$

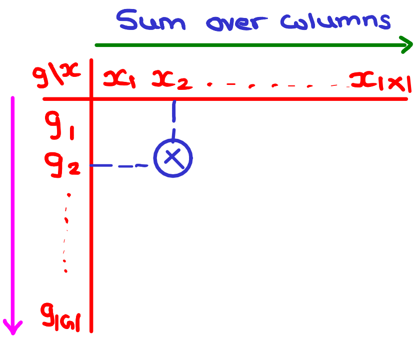
x	C_1	C_2 C_3 C_4 C_5	C_6 C_7 C_8 C_9	C_{10} C_{11}	C_{12} C_{13} C_{14} C_{15}	C_{16}
$\lvert S_x \rvert$	8	2	2	4	2	8

$\Rightarrow$ Colourings

$= \frac{1}{8}(8 \times 1 + 2 \times 4 + 2 \times 4 + 4 \times 2 + 2 \times 4 + 8 \times 1)$

$= \underline{\underline{6}}$

★ # of orbits $= \underbrace{\frac{1}{|G|} \sum_{x \in X} |S_x|}_{\text{Formula 1}} = \underbrace{\frac{1}{|G|} \sum_{g \in G} |Fix(g)|}_{\text{Formula}}$



Where $Fix(g) = \{x \in X \mid g.x = x\}$

eg $\longrightarrow |Fix(e)| = 16$

$|Fix(a)| = 2$

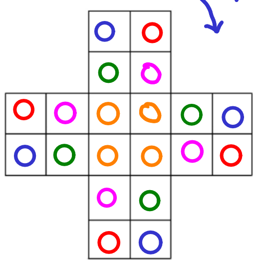
$|Fix(b)| = 4$

→ Poyla's enumeration theorem:

$$\# \text{ Orbits} = \frac{1}{|G|} \sum_{g \in G} k^{c(g)}$$

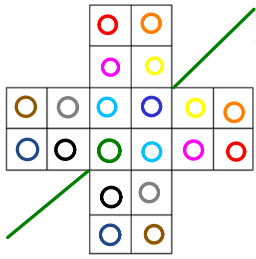
→ $k \rightarrow$ no of colours
$c(g) \rightarrow$ no of cycles due to symmetry

90°

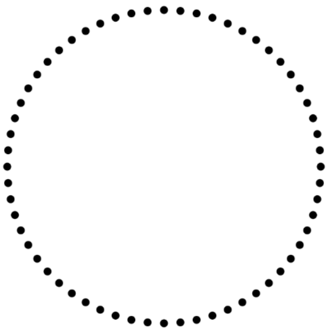


★ Cycles – 5

→ Reflect along Diagonal



★ Cycles – 11



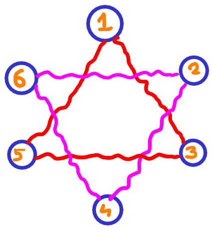
★ Necklace with m beads

$$G = \left\{ \frac{2\pi i}{m} \;\middle|\; i \in [0, m) \right\}$$

$$c(2\pi i/m) = \gcd(m, i)$$

→ k colours

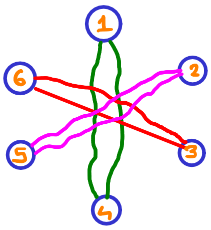
$$★ \quad \frac{1}{m} \sum_{i=0}^{m-1} k^{\gcd(i,m)} \rightarrow \text{No. of colourings}$$



→ 6 beads

$$c\left(\frac{2\pi}{6} \times 2\right) = \gcd(6,2) = 2$$

★ Cycles → (135)(246)



$$\rightarrow c\left(\frac{2\pi}{6} \times 3\right) = \gcd(6,3) = \underline{\underline{3}}$$

★ Cycles → (14)(25)(36)