⑥ Fast Fourier Transform (FFT)

$$A(x) = \sum_{i=0}^{n-1} a_i x^i \quad , \quad B(x) = \sum_{i=0}^{n-1} b_i x^i$$

$$C(x) = A(x) \cdot B(x)$$

$$= c_0 + c_1 x + \dots + c_{2n-2} x^{2n-2}$$

	b_0	b_1	b_2	$\dots$	b_{n-1}
	c_0	c_1			c_{n-1}
a_0	$\cancel{a_0 b_0}$	$\cancel{a_0 b_1}$			
a_1	$\cancel{a_1 b_0}$				
$\vdots$					
a_{n-1}					

★ Naive : $O(n^2)$

★ Polynomial can be represented as coefficient representation.

Vector of coefficients

★ Set of points representation

$$A(x) = \{(x_0, y_0), (x_1, y_1), \dots (x_{n-1}, y_{n-1})\}$$

$$y_k = A(x_k)$$

★ Horner's Rule :

$$A(x) = a_0 + \dots x(a_{n-3} + x(a_{n-2} + x a_{n-1}))$$

↓

Takes $O(n)$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & \cdots & \cdots & \cdots \\ \vdots \\ 1 & x_{n-1} & \cdots & \cdots & x_{n-1}^{n-1} \end{pmatrix} = X \qquad \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = A$$

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = Y$$

* $XA = Y$

$\therefore A = X^{-1}Y$

* Deg $(C(x))$ is $2n-2$

Hence atleast $2n-1$ points required to find its coefficients.

→ Step 1:

$x_0, x_1, \ldots, x_{2n-2}$

find $A(x)$ & $B(x)$ for each x_i : $O(n^2)$

→ Step 2:

$C(x_k) = A(x_k) \cdot B(x_k)$: $O(n)$ ($\because 0 \le k \le 2n-2$)

→ Step 3:

Find c.e. of $C(x)$

$C = X^{-1}Y$: $\Omega(n^2)$

★ $A(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$

$A_{even}(x) = (a_0 + a_2 x + a_4 x^2 + a_{n-2} x^{n/2-1})$

$A_{odd}(x) = (a_1 + a_3 x + a_5 x^2 + \dots + a_{n-1} x^{n/2-1})$

$\therefore A(x) = A_{even}(x^2) + x A_{odd}(x^2)$

★ Note Here we have to evaluate $A(x)$ of deg $n-1$ at $2n$ points Let the time required to do so be $T(n)$

→ Twiddle Factors:

$w_{j,r} = e^{2\pi j/r \cdot i}$

$= (\cos 2\pi j/r, \sin 2\pi j/r)$

→ Let the required $2n$ points be

$w_{0,2n}, w_{1,2n}, \dots, w_{2n-1,2n}$

★ $A(w_{j,2n}) = A_{even}(w_{j,2n}^2) + w_{j,2n} A_{odd}(w_{j,2n}^2)$

$w_{j+n,2n} = e^{2\pi(j+n)/2n \cdot i}$

$= e^{2\pi j/2n \cdot i} \cdot e^{i\pi}$

$= -w_{j,2n}$

$\therefore w_{j+n,2n}^2 = w_{j,2n}^2$ ★

$A(j) = A_{even}(j^2) + j\,A_{odd}(j^2)$

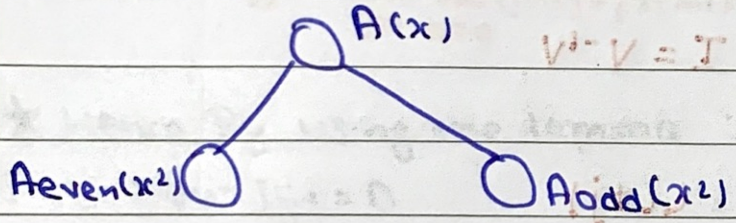
$A(-j) = A_{even}(j^2) - j\,A_{odd}(j^2)$

$A(\omega_{j,2n}) = A_{even}(\omega_{j,2n}^2) + \omega_{j,2n} A_{odd}(\omega_{j,2n}^2) \quad \ldots (1)$

(#★) $A(\omega_{j+n,2n}) = A_{even}(\omega_{j,2n}^2) - \omega_{j,2n} A_{odd}(\omega_{j,2n}^2) \quad \ldots (2)$

$$A(\omega_{j,2n}) \begin{cases} j = 0, 1, \ldots, n-1 & \text{Apply (1)} \\ j = n, \ldots \ldots 2n-1 & \text{Apply (2)} \end{cases}$$

★ So to calculate $A(x)$ for $2n$ values we effectively need to calculate $A_{even}(x^2)$ & $A_{odd}(x^2)$ for only n values



$\therefore T(n) = 2T(n/2) + \underbrace{O(n)}_{\text{Divide}} + \underbrace{O(n)}_{\text{Combine}}$

$\therefore T(n) = O(n \log n)$ for step 1

★ DFT: Discrete Fourier Transform

Represent a polynomial using Twiddle Factors

★ Inverse FFT

$C(x) = A(x) \cdot B(x)$

$\therefore C(\omega_{j,2n}) = \sum_{i=0}^{2n-1} c_i (\omega_{j,2n})^i$

$$\begin{bmatrix} 1 & \omega_{0,2n} & \omega_{0,2n}^2 & \cdots & \omega_{0,2n}^{2n-1} \\ 1 & & & & \vdots \\ \vdots & & & & \vdots \\ 1 & \omega_{2n-1,2n} & \cdots & \cdots & \omega_{2n-1,2n}^{2n-1} \end{bmatrix} = V$$

(Vandermonde' matrix)

★ $V_{jk} = \omega_{j,2n}^k = \omega_{jk,2n}$

$V_{jk}^{-1} = \frac{-\omega_{jk,2n}}{2n}$ } Claim

★ We know $VV^{-1} = I = V^{-1}V$

i.e. $[VV^{-1}]_{ii} = 1$

$[VV^{-1}]_{ij} = 0 \quad (i \neq j)$

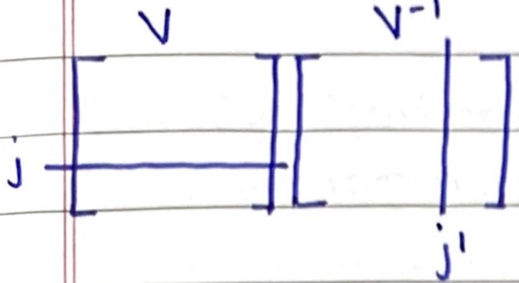
★ Lemma: For $m \geq 1$ and $k \in \mathbb{Z}^+$ which is not a multiple of m, $\sum_{j=0}^{m-1} (\omega_{k,m})^j = 0$

Proof: $\sum_{j=0}^{m-1} (\omega_{k,m})^j = \frac{(\omega_{k,m})^m - 1}{\omega_{k,m} - 1}$

$\because$ k is not a multiple of m $\therefore$ denominator is not zero Hence $(\omega_{k,m})^m - 1 = 0 \Rightarrow$ Given sum $= 0$

↓

mth root of unity



$$[VV^{-1}]_{jj'} = \sum_{k=0}^{2n-1} V_{jk} \cdot V^{-1}_{k,j'}$$

$$= \sum_{k=0}^{2n-1} \omega_{jk,2n} \cdot \frac{-\omega_{j'k,2n}}{2n}$$

$$= \frac{1}{2n} \sum_{k=0}^{2n-1} (\omega_{k,2n})^{j-j'}$$

$\therefore$ If $j=j'$ Then $[VV^{-1}]_{jj'} = 1$

If $j \neq j'$

$$[VV^{-1}]_{jj'} = \frac{1}{2n} \sum_{k=0}^{2n-1} (\omega_{(j-j'),2n})^{k}$$

★ Hence By using the lemma $\because 2n \nmid j-j'$

$\therefore [VV^{-1}]_{jj'} = 0$

$\therefore V^{-1}_{jk} = \frac{-\omega_{jk,2n}}{2n}$

★ $y_k = \sum_{j=0}^{2n-1} a_j \cdot \omega_{kj,2n}$

★ $c_k = \sum_{j=0}^{2n-1} y_j \cdot -\omega_{kj,2n} / 2n$ $\Big\}$ Can be calculated using FFT (inverted version)

in $O(n\log n)$