Baysian Inference using Markov Chain Monte Carlo for Siesmic Data set

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Abstract

In this project, a Metropolis hasting MCMC is performed to sample the parameters from an unknown distribution of a static model used in modeling the relative plate motion in central California. Parameters such as fault slip rate, locking depth, and the fault location are found using the Bayesian inference technique. 200 Observational data, such as cross-fault location and the measured strike-slip displacement, are used to carry on the Bayesian inference via MCMC. The posterior distribution of the parameters of the model is found assuming a uniform prior. Prediction of the slip-distance velocity is made by randomly sampling the parameters from the posterior distribution and evaluating at these points. 95% credible and prediction interval for prediction is also plotted.

1. Siesmic Model

$$v(x;D,x_0,v_0) = \frac{v_0}{\pi}\arctan\left(\frac{x-x_0}{D}\right) \tag{1}$$

where

- v is the strike-slip displacement §,
- x is the cross fault coordinate,
- D is the locking depth,
- x_0 is the location of the fault
- v_0 is the fault slip rate.

The parameters of the model are sampled using bayesian inference, with the help of observational data. We assume that the error in the observation is normally distributed, and hence we write the observational displacement in terms of the model as

$$y_i = v_i(x_i;D,x_0,v_0) + \varepsilon_i \tag{2}$$

$$\varepsilon_i \sim \mathcal{N}(0,1) \tag{3}$$

2. Derivation of Posterior Distribution

We know that the strike-slip displacement observation is related to the model as depicted by equation 3. From this we can calculate the likelihood function calculated by

$$\pi(y_1,\ldots,y_n \mid x_0,v_0,D) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}} \times \exp\left(-\frac{1}{2\sigma^2}\left(y_i - \frac{v_0}{\pi}\arctan\left(\frac{x_i-x_0}{D}\right)\right)^2\right) \tag{4}$$

Therefore, the posterior is given by

$$P(x_0,v_0,D \mid y_1,\ldots,y_n) = \frac{\pi(y_1,\ldots,y_n \mid x_0,v_0,D)\cdot P(x_0,v_0,D)}{\int_{x_0,v_0,D} \pi(y_1,\ldots,y_n \mid x_0,v_0,D)\cdot P(x_0,v_0,D)\,dx_0\,dv_0\,dD} \tag{5}$$

This is the distribution through which we sample the parameters x_0, v_0, D, instead of the frequentist approach that has fixed values for these parameters based on the observational data. This ensures the predicted slip-displacement is much more robust when the data is limited.

Assumptions made

- observational error is normally distributed across the data points
- Error is independent of the parameters x_0, v_0, D chosen;
- The prior is uniform and the parameters x_0, v_0, D are independent of each other.

3. Sampling from the posterior via Metropolis-Hastings Markov Chain Monte Carlo

Equation 5 is pretty much difficult to evaluate due to a high-dimensional integral in the denominator. Therefore, instead of trying to assess the distribution, it is enough to sample the parameters from the distribution, and hence, an efficient sampling technique likethe MCMC algorithm will be used to sample the parameters from the seemingly complicated distribution.

3.1. *Metropolis hastings*

Metropolis-Hastings is an MCMC algorithm used to sample from the posterior distribution. At each step, it proposes a new sample using a proposal distribution $q(x' \mid x)$, and accepts it with probability α.

Algorithm 1: Metropolis-Hastings Algorithm

1: **Input:** Target density $\pi(X)$ (up to normalization), proposal distribution $q(X' \mid X)$, number of samples T
2: **Initialize:** Choose initial state X_0
3: **for** $t = 1$ to T **do**
4: Sample $X' \sim q(X' \mid X_{t-1})$ {Propose a new candidate}
5: Compute acceptance ratio:

$$\alpha = \min\left(1, \frac{\pi(X') \cdot q(X_{t-1} \mid X')}{\pi(X_{t-1}) \cdot q(X' \mid X_{t-1})}\right)$$

6: Sample $u \sim \text{Uniform}(0,1)$
7: **if** $u < \alpha$ **then**
8: Accept: $X_t = X'$
9: **else**
10: Reject: $X_t = X_{t-1}$
11: **end if**
12: **end for**
13: **Return:** Samples $\{X_1, X_2, \ldots, X_T\}$ =0

3.2. Random Walk Metropolis

If the assumption for the proposal distribution is made to be Gaussian normal, then this class of Metropolis-Hastings is called Random Walk Metropolis. In this project, a Random Walk Metropolis is used to sample from the posterior distribution. Therefore, the algorithm 1 simplifies to the algorithm shown in 2

Algorithm 2: Random Walk Metropolis Algorithm

1: **Input:** Target density $\pi(X)$ (unnormalized), number of samples T
2: **Initialize:** Choose initial state X_0
3: **for** $t = 1$ to T **do**
4: Sample $X' \sim \mathscr{N}(X_{t-1}, \Sigma)$ {Propose a new candidate}
5: Compute acceptance ratio: $\alpha = \min\left(1, \frac{\pi(X')}{\pi(X_{t-1})}\right)$
6: Sample $u \sim \mathscr{U}(0,1)$
7: **if** $u < \alpha$ **then**
8: Accept: $x_t = X'$
9: **else**
10: Reject: $X_t = X_{t-1}$
11: **end if**
12: **end for**
13: **Return:** Samples $\{X_1, X_2, \ldots, X_T\}$ =0

Few Observations from the Random Walk Metropolis Algorithm

- The proposal distribution is a Gaussian normal, by construction it is a symmetric distribution, i.e, $q(X_{t-1} \mid X') = q(x' \mid X_{t-1})$. This explains the simplification of acceptance ratio in step 5 of 2
- The variance of the proposal distribution can be found by maximizing the likelihood function given in 4 with the variance. However, this is not essential,and a simple diagonal co-variance assumption can be made
- The initial guess X_0 is also made by maximizing the likelihood with respect to the parameters of the model. This is essential as it can reduce the amount of burnout period.
- A note to be made is that X's in the algorithm refers to the multi-dimensional random variable corresponding to the three parameters x_0, v_0, D
- Target density $\pi(X)$ refers to 5

4. Results

In this section, detailed results of the outcome of RWC is explained. The initial state and co-variance matrix are assigned based on the likelihood function. A total of 10,000 iterations are being carried, out of which first few around 1000 are being rejected

References