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* Greedy Algorithm

① JOB SCHEDULING

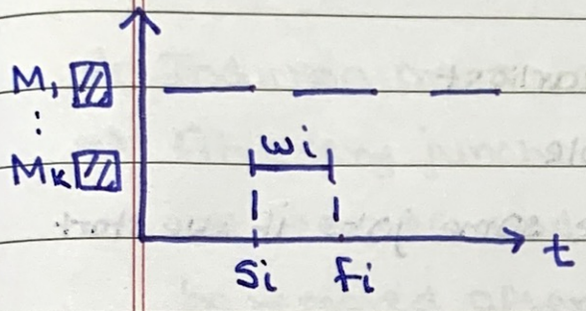
Homogeneous machines : same performance

Heterogeneous machines : different performance

* No preemption: Machine is occupied from the start time of the job till the job ends.

* Mutual Compatibility: No overlapping in jobs

→ Time interval is contiguous.



|S| = n

K : machines

wi : profit with ith job

* Optimal Solution:

① Cast the optimization problem as one in which we make a choice and are left with one subproblem to solve.

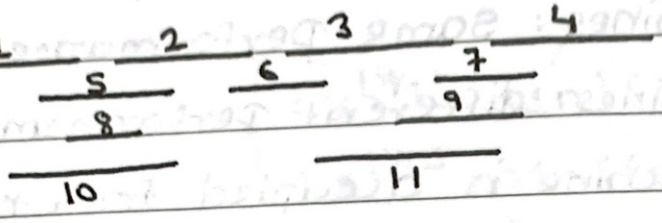
② Check whether it is feasible

③ Check the optimality:

By choosing any subset of S you cannot better the solution / profit

★ JOB SCHEDULING:

1 Machine, n Jobs (Finish max no. of jobs)



$S^1 = \{1, 2, 3, 4\}$: Optimal & feasible (Optimal solution)

$S^{11} = \{1, 3, 4\}$: Only feasible

$S^{111} = \{1, 2, 3, \ldots, 11\}$: Optimal but not feasible

S1: Select the job that starts the earliest

_____ : Counter example

Reason to pick: We might lose some jobs if we start late.

S2: Select the shortest job

_____ _____ : Counter example

Reason: Machine will have more free time

S3: Select the job with fewest conflicts:

Counter example: 1st dia on page 11 segment

Reason: Need to remove the fewest no. of jobs.

S4: Select the job with earliest finish time.

Reason: Machine could be freed earlier.

(1) Maintain two arrays

(one with jobs sorted in finish time order and second with the corresponding start times of array 1)

$f_1\ f_2 \ldots\ldots f_n$ (sorted)

$s_1\ s_2 \ldots\ldots s_n$

Greedy Strategy: Earliest finish time

★ Induction on no. of jobs in the output set

→ At every juncture of the algorithm, no compatible jobs get removed and every incompatible job is guaranteed to be removed after placing a job in output array.

Induction Basis:

First job with earliest finish time inserted in the set and pruning is done correctly.

f_1 ✓ f_2 ✗ f_3 ✗ f_4 ✓ f_5 f_6

s_1 s_2 s_3 s_4 s_5 s_6

Let f_1 be the job with earliest finish time and let $s_2, s_3 < f_1$, Hence we cant choose job 2 & job 3 in output set if $s_4 > f_1$ insert job 4 in output set. Note in case $s_i < f_1$ for $i > 4$ it is anyways going to be less than f_4.

Hence pruned correctly.

Induction Hypothesis:

$S' = \{j_1, j_2, \ldots, j_i, \ldots\}$

upto i^{th} job we have correctly choosen (followed greedy strategy)

Inductive Step

Next job would be choosen with earliest finish time

* The solution output by greedy algorithm is feasible

i.e. every two jobs in S' are compatible.

Let two jobs of S' not be compatible

say j_p and j_r

WLG let $p < r$ $\therefore f_p > s_r$

Consider q such that $p \leq q < r$

such that: $f_q \geq f_p > s_r$

But when we insert j_q in set S'

we must remove j_r. Hence contradiction

★ STAYS AHEAD ARGUMENT:

∃ a Optimal Solution # Naive : $\Omega(2^n)$: O

Greedy output : S'

Let n' be the number of jobs in S'. For every $1 <= i \leq n'$ i^{th} job chosen into S' stays ahead of i^{th} job in the optimal solution O when jobs in O are sorted according to their finish times.

Induction on no. of jobs selected in S'

Induction Base':

$f_1 \leq f_1'$ (# Greedy Strategy)

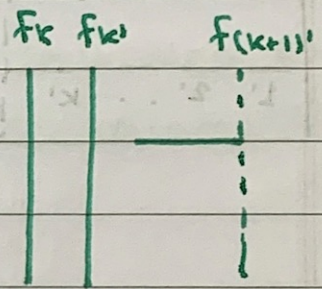
Induction Hypo:

$f_1 \leq f_1'$, $f_2 \leq f_2'$ . . . . , $f_k \leq f_k'$

Inductive Step:

TPT: $f_{k+1} \leq f_{(k+1)'}$

We can choose $(k+1)^{th}$ job such that $f_{k+1} \leq f_{(k+1)'}$



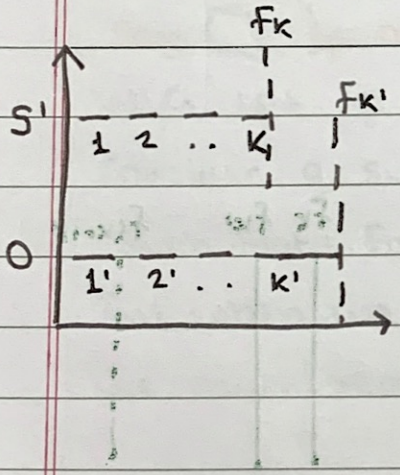
Hence by Induction S' stays ahead of any optimal solution O.

* Given that the solution output by greedy algo S' 'stays ahead' of any optimal solution O, then S' must be an optimal solution.

If $|S'| < |O|$ consider jobs with ids $|S'|+1, \ldots |O|$ in the sorted order of jobs in O, let it be O'

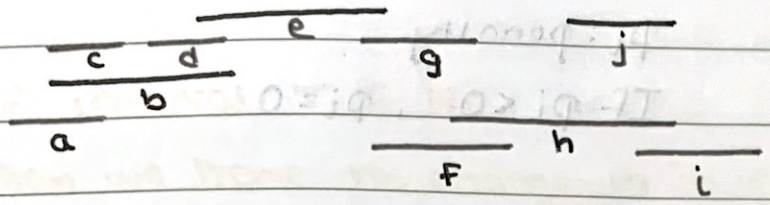
Every job in O' is compatible with every job in S' & hence available for greedy algo after it had choosen set S' of jobs.

* Note greedy algo stops only when there are no compatible jobs to the ones which are already chosen.



* Find Minimum no. of machines required to schedule all the tasks.

→ Duration of n tasks given



* Pool of machines : S

Assign a machine at the start of the task

If no machine among the pool is available push a new machine to the pool.

As soon as a task is completed the machine performing that task is free.

* Hence according to input instance any algo should use atleast the maximum no. of conflicting machines.

* MEETING THE LOWER BOUND:

Input instance says any algo should use this much amount of minimum resources and our algo matchs it.

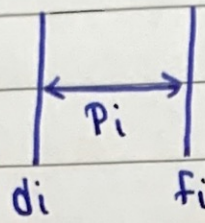
→ Algo: Push all start times & finish times in a vector and sort according to their values & maintain counter for no. of overlaps.

O(nlogn)

★ Deadlines

Minimise the maximum penalty

Each task given length & deadline



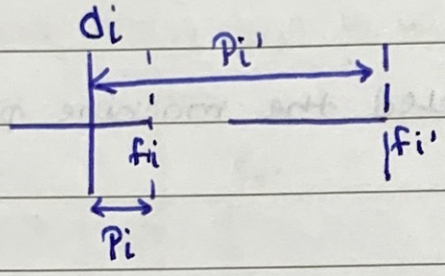
p_i : penalty

If $p_i < 0$, $p_i = 0$

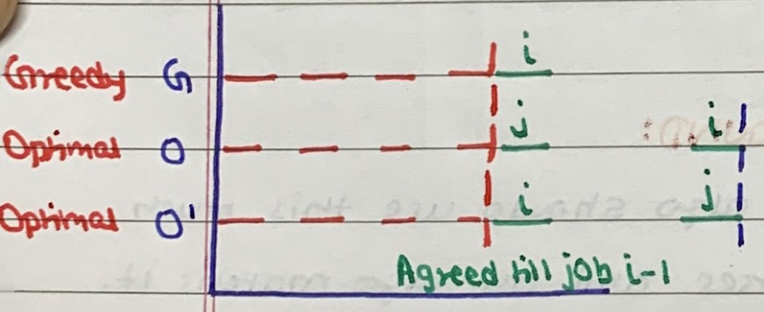
★ GREEDY STRATEGY:

Earliest deadline first

Reason: More free time to the machine



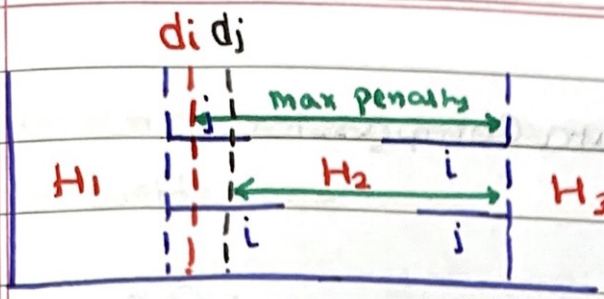
★ EXCHANGE ARGUMENT:



O' agrees with G till i^{th} job.

$d_j > d_i$ (# Greedy Strategy)

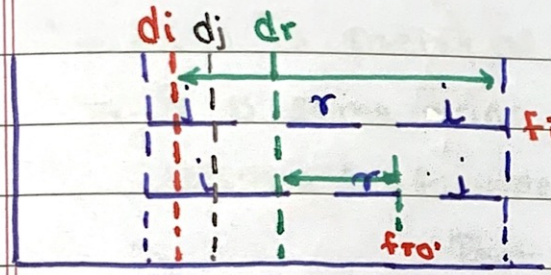
★ Optimal solution exists can be found in O(n!)



∴ max penalty (O') ≤ max penalty (O)

* If interval r ∈ H_1
then we have no change in its penalty
or so for r ∈ H_3

* Let r ∈ H_2



$d_r > d_i$
(# O' follows Greedy till i)

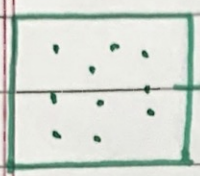
and $f_{io} > f_{ro'}$

∴ max penalty (O') ≤ max penalty (O)

* Hence we keep on making optimal solution more coherent with the greedy solution hence our Greedy solution is optimal.

★ Huffman Codes and Data Compression.

$S = \{a, b, \ldots, z\}$

(Text) T →  → Symbols from S

★ Also freq of each symbol is given

a b c .... z

f_a f_b f_c .... f_z

γ : Code

$\gamma(\alpha)$: Codeword

★ Fixed length code

$> 2^6$ $< 2^7$ symbols

7 bits for each symbol

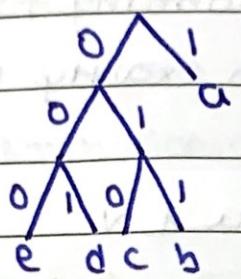
★ Avg bits for symbol = $\sum_{x \in S} f_x |\gamma(x)|$

★ Prefix codes

No code is prefix of any other code.

a : 11 b : 01 c : 001 d : 10 e : 000

* Given a binary tree + leaves associated with symbols
→ prefix code



a : 1 $f_a = 0.32$

b : 011 $f_b = 0.25$

c : 010 $f_c = 0.20$

d : 001 $f_d = 0.18$

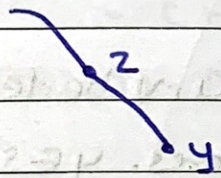
e : 000 $f_e = 0.05$

→ Proof by contradiction

Let $\gamma(z)$ be prefix of $\gamma(y)$ where $z \neq y$

$\therefore \gamma(z)$ is prefix of $\gamma(y)$

$\Rightarrow$ z is some node in the path from root to the leaf node y



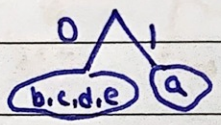
But z must be a leaf node

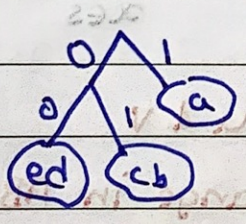
$\therefore z = y$

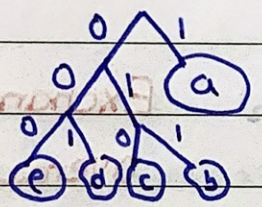
to $z \neq y$

* Prefix code given can construct the tree

a : 1 b : 011 c : 010 d : 001 e : 000







* Binary Tree corresponding to optimal prefix code is full.

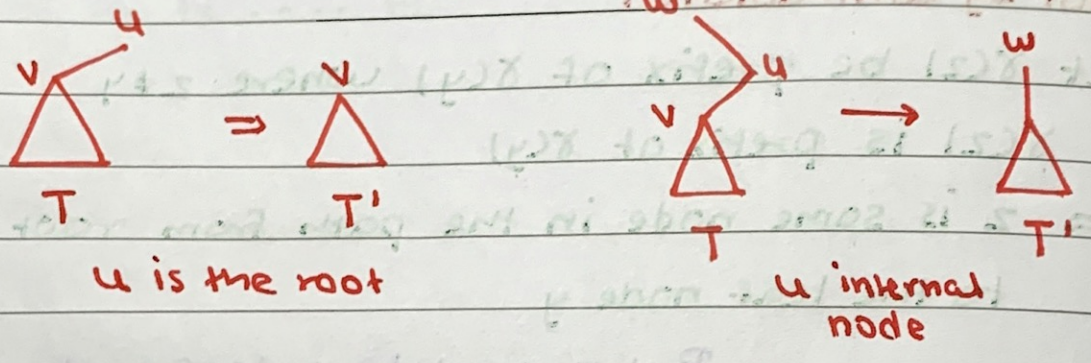
Proof by Exchange Argument

Let T be the binary tree corresponding to optimal prefix code and ∃ a node u with exactly one child v.

Now convert T → T' by replacing u & v

∴ T' has smaller avg no. of bits.

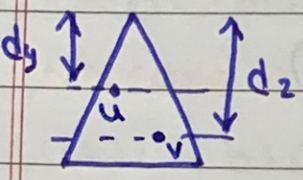
contradicts optimality of T



* γ^* optimal corresponds to T^* & u, v be leaves ∃ depth(u) < depth(v) If u corres. $y \in S$ & v corres. to $z \in S$ then $f_y \geq f_z$

Proof by Exchange Argument

Avg Bits per Symbol (ABS) = $\sum_{x \in S} f_x \, \text{depth}(x)$



Exchange u & v

Overall change in ABS

i.e. $\Delta ABS = (\text{depth}(v) - \text{depth}(u))(f_y - f_z)$

If $f_z > f_y$ ∴ ΔABS -ve # to optimality of T^*.

★ There is an optimal prefix code comes to T^* where two lowest freq letters are assigned to leaves that are siblings in T^*

→ The choice of assignment among leaves at same depth doesn't affect the avg no. of bits per symbol.

★ Algo for Optimal prefix code:

Let y^* & z^* be having two lowest freq in S

parent w^* : meta letter with freq

$$f_{w^*} = f_{y^*} + f_{z^*}$$

Replace y^* & z^* with this w^*

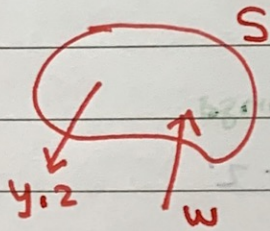
$$ABS(T') = C + f_{w^*} d_{T'}(w^*)$$

$$ABS(T) = C + f_{y^*} d_T(y^*) + f_{z^*} d_T(z^*)$$

$$= C + f_{w^*}(d_{T'}(w^*) + 1)$$

$$\therefore ABS(T) - ABS(T') = f_{y^*} + f_{z^*} = f_{w^*}$$

★ Huffman's prefix code



$$S' = S - \{y, z\} \cup \{w\}$$

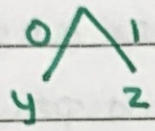
★ Binary tree # at every step two symbols removed & hence every node two children (internal node)

* Optimality

→ Induction on no. of symbols

Induction Basis

For 2 symbols



Optimal # only one bit per symbol

Induction Hypo.

$|S| = k-1$ (Optimal)

T.P.T $|S| = k$ also optimal

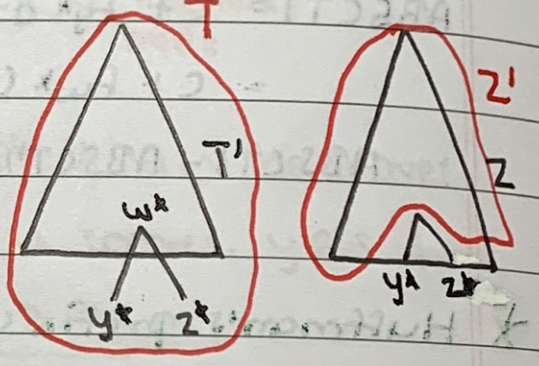
Induction step

Let S' be set of symbols where $\{y^*, z^*\}$ replaced by w^* which is optimal ⬭ I.H.

Let the tree be T'

attach leaves y^* & z^* to w^*

$ABL(T') = ABL(T) - f_w$



Now let T not be optimal

$\therefore \exists\ Z \ni ABS(Z) < ABS(T)$

there is a tree Z $\ni$ y^*, z^* are siblings

If we delete y^*, z^* from Z to get Z'

$\therefore ABS(Z') = ABS(Z) - f_w$

$\therefore ABS(Z) < ABS(T)$

$\Rightarrow ABS(Z') < ABS(T')$ # to optimality of T'

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* Heap with n symbols
extract two with min freq
& insert their sum
Build O(n)
Extract O(nlog n)
Insert O(nlog n)

* ∴ T.C. O(nlogn)

* Greedy Choice Property:
Many choices choose the one which looks best at the moment.

* Optimal Substructure Property
Optimal Solution to problem P within it contains optimal substructure (solutions) to subproblems of P.