MSc in CSTE
Computational Methods

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Hand in date: 21/11/2022 (FT), 05/12/2022 (PT), 09:30am

1 Introduction

In this assignment you are asked to examine the application of numerical schemes for the solution of partial differential equations as discussed in the Computational Methods lectures, using C++ Object Oriented and functional programming practices discussed in the C++ lectures. In order to do this, we will study the following problem.

Consider the first order wave equation:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

where u, the speed of sound, is $250m/sec$. Assume that a disturbance is introduced in a one-dimensional long tube of length $L = 400m$ with both ends closed. The imposed boundary conditions are:

$$\begin{aligned} x &= 0 & f(0,t) &= 0 \\ x &= L & f(L,t) &= 0 \end{aligned}$$

Assume that at time $t = 0$, a disturbance of half sinusoidal shape has been generated. The initial condition is shown in Figure 1 and specified as:

$$\begin{aligned} f(x,0) &= 0 & 0 \le x \le 50 \\ f(x,0) &= 100\left\{\sin\left[\pi\left(\frac{x-50}{60}\right)\right]\right\} & 50 \le x \le 110 \\ f(x,0) &= 0 & 110 \le x \le L \end{aligned}$$

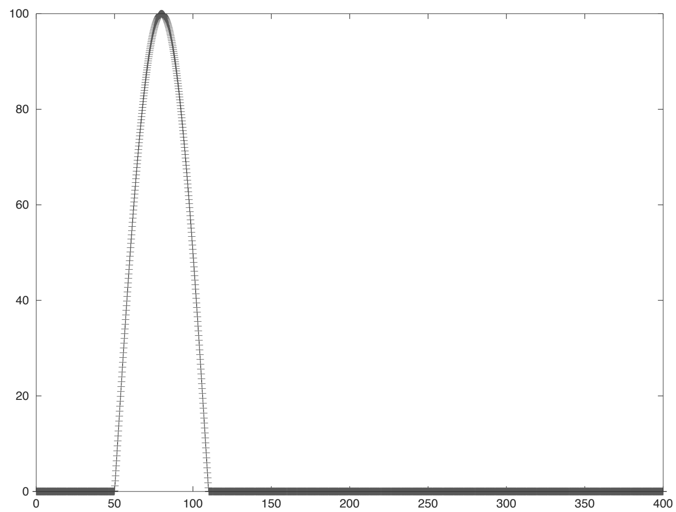

Figure 1: Initial condition at t=0 sec

2 Tasks

1. Write a C++ program which solves the above problem on a uniform grid with the prescribed initial and boundary conditions using the following methods:
   - Explicit Upwind FTBS (Forward time, Backward space)
   - Implicit Upwind FTBS (Forward time, Backward space)
   - Lax-Wendroff
   - Implicit FTCS (Forward time, Central space)
2. The analytical solution of this problem, subject to the imposed initial and boundary conditions, is

$$
\begin{aligned}
f(x,t) &= 0 && 0 \le x \le 50 + 250t \\
f(x,t) &= 100\left\{\sin\left[\pi\left(\frac{x-50-250t}{60}\right)\right]\right\} && 50+250t \le x \le 110+250t \\
f(x,t) &= 0 && 110+250t \le x \le L
\end{aligned}
$$

   Use the analytical solution to compare/validate the results of the above numerical methods. Comparisons should be both *qualitative* and *quantitative.*

   In all cases the solution is to be printed and plotted for all x locations every $0.1sec$ time intervals from $0.0sec$ to $0.5sec$.

3. Investigate the effect of step size on the accuracy of the solution and required computation time using the following three scenarios:

   (a) $\Delta x = 5.0m$ and $\Delta t = 0.02sec$

   (b) $\Delta x = 5.0m$ and $\Delta t = 0.01sec$

   (c) $\Delta x = 5.0m$ and $\Delta t = 0.005sec$

4. Explain the behaviour of the solutions of the above numerical methods in terms of the expected properties of the numerical methods involved.

5. In particular for the implicit FTCS method, study the accuracy and stability properties in depth and include your detailed mathematical calculations in the Appendix of your report.

Think about the design of your solution before jumping into coding. Remember there is no right or wrong answer when it comes to the design but there are better and worse designs. In terms of software, you are aiming for a clean and effective solution to the problem.

3 Source Code and Report Requirements

The source code must compile on the IT lab PCs using Visual Studio or Intel/GNU compilers on Linux, without any other external dependencies/libraries/source codes of third parties.

Write a report to present and discuss your findings. The report should be no less than 2,000 words and must not exceed 4,000 words. The report can contain any number of figures/tables, however all figures/tables should be numbered and discussed. The report should include a description of the design of your solution. The source code should be included as an Appendix to the report. The report should finally contain an Appendix, called Individual Contributions, where each group member will present their opinion of the contribution of every group member towards this assessment.

4 Assignment Submission

The source code should be submitted electronically via the **Technical Work submission point** by 9:30am on the 21^{st} November (full-time students) or the 5^{th} December (part-time students).

The report should be submitted electronically via the **TurnItInUK submission point** by the prescribed deadline, for the assignment submission to be considered complete.

This is a group assessment, only one submission per group is required.

5 Marking

The assignment will be assessed based on the following marking scheme:

- 20% Introduction, methodology, conclusions
- 40% Source code
- 30% Analysis of the results and derivation of theoretical properties
- 10% Report structure, presentation, references

6 References

1. K.A. Hoffmann and S.T. Chiang, ‘Computational Fluid Dynamics’, Fourth Edition, Vol. I, Engineering Education System Books, pp. 486, 2000.

2. S. Scott Collis, ‘An Introduction to Numerical Analysis for Computational Fluid Dynamics’, Technical Report SAND2005-2745, Sandia National Laboratories, 2005.

Links to the following two documents can be found on the C++ Canvas page under external links:

3. Pras Pathmanathan, ‘Numerical Methods and Object-oriented Design’.

4. Dr O Gloth, ‘Object Oriented Techniques and Numerics’.